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FOCUSING OF SPHERICAL NONLINEAR PULSES IN $\boldsymbol{R}^{1+3}$, III. SUB AND SUPERCRITICAL CASES

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Abstract. We study the validity of geometric optics in L^∞ for nonlinear wave equations in three space dimensions whose solutions, pulse like, focus at a point. If the amplitude of the initial data is subcritical, then no nonlinear effect occurs at leading order. If the amplitude of the initial data is sufficiently big, then strong nonlinear effects occur; we study the cases where the equation is either dissipative or accretive. When the equation is dissipative, pulses are absorbed before reaching the focal point. When the equation is accretive, the family of pulses becomes unbounded.

1. Introduction. This paper is the last of a series of three, after [2] and [4]. In these three papers, we consider the asymptotic behavior as $\varepsilon \to 0$ of solutions of the initial value problem

$$\begin{cases} \Box \mathbf{u}^\varepsilon + a|\partial_t \mathbf{u}^\varepsilon|^{p-1}\partial_t \mathbf{u}^\varepsilon \;=\; 0\,, & (t,x) \in [0,T] \times \boldsymbol{R}^3\,, \\ \mathbf{u}^\varepsilon\big|_{t=0} \;=\; \varepsilon^{J+1} U_0\Big(r\,, \dfrac{r-r_0}{\varepsilon}\Big)\,, & \\ \partial_t \mathbf{u}^\varepsilon\big|_{t=0} \;=\; \varepsilon^{J} U_1\Big(r\,, \dfrac{r-r_0}{\varepsilon}\Big)\,, & \end{cases} \tag{1.1}$$

where $\Box := \partial_t^2 - \Delta_x$, a is a complex number, $r = |x|$, $r_0 > 0$, and $1 < p < \infty$. The functions U_0 and U_1 are real-valued, infinitely differentiable, bounded, and there is a $z_0 > 0$ such that for all $r \geq 0$,

$$\operatorname{supp} U_j(r,\cdot) \subset [-z_0, z_0]\,. \tag{1.2}$$

The last assumption implies that at time $t = 0$ the solutions are families of spherical pulses supported in a $O(\varepsilon)$ neighborhood of $r = r_0$. The initial data are spherically symmetric, thus in the limit $\varepsilon \to 0$, a caustic is formed, reduced to the focal point $(t,x) = (r_0, 0)$. Before going further into details, we rescale our parameters as in [4]. Introduce $\varepsilon^{-J}\mathbf{u}^\varepsilon =: u^\varepsilon$ instead of $\mathbf{u}^\varepsilon$ so that the solutions have derivatives of order $O(1)$ away from the caustic. Define $\alpha := (p-1)J$. The initial value problem (1.1) is transformed to

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$$\begin{cases} \Box u^\varepsilon + a\varepsilon^\alpha |\partial_t u^\varepsilon|^{p-1}\,\partial_t u^\varepsilon = 0\,, & (t,x) \in [0,T] \times \boldsymbol{R}^3\,, \\ u^\varepsilon\big|_{t=0} = \varepsilon U_0\Big(r, \dfrac{r-r_0}{\varepsilon}\Big)\,, & \partial_t u^\varepsilon\big|_{t=0} = U_1\Big(r, \dfrac{r-r_0}{\varepsilon}\Big)\,. \end{cases} \tag{1.3}$$

In [4], formal arguments, inspired by the linear case $a = 0$, led to the following distinctions, in the spirit of those computed formally in [5],

	$\alpha + 2 > p$	$\alpha + 2 = p$	$\alpha + 2 < p$
$\alpha > 0$	linear caustic, linear propagation	nonlinear caustic, linear propagation	supercritical caustic, linear propagation
$\alpha = 0$	linear caustic, nonlinear propagation	nonlinear caustic, nonlinear propagation	supercritical caustic, nonlinear propagation

In [2], we studied the case “linear caustic, nonlinear propagation”; we proved that nonlinear geometric optics provides a good approximation of $\partial_t \mathbf{u}^\varepsilon$ away from the focal point $(t, r) = (r_0, 0)$, and that the nonlinear term is negligible near the focus. In [4], we analyzed the case “nonlinear caustic, linear propagation”. In some sense, it is the exact opposite of the previous case; the nonlinear term is negligible outside the focal point, but has a relevant influence near $(r_0, 0)$, which is described by a nonlinear scattering operator. Moreover, this scattering operator broadens the pulses (at least if U_0 and U_1 are small), which leave the focus with algebraically decaying tails.

In this paper, we discuss the remaining cases of the above table. In the last three cases, we treat only the case of a real, that is, when Equation (1.3) is dissipative or accretive (in particular, we do not treat the case of conservative equations).

The first case, “linear caustic, linear propagation”, suggests that the nonlinear term is everywhere negligible; we prove that this is so. In the last three cases, we assume that the coupling constant a is real. For the “supercritical caustic” cases, strong nonlinear effects are expected near the focal point. We prove that when the equation is dissipative ($a > 0$), then the dissipation is so strong near the focus that the pulses are absorbed. This result is the pulse analogue of [6] and [7], which proved absorption in the case of wave trains (the initial profiles U_j are assumed to be periodic with respect to their last variable instead of compactly supported). More precisely, in [6] and [7], it is proved that the exact solution u^ε of the dissipative ($a > 0$) wave equation (1.1) with $J = 0$, is approximated as follows,

$$\partial_t u^\varepsilon(t,x) \underset{\varepsilon\to 0}{\sim} \partial_t \underline{u}(t,x) + U_-\Big(t, x, \frac{t+|x|}{\varepsilon}\Big) + U_+\Big(t, x, \frac{t-|x|}{\varepsilon}\Big)\,,$$

where the profiles $U_\pm$ are periodic with respect to their last variable, with mean value zero. The absorption of oscillations is given by $U_\pm \equiv 0$ past the caustic. Thus, only the average term remains, included in $\partial_t \underline{u}$. For an almost periodic function, the notion of average is given by

$$\underline{f} = \lim_{T\to+\infty} \frac{1}{2T} \int_{-T}^{T} f(\theta)d\theta\,.$$

When f is compactly supported, the above limit is zero, and pulses formally have mean value zero. Thus, the absorption of pulses is the formal analogue of the absorption of oscillations.

Our present framework makes it possible to analyze very precisely the corresponding phenomenon for pulses; they are absorbed when approaching the caustic, that is even *before reaching it*. We prove that this phenomenon occurs for the last three cases of the table, "supercritical caustic, linear/nonlinear propagation" and "nonlinear caustic, nonlinear propagation".

The present paper along with [2] and [4] prove that the distinctions derived formally in [4] and recalled in the above tables are correct. Let us give an interpretation of these results when the nonlinearity is fixed, and when one modulates the amplitude of the initial data in (1.1). Consider a fixed $p > 2$, and modify the value of J. For a unified complete presentation, we assume that the equation is dissipative, $a > 0$.

(1) If $J > (p-2)/(p-1)$, then the pulse is not affected by the nonlinearity at leading order. It remains too small to ignite the nonlinearity.

(2) If $J = (p-2)/(p-1)$, then the nonlinear term is negligible away from the focus, but the caustic crossing, described by a scattering operator, has enlarged the support of the pulse, and decreased its amplitude. The pulse is too small to see the nonlinearity outside the focal point, but the amplification near the caustic makes the nonlinear term relevant there.

(3) If $J < (p-2)/(p-1)$, then the pulse is absorbed at the focus. It is sufficiently big to make the nonlinear effects so strong that the dissipation is complete before the focus. When $p = 2$, the nonlinear term is negligible if $J > 0$, and if $J = 0$, the pulses are absorbed. When $1 < p < 2$, the same method would prove that the nonlinear term is negligible if $J > 0$, and the case $J = 0$ was treated in [2].

