

FOKKER-PLANCK-KOLMOGOROV EQUATIONS ASSOCIATED WITH TIME-CHANGED FRACTIONAL BROWNIAN MOTION

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ABSTRACT. In this paper Fokker-Planck-Kolmogorov type equations associated with stochastic differential equations driven by a time-changed fractional Brownian motion are derived. Two equivalent forms are suggested. The time-change process considered is the first hitting time process for either a stable subordinator or a mixture of stable subordinators. A family of operators arising in the representation of the Fokker-Planck-Kolmogorov equations is shown to have the semigroup property.

1. INTRODUCTION

In this paper we establish Fokker-Planck-Kolmogorov type (FPK) equations, also called governing equations, associated with stochastic differential equations driven by a time-changed fractional Brownian motion (fBM). A one-dimensional fBM B_t^H is a zero-mean Gaussian process with continuous paths and covariance function

$$(1.1) \quad R_H(s, t) = E(B_s^H B_t^H) = \frac{1}{2}(s^{2H} + t^{2H} - |s - t|^{2H}),$$

where the Hurst parameter H takes values in $(0, 1)$. If $H = \frac{1}{2}$, then B_t^H becomes a standard Brownian motion. Stochastic processes driven by an fBM are of increasing interest for both theorists and applied researchers due to their wide application in fields such as mathematical finance [6, 33], solar activities [34], turbulence [10], etc.

Fractional Brownian motion B_t^H , like standard Brownian motion, has nowhere differentiable sample-paths and stationary increments, but it does not have independent increments. Namely, the covariance between increments over non-overlapping intervals is positive, if $\frac{1}{2} < H < 1$, and negative, if $0 < H < \frac{1}{2}$. In particular, when $\frac{1}{2} < H < 1$, increments of B_t^H exhibit long range dependence. B_t^H has the integral representation $B_t^H = \int_0^t K_H(t, s) dB_s$, where B_t is a Brownian motion. We refer the reader to [5, 30] for details of the above properties, including forms for $K_H(t, s)$.

Fractional Brownian motion is not a semimartingale unless $H = \frac{1}{2}$ ([5, 30]), so the usual Itô's stochastic calculus is not valid. Nevertheless, there are several approaches [3, 5, 8, 30] to a stochastic calculus in order to interpret in a meaningful

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way a stochastic differential equation (SDE) driven by an m -dimensional fBM B_t^H of the form

$$(1.2) \quad X_t = X_0 + \int_0^t b(X_s) ds + \int_0^t \sigma(X_s) dB_s^H,$$

where mappings $b : \mathbb{R}^n \rightarrow \mathbb{R}^n$ and $\sigma : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$ are Lipschitz continuous and bounded; X_0 is a random variable independent of B_t^H . We do not discuss in this paper these approaches, but refer the interested reader to [5, 7, 30]. Instead, we focus our attention on the FPK equation associated with SDE (1.2) driven by fBM whose generic form is given by

$$(1.3) \quad \frac{\partial u(t, x)}{\partial t} = B(x, D_x)u(t, x) + Ht^{2H-1}A(x, D_x)u(t, x),$$

where $B(x, D_x) = \sum_{j=1}^n b_j(x) \frac{\partial}{\partial x_j}$, a first order differential operator, and $A(x, D_x)$ is a second order elliptic differential operator

$$(1.4) \quad A(x, D_x) = \sum_{j,k=1}^n a_{jk}(x) \frac{\partial^2}{\partial x_j \partial x_k}.$$

Functions $a_{jk}(x)$, $j, k = 1, \dots, n$, are entries of the matrix $\mathcal{A}(x) = \sigma(x) \times \sigma^T(x)$, where $\sigma^T(x)$ is the transpose of matrix $\sigma(x)$. By definition $\mathcal{A}(x)$ is positive definite: for any $x \in \mathbb{R}^n$ and $\xi \in \mathbb{R}^n$ one has $\sum_{j,k=1}^n a_{jk}(x) \xi_j \xi_k \geq C|\xi|^2$, where C is a positive constant. The operator $A(x, D_x)$ can also be given in the divergent form

$$(1.5) \quad A(x, D_x) = \sum_{j,k=1}^n \frac{\partial}{\partial x_j} \left(a_{jk}(x) \frac{\partial}{\partial x_k} \right).$$

The right hand side of (1.3) depends on the time variable t , which, in fact, reflects the presence of correlation. Additionally, $u(t, x)$ in equation (1.3) satisfies the initial condition

$$(1.6) \quad u(0, x) = \varphi(x), \quad x \in \mathbb{R}^n,$$

where $\varphi(x)$ belongs to some function space or is a generalized function. In the particular case of FPK equation associated with SDE (1.2), $\varphi(x) = f_{X_0}(x)$, the density function of X_0 . If $X_0 = x_0 \in \mathbb{R}^n$, then $\varphi(x) = \delta_{x_0}(x)$, Dirac's delta with mass on x_0 . In this case the solution to the FPK equation is understood in the weak sense.

In the one-dimensional case with $H \in (\frac{1}{4}, 1)$, paper [2] establishes that $u(t, x) = E_x[\varphi(X_t)]$ solves the equation (1.3) with initial condition (1.6) when X_t solves SDE (1.2) with $b = 0$ and a stochastic integral in the sense of Stratanovich. The operator $A(x, D_x)$ appearing in (1.3) is expressed in the divergence form (1.5) (see Example 28 in [2]). Paper [11] derives an FPK equation with $A(x, D_x)$ in the form (1.4). However, the derivation of the FPK equation in this paper is based on the Itô formula obtained in [3] for fBM (that is, for $b = 0$ and $\sigma = I$, the identity matrix). Therefore, their derivation might require a modification. In the general setting of (1.2) and (1.3), it is not known to us whether $u(t, x) = E_x[\varphi(X_t)]$ solves (1.3) with initial condition (1.6) when X_t solves (1.2).

In the sequel we use the following notation:

$$(1.7) \quad L_\gamma(t, x, D_x) = B(x, D_x) + \frac{\gamma+1}{2} t^\gamma A(x, D_x),$$

where $\gamma = 2H - 1$. The introduction of γ is made so that the operators G_γ arising in Section 3 will have the semigroup property. If $\gamma = 0$, equivalently $H = \frac{1}{2}$, then the operator $L_0(t, x, D_x)$ has a form with coefficients not depending on t :

$$(1.8) \quad L_0(t, x, D_x) \equiv L(x, D_x) = B(x, D_x) + \frac{1}{2}A(x, D_x),$$

and equation (1.3) coincides with the FPK equation associated with the SDE driven by Brownian motion B_t (see, e.g. [36]):

$$(1.9) \quad \frac{\partial u(t, x)}{\partial t} = L(x, D_x)u(t, x).$$

The fractional FPK equation is obtained from equation (1.9) upon replacing the first order derivative on its left hand side by the time-fractional derivative D_*^β in the sense of Caputo-Djrbashian [12]. By definition, the Caputo-Djrbashian derivative of order β is given by

$$(1.10) \quad D_*^\beta f(t) = \frac{1}{\Gamma(1-\beta)} \int_0^t \frac{f'(\tau)d\tau}{(t-\tau)^\beta}, \quad 0 < \beta < 1,$$

where $\Gamma(\cdot)$ stands for Euler's Gamma function. Introducing the fractional integration operator

$$J^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad \alpha > 0,$$

one can represent D_*^β in the form $D_*^\beta = J^{1-\beta} \frac{d}{dt}$. We also write $D_{*,t}^\beta$ or J_t^α emphasizing that the fractional operators act with respect to the variable t . An equivalent but slightly different representation of the fractional FPK equation is possible through the Riemann-Liouville derivative also; see e.g. [35]. The obtained initial value problem for the time-fractional FPK equation

$$(1.11) \quad D_{*,t}^\beta v(t, x) = L(x, D_x)v(t, x), \quad t > 0, \quad x \in \mathbb{R}^n,$$

$$(1.12) \quad v(0, x) = \varphi(x), \quad x \in \mathbb{R}^n,$$

describes the dynamics of a stochastic process driven by a time-changed Brownian motion (see Section 2). Such equations appear in many fields, including statistical physics [29, 40], finance [15], hydrology [4], cell biology [31], etc. Existence and uniqueness theorems related to the initial value problem for fractional differential equations, as well as more general distributed order equations, can be found in [9, 19, 26, 38]. Instead, we focus on how fractional order FPK equations are obtained from non-fractional FPK equations.

