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Folding instability of a layered viscoelastic medium under compression

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A preliminary theory is established for the stability of a viscoelastic layer sandwiched in an infinitely extended medium of another viscoelastic material when a compressive force is acting in a direction parallel with the layer. The instability is manifested by a folding of the layer. It is shown that in general there exists a lower and a higher critical load between which folding occurs at a finite rate with a dominant wavelength. This is the wavelength whose amplitude increases at the fastest rate. Special cases are also discussed in more detail such as that of a purely viscous layer in a viscous fluid, an elastic layer in a viscous fluid, a viscous layer in an elastic medium, and of two Maxwell materials. Results indicate that the ratio of the relaxation times of the two materials is an important parameter.

1. Introduction

The problem under consideration deals with the deformation of a composite medium made up of a layer sandwiched in another medium of infinite extent or lying at the surface of another semi-infinite medium. We are also concerned with a particular feature of the deformation, namely, the folding of the layer as a result of instability when the medium is subject to a compression in a direction parallel with the layer. When the materials are purely elastic the theory corresponds to the buckling of an axially loaded beam lying on an elastic continuum. It is known in this case that under a critical compression load a folding of the beam occurs suddenly with a wavelength characteristic of the geometry and the elastic constants of the materials.

It is clear that for viscoelastic materials such sudden deformation will generally not occur and that we shall not be able to define a unique critical load.

We have attempted here to establish a systematic analysis of the behaviour under compression of the stratified viscoelastic medium, of two distinct materials. Before such an analysis could be undertaken on a general basis, a sufficiently general approach to stress relations in viscoelastic materials had to be developed. This was done by the writer (Biot 1954). The results are of interest to the geologist as they should throw some light on the mechanism of folding of sedimentary rock and other tectonic features of stratified geological formations, and also furnish the beginnings of a quantitative approach to these problems. Some technological applications may also result in the field of plastics and sandwich panels, and in the behaviour of metals under creep conditions at elevated temperature.

Some of the cases investigated may also be considered as qualitative models for the approximate representation of analogous configurations encountered in technological problems as exemplified by the case in § 5 which is related to the buckling of an oilwell drill rod.

In § 2 is established the basic differential equation for the layer instability. The procedure is based on certain approximations of the type encountered in the elastic beam theory. In fact the theory is formally the same, and the results are obtained from the elastic equation by the rule of correspondence as formulated in more general form by the present writer in previous publications (Biot 1955 *a*, *b*). By this correspondence rule, we may replace the elastic constants by their corresponding operational expressions for the viscoelastic case, in a wide class of problems including those which have been solved by variational or elementary strength-of-materials methods. Also derived previously is the nature of these operational expressions from the general principles of irreversible thermodynamics (Biot 1954).

The theory is confined to the linear case from the standpoint of the physical properties involved as well as the assumption of infinitesimal deformations, and we shall have to keep this in mind when evaluating its applicability to actual configurations and materials. In addition to approximations of the beam-theory type, we assume that the compressive force acts only in the layer, i.e. we neglect the compressive pre-stress in the infinite medium. This, however, is generally justified if the stiffness or viscosity of the layer is sufficiently high, but in principle the stress in the infinite medium cannot be overlooked. The effects of adhesion and friction between layer and medium are also neglected. A more exact but more elaborate theory which does not introduce these assumptions has been developed and the results will be presented later.

Section 3 discusses the general case and derives general conclusions and formulae. The possibility of the existence of a lower and a higher critical load is established. It is also shown that, in general, multiple folding can occur; by that, is meant the appearance of several wavelengths in the folding of the layer depending on the time history of the load. This in turn can lead to a folding of variable amplitude characterized by the appearance of a beat phenomenon in the spatial distribution of the wave shape.

The following sections deal with special cases. In § 4 we assume both materials to be viscous fluids. Section 5 considers the folding of an elastic layer in a viscous fluid. This may be looked upon as a qualitative mathematical model for the stability of a drill rod in an oil well. Section 6 discusses the viscous layer in an elastic medium, and § 7, the case of two Maxwell-type materials. One of the salient results is that the ratio of relaxation times of the two materials is an important parameter. If this ratio is unity, as it is if the materials are both elastic or incompressible viscous fluids, the wavelength of the folds does not depend on the magnitude of the load. If this ratio is not unity this is not the case and multiple folding may take place.

