Turkish Journal of Mathematics

http://journals.tubitak.gov.tr/math/

Research Article

Turk J Math
(2013) 37: 348 – 359
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doi:10.3906/mat-1102-36

Foliations and a class of metrics on tangent bundle

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Received: 23.02.2011 • **Accepted:** 12.11.2011 • **Published Online:** 19.03.2013 • **Printed:** 22.04.2013

Abstract: Let M be a smooth manifold with Finsler metric F, and let TM° be the slit tangent bundle of M with a generalized Riemannian metric G, which is induced by F. In this paper, we extract many natural foliations of (TM°, G) and study some of their geometric properties. Next we use this approach to obtain new characterizations of Finsler manifolds with positive constant curvature.

Key words: Finsler manifold, foliation, constant curvature, Riemannian metric

1. Introduction

Several monographs present methods of differential geometry used in the study of Finsler manifolds [1, 2, 3, 5, 6]. As the geometric objects that occur in Finsler geometry depend on both point and direction, the tangent bundle of a Finsler manifold plays a major role in this study. To emphasize this Bejancu and Farran in [4], by using Sasaki-Finsler metric G_S, initiate a study of interrelations between the geometry of foliations on the tangent bundle of a Finsler manifold and the geometry of the Finsler manifold itself. Then, Peyghan and Tayebi introduce new metric G on slit bundle of Finsler manifold and they study geometric properties of this metric [10]. In this paper, we use this metric on TM° and we show that the vertical and horizontal Liouville vector fields L and L^* determine three totally geodesic foliations on (TM°, G). Finally, the main properties of the two foliations defined by F on (TM°, G) are presented in Propositions 1 and 2. In the last section, for any $c > 0$ we consider the indicatrix-bundle $IM(c)$ and by using the horizontal Liouville foliation on $(IM(c), G)$ and the curvature-angular form we obtain three new characterizations of Finsler manifolds of positive constant curvature.

2. Preliminaries

Let (M, F) be a Finsler manifold, where M is a real n-dimensional smooth manifold and F is the fundamental function of (M, F) [2]. Consider $TM^\circ = TM \setminus \{0\}$ and denote by VTM° the vertical vector bundle over TM°, that is, $VTM^\circ = \ker \pi_*$, where π_* is the tangent mapping of the canonical projection $\pi : TM^\circ \to M$. We may think of the Finsler metric $(g = g_{ij}(x, y))$, where we set $g_{ij}(x, y) = \frac{1}{2}\frac{\partial^2 F^2}{\partial y^i \partial y^j}$ as a Riemannian metric on VTM°. The canonical nonlinear connection $HTM^\circ = (N_i^j(x, y))$ of (M, F) is given by $N_i^j = \frac{\partial G^j}{\partial y^i}$, where $G^j = \frac{1}{4} g^{jh}(\frac{\partial^2 F^2}{\partial y^h \partial x^k} y^k - \frac{\partial F^2}{\partial x^h})$. Then on any coordinate neighborhood $\mathfrak{u} \subset TM^\circ$ the vector fields

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2000 *AMS Mathematics Subject Classification:* Primary 53C20, 53C22.

$\frac{\delta}{\delta x^i} = \frac{\partial}{\partial x^i} - N_i^j \frac{\partial}{\partial y^j}$, $i = 1, \cdots, n$ form a basis for $\Gamma(HTM^\circ{}_{|\mathfrak{u}})$. By straightforward calculation, we obtain the following Lie brackets:

$$\left[\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right] = R^k{}_{ij} \frac{\partial}{\partial y^k}, \quad \left[\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right] = G^k_{ij} \frac{\partial}{\partial y^k}, \tag{2.1}$$

where $R^k{}_{ij} = \frac{\delta N_i^k}{\delta x^j} - \frac{\delta N_j^k}{\delta x^i}$ and $G^k_{ij} = \frac{\partial N_i^k}{\partial y^j}$. Note that $R^k{}_{ij}$ is a skew symmetric Finsler tensor field of type (1,2) while G^k_{ij} are the local coefficients of the Berwald connection associated to (M, F). Some other Finsler tensor fields defined by $R^k{}_{ij}$ will be useful in the study of Finsler manifolds of constant flag curvature (see [4])

$$(i)\ R_{hij} = g_{hk} R^k{}_{ij}, \quad (ii)\ R_{hj} = R_{hij} y^i, \quad (iii)\ R^k{}_j = g^{kh} R_{hj}. \tag{2.2}$$