Before stating precisely our results, we make a change of unknown, as in [2] and [4]. Since the initial data are spherical, so is the solution. With the usual abuse of notation,

$$u^\varepsilon(t,x) \;=\; u^\varepsilon(t,|x|)\,, \quad u^\varepsilon(t,|x|) \;\in\; C^\infty_{\text{even in } r}(\boldsymbol{R}_t \times \boldsymbol{R}_r)\,.$$

Introduce $v^\varepsilon := (v^\varepsilon_-, v^\varepsilon_+)$, where

$$\tilde{u}^\varepsilon(t,r) := r u^\varepsilon(t,r)\,, \quad v^\varepsilon_\mp := (\partial_t \pm \partial_r)\tilde{u}^\varepsilon\,, \quad v^\varepsilon_\mp \in C^\infty(\boldsymbol{R}_t \times \boldsymbol{R}_r)\,. \tag{1.4}$$

Then (1.3) becomes

$$\begin{cases} (\partial_t \pm \partial_r) v^\varepsilon_\pm = \varepsilon^\alpha r^{1-p} g(v^\varepsilon_- + v^\varepsilon_+)\,, \quad g(y) := -a 2^{-p}|y|^{p-1}y\,, \\ (v^\varepsilon_- + v^\varepsilon_+)\big|_{r=0} = 0\,, \\ v^\varepsilon_\mp\big|_{t=0} = P_\mp\left(r, \dfrac{r-r_0}{\varepsilon}\right) \pm \varepsilon P_1\left(r, \dfrac{r-r_0}{\varepsilon}\right), \end{cases} \tag{1.5}$$

where

$$\begin{aligned} P_\mp(r,z) &:= rU_1(r,z) \pm r\partial_z U_0(r,z)\,, \\ P_1(r,z) &:= U_0(r,z) + r\partial_r U_0(r,z)\,. \end{aligned}$$

We prove asymptotics for v^ε; asymptotics for $\partial_t u^\varepsilon$ are deduced by (1.4),

$$\partial_t u^\varepsilon(t,r) = \frac{v^\varepsilon_- + v^\varepsilon_+}{2r}\,.$$

For the subcritical case, introduce the solution of the linear equation

$$\begin{cases} (\partial_t \pm \partial_r)(v_\pm^\varepsilon)_{\mathrm{free}} = 0\,, \\ ((v_-^\varepsilon)_{\mathrm{free}} + (v_+^\varepsilon)_{\mathrm{free}})\big|_{r=0} = 0\,, \\ (v_\mp^\varepsilon)_{\mathrm{free}}\big|_{t=0} = P_\mp\Big(r, \dfrac{r-r_0}{\varepsilon}\Big)\,. \end{cases} \tag{1.6}$$

It is given explicitly by the formulae,

$$\begin{aligned} (v_-^\varepsilon)_{\mathrm{free}}(t,r) &= P_-\Big(r+t, \frac{r+t-r_0}{\varepsilon}\Big)\,, \\ (v_+^\varepsilon)_{\mathrm{free}}(t,r) &= P_+\Big(r-t, \frac{r-t-r_0}{\varepsilon}\Big) - P_-\Big(t-r, \frac{t-r-r_0}{\varepsilon}\Big)\,. \end{aligned}$$

The pulse $(v_-^\varepsilon)_{\mathrm{free}}$ corresponds to an incoming wave, and $(v_+^\varepsilon)_{\mathrm{free}}$ is the sum of two outgoing waves, one from P_+, and the other from the focusing of the incoming wave.

THEOREM 1.1 (Subcritical case). *Assume that* $\alpha > \max(0, p-2)$.

(1) *If* $\alpha > p-2 > 0$, *then there exists* $\varepsilon_0 > 0$ *such that for any* $\varepsilon \in]0, \varepsilon_0]$, (1.5) *has a unique global solution* $v^\varepsilon \in C^1([0,\infty[\times \boldsymbol{R}_+)$. *Moreover, the following asymptotics holds in* $L^\infty(\boldsymbol{R}_+ \times \boldsymbol{R}_+)$,

$$\begin{aligned} v_\pm^\varepsilon(t,r) &= (v_\pm^\varepsilon)_{\mathrm{free}}(t,r) + O(\varepsilon^{\min(1,\alpha+2-p)})\,, \\ \varepsilon\partial_t v_\pm^\varepsilon(t,r) &= \varepsilon\partial_t (v_\pm^\varepsilon)_{\mathrm{free}}(t,r) + O(\varepsilon^{\min(1,\alpha+2-p)})\,. \end{aligned}$$

(2) *If* $\alpha > 0$ *and* $1 < p \le 2$, *let* $T > 0$. *Then there exists* $\varepsilon(T) > 0$ *such that for any* $\varepsilon \in]0, \varepsilon(T)]$, (1.5) *has a unique solution* $v^\varepsilon \in C^1([0,T]\times \boldsymbol{R}_+)$. *Moreover, the following asymptotics holds in* $L^\infty([0,T]\times \boldsymbol{R}_+)$,

$$\begin{aligned} v_\pm^\varepsilon(t,r) &= (v_\pm^\varepsilon)_{\mathrm{free}}(t,r) + O(\varepsilon^{\min(1,\alpha)}) \quad (O(\varepsilon + \varepsilon^\alpha|\log\varepsilon|)\ \textit{if}\ p=2)\,, \\ \varepsilon\partial_t v_\pm^\varepsilon(t,r) &= \varepsilon\partial_t (v_\pm^\varepsilon)_{\mathrm{free}}(t,r) + O(\varepsilon^{\min(1,\alpha)}) \quad (O(\varepsilon + \varepsilon^\alpha|\log\varepsilon|)\ \textit{if}\ p=2)\,. \end{aligned}$$

THEOREM 1.2 (Supercritical case). *Assume that* $0 \le \alpha < p-2$ *or* $\alpha = 0 = p-2$.

(1) *If the equation is dissipative*, $a > 0$, *then the pulses are absorbed before reaching the focus. If* $T \ge r_0$,

$$\limsup_{\varepsilon\to 0}(\|v_-^\varepsilon(T)\|_{L^\infty(0\le r\le T)} + \|v_+^\varepsilon(T)\|_{L^\infty(0\le r\le T)}) = 0\,.$$

More precisely, for $\lambda > 0$, *define* $T(\lambda,\varepsilon)$ *as follows; if* $0 \le \alpha < p-2$, *then* $T(\lambda,\varepsilon) := r_0 - z_0\varepsilon - \lambda\varepsilon^{\alpha/(p-2)}$, *and if* $\alpha = 0 = p-2$, *then* $T(\lambda,\varepsilon) := r_0 - z_0\varepsilon - \lambda$. *For any* $T = T(\varepsilon) \ge T(\lambda,\varepsilon)$,

$$\lim_{\lambda\to 0}\limsup_{\varepsilon\to 0}(\|v_-^\varepsilon(T)\|_{L^\infty(0\le r\le T)} + \|v_+^\varepsilon(T)\|_{L^\infty(0\le r\le T)}) = 0\,.$$

(2) *If the equation is accretive*, $a < 0$, *there exists* $T^* \le r_0$ *such that the family* $(v_-^\varepsilon, v_+^\varepsilon)$ *is not bounded in* $L^\infty([0,T^*]\times \boldsymbol{R}_+)^2$.

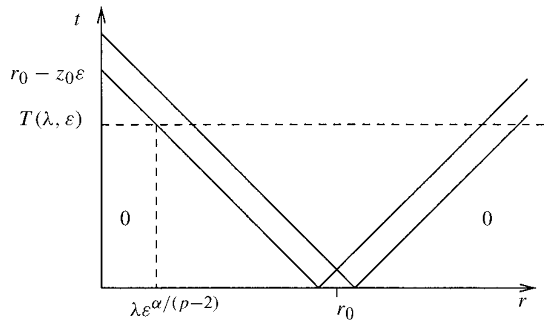

FIGURE 1. Geometry of the propagation in the super-critical case.

REMARK. For $0 < \varepsilon \leq 1$ and λ positive, $T(\lambda, \varepsilon) < r_0 - z_0\varepsilon$, and so the first part of the theorem shows that the absorption mechanism takes place before the incoming wave reaches the focus. Indeed, the pulses are initially supported in $\{|r - r_0| \leq z_0\varepsilon\}$, and so by finite speed of propagation, they do not reach the origin before $t = r_0 - z_0\varepsilon$ (see Fig. 1). Notice that in the dissipative case, our estimates are in $\{t \geq r\}$, which includes the focusing region, and its domain of influence. The key to the proof of Theorem 1.2 is to construct an approximate solution which is more accurate than the approximation of nonlinear geometric optics and which permits us to penetrate with high accuracy to small distances from the focal point $r = 0$.

REMARK. If we considered wave trains, that is, $U_j(r, \cdot)$ periodic instead of compactly supported, Theorem 1.1 would still hold. The proof we give works in both cases. On the other hand, the proof of Theorem 1.2 relies on the compact support assumption. We construct approximate solutions that solve ordinary differential equations along the rays of geometrical optics (see Section 5), and can be computed explicitly. In the dissipative case, these approximate solutions are absorbed before they stop being good approximations, proving thereby the absorption of the exact solution. In the case of wave trains, the computation of the counterpart of these approximations is a project for the future.

REMARK. As we mentioned above, in the supercritical cases, our framework is restricted. We assume that the coupling constant a is real, while we do not make this assumption in Theorem 1.1. It would be interesting to know what happens when, for instance, a is pure imaginary; no absorption can happen, for the equation in that case is conservative. A partial answer is given in [1], in the case $\alpha = 0 = p - 2$, on a system which is a simplified model for (1.3); an arbitrary phase shift appears, varying like $\log \varepsilon$. In the supercritical framework, $\alpha = 0$ and $p > 2$, one may expect even more pathological behaviors.

In Section 2, we prove two stability results. In Section 3, we discuss existence results, and establish estimates for the subcritical case. Theorem 1.1 is proved in Section 4, and Theorem 1.2 is proved in Section 5.

The results of Theorem 1.2 were announced in [3].

2. General stability results. In this section, we state two general approximation arguments. The first one will allow us to prove Theorem 1.1 and the second one, Theorem 1.2. Our first result is an easy estimate, proved in [4].