By definition, a time-change process is a stochastic process with continuous non-decreasing sample paths starting at 0. For details concerning general time-changed stochastic processes, see [17]. Let E be the time-change process given by the first hitting time process for a β -stable subordinator, with $\beta \in (0, 1)$, independent of B_t^H . Suppose $H = \frac{1}{2}$ and the stochastic integral in SDE (1.2) is understood in the sense of Itô. If the driving process in (1.2) is replaced by a composition of the driving process with E , then the left hand side of equation (1.9) becomes the fractional derivative of order β and the right hand side remains unchanged. For details see [16, 21]. As we will see, *this is not the case for fractional FPK equations associated with SDEs driven by time-changed fBM* (Section 3). The abstract version of equation (1.11) with the infinitesimal generator of a strongly continuous semigroup in place of L was obtained first in [23]; see also [27].

Fractional FPK equations associated with SDEs driven by a time-changed fBM (see equation (1.13) below) have not yet been determined. Meerschaert et al. [25] studied the continuous time random walk (CTRW) limits for certain correlated random variables, which include linear fractional stable motions, and in particular, fBM. For the latter, the scaling limits represent time-changed fBM, where the time-change process is the inverse to a stable subordinator. Authors of that paper write, “An interesting open question is to establish the governing equation for the CTRW scaling limit.” A particular case of our Theorem 3.1 answers that question. (See also Remark 3.7 (3).)

There are several approaches for deriving equation (1.11), including via semigroup theory [1, 16], master equations [24, 32], and continuous time random walks [13, 14, 28, 39]. In this paper we use a different technique, which can be extended for equations with a time dependent right hand side as well, including equations of the form (1.3). This technique is close to the method used in [20].

SDEs driven by fBM are studied by several authors using different approaches; for references we refer the reader to [5]. SDEs driven by time-changed Brownian motion are discussed in [18]. The associated fractional FPK equations driven by time-changed Lévy processes when the time-change process is the inverse to an arbitrary mixture of stable subordinators are studied by Hahn et al. in [16]. Note that any time-changed semimartingale is again a semimartingale [17]. However, since fBM is not a semimartingale if $H \neq \frac{1}{2}$, the methods used in [16] and [18] are not applicable in this case. We plan to discuss a possible interpretation of SDEs driven by a time-changed fractional Brownian and linear fractional stable motion in a separate paper. Instead, in the present paper we derive FPK type equations associated with the SDE

$$(1.13) \quad X_t = x_0 + \int_0^t b(X_s) dE_s + \int_0^t \sigma(X_s) dB_{E_s}^H,$$

where E_t is the inverse to an arbitrary mixture of stable subordinators with indices in $(0, 1)$. Throughout the paper we assume that E_t is independent of the driving process B_t^H . An important particular case is when E_t is the inverse to a single stable subordinator. The main ideas used in this paper will be illustrated in this simpler case. The associated FPK equation can be represented as a time-fractional order differential equation, but the right hand side does not coincide with the right hand side of equation (1.3) unless $\gamma = 0$ ($H = \frac{1}{2}$). However, in the case of zero drift (i.e. $b(x) \equiv 0$), the FPK equation can be obtained with the same operator as on the right hand side of (1.3), but in this case the left hand side is *not* a time-fractional differential operator. This difference of FPK equations is an essential consequence of the correlation of the increments of the fBM that is the driving process of the corresponding SDEs.

The paper is organized as follows. Section 2 illustrates the method of this paper when the driving process is a time-changed Brownian motion. The results obtained further clarify properties of density functions of processes which are inverses of arbitrary mixtures of stable subordinators. In Section 3, two equivalent FPK equations associated with SDEs driven by time-changed fBM are obtained, extending the technique used in Section 2. Furthermore, the family of operators appearing in the FPK equations is shown to have the semigroup property.

2. FPK EQUATIONS ASSOCIATED WITH SDEs DRIVEN BY A TIME-CHANGED BROWNIAN MOTION

Consider an SDE driven by a time-changed Brownian motion:

$$(2.1) \quad X_t = x_0 + \int_0^t b(X_s)dE_s + \int_0^t \sigma(X_s)dB_{E_s}, \quad t > 0,$$

where $b(x)$ and $\sigma(x)$ are Lipschitz continuous mappings and E_t is the first hitting time process for a stable subordinator W_t with stability index $\beta \in (0, 1)$. The process E_t is also called an inverse to W_t . The relation between E_t and W_t can be expressed as $E_t = \min\{\tau : W_\tau \geq t\}$. The process $W_t, t \geq 0$, is a self-similar Lévy process with $W_0 = 0$, that is, $W_{ct} = c^{1/\beta}W_t$ as processes in the sense of finite dimensional distributions, and its Laplace transform is $\mathbb{E}(e^{-sW_t}) = e^{-ts^\beta}$. The density $f_{W_1}(\tau)$ of W_1 is infinitely differentiable on $(0, \infty)$, with the following asymptotics at zero and infinity [22, 37]:

$$(2.2) \quad f_{W_1}(\tau) \sim \frac{(\frac{\beta}{\tau})^{\frac{2-\beta}{2(1-\beta)}}}{\sqrt{2\pi\beta(1-\beta)}} e^{-(1-\beta)(\frac{\tau}{\beta})^{-\frac{\beta}{1-\beta}}}, \quad \tau \rightarrow 0;$$

$$(2.3) \quad f_{W_1}(\tau) \sim \frac{\beta}{\Gamma(1-\beta)\tau^{1+\beta}}, \quad \tau \rightarrow \infty.$$

Since W_t is strictly increasing, its inverse process E_t is continuous and nondecreasing, but not a Lévy process. Likewise the time-changed process B_{E_t} is also not a Lévy process (see details in [16]). The associated FPK equation for SDE (2.1) has the form

$$(2.4) \quad D_*^\beta v(t, x) = L(x, D_x)v(t, x),$$

with the initial condition $v(0, x) = \delta_{x_0}(x)$, where $L(x, D_x)$ is defined in (1.8) and D_*^β is the fractional derivative in the sense of Caputo-Djrbashian.