From their properties, we mention the following:

$$(i)\ y^h R_{hij} = 0, \quad (ii)\ y^h R_{hj} = 0, \quad (iii)\ R_{ij} = R_{ji}, \quad (iv)\ R^k{}_{ij} = \frac{1}{3}\left\{\dot{\partial}_i R^k{}_j - \dot{\partial}_j R^k{}_i\right\}. \tag{2.3}$$

We also need the angular metric h_{ij} of (M, F) given by

$$h_{ij} = g_{ij} - l_i l_j, \tag{2.4}$$

where $l_i = \frac{y_i}{F}$ and $y_i = g_{ij} y^j$. Moreover, we have the following theorem:

Theorem 1 ([7]) A Finsler manifold (M, F) is of constant curvature k if and only if the following holds

$$R_{ij} = kF^2 h_{ij}, \quad i, j = 1, \ldots, n. \tag{2.5}$$

Consider now the energy density $2t(x, y) = F^2 = g_{ij}(x, y) y^i y^j$ defined by the Finsler metric F and also the smooth functions $u, v : [0, \infty) \to \mathbb{R}$ such that $u + 2tv > 0$ for every t. The above conditions assure that the symmetric $(0,2)$-type tensor field of TM°, $G_{ij} = u(t) g_{ij} + v(t) y_i y_j$ is positive definite. The inverse of this matrix has the entries $H^{kl} = \frac{1}{u} g^{kl} + \omega(t) y^k y^l$, where (g^{kl}) are the components of the inverse of the matrix (g_{ij}) and $\omega(t) = -\frac{v}{u(u+2tv)}$. The components H^{kl} define symmetric $(0,2)$-type tensor field of TM°. It is easy to see that if the matrix (G_{ij}) is positive definite, then matrix H^{kl} is positive definite, too. We use also the components H_{ij} of symmetric $(0,2)$-type tensor field of TM° obtained from the components H^{kl} by "lowering" the indices $H_{ij} = g_{ik} H^{kl} g_{lj} = \frac{1}{u} g_{ij} + \omega y_i y_j$, where $y_i = g_{ik} y^k$. The following Riemannian metric may be considered on TM° (cf. [8]):

$$G\left(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}\right) = G_{ij}, \quad G\left(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}\right) = H_{ij}, \quad G\left(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}\right) = G\left(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}\right) = 0. \tag{2.6}$$

If $u = 1$ and $v(t) = 0$, then the above metric gives us the Sasaki-Finsler metric G_S as follows [4]:

$$G_S(\frac{\delta}{\delta x^i}, \frac{\delta}{\delta x^j}) = G_S(\frac{\partial}{\partial y^i}, \frac{\partial}{\partial y^j}) = g_{ij}, \quad G_S(\frac{\delta}{\delta x^i}, \frac{\partial}{\partial y^j}) = G_S(\frac{\partial}{\partial y^i}, \frac{\delta}{\delta x^j}) = 0. \tag{2.7}$$

Lemma 1 *The Levi-Civita connection of the Riemannian metric G defined by (2.6) is as follows:*