The stability of an embedded layer for the case where the two materials are purely elastic has been the object of numerous studies in the engineering literature particularly in connexion with the properties of sandwich panels. The problem has been considered by Bijlaard (1946, 1947) who refers to its application to sandwich panels and the problem of wrinkling of road surfaces, and also by Gough, Elam & DeBruyne (1940).

2. Basic equations of flexural instability

We shall consider a layer of viscoelastic material embedded in an infinite medium which is also viscoelastic. The layer of thickness h is subject to a compression P (figure 1). The problem is to determine the folding of this layer induced by the compressive stress.

We will first establish the equations for the layer itself independently of the surrounding medium. The problem is that of bending of a plate under initial compression. The deformation is assumed to be cylindrical, the folding occurring perpendicularly to the x direction which is also the direction of the compressive load. The effect of the surrounding medium is represented by a lateral load q per unit length acting on the plate. The simplifying assumption is here introduced that no tangential forces are exerted between the plate and the surrounding medium.

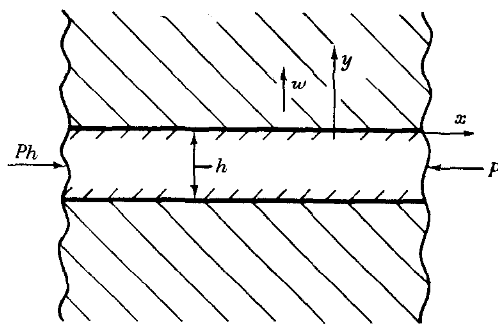

Figure 1. Layer embedded in an infinite medium.

The equations for the flexural deformation of a viscoelastic plate may be obtained (Biot 1955 *a*, *b*) from a correspondence rule, by considering first an elastic plate and replacing the Lamé constant by the corresponding operators. The flexural deflexion w of an elastic plate under a compressive P stress and a lateral load q is

$$\frac{Eh^3}{12(1-\nu^2)}\frac{\mathrm{d}^4w}{\mathrm{d}x^4}+Ph\frac{\mathrm{d}^2w}{\mathrm{d}x^2}=q, \tag{2·1}$$

where E is Young's modulus and ν Poisson's ratio. The elastic coefficients in this equation can be written in terms of the Lamé constants λ and G as follows

$$\frac{E}{1-\nu^2}=\frac{4G(G+\lambda)}{2G+\lambda}. \tag{2·2}$$

The shear modulus is designated by G to distinguish it from the viscosity coefficient μ.

The stress-strain relations for an isotropic material are written operationally (Biot 1954)

$$\sigma_{ij}=2Q(p)\,e_{ij}+\delta_{ij}R(p)\,e \tag{2·3}$$

with δ_{ij} the unit matrix and e_{ij} defined in terms of the displacement u_i as

$$\left.\begin{aligned} e_{ij}&=\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right),\\ e&=e_{xx}+e_{yy}+e_{zz}.\end{aligned}\right\} \tag{2·4}$$

The operators are
$$\left.\begin{aligned} Q(p) &= p\int_0^\infty \frac{Q''(r)}{p+r}\gamma(r)\,\mathrm{d}r + Q + Q'p \\ R(p) &= p\int_0^\infty \frac{R''(r)}{p+r}\gamma(r)\,\mathrm{d}r + R + R'p, \end{aligned}\right\} \tag{2·5}$$

with
$$p = \frac{\mathrm{d}}{\mathrm{d}t} \quad \text{or} \quad \frac{\partial}{\partial t}.$$

According to the correspondence rule we must replace in equation (2·1) the Lamé constants G and λ by their corresponding operators $Q(p)$ and $R(p)$. Hence, the equation for the flexural deformation of the viscoelastic plate is

$$B(p)\frac{h^3}{12}\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + Ph\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = q, \tag{2·6}$$

with the operator
$$B(p) = \frac{4Q(p)\,[Q(p)+R(p)]}{2Q(p)+R(p)}. \tag{2·7}$$

The load q exerted by the surrounding medium on the plate is still unknown and depends on the deflexion w. In order to find an expression for this load we assume that the deflexion is sinusoidal along x,

$$w = w_0 \cos lx. \tag{2·8}$$

We must therefore determine the load $-q$ necessary to produce such a sinusoidal deflexion of a half space. The load will also be sinusoidal,

$$-q' = q_0 \cos lx. \tag{2·9}$$

In order to find the relation between w and q' we again consider the purely elastic case. This problem has been solved for two-dimensional stress (Biot 1937). The present case is one of two-dimensional strain which may be deduced from the result for two-dimensional stress by replacing E by $E/(1-\nu^2)$.