Notice that solutions to equations (2.4) and (1.9) are connected by a certain relationship. Namely, a solution $v(t, x)$ to equation (2.4) satisfying the initial condition (1.6) can be represented through the solution $u(t, x)$ to equation (1.9), satisfying the same initial condition (1.6), by the formula

$$(2.5) \quad v(t, x) = \int_0^\infty f_t(\tau)u(\tau, x)d\tau,$$

where $f_t(\tau)$ is the density function of E_t for each fixed $t > 0$. If $f_{W_1}(t)$ is the density function of W_1 , then

$$(2.6) \quad \begin{aligned} f_t(\tau) &= \frac{\partial}{\partial \tau}P(E_t \leq \tau) = \frac{\partial}{\partial \tau}(1 - P(W_\tau < t)) = -\frac{\partial}{\partial \tau}P(W_1 < \frac{t}{\tau^{1/\beta}}) \\ &= -\frac{\partial}{\partial \tau}[Jf_{W_1}](\frac{t}{\tau^{1/\beta}}) = -\frac{\partial}{\partial \tau} \int_0^{\frac{t}{\tau^{1/\beta}}} f_{W_1}(u)du = \frac{t}{\beta\tau^{1+\frac{1}{\beta}}} f_{W_1}(\frac{t}{\tau^{1/\beta}}). \end{aligned}$$

Since $f_{W_1}(u) \in C^\infty(0, \infty)$, it follows from representation (2.6) that $f_t(\tau) \in C^\infty(\mathbb{R}_+^2)$, where $\mathbb{R}_+^2 = (0, \infty) \times (0, \infty)$. Further properties of $f_t(\tau)$ are represented in the following lemma.

Lemma 2.1. *Let $f_t(\tau)$ be the function given in (2.6). Then*

- (a) $\lim_{t \rightarrow +0} f_t(\tau) = \delta_0(\tau)$ in the sense of the topology of the space of tempered distributions $\mathcal{D}'(\mathbb{R})$;

- (b) $\lim_{\tau \rightarrow +0} f_t(\tau) = \frac{t^{-\beta}}{\Gamma(1-\beta)}$, $t > 0$;
 (c) $\lim_{\tau \rightarrow \infty} f_t(\tau) = 0$, $t > 0$;
 (d) $\mathcal{L}_{t \rightarrow s}[f_t(\tau)](s) = s^{\beta-1}e^{-\tau s^\beta}$, $s > 0$, $\tau \geq 0$,

where $\mathcal{L}_{t \rightarrow s}$ denotes the Laplace transform with respect to the variable t .

Proof. (a) Let $\psi(\tau)$ be an infinitely differentiable function rapidly decreasing at infinity. We have to show that $\lim_{t \rightarrow +0} \langle f_t, \psi \rangle = \psi(0)$. Here $\langle f_t, \psi \rangle$ denotes the value of $f_t \in \mathcal{D}'(\mathbb{R})$ on ψ . We have

$$\begin{aligned} \lim_{t \rightarrow +0} \langle f_t(\tau), \psi(\tau) \rangle &= \lim_{t \rightarrow +0} \int_0^\infty f_t(\tau) \psi(\tau) d\tau = \lim_{t \rightarrow +0} \int_0^\infty f_{W_1}(u) \psi\left(\left(\frac{t}{u}\right)^\beta\right) du \\ &= \psi(0) \int_0^\infty f_{W_1}(u) du = \psi(0). \end{aligned}$$

Parts (b) and (c) follow from asymptotic relations (2.3) and (2.2), respectively. Part (d) is straightforward; just compute the Laplace transform of $f_t(\tau)$ using the representation $f_t(\tau) = -\frac{\partial}{\partial \tau} [Jf_{W_1}]\left(\frac{t}{\tau^{1/\beta}}\right)$. \square

Due to part (b) of Lemma 2.1, $f_t \in C^\infty(0, \infty)$ for each fixed $\tau \geq 0$. Hence, the fractional derivative $D_{*,t}^\beta f_t(\tau)$ in the variable t is meaningful and is a generalized function of the variable τ .

Lemma 2.2. *Function $f_t(\tau)$ defined in (2.6) for each $t > 0$ satisfies the equation*

$$(2.7) \quad D_{*,t}^\beta f_t(\tau) = -\frac{\partial}{\partial \tau} f_t(\tau) - \frac{t^{-\beta}}{\Gamma(1-\beta)} \delta_0(\tau),$$

in the sense of tempered distributions.

Proof. The Laplace transform (in variable t) of the left hand side, using the definition (2.6) of $f_t(\tau)$, equals

$$\begin{aligned} \mathcal{L}_{t \rightarrow s}[D_{*,t}^\beta f_t(\tau)](s) &= s^\beta \mathcal{L}_{t \rightarrow s}[f_t(\tau)](s) - s^{\beta-1} \lim_{t \rightarrow +0} f_t(\tau) \\ &= s^{2\beta-1} e^{-\tau s^\beta} - s^{\beta-1} \delta_0(\tau), \quad s > 0. \end{aligned}$$

On the other hand, the Laplace transform of the right hand side equals

$$\begin{aligned} \mathcal{L}_{t \rightarrow s} \left[-\frac{\partial}{\partial \tau} f_t(\tau) - \frac{t^{-\beta}}{\Gamma(1-\beta)} \delta_0(\tau) \right](s) &= \frac{\partial^2}{\partial \tau^2} \left(\frac{1}{s} e^{-\tau s^\beta} \right) - s^{\beta-1} \delta_0(\tau) \\ &= s^{2\beta-1} e^{-\tau s^\beta} - s^{\beta-1} \delta_0(\tau), \quad s > 0, \end{aligned}$$

completing the proof. \square

Derivation of fractional FPK equation. Now it is easy to show the derivation of the fractional order FPK equation (2.4), a solution of which is given by $v(t, x)$ in (2.5). We have

$$\begin{aligned} D_{*,t}^\beta v(t, x) &= \int_0^\infty D_{*,t}^\beta f_t(\tau) u(\tau, x) d\tau = - \int_0^\infty \left[\frac{\partial}{\partial \tau} f_t(\tau) + \frac{t^{-\beta}}{\Gamma(1-\beta)} \delta_0(\tau) \right] u(\tau, x) d\tau \\ &= - \lim_{\tau \rightarrow \infty} [f_t(\tau) u(\tau, x)] + \lim_{\tau \rightarrow 0} [f_t(\tau) u(\tau, x)] \\ &\quad + \int_0^\infty f_t(\tau) \frac{\partial}{\partial \tau} u(\tau, x) d\tau - \frac{t^{-\beta}}{\Gamma(1-\beta)} u(0, x). \end{aligned}$$

Due to Lemma 2.1, part (c) implies that the first term vanishes since $u(\tau, x)$ is bounded, while part (b) implies that the second and last terms cancel. Taking into account (1.9),

$$(2.8) \quad D_{*,t}^\beta v(t, x) = \int_0^\infty f_t(\tau) L(x, D_x) u(\tau, x) d\tau = L(x, D_x) v(t, x).$$

Moreover, by property (a) of Lemma 2.1,

$$\lim_{t \rightarrow +0} v(t, x) = \langle \delta_0(\tau), u(\tau, x) \rangle = u(0, x) = \delta_{x_0}(x). \quad \square$$

Our technique extends to the more general case when the time-change process is the first hitting time for an arbitrary mixture of independent stable subordinators. Let $\rho(s) = \int_0^1 s^\beta d\mu(\beta)$, where μ is a finite measure with $\text{supp } \mu \subset (0, 1]$. Let W_t^μ be a nonnegative stochastic process satisfying $\mathbb{E}(e^{-sW_t^\mu}) = e^{-t\rho(s)}$, and let $E_t^\mu = \min\{\tau : W_\tau^\mu \geq t\}$. The process W_t^μ represents a mixture of independent stable subordinators with a mixing measure μ (see [16]).