$$\begin{aligned}\widetilde{\nabla}_{\partial_{\bar{i}}}\partial_{\bar{j}} &= \frac{1}{u^2}(-F^s_{ij}+G^s_{ij})\frac{\delta}{\delta x^s}+(C^s_{ij}+\alpha_1 g_{ij}y^s-\alpha_2(y_i\delta^s_j+y_j\delta^s_i)\\ &+\alpha_3 y_i y_j y^s)\frac{\partial}{\partial y^s}, \qquad (2.8)\\ \widetilde{\nabla}_{\delta_i}\delta_j &= F^s_{ij}\frac{\delta}{\delta x^s}+(-u^2C^s_{ij}+\alpha_4(y_j\delta^s_i+y_i\delta^s_j)+\alpha_5 y_i y_j y^s+\alpha_6 g_{ij}y^s\\ &+\frac{1}{2}R^s_{ij})\frac{\partial}{\partial y^s}, \qquad (2.9)\\ \widetilde{\nabla}_{\partial_{\bar{i}}}\delta_j &= (C^s_{ij}+\alpha_7 y_i y_j y^s+\alpha_8 g_{ij}y^s+\alpha_2 y_i\delta^s_j+\alpha_9 y_j\delta^s_i+\frac{1}{2u}R_{ikj}H^{ks})\frac{\delta}{\delta x^s}\\ &+(F^s_{ij}-G^s_{ij})\frac{\partial}{\partial y^s}, \qquad (2.10)\\ \widetilde{\nabla}_{\delta_i}\partial_{\bar{j}} &= (C^s_{ij}+\alpha_7 y_i y_j y^s+\alpha_8 g_{ij}y^s+\alpha_2 y_j\delta^s_i+\alpha_9 y_i\delta^s_j+\frac{1}{2u}R_{jki}H^{ks})\frac{\delta}{\delta x^s}\\ &+F^s_{ij}\frac{\partial}{\partial y^s}, \qquad (2.11)\end{aligned}$$

where $\alpha_1=\frac{u'u+2tu'v+2wu^2(u+2tv)}{2u^2}$, $\alpha_2=\frac{u'}{2u}$, $\alpha_3=\frac{-2u'v+w'u^2(u+2tv)}{2u^2}$, $\alpha_4=-\frac{vu}{2}$, $\alpha_5=-\frac{v'(u+2tv)+2v^2}{2}$, $\alpha_6=-\frac{u'(u+2tv)}{2}$, $\alpha_7=\frac{wu'u+wvu+v'(1+2twu)}{2u}$, $\alpha_8=\frac{v(1+2twu)}{2u}$, $\alpha_9=\frac{v}{2u}$ *and* $C^s_{ij|t}$ *is the* h*-covariant derivative of* C^s_{ij} *with respect to Cartan connection.*

3. Foliations on (TM°, G)

In this section, we shall study various kinds of foliation which are naturally associated to (TM°, G). For this purpose, we consider two globally defined vector fields on TM° locally given by

$$L=y^i\dot{\partial}_i, \qquad (3.12)$$

$$L^*=y^i\delta_i. \qquad (3.13)$$

L and L^* are called the *vertical* and *horizontal Liouville vector fields*, respectively. The line distribution $\mathcal{L}=\mathrm{span}\{L\}$ and $\mathcal{L}^*=\mathrm{span}\{L^*\}$ are called the *vertical* and *horizontal Liouville distributions*, respectively.

Theorem 2 *Let (M,F) be a Finsler manifold. Then we have the following assertions:*

(i) The vertical Liouville vector field determines a totally geodesic foliation on (TM°, G).

(ii) The horizontal Liouville vector field determines a totally geodesic foliation on (TM°, G) if and only if u and v satisfy in

$$uv+\frac{1}{2}u'u+t(vu'+v'u)+2t^2vv'+2tv^2=0. \qquad (3.14)$$

(iii) The distribution $\Gamma(\mathcal{L}\oplus\mathcal{L}^*)$ is integrable and its tangent foliation is totally geodesic on (TM°, G).

Proof By using Lemma 1, we get

$$\widetilde{\nabla}_L L = \Big(1 + 2t(\alpha_1 - 2\alpha_2 + 2t\alpha_3)\Big) L, \tag{3.15}$$

$$\widetilde{\nabla}_{L^*} L^* = \Big(2t(2\alpha_4 + \alpha_6 + 2t\alpha_5)\Big) L, \tag{3.16}$$

$$\widetilde{\nabla}_{L^*} L = \Big(2t(\alpha_2 + \alpha_9 + \alpha_8 + 2t\alpha_7)\Big) L^*, \tag{3.17}$$

$$\widetilde{\nabla}_L L^* = \Big(1 + 2t(\alpha_2 + \alpha_9 + \alpha_8 + 2t\alpha_7)\Big) L^*. \tag{3.18}$$