We derive for the elastic case

$$\frac{w}{-q'} = \frac{2(1-\nu_1^2)}{E_1 l}, \tag{2·10}$$

where E_1 and ν_1 are Young's modulus and Poisson's ratio of the medium. As before, we use the correspondence rule and replace the Lamé constants by the corresponding operators $Q_1(p)$ and $R_1(p)$ of the surrounding medium. Now q' is the load exerted by the surrounding medium of one side of the layer. The total load q due to both sides is

$$q = 2q'. \tag{2·11}$$

With an operator
$$B_1(p) = \frac{4Q_1(p)\,[Q_1(p)+R_1(p)]}{2Q_1(p)+R_1(p)}, \tag{2·12}$$

we find
$$-q = B_1 lw. \tag{2·13}$$

Substituting q in the flexural equation (2·6) yields finally

$$\frac{h^3}{12}B(p)\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} + Ph\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + B_1(p)\,lw = 0. \tag{2·14}$$

This is an operational equation valid for sinusoidal folding of the layer with a wavelength

$$L = 2\pi/l. \tag{2·15}$$

This equation contains two operators $B(p)$ and $B_1(p)$ which characterize respectively the viscoelastic properties of the layer and that of the surrounding medium.

It is clear that the problem of a viscoelastic layer lying on top of a semi-infinite viscoelastic continuum as illustrated in figure 2 is the same as the one above, under the approximations introduced. The lateral restraint of the folding layer is simply divided by 2. The equation of the deflexion is

$$\frac{h^3}{12}B(p)\frac{d^4w}{dx^4}+Ph\frac{d^2w}{dx^2}+\tfrac{1}{2}B_1(p)\,lw = 0. \tag{2·16}$$

This is the same as equation (2·14) with the factor $\tfrac{1}{2}$ introduced in the last term. All results obtained hereafter for the embedded layer will be applicable to the surface layer if we replace $B_1(p)$ by $\tfrac{1}{2}B_1(p)$.

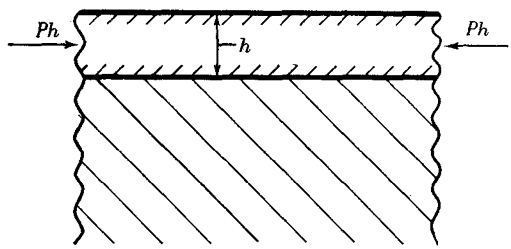

FIGURE 2. Layer lying on top of a semi-infinite medium.

3. The general nature of the instability

We shall now discuss the various types of folding instability which will arise in the general case. If we substitute into the differential equation (2·14) the sinusoidal distribution (2·8) of the deflexion w we obtain the basic characteristic equation

$$\tfrac{1}{12}B(p)\,l^2h^2+\frac{B_1(p)}{lh} = P, \tag{3·1}$$

which is a relation between the compressive load P, the wavelength $L = 2\pi/l$, and the time parameter p. The significance of positive values of the parameter p arises from the fact that such values correspond to deflexions w containing a time factor e^{pt} so that p is a measure of the instability and the rate of growth of the folding amplitude. The load P may be considered a function of lh and the parameter p. We now consider the physical significance of the operator $B(p)$. It yields the stress $\sigma_{xx} = B(p)\,e_{xx}$ for a one-dimensional deformation e_{xx} in which $e_{zz} = 0$ and $\sigma_{yy} = 0$. Hence from thermodynamics (Biot 1954) $B(p)$ may be written in the same form as (2·5) for Q/p, with all terms positive. As a consequence $B(p)$ and $B_1(p)$ are monotonically increasing functions of p. Therefore if we plot P as a function of lh we obtain a family of curves as illustrated in figure 3.