DEFINITION 2.1. For $t > 0$, we denote by Γ_-^t (resp. Γ_+^t) the set of all speed minus one (resp. plus one) characteristics connecting points on the initial line $\{t = 0\}$ to points at time t. We also denote $\Gamma^t = \Gamma_-^t \cup \Gamma_+^t$.

For a characteristic $\gamma \in \Gamma^t$, we use the convention that

$$\int_\gamma f$$

stands for the integral of f along γ, parameterized by the time variable,

$$\int_\gamma f = \int_0^t f(s, r(s))ds\,,$$

where $(s, r(s))_{s\in[0,t]}$ is a parametrization of γ. In particular, if f is nonnegative, then so is the above integral.

LEMMA 2.2 ([4], Lemma 2.1). *Suppose that w and $f = (f_+, f_-)$ are bounded continuous functions on $[0, T] \times [0, \infty[$ satisfying in the sense of distributions*

$$(\partial_t \pm \partial_r)w_\pm = f_\pm\,, \quad w_+(t, 0) + w_-(t, 0) = 0 \quad \text{for } 0 \le t \le T\,.$$

Denote by

$$M_\pm(t) \;:=\; \|w_\pm(t)\|_{L^\infty([0,\infty[)}\,.$$

Then for $0 \le t \le T$ one has

$$M_\pm(t) \;\le\; \max\{M_+(0), M_-(0)\} \;+\; \sup_{\gamma_-\in\Gamma_-^t}\int_{\gamma_-}|f_-| + \sup_{\gamma_+\in\Gamma_+^t}\int_{\gamma_+}|f_+|\,.$$

To prove Theorem 1.2, we use a result in the spirit of Gronwall's lemma. The assumption $0 < T < \delta$ guarantees that the support of the solution does not touch $\{r = 0\}$.

PROPOSITION 2.3. *Suppose that $0 < T < \delta$, and $w = (w_-, w_+) \in C \cap L^\infty([0, T] \times \boldsymbol{R}_+)$ satisfies*

$$\begin{cases}(\partial_t \pm \partial_r)w_\pm = f_\pm(t, r)(w_- + w_+) + S_\pm\,,\\ w_{\pm|t=0} = w_{0\pm}\,,\end{cases} \tag{2.1}$$

with $\operatorname{supp} w_{0\pm} \subset [\delta, +\infty[$. *Suppose that*

$$C_1 = C_1(f) := \int_0^T \sup_{\gamma_-\in\Gamma_-^t}|f_-|dt + \int_0^T \sup_{\gamma_+\in\Gamma_+^t}|f_+|dt < \infty\,. \tag{2.2}$$

Then

$$\sup_{0\le t\le T} \|w_\pm(t)\|_{L^\infty} \le C_2 \sum_\pm \left(\|w_{0\pm}\|_{L^\infty} + \sup_{\gamma_\pm\in\Gamma_\pm^T} \int_{\gamma_\pm} |S_\pm| \right),$$

with $C_2 = \max(C_1 e^{2C_1}, C_1^2 e^{3C_1})$.

WARNING. Note that in Hypothesis (2.2), the supremum is inside the integral. The estimate would not be true with the supremum outside.

PROOF. Let $(t, r) \in [0, T]\times]0, \infty[$, and denote by $\gamma_- = \gamma_-(t, r)$ the characteristic from $(0, t + r)$ to (t, r). Duhamel's principle for w_- reads

$$w_-(t, r) = w_{0-}(t + r) + \int_{\gamma_-} f_- \times (w_- + w_+) + \int_{\gamma_-} S_- .$$

Gronwall's lemma, along with assumption (2.2), yields, for any $t \in [0, T]$,

$$\begin{aligned} \|w_-(t)\|_{L^\infty} &\le e^{C_1}\left(\|w_{0-}\|_{L^\infty} + \sup_{\gamma_-\in\Gamma_-^t} \int_{\gamma_-} |f_-.w_+| + \sup_{\gamma_-\in\Gamma_-^t} \int_{\gamma_-} |S_-| \right) \\ &\le e^{C_1}\left(\|w_{0-}\|_{L^\infty} + C_1 \sup_{0\le s\le t} \|w_+(s)\|_{L^\infty} + \sup_{\gamma_-\in\Gamma_-^t} \int_{\gamma_-} |S_-| \right). \end{aligned} \tag{2.3}$$

Similarly,

$$\|w_+(t)\|_{L^\infty} \le e^{C_1}\left(\|w_{0+}\|_{L^\infty} + \sup_{\gamma_+\in\Gamma_+^t} \int_{\gamma_+} |f_+.w_-| + \sup_{\gamma_+\in\Gamma_+^t} \int_{\gamma_+} |S_+| \right). \tag{2.4}$$

For the w_- integrals on the right in (2.4), use estimate (2.3) to find

$$\int_{\gamma_\pm} |f_\pm w_-| \ \le\ \int_{\gamma_\pm} |f_\pm|\, e^{C_1}\left(\|w_-(0)\| + C_1 \max_{0\le\tau\le s} \|w_+(\tau)\| + \sup_{\widetilde{\gamma}_-\in\Gamma_-^s} \int_{\widetilde{\gamma}_-} |S_-| \right) ds\,,$$

where $\gamma_\pm$ is parameterized by the time $s \in [0, t]$. Introduce

$$\mathbf{m}_\pm(t) := \sup_{0\le s\le t} \|w_\pm(s)\|\,,$$

and take the supremum on r in (2.4) to obtain

$$\begin{aligned} \|w_+(t)\| \ &\le C_1 e^{C_1}\left(\sum_\pm \left(\|w_\pm(0)\| + \sup_{\gamma_\pm\in\Gamma_\pm^t} \int_{\gamma_\pm} |S_\pm| \right) \right. \\ &\quad \left. + \sup_{\gamma_+\in\Gamma_+^t} \int_{\gamma_+} |f_+(s, r)|\, \mathbf{m}_+(s) ds + \sup_{\gamma_-\in\Gamma_-^t} \int_{\gamma_-} |f_-(s, r)|\, \mathbf{m}_+(s) ds \right). \end{aligned}$$

Therefore,

$$\begin{aligned}\|\mathbf{m}_+(t)\| &\le C_1 e^{C_1}\Bigg(\sum_{\pm}\bigg(\|w_\pm(0)\| + \sup_{\gamma_\pm\in\Gamma^t_\pm}\int_{\gamma_\pm}|S_\pm|\bigg)\\ &\quad + \sup_{\gamma_+\in\Gamma^t_+}\int_{\gamma_+}|f_+(s,r)|\,\mathbf{m}_+(s)ds + \sup_{\gamma_-\in\Gamma^t_-}\int_{\gamma_-}|f_-(s,r)|\,\mathbf{m}_+(s)ds\Bigg)\\ &\le C_1 e^{C_1}\Bigg(\sum_{\pm}\bigg(\|w_\pm(0)\| + \sup_{\gamma_\pm\in\Gamma^t_\pm}\int_{\gamma_\pm}|S_\pm|\bigg)\\ &\quad + \int_0^t \sup_{\gamma_+\in\Gamma^t_+}|f_+(s,r)|\,\mathbf{m}_+(s)ds + \int_0^t \sup_{\gamma_-\in\Gamma^t_-}|f_-(s,r)|\,\mathbf{m}_+(s)ds\Bigg).\end{aligned}$$

Applying Gronwall's lemma, using assumption (2.2), yields

$$\mathbf{m}_+(T) \le C_1 e^{2C_1}\bigg(\|w_+(0)\| + \|w_-(0)\| + \sup_{\gamma_-\in\Gamma^T_-}\int_{\gamma_-}|S_-| + \sup_{\gamma_+\in\Gamma^T_+}\int_{\gamma_+}|S_+|\bigg).$$

This inequality, along with (2.3), proves the proposition. □

The following example contains the core of the proof of Theorem 1.2.