Theorem 2.3. *Let $u(t, x)$ be a solution of the Cauchy problem*

$$(2.9) \quad \frac{\partial u(t, x)}{\partial t} = L(x, D_x) u(t, x), \quad t > 0, \quad x \in \mathbb{R}^n,$$

$$(2.10) \quad u(0, x) = \varphi(x), \quad x \in \mathbb{R}^n.$$

Then the function $v(t, x) = \int_0^\infty f_t^\mu(\tau) u(\tau, x)$, where $f_t^\mu(\tau)$ is the density function of E_t^μ , satisfies the initial value problem for the distributed order differential equation

$$(2.11) \quad D_\mu v(t, x) \equiv \int_0^1 D_{*,t}^\beta v(t, x) d\mu(\beta) = L(x, D_x) v(t, x), \quad t > 0, \quad x \in \mathbb{R}^n,$$

$$(2.12) \quad v(0, x) = \varphi(x), \quad x \in \mathbb{R}^n.$$

The proof of this theorem requires two lemmas which generalize Lemmas 2.1 and 2.2. Define the function

$$(2.13) \quad \Phi_\mu(t) = \int_0^1 \frac{t^{-\beta}}{\Gamma(1-\beta)} d\mu(\beta), \quad t > 0.$$

Lemma 2.4. *Let $f_t^\mu(\tau)$ be the function defined in Theorem 2.3. Then*

- (a) $\lim_{t \rightarrow +0} f_t^\mu(\tau) = \delta_0(\tau), \quad \tau \geq 0;$
- (b) $\lim_{\tau \rightarrow +0} f_t^\mu(\tau) = \Phi_\mu(t), \quad t > 0;$
- (c) $\lim_{\tau \rightarrow \infty} f_t^\mu(\tau) = 0, \quad t > 0;$
- (d) $\mathcal{L}_{t \rightarrow s}[f_t^\mu(\tau)](s) = \frac{\rho(s)}{s} e^{-\tau\rho(s)}, \quad s > 0, \quad \tau \geq 0.$

Proof. First notice that $f_t^\mu(\tau) = f_{E_t^\mu}(\tau) = -\frac{\partial}{\partial \tau}[Jf_{W_\tau^\mu}](t)$, where J is the usual integration operator. The proofs of parts (a) – (c) are similar to the proofs of parts (a) – (c) of Lemma 2.1. Further, using the definition of W_t^μ ,

$$\mathcal{L}_{t \rightarrow s}[f_t^\mu(\tau)](s) = -\frac{1}{s} \frac{\partial}{\partial \tau} \mathcal{L}_{t \rightarrow s}[f_{W_\tau^\mu}(t)](s) = \frac{\rho(s)}{s} e^{-\tau\rho(s)}, \quad s > 0,$$

which completes the proof. □

Lemma 2.5. *The function $f_t^\mu(\tau)$ defined in Theorem 2.3 satisfies for each $t > 0$ the following equation*

$$(2.14) \quad D_{\mu,t}f_t^\mu(\tau) = -\frac{\partial}{\partial\tau}f_t^\mu(\tau) - \delta_0(\tau)\Phi_\mu(t),$$

in the sense of tempered distributions.

Proof. Integrating both sides of the equation

$$\mathcal{L}_{t \rightarrow s}[D_{*,t}^\beta f_t^\mu(\tau)] = s^\beta \mathcal{L}_{t \rightarrow s}[f_t^\mu(\tau)](s) - s^{\beta-1}\delta_0(\tau)$$

and taking into account part (d) of Lemma 2.4 yield

$$\mathcal{L}_{t \rightarrow s}[D_{\mu,t}f_t^\mu(\tau)] = \frac{\rho^2(s)}{s}e^{-\tau\rho(s)} - \frac{\rho(s)}{s}\delta_0(\tau).$$

It is easy to verify that the latter coincides with the Laplace transform of the right hand side of (2.14). \square

Proof of Theorem 2.3. Using Lemma 2.5, we have

$$\begin{aligned} D_{\mu,t}v(t,x) &= \int_0^\infty D_{\mu,t}f_t^\mu(\tau)u(\tau,x)d\tau = -\lim_{\tau \rightarrow \infty}[f_t^\mu(\tau)u(\tau,x)] + \lim_{\tau \rightarrow 0}[f_t^\mu(\tau)u(\tau,x)] \\ &\quad + \int_0^\infty f_t^\mu(\tau)\frac{\partial}{\partial\tau}u(\tau,x)d\tau - \Phi_\mu(t)u(0,x) = \int_0^\infty f_t^\mu(\tau)\frac{\partial}{\partial\tau}u(\tau,x)d\tau, \end{aligned}$$

since all the limit expressions vanish due to parts (b) and (c) of Lemma 2.4. Now taking into account equation (2.9),

$$(2.15) \quad D_{\mu,t}v(t,x) = \int_0^\infty f_t^\mu(\tau)L(x,D_x)u(\tau,x)d\tau = L(x,D_x)v(t,x).$$

The initial condition (2.10) is also verified by using property (a) of Lemma 2.4:

$$\lim_{t \rightarrow +0} v(t,x) = \langle \delta_0(\tau), u(\tau,x) \rangle = u(0,x) = \varphi(x),$$

which completes the proof. \square

Remark 2.6. The equivalent version of formula (2.8) in terms of Riemann-Liouville fractional derivatives was proven in [1] in a more general setting. Theorem 2.3, Lemma 2.4 (d), and Lemma 2.5 are special cases of theorems and equations proven in [28]. However, our proofs are simpler and significantly different from the treatment in [28].

3. FPK EQUATIONS ASSOCIATED WITH SDES DRIVEN BY TIME-CHANGED FBM

Now let us focus on the FPK equation associated with the SDE (1.13) driven by a time-changed fBM $B_{E_t}^H$. Recall that the FPK equation associated with an SDE driven by an fBM (without time-change) has the form [2, 11]

$$(3.1) \quad \frac{\partial u(t,x)}{\partial t} = L_\gamma(t,x,D_x)u(t,x),$$

where $L_\gamma(t,x,D_x)$ is defined in (1.7) and the Hurst parameter H is connected with γ via $2H - 1 = \gamma$. Again for simplicity, we consider a time-change process E_t inverse to a single stable subordinator W_t , though mixtures of stable subordinators can be treated similarly. Hence, the density function $f_t(\tau)$ of E_t possesses all the properties mentioned in Lemmas 2.1 and 2.2.

Theorem 3.1. *Let $u(t, x)$ be a solution to the initial value problem*

$$(3.2) \quad \frac{\partial u(t, x)}{\partial t} = B(x, D_x)u(t, x) + \frac{\gamma + 1}{2}t^\gamma A(x, D_x)u(t, x), \quad t > 0, \quad x \in \mathbb{R}^n,$$

$$(3.3) \quad u(0, x) = \varphi(x), \quad x \in \mathbb{R}^n.$$

Let $f_t(\tau)$ be the density function of the process inverse to a stable subordinator of index β . Then $v(t, x) = \int_0^\infty f_t(\tau)u(\tau, x)d\tau$ satisfies the initial value problem for a fractional order differential equation,

$$(3.4) \quad D_{*t}^\beta v(t, x) = B(x, D_x)v(t, x) + \frac{\gamma + 1}{2}G_{\gamma,t}A(x, D_x)v(t, x), \quad t > 0, \quad x \in \mathbb{R}^n,$$

$$(3.5) \quad v(0, x) = \varphi(x), \quad x \in \mathbb{R}^n,$$

where the operator $G_{\gamma,t}$ acts on the variable t and is defined by

$$(3.6) \quad G_{\gamma,t}v(t, x) = \beta\Gamma(\gamma + 1)J_t^{1-\beta}\mathcal{L}_{s \rightarrow t}^{-1} \left[\frac{1}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{\tilde{v}(z, x)}{(s^\beta - z^\beta)^{\gamma+1}} dz \right] (t),$$

with $0 < C < s$ and $z^\beta = e^{\beta \text{Ln}(z)}$, with $\text{Ln}(z)$ being the principal value of the complex $\ln(z)$ with cut along the negative real axis.