This diagram shows that for a given load there is a wavelength of fastest rate of growth, i.e. highest value of p. This will be called the *dominant wavelength* L_d. It is determined by a value $l_d h$ of lh for which P is minimum under constant value of p. Hence, since

$$l_d h = \sqrt[3]{\frac{6B_1(p)}{B(p)}}, \tag{3·2}$$

the dominant wavelength is

$$L_d = 2\pi h \sqrt[3]{\frac{B(p)}{6B_1(p)}}. \tag{3·3}$$

The corresponding load P at which it appears is

$$P = \tfrac{3}{2}B_1(p) \sqrt[3]{\frac{B(p)}{6B_1(p)}}. \tag{3·4}$$

By plotting L_d against P for values of p varying from 0 to ∞ we obtain the dominant wavelength as a function of the load. The dominant wavelength may increase or decrease with the load depending on whether the ratio $B_1(p)/B(p)$ is an increasing or decreasing function of p.

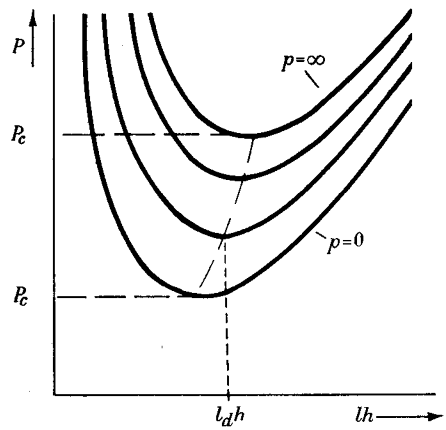

FIGURE 3. Stability diagram relating the compressive load P, the wavelength $L = 2\pi/l$ of the folding, and the rate of folding p.

The dominant wavelength will be independent of the load if this ratio is constant. Such will be the case for instance if the medium is homogeneous with respect to its time constants. We can call this the case of a *homogeneous spectrum* and assume that the operators for the two media are of the type

$$\left.\begin{aligned} Q(p) &= Cf(p), & R &= Df(p), \\ Q_1(p) &= C_1 f(p), & R_1 &= D_1 f(p), \end{aligned}\right\} \tag{3·5}$$

with constants C, C_1, D, D_1, and a single operator

$$f(p) = \int_0^\infty \frac{p}{p+r}\gamma(r)\,\mathrm{d}r + \alpha + \beta p. \tag{3·6}$$

We have then
$$\frac{B}{B_1} = \frac{C(C+D)(2C_1+D_1)}{C_1(C_1+D_1)(2C+D)}, \tag{3·7}$$

a constant value independent of the load.

Let us consider the behaviour of the layer in the most general case when a compressive load is gradually applied. From small enough values of the load it is possible that no instability occurs. In this case incipient folding will appear at a lower critical load P_c corresponding to $p = 0$. This load is

$$P_c = \tfrac{3}{2}B_1(0)\sqrt[3]{\frac{B(0)}{6B_1(0)}}. \tag{3·8}$$

For this load the instability is incipient and the rate of growth of the folding theoretically vanishes. As the load increases we may reach a point for which $p = \infty$, i.e. the folding may occur at an infinite rate and is purely elastic.* This upper critical load is a buckling load.

$$P'_c = \tfrac{3}{2}B_1(\infty)\sqrt[3]{\frac{B(\infty)}{6B_1(\infty)}}. \tag{3·9}$$

It may happen of course that the lower critical load is zero and that the instability occurs no matter how small the load. Also in some cases the upper critical load may be infinite and sudden buckling never appears. It is easily seen from the nature of the operators in which cases such behaviour will occur. For instance if for the layer we have

$$Q(p) = Q''\frac{p}{p+r}+Q'p, \tag{3·10}$$

the lower critical load is zero and the folding occurs under any load no matter how small provided we wait a sufficiently long time. On the other hand, if either $Q' \neq 0$ or $Q'_1 \neq 0$, the upper critical load is infinite. This corresponds to the existence of a Newtonian viscosity term. Of course, while this may prevent the instantaneous buckling to occur theoretically there may be in effect a finite buckling load unless the viscosity is sufficiently high.

A feature of great interest is the possibility of folding with multiple wavelength. As, in general, the dominant wavelength varies with the load, if the load is applied in two successive steps with different values, the dominant wavelength of each load will both be present and will be superposed. We may envisage the appearance of beats in the folding if the two wavelengths are not far apart. The appearance of the folding will therefore depend on the time history of the load, a fact which should be of value in geological studies. Finally, it should be remarked that the validity of the theory is limited to cases where the wavelength is sufficiently large relative to the thickness so that the beam theory applies. According to (3·3) if the layer and the surrounding material are the same, $B = B_1$ we would observe a dominant wavelength $L_d = 2\pi h/\sqrt[3]{6}$, which is a physical impossibility. However, in this case the wavelength is too short for the elementary beam theory to be valid. For the same reason the validity of the diagram of figure 3 should be limited to a region lying to the left of a certain vertical line whose abscissa represents a value of lh beyond which the beam theory is not applicable. This point will be discussed in the more complete theory now under study.