EXAMPLE 2.4. Consider the case

$$f^\varepsilon_\pm(t,r) = \varepsilon^\alpha r^{1-p}|\tilde{v}^\varepsilon_-(t,r)|^{p-1},$$

where $\tilde{v}^\varepsilon_-$ solves an initial value problem of the form (compare with (5.1) below),

$$(\partial_t - \partial_r)\tilde{v}^\varepsilon_- = F^\varepsilon(\tilde{v}^\varepsilon_-)\,,\quad \tilde{v}^\varepsilon_{-|t=0} = P_-\left(r, \frac{r-r_0}{\varepsilon}\right).$$

Recall that $\operatorname{supp} P_-(r,\cdot) \subset [-z_0, z_0]$. For $w^\varepsilon_{0_\pm}$, consider pulses with the same support as $\tilde{v}^\varepsilon_{-|t=0}$. Then by finite speed of propagation, we can take $\delta^\varepsilon = r_0 - z_0\varepsilon$ in Proposition 2.3. For $0 < \underline{t} < \delta^\varepsilon$, the maximum of $f^\varepsilon_\pm$ at time $\underline{t}$ is estimated by

$$\varepsilon^\alpha \underline{r}^{1-p}\|\tilde{v}^\varepsilon_-\|^{p-1}_{L^\infty(0\le t\le \underline{t})},$$

where $\underline{r}$ is such that $\underline{r} + \underline{t} = \delta^\varepsilon = r_0 - z_0\varepsilon$. Therefore we have

$$\begin{aligned}\int_0^{\underline{t}} \sup_{\gamma_-\in\Gamma^t_-}|f^\varepsilon_-|dt + \int_0^{\underline{t}} \sup_{\gamma_+\in\Gamma^t_+}|f^\varepsilon_+|dt &\le 2\|\tilde{v}^\varepsilon_-\|^{p-1}_{L^\infty(0\le t\le \underline{t})}\int_0^{\underline{t}}\varepsilon^\alpha(\delta^\varepsilon - t)^{1-p}dt\\ &\le C_p\varepsilon^\alpha\|\tilde{v}^\varepsilon_-\|^{p-1}_{L^\infty(0\le t\le \underline{t})}\times\begin{cases}(\delta^\varepsilon - \underline{t})^{2-p} \text{ if } p>2\,,\\ |\log(\delta^\varepsilon - \underline{t})| \text{ if } p=2\,.\end{cases}\end{aligned}$$

Thus, when $\tilde{v}^\varepsilon_-$ remains bounded and T^ε is chosen so that $\varepsilon^\alpha(\delta^\varepsilon - T^\varepsilon)^{2-p}$ (resp. $\varepsilon^\alpha|\log(\delta^\varepsilon - T^\varepsilon)|$ if $p=2$) is bounded independent of ε, then we can use Proposition 2.3. This is the case in particular if $T^\varepsilon = T_{\lambda,\varepsilon}$ defined in Theorem 1.2.

3. Existence results. In this section, we prove two kinds of results concerning the existence of solutions to (1.5); local existence in L^∞ before the wave meets the boundary $\{r = 0\}$, and global existence in $W^{1,\infty}$ when the boundary condition has to be taken into account. The reason appears in Theorems 1.1 and 1.2; in Theorem 1.2, the phenomena we want to prove occur *before* the pulses reach the boundary, while Theorem 1.1 includes the caustic crossing.

In the first case, we are interested in a problem

$$
\begin{cases} (\partial_t \pm \partial_r) v_\pm^\varepsilon = \varepsilon^\alpha r^{1-p} g(v_-^\varepsilon + v_+^\varepsilon)\,, \quad r > 0\,, \\ v_\mp^\varepsilon\big|_{t=0} = v_{0\mp}^\varepsilon\,, \end{cases} \tag{3.1}
$$

where $\operatorname{supp} v_{0\mp}^\varepsilon \subset [\delta^\varepsilon, +\infty[$ for some $\delta^\varepsilon > 0$, and $v_{0\mp}^\varepsilon \in L^\infty(\boldsymbol{R}_+)$. Then by finite speed of propagation, the term r^{1-p} is harmless at least up to time δ^ε. In that case, local in time existence of solutions to (3.1) is easy.

LEMMA 3.1. *Fix $\alpha \geq 0$, $p > 1$. Let $v_{0\mp}^\varepsilon \in L^\infty(\boldsymbol{R}_+)$ such that* $\operatorname{supp} v_{0\mp}^\varepsilon \subset [\delta^\varepsilon, +\infty[$ *for some $\delta^\varepsilon > 0$. Then there exists T^ε, with $0 < T^\varepsilon < \delta^\varepsilon$, and a unique solution $(v_-^\varepsilon, v_+^\varepsilon) \in L^\infty \cap C([0, T^\varepsilon] \times \boldsymbol{R}_+)^2$ to the initial value problem* (3.1).

When the incoming wave v_-^ε reaches the origin $\{r = 0\}$, a boundary condition is needed in order to solve the above system. We are interested in that given in (1.5), that is,

$$
(v_-^\varepsilon + v_+^\varepsilon)\big|_{r=0} = 0\,.
$$

Consider the mixed problem

$$
\begin{cases} (\partial_t \pm \partial_r) v_\pm^\varepsilon = \varepsilon^\alpha r^{1-p} g(v_-^\varepsilon + v_+^\varepsilon)\,, \quad r > 0\,, \\ (v_-^\varepsilon + v_+^\varepsilon)\big|_{r=0} = 0\,, \\ v_\mp^\varepsilon\big|_{t=0} = v_{0\mp}^\varepsilon\,. \end{cases} \tag{3.2}
$$

The boundary condition compensates the singularity r^{1-p} when r gets close to zero. Indeed, Taylor's formula yields, for C^1 solutions,

$$
(v_-^\varepsilon + v_+^\varepsilon)(t, r) = r\,\partial_r (v_-^\varepsilon + v_+^\varepsilon)(t, r) + o(r)\,, \quad \text{as } r \to 0\,.
$$

Now from the differential equation, we also have

$$
\partial_r (v_-^\varepsilon + v_+^\varepsilon) = \partial_t (v_-^\varepsilon - v_+^\varepsilon)\,,
$$

and so if we know that the time derivatives of v^ε remain bounded, then the singularity r^{1-p} is compensated. We have precisely,

$$
|(v_-^\varepsilon + v_+^\varepsilon)(t, r)| \leq \frac{4r}{r + \varepsilon}(|v_-^\varepsilon| + |v_+^\varepsilon| + |\varepsilon \partial_t v_-^\varepsilon| + |\varepsilon \partial_t v_+^\varepsilon|)(t, r) \quad \text{for all } t, r \geq 0\,. \tag{3.3}
$$

This is the strategy we used in [4], Proposition 3.4, to prove local existence, in the case $p > 2$. Notice that at this stage, the dependence upon ε is unimportant, and that $p > 1$ suffices for local existence.

LEMMA 3.2. *Fix* $\alpha \geq 0$, $p > 1$. *Let* $v^\varepsilon_{0\mp} \in W^{1,\infty}(\boldsymbol{R}_+)$ *such that*

$$v^\varepsilon_{0-}(0) + v^\varepsilon_{0+}(0) = 0\,, \quad \partial_r v^\varepsilon_{0-}(0) - \partial_r v^\varepsilon_{0+}(0) = 0\,. \tag{3.4}$$

Then there exists $T^\varepsilon > 0$ *and a unique solution* $(v^\varepsilon_-, v^\varepsilon_+) \in C^1 \cap W^{1,\infty}([0, T^\varepsilon] \times \boldsymbol{R}_+)^2$ *of* (3.2).

As recalled in the beginning of this section, such a result will be needed only in the proof of Theorem 1.1, and not in the proof of Theorem 1.2. From now on in this section, we assume $\alpha > \max(0, p-2)$. In [4], we also proved that the solutions of (3.2) with $\varepsilon = 1$ and $p > 2$ are global, provided that the initial data $v_{0\pm}$ are sufficiently small.

PROPOSITION 3.3 ([4], Proposition 3.5). *Fix* $\alpha \geq 0$, $p > 2$. *There are constants* K_1 *and* $K_1' > 0$ *such that for all initial data* $\psi_0 \in C^1([0,\infty))$ *satisfying*

$$\|\psi_0, \partial_r \psi_0\|_{L^\infty([0,\infty[)} \leq K_1\,,$$

and the compatibility conditions

$$\psi_{0+}(0) + \psi_{0-}(0) = 0\,, \quad \partial_r \psi_{0+}(0) - \partial_r \psi_{0-}(0) = 0\,,$$

there is a unique solution $\psi \in C^1([-\infty,\infty] \times [0,\infty[)$ *of*

$$\begin{cases} (\partial_t \pm \partial_r)\psi_\pm = r^{1-p} g(\psi_- + \psi_+)\,, \\ \psi_-(t,0) + \psi_+(t,0) = 0\,, \\ \psi_\mp|_{t=0} = \psi_{0\mp}\,. \end{cases} \tag{3.5}$$

In addition,

$$\|\psi, \partial_t \psi\|_{L^\infty([-\infty,\infty]\times[0,\infty[)} \leq K_1' \,\|\psi_0, \partial_r \psi_0\|_{L^\infty([0,\infty[)}\,.$$

The idea is then to find a scaling such that we can use the above proposition to prove global existence for (1.5) when $\alpha > p-2 > 0$, along with useful estimates. Try

$$v^\varepsilon_\pm(t,r) = \varepsilon^\gamma \psi^\varepsilon_\pm(\tau,\rho)\big|_{\tau=t/\varepsilon,\rho=r/\varepsilon}\,, \quad \psi^\varepsilon_\pm(\tau,\rho) = \varepsilon^{-\gamma} v^\varepsilon_\pm(\varepsilon\tau, \varepsilon\rho)\,. \tag{3.6}$$

Then $\psi^\varepsilon_\pm$ solve the differential equations

$$(\partial_\tau \pm \partial_\rho)\psi^\varepsilon_\pm = \varepsilon^\alpha (\varepsilon\rho)^{1-p} \varepsilon^{(p-1)\gamma} g(\psi^\varepsilon_- + \psi^\varepsilon_+)\,.$$

Choose γ so that the powers of ε cancel,

$$\alpha + 1 - p + (p-1)\gamma = 0 \Leftrightarrow \gamma = 1 - \frac{\alpha}{p-1}\,.$$

Since γ is negative,

$$\|\psi^\varepsilon_\pm(0)\|_{L^\infty_\rho}\,, \quad \|\partial_\tau \psi^\varepsilon_\pm(0)\|_{L^\infty_\rho} \to 0 \quad \text{as } \varepsilon \to 0\,,$$

so that we can apply Proposition 3.3, for ε sufficiently small. Moreover, it provides L^∞ estimates for $v^\varepsilon_\pm, \varepsilon\partial_t v^\varepsilon_\pm$. We deduce that $\partial_r v^\varepsilon_\pm \in L^\infty$ from the differential equations and (3.3).