Relation (3.15) tell us that $\mathcal{L}$ is totally geodesic. Also, from (3.16) we derive that $\mathcal{L}^*$ is totally geodesic if and only if (3.14) holds. Relations (3.17) and (3.18) give us $[L, L^*] = L^* \in \Gamma(\mathcal{L} \oplus \mathcal{L}^*)$. Now let $X = XL + X^*L^*$ and $Y = YL + Y^*L^*$ belong to $\Gamma(\mathcal{L} \oplus \mathcal{L}^*)$, then by Dirac calculation we obtain

$$\begin{aligned}\widetilde{\nabla}_X Y = &\Big(XL(Y) + X^*L^*(Y) + XY(1 + 2t(\alpha_1 - 2\alpha_2 + 2t\alpha_3))\\ &+2tX^*Y^*(2\alpha_4 + \alpha_6 + 2t\alpha_5)\Big)L + \Big(XL(Y^*) + X^*L^*(Y^*)\\ &+XY^*(1 + 2t(\alpha_2 + \alpha_9 + \alpha_8 + 2t\alpha_7))\\ &+2tX^*Y(\alpha_2 + \alpha_9 + \alpha_8 + 2t\alpha_7)\Big)L^*.\end{aligned}$$

Hence we derive that $\widetilde{\nabla}_X Y \in \Gamma(\mathcal{L} \oplus \mathcal{L}^*)$ for any $X, Y \in \Gamma(\mathcal{L} \oplus \mathcal{L}^*)$. Therefore the foliation determined by $\Gamma(\mathcal{L} \oplus \mathcal{L}^*)$ is totally geodesic. □

Remark 1 It is remarkable that the foliation in (ii) is also totally geodesic with respect to the Sasaki-Finsler metric (cf. [4]).

Also, by using (2.8)–(2.9) we can conclude the following:

Lemma 2 *Let (M, F) be a Finsler manifold. Then we have*

$$\begin{aligned}\widetilde{\nabla}_X L = &X^i\Big((2t\alpha_7 + \alpha_8 + \alpha_9)y_i y^k + 2t\alpha_2\delta_i^k\Big)\delta_k\\ &+\dot{X}^i\Big((\alpha_1 - \alpha_2 + 2t\alpha_3)y_i y^k + (1 - 2t\alpha_2)\delta_i^k\Big)\dot{\partial}_k,\end{aligned} \tag{3.19}$$

$$\begin{aligned}\widetilde{\nabla}_X L^* = &\dot{X}^i\Big((1 + 2t\alpha_9)\delta_i^k + (2t\alpha_7 + \alpha_8 + \alpha_2)y_i y^k + \frac{1}{2u} y^j R_{isj} H^{ks}\Big)\delta_k\\ &+X^i\Big(\frac{1}{2} y^j R^k{}_{ij} + 2t\alpha_4\delta_i^k + (\alpha_4 + 2t\alpha_5 + \alpha_6)y_i y^k\Big)\dot{\partial}_k,\end{aligned} \tag{3.20}$$

where $X = X^i\delta_i + \dot{X}^i\dot{\partial}_i \in \Gamma(TTM^\circ)$.

To introduce two more foliations on (TM°, G) we denote by $\mathcal{L}'$ and $\mathcal{L}^\perp$ the complementary orthogonal distributions to $\mathcal{L}$ in VTM° and TTM°, respectively. Now, let $X, Y \in \Gamma(\mathcal{L}^\perp)$. Since G is parallel with respect to $\widetilde{\nabla}$, then we get

$$G([X,Y],L) = G(\widetilde{\nabla}_X Y, L) - G(\widetilde{\nabla}_Y X, L) = G(X, \widetilde{\nabla}_Y L) - G(Y, \widetilde{\nabla}_X L). \tag{3.21}$$

By using (3.19), we derive

$$\begin{aligned} G(X, \widetilde{\nabla}_Y L) = & \Big[[(\alpha_9 + \alpha_8 + 2t\alpha_7)(u+2tv) + 2tv\alpha_2]y_i y_j + 2tu\alpha_2 g_{ij}\Big] X^i Y^j \\ & + \Big[(\alpha_1 - \alpha_2 + 2t\alpha_3)(\frac{1}{u} + 2tw) + w(1-2t\alpha_2)]y_i y_j \\ & + \frac{1}{u}(1-2t\alpha_2)g_{ij}\Big]\dot{X}^i \dot{Y}^j. \end{aligned} \tag{3.22}$$