* I.e. if we neglect the inertia of the system. The dynamical effects may easily be included in the present theory if desired.

4. Two viscous fluids

Consider first a layer and a surrounding medium which are both incompressible fluids with Newtonian viscosity. The stress-strain relations for both fluids may be written

$$\sigma_{ij} - \delta_{ij}\sigma = 2\mu\dot{e}_{ij}, \tag{4·1}$$

where μ is the viscosity and σ the hydrostatic component of the stress. This falls in the general case above by putting $R = R_1 = \infty$ and $e = 0$ in such a way that we have the limiting value $Re = \sigma$. Moreover, denoting by μ and μ_1 the viscosity coefficients of the layer and the surrounding medium respectively we have

$$\left.\begin{aligned} Q(p) &= \mu p, \\ Q_1(p) &= \mu_1 p. \end{aligned}\right\} \tag{4·2}$$

The other operators become

$$\left.\begin{aligned} B(p) &= 4\mu p, \\ B_1(p) &= 4\mu_1 p. \end{aligned}\right\} \tag{4·3}$$

This case belongs to what we have already called the case of a homogeneous spectrum. Here the relaxation times of both media are infinite. The dominant wavelength is independent of the load and equal to

$$L_d = 2\pi h \sqrt[3]{\frac{\mu}{6\mu_1}}. \tag{4·4}$$

The lower critical load is zero and the upper critical load infinite. The folding appears as soon as the load is applied and the dominant wavelength is independent of the load. Equation (4·4) shows that a layer viscosity six times higher than the medium generates a wavelength 2π times the layer thickness.*

It is of interest to examine the effect of the compressibility of the fluids. Assuming perfect elasticity in compression, we denote by K and K_1, the elastic bulk moduli of the layer and the infinite medium, respectively. We may write (Biot 1954)

$$\left.\begin{aligned} R(p) &= K - \tfrac{2}{3}Q(p) = K - \tfrac{2}{3}\mu p, \\ R_1(p) &= K_1 - \tfrac{2}{3}Q_1(p) = K_1 - \tfrac{2}{3}\mu_1 p. \end{aligned}\right\} \tag{4·5}$$

Hence

$$\frac{B(p)}{B_1(p)} = \frac{\mu}{\mu_1}\frac{(\mu p + 3K)}{(\mu_1 p + 3K_1)}\frac{(4\mu_1 p + 3K_1)}{(4\mu p + 3K)}, \tag{4·6}$$

leading to a slight variation of the wavelength with the load. The lower critical load remains zero and there is no buckling load.

5. The elastic layer in a viscous fluid

Consider now the case of a purely elastic layer. The operators for the layer become the elastic moduli, i.e. the Lamé constants

$$\left.\begin{aligned} Q(p) &= G, \\ R(p) &= \lambda. \end{aligned}\right\} \tag{5·1}$$

* Results obtained from a more elaborate theory indicate that the instability becomes significant only for values of μ of the order of about $60\mu_1$.

The constant G is the elastic shear modulus. The fluid medium is assumed incompressible of viscosity μ_1. Hence

$$\left.\begin{aligned} Q_1(p) &= \mu_1 p, \\ R_1(p) &= \infty. \end{aligned}\right\} \tag{5·2}$$

The operator B becomes a constant modulus

$$B = \frac{4G(G+\lambda)}{2G+\lambda} = \frac{E}{1-\nu^2}, \tag{5·3}$$

where E is Young's modulus of the layer and ν its Poisson's ratio. The operator B_1, becomes

$$B_1(p) = 4\mu_1 p. \tag{5·4}$$

The dominant wavelength (3·3) becomes

$$L_d = \pi h \sqrt[3]{\frac{B}{3\mu_1 p}} \tag{5·5}$$

and the corresponding load

$$P = \frac{3}{\sqrt[3]{3}} B^{\frac{1}{3}}(\mu_1 p)^{\frac{2}{3}}. \tag{5·6}$$

Eliminating p we find

$$L_d = \pi h \sqrt{\frac{B}{P}}. \tag{5·7}$$

The dominant wavelength depends on the load. With increasing load it decreases like $1/\sqrt{P}$. The higher the load the shorter the wavelength. This effect is further accentuated if we take into account plasticity in the layer, so that the effective modulus B decreases with the load. We note that the dominant wavelength is independent of the viscosity of the fluid. Unless P is large, i.e. of the order of the elastic molulus B the wavelength will also be large.