COROLLARY 3.4. *Assume* $\alpha > p - 2 > 0$. *Then there exists* $\varepsilon_0 > 0$ *such that for* $0 < \varepsilon \leq \varepsilon_0$, (1.5) *has a unique solution* $(v_-^\varepsilon, v_+^\varepsilon) \in C^1 \cap W^{1,\infty}(\boldsymbol{R}_+ \times \boldsymbol{R}_+)^2$. *Moreover*, *there exists* C *such that for any* $\varepsilon \in]0, \varepsilon_0]$,

$$\|v_\pm^\varepsilon, \varepsilon \partial_t v_\pm^\varepsilon\|_{L^\infty(\boldsymbol{R}_+ \times \boldsymbol{R}_+)} \leq C\,.$$

We now have to prove global existence when $1 < p \leq 2$ and $\alpha > 0$. Since local existence is known (Lemma 3.2), global existence is a consequence of *a priori* estimates, which follow from (3.3) and Lemma 2.2. Define

$$\mathbf{m}_\pm^\varepsilon(t) = \sup_{0 \leq s \leq t} \left(\|v_\pm^\varepsilon(s)\|_{L^\infty} + \|\varepsilon \partial_t v_\pm^\varepsilon(s)\|_{L^\infty}\right).$$

By assumption, $\mathbf{m}_\pm^\varepsilon(0)$ are bounded independent of $\varepsilon \in]0, 1]$. Lemma 2.2 and (3.3) yield, for $T > 0$,

$$\begin{aligned}\mathbf{m}_-^\varepsilon(T) + \mathbf{m}_+^\varepsilon(T) &\leq C(\mathbf{m}_-^\varepsilon(0) + \mathbf{m}_+^\varepsilon(0) + h_p(\varepsilon, T)(\mathbf{m}_-^\varepsilon(T) + \mathbf{m}_+^\varepsilon(T))^p) \\ &\leq C_0 + C(h_p(\varepsilon, T)(\mathbf{m}_-^\varepsilon(T) + \mathbf{m}_+^\varepsilon(T))^p)\,,\end{aligned}$$

with

$$h_p(\varepsilon, T) = \begin{cases} \varepsilon^\alpha T^{2-p} & \text{if } 1 < p < 2\,, \\ \varepsilon^\alpha \log(1 + T/\varepsilon) & \text{if } p = 2\,. \end{cases}$$

In particular, for fixed $T > 0$, $h_p(\varepsilon, T) \to 0$ as $\varepsilon \to 0$. Therefore, for any $T > 0$, there exists $\varepsilon(T) > 0$ such that for any $\varepsilon \in]0, \varepsilon(T)]$, v^ε exists and

$$\mathbf{m}_-^\varepsilon(T) + \mathbf{m}_+^\varepsilon(T) \leq 4C_0\,.$$

If $\mathbf{m}_-^\varepsilon(T) + \mathbf{m}_+^\varepsilon(T) \leq 4C_0$, then it follows that

$$\mathbf{m}_-^\varepsilon(T) + \mathbf{m}_+^\varepsilon(T) \leq C_0 + C h_p(\varepsilon, T)(4C_0)^p\,,$$

and therefore for $0 < \varepsilon \leq \varepsilon(T)$,

$$\mathbf{m}_-^\varepsilon(T) + \mathbf{m}_+^\varepsilon(T) \leq 2C_0\,.$$

This proves that in fact $\mathbf{m}_-^\varepsilon(T) + \mathbf{m}_+^\varepsilon(T) < 4C_0$, since if that were not true, there would be a first T_* where $\mathbf{m}_-^\varepsilon(T) + \mathbf{m}_+^\varepsilon(T) = 4C_0$, and at that value of T the above estimate leads to the contradiction $4C_0 < 2C_0$.

PROPOSITION 3.5. *Assume that* $\alpha > 0$ *and* $1 < p \leq 2$. *Let* $T > 0$. *Then there exists* $\varepsilon(T) > 0$ *such that for* $0 < \varepsilon \leq \varepsilon(T)$, (1.5) *has a unique solution*

$$(v_-^\varepsilon, v_+^\varepsilon) \in C^1 \cap W^{1,\infty}([0, T] \times \boldsymbol{R}_+)^2\,.$$

Moreover, *there exists* C *such that for any* $\varepsilon \in]0, \varepsilon(T)]$,

$$\|v_\pm^\varepsilon, \varepsilon \partial_t v_\pm^\varepsilon\|_{L^\infty([0,T] \times \boldsymbol{R}_+)} \leq C\,.$$

4. The subcritical case. Now Theorem 1.1 is a straightforward consequence of Lemma 2.2, Corollary 3.4 and Proposition 3.5. In the statement of Theorem 1.1, we distinguished three cases; $p > 2$, $p = 2$ and $1 < p < 2$. The distinction between $p > 2$ and

$p \leq 2$ appeared in the previous section, in Corollary 3.4 and Proposition 3.5. It corresponds to the question of the integrability at infinity of the mapping $r \mapsto r^{1-p}$. The further distinction $p = 2$ corresponds to the local integrability of this mapping.

Define the remainder $w_\pm^\varepsilon = v_\pm^\varepsilon - (v_\pm^\varepsilon)_{\text{free}}$. It solves the mixed problem,

$$\begin{cases} (\partial_t \pm \partial_r) w_\pm^\varepsilon = \varepsilon^\alpha r^{1-p} g(v_-^\varepsilon + v_+^\varepsilon)\,, \\ (w_-^\varepsilon + w_+^\varepsilon)\big|_{r=0} = 0\,, \\ w_\mp^\varepsilon\big|_{t=0} = \pm\varepsilon P_1\left(r, \dfrac{r - r_0}{\varepsilon}\right). \end{cases} \tag{4.1}$$

Let I be an interval of the form $[0, T[$, with $T \in \boldsymbol{R}_+ \cup \{+\infty\}$. From Lemma 2.2, we have

$$\|w_\pm^\varepsilon\|_{L^\infty(I\times\boldsymbol{R}_+)} \leq 2\varepsilon\|P_1\|_{L^\infty} + 2\sup_{t\in I}\sup_{\gamma\in\Gamma^t}\int_\gamma \varepsilon^\alpha r^{1-p} g(v_-^\varepsilon + v_+^\varepsilon)\,.$$

Using (3.3), we also have

$$\|w_\pm^\varepsilon\|_{L^\infty(I\times\boldsymbol{R}_+)} \leq C\varepsilon + C\left(\sup_{t\in I}\sup_{\gamma\in\Gamma^t}\int_\gamma \frac{\varepsilon^\alpha}{(r+\varepsilon)^{p-1}}\right)\|v^\varepsilon, \varepsilon\partial_t v^\varepsilon\|^p_{L^\infty(I\times\boldsymbol{R}_+)}\,. \tag{4.2}$$

Differentiating (4.1) with respect to time yields, using the differential equation (4.1), to find the initial data

$$\begin{cases} (\partial_t \pm \partial_r)\varepsilon\partial_t w_\pm^\varepsilon = \varepsilon^\alpha r^{1-p} g'(v_-^\varepsilon + v_+^\varepsilon)\varepsilon\partial_t(v_-^\varepsilon + v_+^\varepsilon)\,, \\ \varepsilon\partial_t(w_-^\varepsilon + w_+^\varepsilon)\big|_{r=0} = 0\,, \\ \varepsilon\partial_t w_\mp^\varepsilon\big|_{t=0} = \varepsilon(\varepsilon\partial_r P_1 + \partial_z P_1)\left(r, \dfrac{r-r_0}{\varepsilon}\right) + \varepsilon^\alpha r^{1-p} g(v_-^\varepsilon + v_+^\varepsilon)\big|_{t=0}\,. \end{cases}$$

Since the initial data for $v_\pm^\varepsilon$ are supported in $|r - r_0| \leq z_0\varepsilon$, the term r^{1-p} in the initial data for $\varepsilon\partial_t w_\mp^\varepsilon$ is harmless regarding to L^∞ estimates. From Lemma 2.2 and (3.3), we have

$$\|\varepsilon\partial_t w_\pm^\varepsilon\|_{L^\infty(I\times\boldsymbol{R}_+)} \lesssim \varepsilon^{\min(1,\alpha)} + \left(\sup_{t\in I}\sup_{\gamma\in\Gamma^t}\int_\gamma \frac{\varepsilon^\alpha}{(r+\varepsilon)^{p-1}}\right)\|v^\varepsilon, \varepsilon\partial_t v^\varepsilon\|^p_{L^\infty(I\times\boldsymbol{R}_+)}\,.$$

The case $p > 2$. Assume $\alpha > p - 2 > 0$. Then we have

$$\sup_{t\geq 0}\sup_{\gamma\in\Gamma^t}\int_\gamma \frac{\varepsilon^\alpha}{(r+\varepsilon)^{p-1}} \leq C\varepsilon^{\alpha+2-p}\,,$$

and from Corollary 3.4, there exists ε_0 such that if $\varepsilon \in]0, \varepsilon_0]$, $\|v^\varepsilon, \varepsilon\partial_t v^\varepsilon\|_{L^\infty(\boldsymbol{R}_+\times\boldsymbol{R}_+)}$ is bounded. This proves the first part of Theorem 1.1 .