Proof. Using the properties of $f_t(\tau)$, we obtain as in the proof of (2.8)

$$D_{*t}^\beta v(t, x) = B(x, D_x)v(t, x) + \frac{\gamma + 1}{2}A(x, D_x)G_{\gamma,t}v(t, x),$$

where

$$(3.7) \quad G_{\gamma,t}v(t, x) = \int_0^\infty f_t(\tau)\tau^\gamma u(\tau, x)d\tau.$$

It follows from the definition (2.5) of $v(t, x)$ that if $\gamma = 0$, then $G_{0,t}$ is the identity operator. To show representation (3.6) in the case $\gamma \neq 0$, we find the Laplace transform of $G_{\gamma,t}v(t, x)$. In accordance with the property (d) of Lemma (2.1), we have

$$\mathcal{L}[G_{\gamma,t}v(t, x)](s) = s^{\beta-1} \int_0^\infty e^{-\tau s^\beta} \tau^\gamma u(\tau, x)d\tau = s^{\beta-1}\mathcal{L}[\tau^\gamma u(\tau, x)](s^\beta).$$

Obviously, if $\gamma = 0$, then $\mathcal{L}[G_{0,t}v(t, x)](s) = s^{\beta-1}\tilde{u}(s^\beta, x)$, which implies $\tilde{v}(s, x) = s^{\beta-1}\tilde{u}(s^\beta, x)$. If $\gamma \neq 0$, then

$$(3.8) \quad \mathcal{L}[t^\gamma u(t, x)](s) = \mathcal{L}[t^\gamma](s) * \tilde{u}(s, x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(\gamma + 1)}{(s - z)^{\gamma+1}} \tilde{u}(z, x)dz,$$

where $*$ stands for the convolution of Laplace images of two functions and $0 < c < s$. Now using the substitution $z = e^{\beta \text{Ln}(\zeta)}$, with $\text{Ln}(\zeta)$ the principal part of the complex function $\ln(\zeta)$, the right hand side of (3.8) reduces to

$$(3.9) \quad \begin{aligned} \mathcal{L}[t^\gamma u(t, x)](s) &= \frac{\beta}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{\Gamma(\gamma + 1)}{(s - \zeta^\beta)^{\gamma+1}} \zeta^{\beta-1} \tilde{u}(\zeta^\beta, x)d\zeta \\ &= \frac{\beta}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{\Gamma(\gamma + 1)}{(s - \zeta^\beta)^{\gamma+1}} \tilde{v}(\zeta, x)d\zeta. \end{aligned}$$

The last equality uses the relation $\tilde{v}(\zeta, x) = \zeta^{\beta-1}\tilde{u}(\zeta^\beta, x)$. Further, replacing s by s^β and taking the inverse Laplace transform in (3.9) yields the desired representation (3.6) for the operator $G_{\gamma,t}$ since $\mathcal{L}[J^{1-\beta}f](s) = s^{\beta-1}\tilde{f}(s)$. In accordance with part (a) of Lemma 2.1 we have $v(0, x) = u(0, x)$, which completes the proof. \square

In the more general case when the time-change process E_t^μ is the inverse to W_t^μ , the mixture of stable subordinators with the mixing measure μ , a representation for the FPK equation is given in the following theorem.

Theorem 3.2. *Let $u(t, x)$ be a solution to the initial value problem (3.2)–(3.3). Let $f_t^\mu(\tau)$ be the density function of the process inverse to W_t^μ . Then $v(t, x) = \int_0^\infty f_t^\mu(\tau)u(\tau, x)d\tau$ satisfies the following initial value problem for a fractional order differential equation:*

$$(3.10) \quad D_\mu v(t, x) = B(x, D_x)v(t, x) + \frac{\gamma + 1}{2}G_{\gamma,t}^\mu A(x, D_x)v(t, x), \quad t > 0, \quad x \in \mathbb{R}^n,$$

$$(3.11) \quad v(0, x) = \varphi(x), \quad x \in \mathbb{R}^n.$$

The operator $G_{\gamma,t}^\mu$ acts on the variable t and is defined by

$$(3.12) \quad G_{\gamma,t}^\mu v(t, x) = \Phi_\mu(t) * \mathcal{L}_{s \rightarrow t}^{-1} \left[\frac{\Gamma(\gamma + 1)}{2\pi i} \int_{C-i\infty}^{C+i\infty} \frac{m_\mu(z)\tilde{v}(z, x)}{(\rho(s) - \rho(z))^{\gamma+1}} dz \right] (t),$$

where $*$ denotes the usual convolution of two functions, $0 < C < s$, $\rho(z) = \int_0^1 e^{\beta \text{Ln}(z)} d\mu(\beta)$, $m_\mu(z) = \frac{\int_0^1 \beta z^\beta d\mu(\beta)}{\rho(z)}$, and $\Phi_\mu(t)$ is defined in (2.13).

Proof. The proof is similar to the proof of Theorem 3.1. We only sketch how to obtain representation (3.12) for the operator $G_{\gamma,t}^\mu v(t, x) = \int_0^\infty f_t^\mu(\tau)\tau^\gamma u(\tau, x)d\tau$. The Laplace transform of $G_{\gamma,t}^\mu v(t, x)$, due to part (d) of Lemma 2.4, is

$$\mathcal{L}_{t \rightarrow s} [G_{\gamma,t}^\mu v(t, x)](s) = \frac{\rho(s)}{s} \mathcal{L}[t^\gamma u(t, x)](\rho(s)), \quad s > 0.$$

Since $\mathcal{L}[\Phi_\mu](s) = \frac{\rho(s)}{s}$, $s > 0$, we have

$$G_{\gamma,t}^\mu v(t, x) = \Phi_\mu(t) * \mathcal{L}_{s \rightarrow t}^{-1} [\mathcal{L}[t^\gamma u(t, x)](\rho(s))](t).$$

Further, replacing s by $\rho(s)$ in (3.8), followed by the substitution $z = \rho(\zeta) = \int_0^1 e^{\beta \text{Ln}(\zeta)} d\mu(\beta)$ in the integral on the right side of (3.8), yields the form (3.12). \square

The following theorem represents the general case when the time-change process E_t is not necessarily the first hitting time process for a stable subordinator or their mixtures.