We may compare (5·7) with the Euler formula for the two-dimensional buckling of a pinned plate of length L_e with no lateral restraint. The buckling of such a plate requires a load P satisfying the relation

$$2L_e = \frac{1}{\sqrt{3}} \pi h \sqrt{\frac{B}{P}}. \tag{5·8}$$

Comparing with (5·7) we see that the dominant wavelength is $\sqrt{3}$ times the wavelength $2L_e$ of the Euler buckling of the free plate under the same load.

6. The viscous layer in an elastic medium

This is the reverse of the previous case. We assume a layer of incompressible fluid with Newtonian viscosity μ embedded in a purely elastic medium. The operator B_1, reduces to a characteristic modulus of the elastic medium expressed in terms of the Lamé constants λ_1, G_1, or Young's modulus E_1, and Poisson's ratio ν_1, by

$$B_1 = \frac{4G_1(G_1+\lambda_1)}{2G_1+\lambda_1} = \frac{E_1}{1-\nu_1^2}. \tag{6·1}$$

The operator $B(p)$ for the layer is obtained by putting $R(p) = \infty$ and $Q(p) = \mu p$ in (2·7); hence,

$$B(p) = 4\mu p. \tag{6·2}$$

The dominant wavelength as given by (3·3) becomes

$$L_d = 2\pi h \sqrt[3]{\frac{2\mu p}{3B_1}} \tag{6·3}$$

and the corresponding load (3·4) is

$$P = \tfrac{3}{2}B_1 \sqrt[3]{\frac{2\mu p}{3B_1}}. \tag{6·4}$$

Elimination of p gives

$$L_d = \tfrac{4}{3}\pi h \frac{P}{B_1}. \tag{6·5}$$

In this case the dominant wavelength increases proportionately to the load. Of course, for this theory to be valid the wavelength must be sufficiently large compared to the thickness, which requires the modulus B_1 of the elastic medium to be sufficiently low compared to the load P. This also requires the viscosity μ of the layer to be large enough to sustain the load P without collapse. With these limitations in mind we may state that as the load is gradually increased the instability first appears when the dominant wavelength becomes appreciably larger than the thickness. From then on the folding will continue with the possible appearance of increasing wavelengths.

7. Two Maxwell-type media

Finally, we shall consider the case where both the layer and the surrounding medium are incompressible Maxwell-type materials. Their stress-strain relations are written respectively.

$$\left.\begin{aligned} \sigma_{ij} - \delta_{ij}\sigma &= \frac{2Gp}{p+r}e_{ij}, \\ \sigma_{ij} - \delta_{ij}\sigma &= \frac{2G_1 p}{p+r_1}e_{ij}. \end{aligned}\right\} \tag{7·1}$$

This is equivalent to putting

$$\left.\begin{aligned} Q(p) &= \frac{Gp}{p+r} = \frac{G\mu p}{\mu p+G}, \\ Q_1(p) &= \frac{G_1 p}{p+r_1} = \frac{G_1\mu_1 p}{\mu_1 p+G_1}, \end{aligned}\right\} \tag{7·2}$$

with

$$R = R_1 = \infty.$$

These materials are elastic for fast deformations ($p = \infty$). The shear moduli for fast deformation are G and G_1. For slow deformations (p small) these materials exhibit Newtonian viscosity with corresponding viscosity coefficients μ and μ_1.

The relaxation constants are

$$r = \frac{G}{\mu}, \quad r_1 = \frac{G_1}{\mu_1}. \tag{7·3}$$

The relaxation times are the inverse of these quantities.

The characteristic operators are

$$\left.\begin{aligned} B(p) &= \frac{4Gp}{p+r}, \\ B_1(p) &= \frac{4G_1 p}{p+r_1}. \end{aligned}\right\} \tag{7·4}$$