The case $p = 2$. If $\alpha > p - 2 = 0$, then from Prop. 3.5, for every $T > 0$, there exists $\varepsilon(T)$ such that if $\varepsilon \in]0, \varepsilon(T)]$, $\|v^\varepsilon, \varepsilon\partial_t v^\varepsilon\|_{L^\infty([0,T[\times\boldsymbol{R}_+)}$ remains bounded. Moreover, for any fixed $T > 0$, we have

$$\sup_{\gamma\in\Gamma^T}\int_\gamma \frac{\varepsilon^\alpha}{r+\varepsilon} \leq C\varepsilon^\alpha \log\frac{T}{\varepsilon}\,.$$

The case $1 < p < 2$. If $\alpha > 0$ and $1 < p < 2$, then the only difference with the previous case is that the mapping $r \mapsto r^{1-p}$ is locally integrable, and hence the bound

$$\sup_{\gamma \in \Gamma^T} \int_\gamma \frac{\varepsilon^\alpha}{(r+\varepsilon)^{p-1}} \le \sup_{\gamma \in \Gamma^T} \int_\gamma \frac{\varepsilon^\alpha}{r^{p-1}} \le C\varepsilon^\alpha T^{2-p} .$$

This completes the proof of Theorem 1.1. □

5. The supercritical case. We conclude by proving Theorem 1.2, using Proposition 2.3. Before going into details, we explain how to construct the approximate solutions that lead to the result. Assume for instance that $\alpha > 0$ and, since we are in a supercritical case, $p - 2 > \alpha$. The hypothesis $\alpha > 0$ suggests that the nonlinear term in (1.5) is negligible, when r is not too small. This is the outline of the proof of Theorem 1.1, and we could prove in this way that $(v^\varepsilon_\pm)_{\text{free}}$ is a good approximation for $v^\varepsilon_\pm$ at least for $r_0 - t \gg \varepsilon^{\alpha/(p-2)}$. This boundary layer is larger than in the critical case $\alpha = p - 2 > 0$ studied in [4], and nonlinear effects possibly occur sooner. Considering the case $\alpha = 0$ gives us a further hint. We proved in [2] that a good approximation at leading order was given by

$$(\partial_t \pm \partial_r)(v^\varepsilon_\pm)_{\text{app}} = r^{1-p} g((v^\varepsilon_\pm)_{\text{app}}) ,$$

that is, solutions of ordinary differential equations along the rays of geometrical optics. A natural generalization of this approach to the case $\alpha \ge 0$ leads to

$$\begin{cases} (\partial_t \pm \partial_r)(v^\varepsilon_\pm)_{\text{app}} = \varepsilon^\alpha r^{1-p} g((v^\varepsilon_\pm)_{\text{app}}) , \quad (t,r) \in ([0, r_0 - z_0\varepsilon[\times \boldsymbol{R}^*_+) , \\ (v^\varepsilon_\pm)_{\text{app}}\big|_{t=0} = P_\pm\left(r, \dfrac{r-r_0}{\varepsilon}\right) . \end{cases} \tag{5.1}$$

We consider the region $t < r_0 - z_0\varepsilon$ so that no boundary condition is needed on $\{r = 0\}$, since the compact support of $P_\pm$ and the finite speed of propagation make $(v^\varepsilon_\pm)_{\text{app}}$ zero in the region $\{r + t < r_0 - z_0\varepsilon\}$ (see Figure 1).

Introduce the function F_p, defined for $y \ge x > 0$, by

$$F_p(x,y) = \int_x^y \frac{ds}{s^{p-1}} = \begin{cases} \dfrac{1}{p-2}(x^{2-p} - y^{2-p}) , & \text{if } p > 2 , \\ \log \dfrac{y}{x} , & \text{if } p = 2 . \end{cases}$$

Then (5.1) can be solved explicitly,

$$(v^\varepsilon_-)_{\text{app}}(t,r) = \frac{P_-(r+t,z)}{(1 + a2^{-p}(p-1)\varepsilon^\alpha F_p(r, r+t)|P_-(r+t,z)|^{p-1})^{1/(p-1)}}\Bigg|_{z=(r+t-r_0)/\varepsilon} ,$$

$$(v^\varepsilon_+)_{\text{app}}(t,r) = \frac{P_+(r-t,z)}{(1 + a2^{-p}(p-1)\varepsilon^\alpha F_p(r-t, r)|P_+(r-t,z)|^{p-1})^{1/(p-1)}}\Bigg|_{z=(r-t-r_0)/\varepsilon} .$$

We have

$$\operatorname{supp}(v^\varepsilon_-)_{\text{app}} = \{|r + t - r_0| \le z_0\varepsilon\} , \quad \operatorname{supp}(v^\varepsilon_+)_{\text{app}} = \{|r - t - r_0| \le z_0\varepsilon\} .$$

Since for any $y > 0$ and any $p \geq 2$, $F_p(x, y) \to +\infty$ as $x \to 0$, the sign of a is crucial.

(1) If $a > 0$ (dissipative case), then $(v_-^\varepsilon)_{\text{app}}$ tends to zero before reaching the focus.

(2) If $a < 0$ (accretive case), then both $(v_-^\varepsilon)_{\text{app}}$ and $(v_+^\varepsilon)_{\text{app}}$ blow up in finite time.

In the accretive case, $(v_+^\varepsilon)_{\text{app}}$ may blow up at time $T^* < r_0$, that is, before any focusing. To avoid useless distinctions, we assume $P_+ \equiv 0$, so that the only "interesting" phenomena occur when t approaches r_0.

The explicit formulae show that when $p > 2$, for $\varepsilon^\alpha r^{2-p} \ll 1$,

$$(v_\pm^\varepsilon)_{\text{app}}(t, r) = (v_\pm^\varepsilon)_{\text{free}}(t, r) + O(\varepsilon^\alpha r^{2-p}) .$$

Assume first $p - 2 > \alpha \geq 0$. Let $\lambda > 0$. We prove that $(v_\pm^\varepsilon)_{\text{app}}$ gives a good approximation of the exact solution in the region $r \geq \lambda \varepsilon^{\alpha/(p-2)}$ before the focus. Notice that in this region, $(v_\pm^\varepsilon)_{\text{app}}$ and $(v_\pm^\varepsilon)_{\text{free}}$ have ceased to be close to each other; the nonlinear effects are significant. By finite propagation speed, this area is defined by

$$t \leq r_0 - \lambda \varepsilon^{\alpha/(p-2)} - z_0 \varepsilon = T_{\lambda,\varepsilon} .$$

We prove that $(v_\pm^\varepsilon)_{\text{app}}$ remains a good approximation of the exact solution up to time $T_{\lambda,\varepsilon}$.

PROPOSITION 5.1. *Assume $p - 2 > \alpha \geq 0$, and let $\lambda > 0$. Then $(v_\pm^\varepsilon)_{\text{app}}$ is a good approximation of the exact solution at least for $t \in [0, T_{\lambda,\varepsilon}]$,*

$$\| v_\pm^\varepsilon - (v_\pm^\varepsilon)_{\text{app}} \|_{L^\infty([0,T_{\lambda,\varepsilon}]\times \boldsymbol{R}_+)} = O(\varepsilon^{1-\alpha/(p-2)}) \quad (O(\varepsilon) \text{ if } p = 2) .$$

PROOF OF PROPOSITION 5.1. The proof relies on Proposition 2.3, which we apply as in Example 2.4.