Similarly we have

$$\begin{aligned} G(Y, \widetilde{\nabla}_X L) = & \Big[[(\alpha_9 + \alpha_8 + 2t\alpha_7)(u+2tv) + 2tv\alpha_2]y_i y_j + 2tu\alpha_2 g_{ij}\Big] Y^i X^j \\ & + \Big[(\alpha_1 - \alpha_2 + 2t\alpha_3)(\frac{1}{u} + 2tw) + w(1-2t\alpha_2)]y_i y_j \\ & + \frac{1}{u}(1-2t\alpha_2)g_{ij}\Big]\dot{Y}^i \dot{X}^j. \end{aligned} \tag{3.23}$$

Since i, j, k in (3.22) and (3.23) are summation indices, then (3.22) is equal (3.23). Therefore, by according to (3.21) we infer

$$G([X,Y],L) = 0. \tag{3.24}$$

Hence $[X,Y] \in \Gamma(\mathcal{L}^\perp)$, that is, $\mathcal{L}^\perp$ is integrable. It is obvious that $\mathcal{L}'$ is integrable, too. Therefore, we have the following theorem.

Theorem 3 *Let (M,F) be a Finsler manifold. Then both distributions $\mathcal{L}^\perp$ and $\mathcal{L}'$ are integrable.*

Also, similar to the proof of Proposition 2.1 in [4], we can prove the following:

Proposition 1 *(i) The fundamental foliation $\mathcal{F}_F$ determined by the level hypersurfaces of the fundamental function F of the Finsler manifold (M,F) is just the foliation determined by the integrable distribution $\mathcal{L}^\perp$.*
(ii) The vertical Liouville vector field is orthogonal to foliation $\mathcal{F}_F$.
(iii) The horizontal Liouville vector field is tangent to foliation $\mathcal{F}_F$.

Next, we consider a fixed point $x_0 = (x_0^i)$ in M and the hypersurfaces $I_{x_0}M(c)$ in $T_{x_0}M^\circ = T_{x_0}M - \{0\}$ given by the equation

$$F(x_0, y) = c, \quad \forall y \in T_{x_0}M^\circ,$$

where c is a positive constant. We call it the c-indicatrix of (M,F) at x_0. Then the set of all c-indicatrices at x_0 determines a foliation of codimension one of the m-dimensional Riemannian manifold $(T_{x_0}M^\circ, g_{x_0})$, where $g_{x_0} = \Big(g_{ij}(x_0,y)\Big)$ (see [4]). Now, let

$$l = \frac{1}{F}\sqrt{u+2tv}L = \sqrt{u+2tv}\, l^i \dot{\partial}_i, \qquad l^i = \frac{y^i}{F},$$

then we have $G(l,l)=1$. Also, we denote by the same symbol g_{x_0} the induced Riemannian metric by g_{x_0} on $I_{x_0}M(c)$. Now, we put

$$\widetilde{\nabla}_X Y = \nabla'_X Y + h(X,Y), \tag{3.25}$$

$$\nabla'_X Y = \nabla''_X Y + B(X,Y)l, \tag{3.26}$$

for any $X, Y \in \Gamma(TI_{x_0}M(c))$, where ∇' and ∇'' are the Levi-Civita connections on $(T_{x_0}M^\circ, g_{x_0})$ and $(I_{x_0}M(c), g_{x_0})$, respectively, while $h(\cdot,\cdot)$ and $B(\cdot,\cdot)l$ are the second fundamental forms of $T_{x_0}M^\circ$ and $I_{x_0}M(c)$ as submanifolds of (TM°, G) and $(T_{x_0}M^\circ, g_{x_0})$, respectively. Since l is orthogonal to $I_{x_0}M(c)$, then we have $g_{x_0}(\nabla''_X Y, l) = 0$. Hence by using (3.26), we obtain

$$\begin{aligned} g_{x_0}(\nabla'_X Y, l) &= g_{x_0}(B(X,Y)l, l) = \frac{u+2tv}{2t} B(X,Y) y^i y^j g_{ij} \\ &= (u+2tv)B(X,Y). \end{aligned} \tag{3.27}$$