Theorem 3.3. *Let $\gamma \in (-1, 1)$. Let E_t be a time-change process and assume that its density $K(t, \tau) = f_{E_t}(\tau)$ satisfies the hypotheses:*

- i) $\lim_{\tau \rightarrow +0} [K(t, \tau)\tau^{-\gamma}] < \infty$ for all $t > 0$;
- ii) $\lim_{\tau \rightarrow \infty} [K(t, \tau)\tau^{-\gamma}u(\tau, x)] = 0$ for all $t > 0$ and $x \in \mathbb{R}^n$,

where $u(t, x)$ is a solution to the initial value problem (3.2)–(3.3). Let H_t be an operator acting in the variable t such that

$$(3.13) \quad H_t K(t, \tau) = -\frac{\partial}{\partial \tau} \left[K(t, \tau) \left(\frac{t}{\tau}\right)^\gamma \right] - \delta_0(\tau) \lim_{\tau \rightarrow +0} \left[\left(\frac{t}{\tau}\right)^\gamma K(t, \tau) \right].$$

Then the function $v(t, x) = \int_0^\infty K(t, \tau)u(\tau, x)d\tau$ satisfies the initial value problem

$$(3.14) \quad H_t v(t, x) = t^\gamma \bar{G}_{-\gamma,t} B(x, D_x)v(t, x) + \frac{\gamma + 1}{2}t^\gamma A(x, D_x)v(t, x), \quad \tau > 0,$$

$$(3.15) \quad v(0, x) = u(0, x), \quad x \in \mathbb{R}^n,$$

where $\bar{G}_{-\gamma,t} v(t, x) = \int_0^\infty K(t, \tau)\tau^{-\gamma}u(\tau, x)d\tau$.

Remark 3.4. Obviously, if $\gamma \neq 0$, then H_t cannot be a fractional derivative in the sense of Caputo (or Riemann-Liouville). A representation of H_t in cases when E_t is the inverse to a stable subordinator is given in Corollary 3.5.

Proof. We have

$$\begin{aligned}
 H_t v(t, x) &= \int_0^\infty H_t K(t, \tau) u(\tau, x) d\tau \\
 &= - \int_0^\infty \left\{ \frac{\partial}{\partial \tau} \left[K(t, \tau) \left(\frac{t}{\tau}\right)^\gamma \right] + \delta_0(\tau) \lim_{\tau \rightarrow +0} \left[\left(\frac{t}{\tau}\right)^\gamma K(t, \tau) \right] \right\} u(\tau, x) d\tau \\
 (3.16) \quad &= -t^\gamma \lim_{\tau \rightarrow \infty} [K(t, \tau) \tau^{-\gamma} u(\tau, x)] + t^\gamma \lim_{\tau \rightarrow 0+} [K(t, \tau) \tau^{-\gamma} u(\tau, x)] \\
 &\quad + \int_0^\infty K(t, \tau) \left(\frac{t}{\tau}\right)^\gamma \frac{\partial u(\tau, x)}{\partial \tau} d\tau - \lim_{\tau \rightarrow +0} \left[\left(\frac{t}{\tau}\right)^\gamma K(t, \tau) \right] u(0, x).
 \end{aligned}$$

The first term on the right of (3.16) is zero by hypothesis *ii*) of the theorem. The sum of the second and last terms, which exist by hypothesis *i*), also equals zero. Now taking equation (3.2) into account,

$$H_t v(t, x) = t^\gamma B(x, D_x) \int_0^\infty K(t, \tau) \tau^{-\gamma} u(\tau, x) d\tau + \frac{\gamma + 1}{2} t^\gamma A(x, D_x) v(t, x).$$

Further, since $E_0 = 0$ it follows that

$$\lim_{t \rightarrow 0} v(t, x) = \int_0^\infty \delta_0(\tau) u(\tau, x) d\tau = u(0, x),$$

which completes the proof. □

Let Π_γ denote the operator of multiplication by t^γ , i.e. $\Pi_\gamma h(t) = t^\gamma h(t)$, $h \in C(0, \infty)$. Applying Theorem 3.3 to the case $K(t, \tau) = f_t(\tau)$ in conjunction with Theorem 3.1, we obtain the following corollary.

Corollary 3.5. *Let $\gamma \in (-1, 0]$ and $K(t, \tau) = f_t(\tau)$, where $f_t(\tau)$ is defined in (2.6). Then (i) $G_{-\gamma, t} = G_{\gamma, t}^{-1}$; (ii) $H_t = \Pi_\gamma G_{-\gamma, t} D_*^\beta$.*

This corollary yields an equivalent form for FPK equation (3.4) in the case when E_t is the inverse to the stable subordinator with index β and $\gamma \in (-1, 0]$:

$$(3.17) \quad H_t v(t, x) = t^\gamma G_{-\gamma, t} B(x, D_x) v(t, x) + \frac{\gamma + 1}{2} t^\gamma A(x, D_x) v(t, x),$$

with H_t as in Corollary 3.5.

Suppose the operator in the drift term $B(x, D_x) = 0$. Then equation (3.17) takes the form

$$(3.18) \quad H_t v(t, x) = \frac{\gamma + 1}{2} t^\gamma A(x, D_x) v(t, x).$$

Consequently, given an FPK equation associated to an SDE driven by a non-time-changed fBM, the FPK equation for the analogous SDE driven by the time-changed fBM cannot be of the form: retain the right hand side and change the left hand side to a fractional derivative. Moreover, if a fractional derivative is desired on the left hand side in the time-changed case, then (3.4) shows that the right hand side must be a different operator from that in the non-time-changed case.

Notice that FPK equation (3.17) is valid for $\gamma \in (0, 1)$ as well. Indeed, part (ii) of Corollary 3.5 can be rewritten in the form $G_{\gamma, t} = G_{-\gamma, t}^{-1}$ for $\gamma > 0$. For $\gamma < 0$ part (ii) of Corollary 3.5 also implies $(G_{\gamma, t}^{-1})^{-1} = G_{-\gamma, t}^{-1} = G_{\gamma, t}$. Now applying

operators $G_{-\gamma,t}$ and Π_γ consecutively to both sides of (3.4) we obtain (3.17) for all $\gamma \in (-1, 1)$.

Analogously, the FPK equation obtained in Theorem 3.2 with the mixing measure μ can be represented in its equivalent form as

$$(3.19) \quad H_t^\mu v(t, x) = t^\gamma G_{-\gamma,t}^\mu B(x, D_x)v(t, x) + \frac{\gamma + 1}{2} t^\gamma A(x, D_x)v(t, x), \quad t > 0, \tau > 0,$$

where $H_t^\mu = \Pi_\gamma G_{\gamma,t}^\mu D_\mu$. We leave verification of the details to the reader.

The equivalence of equations (3.4) and (3.17) and the equivalence of equations (3.10) and (3.19) are obtained by means of Theorem 3.3. This fact can also be established with the help of the semigroup property of the family of operators $\{G_\gamma, -1 < \gamma < 1\}$,

$$(3.20) \quad G_\gamma g(t) = \int_0^\infty f_t(\tau) \tau^\gamma h(\tau) d\tau = \mathcal{F}_\gamma h(t),$$

where $h \in C^\infty(0, \infty)$ is a non-negative bounded function. Denote the class of such functions by U . Functions g and h in (3.20) are connected through the relation $g(t) = \int_0^\infty f_t(\tau) h(\tau) d\tau = \mathcal{F}h(t)$. It follows from the behavior of $f_t(\tau)$ as a function of t that $g \in C^\infty(0, \infty)$. On the other hand, obviously, operator \mathcal{F} is bounded, $\|\mathcal{F}h\| \leq \|h\|$ in the sup-norm, and one-to-one due to positivity of $f_t(\tau)$. Therefore, the inverse $\mathcal{F}^{-1} : \mathcal{F}U \rightarrow U$ exists. Let a distribution $H(t, \tau)$ with $\text{supp } H \subset \mathbb{R}_+^2$ be such that $\mathcal{F}^{-1}g(t) = \int_0^\infty H(t, \tau)g(\tau) d\tau$. Since $f_t(\tau) \in \mathcal{F}U$ as a function of t for each $\tau > 0$, for an arbitrary $h \in U$ one has

$$\begin{aligned} h(t) &= \mathcal{F}^{-1}\mathcal{F}h(t) = \int_0^\infty H(t, s) \left(\int_0^\infty f_s(\tau) h(\tau) d\tau \right) ds \\ &= \int_0^\infty h(\tau) \left(\int_0^\infty H(t, s) f_s(\tau) ds \right) d\tau = \left\langle \int_0^\infty H(t, s) f_s(\tau) ds, h \right\rangle_\tau. \end{aligned}$$