By finite propagation speed, both the exact and approximate solution are zero in the region

$$\{(t, r) \in [0, T_{\lambda,\varepsilon}] \times \boldsymbol{R}_+ ;\ r + t \leq T_{\lambda,\varepsilon}\} .$$

Define the remainder $w_\pm^\varepsilon = v_\pm^\varepsilon - (v_\pm^\varepsilon)_{\text{app}}$. Before rays reach the boundary $\{r = 0\}$,

$$\begin{cases} (\partial_t \pm \partial_r) w_\pm^\varepsilon = \varepsilon^\alpha r^{1-p} (g(v_-^\varepsilon + v_+^\varepsilon) - g((v_\pm^\varepsilon)_{\text{app}})) , \\ w_\pm^\varepsilon\big|_{t=0} = \mp \varepsilon P_1 \left(r, \dfrac{r - r_0}{\varepsilon} \right) . \end{cases}$$

In order to apply Proposition 2.3, write the right hand side as

$$\begin{aligned} \varepsilon^\alpha r^{1-p} (g(v_-^\varepsilon + v_+^\varepsilon) - g((v_\pm^\varepsilon)_{\text{app}})) = \varepsilon^\alpha r^{1-p} (&g(v_-^\varepsilon + v_+^\varepsilon) - g((v_-^\varepsilon)_{\text{app}} + (v_+^\varepsilon)_{\text{app}}) \\ &+ g((v_-^\varepsilon)_{\text{app}} + (v_+^\varepsilon)_{\text{app}}) - g((v_\pm^\varepsilon)_{\text{app}})) . \end{aligned}$$

Using Taylor's Theorem, the first term satisfies

$$\varepsilon^\alpha r^{1-p} (g(v_-^\varepsilon + v_+^\varepsilon) - g((v_-^\varepsilon)_{\text{app}} + (v_+^\varepsilon)_{\text{app}})) = \varepsilon^\alpha r^{1-p} (w_-^\varepsilon + w_+^\varepsilon) f^\varepsilon(t, r) ,$$

where f^ε is uniformly bounded on any set on which the families v^ε and $(v^\varepsilon)_{\text{app}}$ are uniformly bounded. The point now is that this term has exactly the properties mentioned in Example 2.4; in particular, the assumptions of Proposition 2.3 are satisfied up to time $T = T_{\lambda,\varepsilon}$.

Define the source term

$$S_\pm^\varepsilon(t, r) := \varepsilon^\alpha r^{1-p} (g((v_-^\varepsilon)_{\text{app}} + (v_+^\varepsilon)_{\text{app}}) - g((v_\pm^\varepsilon)_{\text{app}})) .$$

Then supp $S^\varepsilon_\pm \subset \operatorname{supp}(v^\varepsilon_\mp)_{\text{app}}$, and

$$
\begin{cases}
(\partial_t \pm \partial_r) w^\varepsilon_\pm = \varepsilon^\alpha r^{1-p} f^\varepsilon(t,r)(w^\varepsilon_- + w^\varepsilon_+) + S^\varepsilon_\pm\,, \\
w^\varepsilon_\pm\big|_{t=0} = \mp\varepsilon P_1\left(r, \dfrac{r-r_0}{\varepsilon}\right).
\end{cases}
\tag{5.2}
$$

From the explicit expression of $(v^\varepsilon_\pm)_{\text{app}}$, we see that for any fixed $\lambda > 0$, there exists $C_\lambda > 0$ independent of ε such that

$$
\left\| (v^\varepsilon_\pm)_{\text{app}} \right\|_{L^\infty([0,T_{\lambda,\varepsilon}]\times \boldsymbol{R}_+)} \le C_\lambda\,. \tag{5.3}
$$

We shall prove that there is $\varepsilon(\lambda) > 0$ such that for $\varepsilon \in]0, \varepsilon(\lambda)]$,

$$
\|w^\varepsilon_\pm\|_{L^\infty([0,T_{\lambda,\varepsilon}]\times \boldsymbol{R}_+)} \le C_\lambda \varepsilon^{1-\alpha/(p-2)} \quad (C_\lambda\varepsilon \text{ if } p = 2)\,. \tag{5.4}
$$

This implies the error estimate of Proposition 5.1. For the sake of readability, we will omit the distinction $p = 2$, and keep the notation $\varepsilon^{1-\alpha/(p-2)}$, with an obvious convention.

From Lemma 3.1, v^ε, hence w^ε, is defined, bounded and continuous, locally in time. At time $t = 0$, it is of order ε, so there exists $t^\varepsilon > 0$ such that

$$
\|w^\varepsilon_\pm\|_{L^\infty([0,t^\varepsilon]\times \boldsymbol{R}_+)} < 2\varepsilon\|P_1\|_{L^\infty}\,,
$$

and we have, possibly increasing the value of C_λ,

$$
\|w^\varepsilon_\pm\|_{L^\infty([0,t^\varepsilon]\times \boldsymbol{R}_+)} \le C_\lambda \varepsilon^{1-\alpha/(p-2)}\,.
$$

So long as $w^\varepsilon_\pm$ is pointwise bounded by $2C_\lambda$, f^ε remains uniformly bounded. As noticed in Example 2.4, there exists $C_2(\lambda)$ such that

$$
\int_0^{T_{\lambda,\varepsilon}} \sup_{\gamma\in\Gamma^t} \varepsilon^\alpha r^{1-p} \left|f^\varepsilon\right| dt \le C_2(\lambda)\,.
$$

Now consider the source terms. On $\operatorname{supp}(v^\varepsilon_+)_{\text{app}}$ (of size $2z_0\varepsilon$, and transverse to any $\gamma \in \Gamma_-^{T(\lambda,\varepsilon)}$), $r > r_0/2$, therefore the singular term r^{1-p} is bounded and

$$
\sup_{\gamma_-\in\Gamma_-^{T(\lambda,\varepsilon)}} \int_{\gamma_-} |S^\varepsilon_-| \le C\varepsilon^{1+\alpha}\,.
$$

More delicate is the treatment of S^ε_+. Because $\operatorname{supp}(v^\varepsilon_-)_{\text{app}}$ is of size $2z_0\varepsilon$ and transverse to any $\gamma_+ \in \Gamma_+^{T(\lambda,\varepsilon)}$, one has

$$
\sup_{\gamma_+\in\Gamma_+^{T(\lambda,\varepsilon)}} \int_{\gamma_+} |S^\varepsilon_+| \le C\varepsilon^\alpha \int_{\lambda\varepsilon^{\alpha/(p-2)}}^{\lambda\varepsilon^{\alpha/(p-2)}+2z_0\varepsilon} r^{1-p}dr = C\varepsilon^\alpha F_p(\lambda\varepsilon^{\alpha/(p-2)}, \lambda\varepsilon^{\alpha/(p-2)} + 2z_0\varepsilon)\,.
$$

If $p > 2$,

$$
\begin{aligned}
(p-2)\varepsilon^\alpha F_p(\lambda\varepsilon^{\alpha/(p-2)}, \lambda\varepsilon^{\alpha/(p-2)} + 2z_0\varepsilon) &= \varepsilon^\alpha((\lambda\varepsilon^{\alpha/(p-2)})^{2-p} - (\lambda\varepsilon^{\alpha/(p-2)} + 2z_0\varepsilon)^{2-p}) \\
&= \lambda^{2-p}\left(1 - \left(1 + \frac{2z_0}{\lambda}\varepsilon^{1-\alpha/(p-2)}\right)^{2-p}\right) \\
&= O(\lambda^{1-p}\varepsilon^{1-\alpha/(p-2)})\,.
\end{aligned}
$$

The case $p = 2$ yields the same estimate. From Proposition 2.3, $w^\varepsilon_\pm$ remains pointwise bounded by $2C_\lambda$ for $t \in [0, T(\lambda, \varepsilon)]$, for $\varepsilon \in]0, \varepsilon(\lambda)]$. This yields (5.4), and completes the proof of Proposition 5.1. □

Proposition 5.1 describes the behavior of the exact solution v^ε up to time $T(\lambda, \varepsilon)$. The outline of the end of the proof in the dissipative case is as follows. At time $t = T(\lambda, \varepsilon)$, the approximate solution $(v^\varepsilon)_{\text{app}}$ is of order $\lambda^{(p-2)/(p-1)}$ if $p > 2$ ($1/|\log \lambda|$ if $p = 2$). Letting ε go to zero, with λ sufficiently small, shows that v^ε becomes arbitrarily small when approaching the focal point. Since the equation is dissipative, this means that v^ε is absorbed. The end of the proof of Theorem 1.2 relies on energy estimates. For $q \geq 1$, define

$$g_{q-1}(s) := \frac{d}{ds}|s|^q = q|s|^{q-1}\operatorname{sgn} s .$$

Then g_q is a non-increasing odd function of s, which is homogeneous of degree q.