Now, let $\widetilde{\nabla}_X Y = (\widetilde{\nabla}_X Y)^i \delta_i + (\dot{\widetilde{\nabla}}_X Y)^i \dot{\partial}_i$. According to (2.6), we get

$$\begin{aligned} G(\widetilde{\nabla}_X Y, l) &= G((\widetilde{\nabla}_X Y)^i \delta_i + (\dot{\widetilde{\nabla}}_X Y)^i \dot{\partial}_i, \frac{1}{F}\sqrt{u+2tv} y^j \dot{\partial}_j) \\ &= (\dot{\widetilde{\nabla}}_X Y)^i \frac{\sqrt{u+2tv}}{F} y^j (\frac{1}{u} g_{ij} - \frac{v}{u(u+2tv)} y_i y_j) \\ &= \frac{1}{F\sqrt{u+2tv}} (\dot{\widetilde{\nabla}}_X Y)^i y_i. \end{aligned} \tag{3.28}$$

Similarly, we obtain

$$\begin{aligned} g_{x_0}(\widetilde{\nabla}_X Y, l) &= g_{x_0}((\widetilde{\nabla}_X Y)^i \delta_i + (\dot{\widetilde{\nabla}}_X Y)^i \dot{\partial}_i, \frac{1}{F}\sqrt{u+2tv} y^j \dot{\partial}_j) \\ &= (\dot{\widetilde{\nabla}}_X Y)^i \frac{\sqrt{u+2tv}}{F} y^j g_{ij} = (\dot{\widetilde{\nabla}}_X Y)^i \frac{\sqrt{u+2tv}}{F} y_i. \end{aligned} \tag{3.29}$$

The relations (3.27), (3.28) and (3.29) give us

$$\begin{aligned} B(X,Y) &= \frac{1}{u+2tv} g_{x_0}(\nabla'_X Y, l) = G(\widetilde{\nabla}_X Y, l) = -G(Y, \widetilde{\nabla}_X l) \\ &= -G(Y, X(\frac{\sqrt{u+2tv}}{F})L + \frac{\sqrt{u+2tv}}{F} \widetilde{\nabla}_X L) \\ &= -\frac{\sqrt{u+2tv}}{F} G(Y, \widetilde{\nabla}_X L). \end{aligned} \tag{3.30}$$

Let $X = \dot{X}^i \dot{\partial}_i \in \Gamma(TI_{x_0}M(c))$. Since L is orthogonal to $I_{x_0}M(c)$, then we have

$$0 = G(X, L) = G(\dot{X}^i \dot{\partial}_i, y^j \dot{\partial}_j) = \dot{X}^i y^j (\frac{1}{u} g_{ij} - \frac{v}{u(u+2tv)} y_i y_j) = \frac{1}{u+2tv} \dot{X}^i y_i.$$

Hence we infer that

$$\dot{X}^i y_i = 0, \tag{3.31}$$

because $u + 2tv \neq 0$. By using (3.19) and (3.31), we deduce

$$\widetilde{\nabla}_X L = (1 - 2t\alpha_2)X. \tag{3.32}$$

The relation (3.32) in (3.30) implies

$$B(X,Y) = \frac{(2t\alpha_2 - 1)\sqrt{u + 2tv}}{F} G(X,Y). \tag{3.33}$$

But by direct calculation we derive $G(X,Y) = \frac{1}{u} g_{x_0}(X,Y)$. Thus for any $X, Y \in \Gamma(TI_{x_0}M(c))$ we obtain

$$B(X,Y) = \frac{(2t\alpha_2 - 1)\sqrt{u + 2tv}}{uF} g_{x_0}(X,Y).$$

Therefore any c-indicatrix at x_0 is a totally umbilical manifold immersed in $(T_{x_0}M^\circ, g_{x_0})$. Finally, we deduce that the leaves of the integrable distribution $\mathcal{L}'$ are c-indicatrices, because L is the normal vector field to each c-indicatrix.

Proposition 2 *Let (M,F) be a Finsler manifold. Then we have the following assertions:*
(i) At any point $x \in M$, the indicatrix foliation I_xM is a totally umbilical foliation of (T_xM, g_x).
(ii) The leaves of the foliation $\mathcal{F}_{\mathcal{L}'}$ determined by the integrable distribution $\mathcal{L}'$ are c-indicatrices of (M,F).
(iii) The foliation $\mathcal{F}_{\mathcal{L}'}$ is a totally umbilical subfoliation of the vertical foliation $\mathcal{F}_V$.