We write this relation between $H(t, \tau)$ and $f_t(\tau)$ in the form

$$(3.21) \quad \int_0^\infty H(t, s) f_s(\tau) ds = \delta_t(\tau).$$

Proposition 3.6. *Let $-1 < \gamma < 1$, $-1 < \alpha < 1$, and $-1 < \gamma + \alpha < 1$. Then $G_\gamma \circ G_\alpha = G_{\gamma+\alpha}$.*

Proof. The proof uses the following two relations:

- (1) $G_\gamma g(t) = \int_0^\infty \mathcal{F}_{\gamma,t} H(t, s) g(s) ds, \quad \gamma \in (-1, 1)$;
- (2) $\int_0^\infty \mathcal{F}_{\gamma,t} H(t, s) \mathcal{F}_{\alpha,s} H(s, \tau) ds = \mathcal{F}_{\gamma+\alpha,t} H(t, \tau)$, with $-1 < \gamma, \alpha < 1$, and $-1 < \gamma + \alpha < 1$.

Indeed, using (3.20) and changing the order of integration, we obtain the first relation

$$(3.22) \quad \begin{aligned} G_\gamma g(t) &= \int_0^\infty f_t(\tau) \tau^\gamma \left(\int_0^\infty H(\tau, s) g(s) ds \right) d\tau \\ &= \int_0^\infty g(s) \left(\int_0^\infty f_t(\tau) H(\tau, s) \tau^\gamma d\tau \right) ds = \int_0^\infty \mathcal{F}_{\gamma,t} H(t, s) g(s) ds. \end{aligned}$$

It is readily seen that the internal integral in the second line of (3.22) is meaningful, since $f_t(\tau)$ is a function of exponential decay when $\tau \rightarrow \infty$, which follows from (2.2).

Further, in order to show the second relation, we have

$$\begin{aligned} \int_0^\infty \mathcal{F}_{\gamma,t} H(t,s) \mathcal{F}_{\alpha,s} H(s,\tau) ds &= \int_0^\infty \left(\int_0^\infty f_t(p) H(p,s) p^\gamma dp \right) \left(\int_0^\infty f_s(q) H(q,\tau) q^\alpha dq \right) ds \\ &= \int_0^\infty \int_0^\infty f_t(p) H(q,\tau) p^\gamma q^\alpha \left(\int_0^\infty H(p,s) f_s(q) ds \right) dp dq. \end{aligned}$$

Due to (3.21), this equals

$$\int_0^\infty f_t(p) p^\gamma \left(\int_0^\infty H(q,\tau) q^\alpha \delta_p(q) dq \right) dp = \int_0^\infty H(p,\tau) p^\alpha f_t(p) p^\gamma dp = \mathcal{F}_{\gamma+\alpha,t} H(t,\tau).$$

Now we are ready to prove the claimed semigroup property. Making use of the two proved relations,

$$\begin{aligned} (G_\gamma \circ G_\alpha)g(t) &= G_\gamma[G_\alpha g(t)] \\ &= G_\gamma \left[\int_0^\infty \mathcal{F}_{\alpha,t} H(t,s) g(s) ds \right] = \int_0^\infty \mathcal{F}_{\gamma,t} H(t,s) \left[\int_0^\infty \mathcal{F}_{\alpha,s} H(s,\tau) g(\tau) d\tau \right] ds \\ &= \int_0^\infty g(\tau) \int_0^\infty \mathcal{F}_{\gamma,t} H(t,s) \mathcal{F}_{\alpha,s} H(s,\tau) ds d\tau = \int_0^\infty \mathcal{F}_{\gamma+\alpha,t} H(t,\tau) g(\tau) d\tau = G_{\gamma+\alpha} g(t), \end{aligned}$$

which completes the proof. □

Remark 3.7.

- (1) Proposition 3.6 immediately implies that $G_\gamma^{-1} = G_{-\gamma}$ for arbitrary $\gamma \in (-1, 1)$. Indeed, $G_\gamma \circ G_{-\gamma} = G_0 = I$, as well as $G_{-\gamma} \circ G_\gamma = I$, where I is the identity operator. Thus, the statement in Corollary 3.5 is valid for all $\gamma \in (-1, 1)$.
- (2) Proposition 3.6 remains valid for the family $\{G_\gamma^\mu, -1 < \gamma < 1\}$ as well.
- (3) As in [25], if the governing equation for fBM is given by $(0 < H < 1)$

$$\frac{\partial h}{\partial t}(t,x) = 2Ht^{2H-1} a \frac{\partial^2 h}{\partial x^2}(t,x),$$

then Theorem 3.1 and Proposition 3.6 imply that the governing equation for time-changed fBM is either of the following equivalent forms:

$$\begin{aligned} D_*^\beta h(t,x) &= 2HG_{2H-1,t} a \frac{\partial^2 h}{\partial x^2}(t,x), \\ G_{1-2H,t} D_*^\beta h(t,x) &= 2Ha \frac{\partial^2 h}{\partial x^2}(t,x). \end{aligned}$$

The method used in this paper allows extension of results of Theorems 3.1–3.3 to the case of SDEs driven by time-changed linear fractional stable motions. See [25] for CTRW limits of correlated random variables, whose limiting processes are time-changed fractional Brownian or linear fractional stable motions.

- (4) The formula $v(t,x) = \mathcal{F}u(t,x)$ for a solution of FPK equations associated with time-changed fBM provides a useful tool for analysis of properties of a solution to initial value problems (3.4)–(3.5), (3.10)–(3.11), and (3.14)–(3.15).
- (5) It is not necessary for the dependence of coefficients in (3.1) on t to be of the form t^γ . This function can be replaced by $[\nu(t)]^\gamma$, where $\nu(t)$ is a continuous function defined on $[0, \infty)$; however, the results essentially depend on the behavior of $\nu(t)$ near zero and infinity.