In (1.5), multiply the equation satisfied by v^ε_- by $g_{q-1}(v^\varepsilon_-)$, and the equation satisfied by v^ε_+ by $g_{q-1}(v^\varepsilon_+)$. Summing up these yields

$$\begin{aligned}&\partial_t(|v^\varepsilon_-|^q + |v^\varepsilon_+|^q) + \partial_r(|v^\varepsilon_+|^q - |v^\varepsilon_-|^q)\\ &\qquad = -\, a2^{-p}r^{1-p}\left(g_{q-1}(v^\varepsilon_-) + g_{q-1}(v^\varepsilon_+)\right) g_p(v^\varepsilon_- + v^\varepsilon_+)/(p+1) .\end{aligned}$$

The signs of both $g_{q-1}(v^\varepsilon_-) + g_{q-1}(v^\varepsilon_+)$ and $g_p(v^\varepsilon_- + v^\varepsilon_+)$ are equal to the sign of the larger of $v^\varepsilon_\pm$. Therefore, when $a > 0$ (dissipative case),

$$\partial_t(|v^\varepsilon_-|^q + |v^\varepsilon_+|^q) - \partial_r(|v^\varepsilon_-|^q - |v^\varepsilon_+|^q) \leq 0 . \tag{5.5}$$

For a fixed $t > 0$, integrate this inequality from $r = 0$ to $r = t$. Recall that when $r = 0$, we have $|v^\varepsilon_-| = |v^\varepsilon_+|$, and so this yields

$$\int_0^t \partial_t(|v^\varepsilon_-|^q + |v^\varepsilon_+|^q)(t, r)dr - |v^\varepsilon_-|^q(t, t) + |v^\varepsilon_+|^q(t, t) \leq 0 .$$

Therefore,

$$\partial_t \int_0^t (|v^\varepsilon_-|^q + |v^\varepsilon_+|^q)(t, r)dr \leq 2|v^\varepsilon_-|^q(t, t) . \tag{5.6}$$

Let $\lambda > 0$. From Proposition 5.1, for $0 \leq t \leq T_{\lambda,\varepsilon}$,

$$|v^\varepsilon_-|^q(t, t) = |(v^\varepsilon_-)_{\text{app}}|^q(t, t) + O(\varepsilon^{q(1-\alpha/(p-2))}) .$$

For $t > (r_0 + z_0\varepsilon)/2$, $(v^\varepsilon_-)_{\text{app}}(t, t) = 0$ and

$$|v^\varepsilon_-|^q(t, t) = O(\varepsilon^{q(1-\alpha/(p-2))}) .$$

Let $T > T_{\lambda,\varepsilon}$ and integrate (5.6) between $t = T_{\lambda,\varepsilon}$ and $t = T$. Then

$$\begin{aligned}\|v^\varepsilon_-(T)\|^q_{L^q(0,T)} + \|v^\varepsilon_+(T)\|^q_{L^q(0,T)} &\leq \|v^\varepsilon_-(T_{\lambda,\varepsilon})\|^q_{L^q(0,T_{\lambda,\varepsilon})} + \|v^\varepsilon_+(T_{\lambda,\varepsilon})\|^q_{L^q(0,T_{\lambda,\varepsilon})}\\ &\quad + C(\lambda)^q T\varepsilon^{q(1-\alpha/(p-2))} ,\end{aligned}$$

and

$$\|v_-^\varepsilon(T)\|_{L^q(0,T)} + \|v_+^\varepsilon(T)\|_{L^q(0,T)} \le \|v_-^\varepsilon(T_{\lambda,\varepsilon})\|_{L^q(0,T_{\lambda,\varepsilon})} + \|v_+^\varepsilon(T_{\lambda,\varepsilon})\|_{L^q(0,T_{\lambda,\varepsilon})} + C(\lambda)T^{1/q}\varepsilon^{1-\alpha/(p-2)}\,.$$

Letting $q \to \infty$ yields

$$\|v_-^\varepsilon(T)\|_{L^\infty(0,T)} + \|v_+^\varepsilon(T)\|_{L^\infty(0,T)} \le \|v_-^\varepsilon(T_{\lambda,\varepsilon})\|_{L^\infty(0,T_{\lambda,\varepsilon})} + \|v_+^\varepsilon(T_{\lambda,\varepsilon})\|_{L^\infty(0,T_{\lambda,\varepsilon})} + C(\lambda)\varepsilon^{1-\alpha/(p-2)}\,.$$

Using Proposition 5.1 again, and the fact that $\operatorname{supp}(v_+^\varepsilon)_{\mathrm{app}} \subset \{|r-t-r_0| \le z_0\varepsilon\}$, we also have

$$\|v_-^\varepsilon(T)\|_{L^\infty(0,T)} + \|v_+^\varepsilon(T)\|_{L^\infty(0,T)} \le \|(v_-^\varepsilon)_{\mathrm{app}}(T_{\lambda,\varepsilon})\|_{L^\infty(0,T_{\lambda,\varepsilon})} + C(\lambda)\varepsilon^{1-\alpha/(p-2)}\,.$$

From the explicit expression for $(v_-^\varepsilon)_{\mathrm{app}}$,

$$\|(v_-^\varepsilon)_{\mathrm{app}}(T_{\lambda,\varepsilon})\|_{L^\infty(0,T_{\lambda,\varepsilon})} \le C \times \begin{cases} \lambda^{(p-2)/(p-1)} & \text{if } p > 2, \\ 1/|\log\lambda| & \text{if } p = 2, \end{cases} \tag{5.7}$$

where C does not depend on λ. Thus if $p > 2$,

$$\|v_-^\varepsilon(T)\|_{L^\infty(0,T)} + \|v_+^\varepsilon(T)\|_{L^\infty(0,T)} \le C\lambda^{(p-2)/(p-1)} + C(\lambda)\varepsilon^{1-\alpha/(p-2)}\,.$$

Therefore, for any $T \ge r_0$ and any $\lambda > 0$,

$$\limsup_{\varepsilon\to 0}(\|v_-^\varepsilon(T)\|_{L^\infty(0,T)} + \|v_+^\varepsilon(T)\|_{L^\infty(0,T)}) \le C\lambda^{(p-2)/(p-1)}\,. \tag{5.8}$$

Letting $\lambda \to 0$ yields the first part of Theorem 1.2 for $p > 2$. The case $p = 2$ is straightforward.

When the equation is accretive ($a < 0$), the energy estimate (5.5) becomes

$$\partial_t(|v_-^\varepsilon|^q + |v_+^\varepsilon|^q)(t,r) - \partial_r(|v_-^\varepsilon|^q - |v_+^\varepsilon|^q)(t,r) \ge 0\,. \tag{5.9}$$

Since we assumed $P_+ \equiv 0$, $(v_-^\varepsilon)_{\mathrm{app}}$ blows up in finite time, while $(v_+^\varepsilon)_{\mathrm{app}}$ does not. The mechanism occurs quite in the same way as the cancellation of $(v_-^\varepsilon)_{\mathrm{app}}$ in the dissipative case. For a fixed $t > 0$, integrating (5.9) between $r = 0$ and $r = \infty$ yields, by finite speed of propagation,

$$\int_0^\infty \partial_t(|v_-^\varepsilon|^q + |v_+^\varepsilon|^q)(t,r)dr = \partial_t \int_0^\infty (|v_-^\varepsilon|^q + |v_+^\varepsilon|^q)(t,r)dr \ge 0\,.$$

Let $r_0 > T > T_{\lambda,\varepsilon}$ and integrate the above inequality between $t = T_{\lambda,\varepsilon}$ and $t = T$. Then

$$\|v_-^\varepsilon(T)\|^q_{L^q(\boldsymbol{R}_+)} + \|v_+^\varepsilon(T)\|^q_{L^q(\boldsymbol{R}_+)} \ge \|v_-^\varepsilon(T_{\lambda,\varepsilon})\|^q_{L^q(\boldsymbol{R}_+)} + \|v_+^\varepsilon(T_{\lambda,\varepsilon})\|^q_{L^q(\boldsymbol{R}_+)}\,.$$

Letting $q \to \infty$ yields

$$\|v_-^\varepsilon(T)\|_{L^\infty(\boldsymbol{R}_+)} + \|v_+^\varepsilon(T)\|_{L^\infty(\boldsymbol{R}_+)} \ge \|v_-^\varepsilon(T_{\lambda,\varepsilon})\|_{L^\infty(\boldsymbol{R}_+)}\,.$$

From Proposition 5.1, it follows that

$$\liminf_{\varepsilon\to 0}(\|v_-^\varepsilon(T)\|_{L^\infty(\boldsymbol{R}_+)} + \|v_+^\varepsilon(T)\|_{L^\infty(\boldsymbol{R}_+)}) \ge \liminf_{\varepsilon\to 0}\|(v_-^\varepsilon)_{\mathrm{app}}(T_{\lambda,\varepsilon})\|_{L^\infty(\boldsymbol{R}_+)}\,.$$

Letting $\lambda \to 0$ yields the last part of Theorem 1.2 with $T^* = r_0$, using the explicit form of $(v_-^\varepsilon)_{\mathrm{app}}$. As we already mentioned, the result may hold with $T^* < r_0$ if $(v_+^\varepsilon)_{\mathrm{app}}$ blows up before $(v_-^\varepsilon)_{\mathrm{app}}$, in which case the proof is essentially as above.

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