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REFERENCES

1. Baeumer, B., Meerschaert, M. M., Stochastic solutions for fractional Cauchy problems. *Fractional Calculus Appl. Anal.* 4, 481–500, 2001. MR1874479 (2003d:26006)
2. Baudoin, F., Coutin, L. Operators associated with a stochastic differential equation driven by fractional Brownian motions, *Stoch. Proc. Appl.* 117, 5, 550–574, 2007. MR2320949 (2008c:60050)
3. Bender, C. An Itô formula for generalized functionals of a fractional Brownian motion with arbitrary Hurst parameter. *Stoch. Process. Appl.* 104 (1), 81–106, 2003. MR1956473 (2003m:60137)
4. Benson, D. A., Wheatcraft, S. W., Meerschaert, M. M. Application of a fractional advection-dispersion equation. *Water Resour. Res.* 36(6), 1403–1412, 2000.
5. Biagini, F., Hu, Y., Oksendal, B., Zhang, T. *Stochastic calculus for fractional Brownian motion and applications*. Springer, 2008. MR2387368 (2010a:60145)
6. Cheridito, P. Arbitrage in fractional Brownian motion models. *Finance Stochast.* 7, 533–553, 2003. MR2014249 (2004m:60076)
7. Coutin, L., Decreusefond, L. Stochastic differential equations driven by fractional Brownian motion. *Tech. Report*, 1997.
8. Decreusefond, L., Üstünel, A.S. Stochastic analysis of the fractional Brownian motion. *Potential Analysis*, V. 10, 2, 177–214, 1998. MR1677455 (2000b:60133)
9. Eidelman, S. D., Kochubei, A. N. Cauchy problem for fractional diffusion equations, *J. Differential Equations*, 199, 211–255, 2004. MR2047909 (2005i:26014)
10. Fannjiang, A., Komorowski, T. Fractional Brownian motions and enhanced diffusion in a unidirectional wave-like turbulence. *Journal of Statistical Physics*, 100 (5–6), 1071–1095, 2000. MR1798554 (2001i:82060)
11. Gazanfer, Ü., Fokker-Planck-Kolmogorov equation for fBM: derivation and analytical solution. *Math. Physics*, 53–60, Proceedings of the 12th regional conference, Islamabad, 2006. MR2349062 (2008g:82099)
12. Gorenflo, R., Mainardi, F. Fractional calculus: integral and differential equations of fractional order. In A. Carpinteri and F. Mainardi (editors): *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, 223–276, 1997. MR1611585 (99g:26015)
13. Gorenflo, R., Mainardi, F. Random walk models for space-fractional diffusion processes. *Fract. Calc. Appl. Anal.* 1, no. 2, 167–191, 1998. MR1656314 (99m:60117)
14. Gorenflo, R., Mainardi, F. Continuous time random walk, Mittag-Leffler waiting time and fractional diffusion: mathematical aspects. In R. Klages, G. Radons and I. M. Sokolov (editors): *Anomalous Transport: Foundations and Applications*, 93–127, Wiley-VCH, Weinheim, Germany, 2008.
15. Gorenflo, R., Mainardi, F., Scalas, E., Raberto, M. Fractional calculus and continuous-time finance. III. *Mathematical Finance*, 171–180, Trends Math., Birkhäuser, Basel, 2001. MR1882830
16. Hahn, M., Kobayashi, K., Umarov, S. SDEs driven by a time-changed Lévy process and their associated time-fractional order pseudo-differential equations. *J. Theor. Probab.*, DOI 10.1007/S10959-010-0289-4. Published electronically 11 May 2010.
17. Jacod, J. *Calcul Stochastique et Problèmes de Martingales*. Lecture Notes in Mathematics, 714, Springer, Berlin, 1979. MR542115 (81e:60053)
18. Kobayashi, K. Stochastic calculus for a time-changed semimartingale and the associated stochastic differential equations. arXiv:0906.5385v1 [math.PR], 2009.
19. Kochubei, A. N. Distributed order calculus and equations of ultraslow diffusion. *J. Math. Anal. Appl.* 340, 252–281, 2008. MR2376152 (2009i:35177)
20. Kolokoltsov, V. N. Generalized continuous-time random walk (CTRW), subordinating by hitting times and fractional dynamics. *Theor. Prob. Appl.* 53 (4), 2009.

21. Magdziarz, M., Weron, A., Klafter, J. Equivalence of the fractional Fokker-Planck and subordinated Langevin equations: the case of a time-dependent force. *Phys. Rev. Lett.* 101, 210601, 2008.
22. Mainardi, F., Luchko, Y., Pagnini, G. The fundamental solution of the space-time fractional diffusion equation. *Fract. Calc. Appl. Anal.* 4, no. 2, 153–192, 2001. MR1829592 (2002h:26010)
23. Meerschaert, M. M., Benson, D. A., Scheffler, H.-P., Baeumer, B. Stochastic solution of space-time fractional diffusion equations. *Phys. Rev. E* 65, 1103–1106, 2002. MR1917983 (2003d:60165)
24. Meerschaert, M. M., Benson, D. A., Scheffler, H.-P., Becker-Kern, P. Governing equations and solutions of anomalous random walk limits. *Physical Review E* 66 (6), 102R–105R, 2002.
25. Meerschaert, M. M., Nane, E., Xiao Y. Correlated continuous time random walks. *Stat. Probabil. Lett.* 79, 1194–1202, 2009. MR2519002
26. Meerschaert, M. M., Scheffler, H.-P. Stochastic model for ultraslow diffusion. *Stoch. Proc. Appl.* 1215–1235, 2006. MR2251542 (2008g:60136)
27. Meerschaert, M. M., Scheffler, H.-P. Limit theorems for continuous-time random walks with infinite mean waiting times. *J. Appl. Probab.* 41, 623–638, 2004. MR2074812 (2005f:60105)
28. Meerschaert, M. M., Scheffler, H.-P. Triangular array limits for continuous time random walks. *Stoch. Proc. Appl.* 118, 1606–1633, 2008. MR2442372 (2010b:60135)
29. Metzler, R., Klafter, J. The random walk’s guide to anomalous diffusion: a fractional dynamics approach. *Phys. Rep.* 339, no. 1, 1–77, 2000. MR1809268 (2001k:82082)
30. Nualart, D. *The Malliavin calculus and related topics*, 2nd edition. Springer, 2006. MR2200233 (2006j:60004)
31. Saxton, M. J., Jacobson, K. Single-particle tracking: applications to membrane dynamics. *Annu. Rev. Biophys. Biomol. Struct.* 26, 373–399, 1997.
32. Scalas, E., Gorenflo, R., Mainardi, F. Uncoupled continuous-time random walks: Solution and limiting behavior of the master equation. *Phys. Rev. E* 69 (1), 011107, 2000. MR2097999
33. Shiryaev, A. *Essentials of stochastic finance: facts, models, theory*. World Scientific Publishing, 1999. MR1695318 (2000e:91085)
34. Silbergleit, V. M., Gigola, S. V., D’Attellis, C. E., Statistical studies of sunspots. *Acta Geodaetica et Geophysica Hungarica* 49 (3), 278–283, 2007.
35. Sokolov, I. M., Chechkin, A. V., Klafter, J. Distributed-order fractional kinetics. *Acta Physica Polonica B* 35, 1323–1341, 2004.
36. Stroock, D. W. *Markov processes from K. Itô’s perspective*. Princeton University Press, 2003. MR1980149 (2004m:60006)
37. Uchaikin, V. V., Zolotarev, V. M. *Chance and stability. Stable distributions and their applications*. VSP, Utrecht, 1999. MR1745764 (2002a:60002)
38. Umarov, S., Gorenflo, R. Cauchy and nonlocal multi-point problems for distributed order pseudo-differential equations. I. *Z. Anal. Anwendungen*. 24, no. 3, 449–466, 2005. MR2208034 (2006k:26006)
39. Umarov, S., Steinberg, S. Random walk models associated with distributed fractional order differential equations. High Dimensional Probability, IMS Lecture Notes - Monograph Series, 51, 117–127, Inst. Math. Stat., 2006. MR2387764 (2008k:60105)
40. Zaslavsky, G. M. Chaos, fractional kinetics, and anomalous transport. *Phys. Rep.* 371, no. 6, 461–580, 2002. MR1937584 (2003i:70030)

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