4. Finsler manifolds of positive constant curvature

In this section, we give some necessary and sufficient conditions for (M,F) to be of constant curvature.

Let (M,F) be a Finsler manifold and consider the symmetric tensor fields $R = (R_{ij})$ and $h = (h_{ij})$, where R_{ij} and h_{ij} are given by (2.2) and (2.4). We define the symmetric Finsler tensor field $\Lambda = (\Lambda_{ij})$ by

$$\Lambda_{ij} = R_{ij} - h_{ij}. \tag{4.34}$$

We consider Λ as a symmetric bilinear form on the $\mathcal{F}(TM^\circ)$-module $\Gamma(HTM^\circ)$ and call it the curvature-angular form of (M,F) (see [4]).

Proposition 3 *For any $X \in \Gamma(HTM^\circ)$ we have*

$$\Lambda(L^*, X) = 0 = R(L^*, X). \tag{4.35}$$

Proof Let $X = X^i\delta_i \in \Gamma(HTM^\circ)$. Using (ii) of (2.3) and (2.4), we have

$$\begin{aligned} \Lambda(L^*, X) = y^i X^j \Lambda_{ij} &= X^j y^i R_{ij} - X^j y^i g_{ij} + X^j y^i \frac{y_i}{F}\frac{y_j}{F} \\ &= -X^j y_j + X^j y_j = 0. \end{aligned} \tag{4.36}$$

Also, part (ii) of (2.3) gives us

$$R(L^*, X) = X^j y^i R_{ij} = 0. \tag{4.37}$$

The relations (4.36) and (4.37) imply (4.35). □

Next, we consider a leaf $IM(c)$ of the fundamental foliation $\mathcal{F}_F$ on (TM°, G). As we can write

$$IM(c) = \bigcup_{x\in M} I_x M(c),$$

we call $IM(c)$ the c-indicatrix bundle over M. Also, we consider the horizontal Liouville foliation $\mathcal{F}_{L^*}$ determined by the integral curves of L^*. According to Theorem 2, $\mathcal{F}_{L^*}$ is a totally geodesic foliation on (TM°, G) if and only if

$$uv + \frac{1}{2}u'u + t(vu' + v'u) + 2t^2vv' + 2tv^2 = 0.$$

Therefore we infer that $\mathcal{F}_{L^*}$ is totally geodesic on any c-indicatrix bundle $(IM(c), G)$ if and only if

$$uv + \frac{1}{2}u'u + \frac{1}{2}(vu' + v'u) + \frac{1}{2}vv' + v^2 = 0.$$

Here and in the sequel, we denote by the same symbol G the Riemannian metric on $IM(c)$ which is induced by the metric G on TM°.

Theorem 4 *Let (M, F) be a Finsler manifold and $IM(c)$ be a c-indicatrix over M. Then the Riemannian metric G on $IM(c)$ is bundle-like for horizontal Liouville foliation $\mathcal{F}_{L^*}$ on $IM(c)$ if and only if $\Lambda = (1-\frac{1}{u^2})R$ on $IM(c)$.*

Proof First, we note that all the vector bundles in this proof are considered to be over $IM(c)$. Let $\mathcal{L}''$ be the complementary orthogonal distribution to the horizontal Liouville distribution $\mathcal{L}^*$ in HTM°. Then $\mathcal{L}^\perp = \mathcal{L}' \oplus \mathcal{L}'' \oplus \mathcal{L}^*$ is the tangent bundle of $IM(c)$. It is known that the Riemannian metric G is bundle-like for $\mathcal{F}_{L^*}$ on $IM(c)$ if and only if

$$G(\tilde{\tilde{\nabla}}_X Y, L^*) + G(\tilde{\tilde{\nabla}}_Y X, L^*) = 0, \tag{4.38}$$

where $X, Y \in \Gamma(\mathcal{L}' \oplus \mathcal{L}'')$ and $\tilde{\tilde{\nabla}}$ is the Levi-Civita connection on $(IM(c), G)$. Since $\tilde{\tilde{\nabla}}$ is parallel with respect to G and $G(X, L^*) = G(Y, L^*) = 0$, then we have $G(\tilde{\tilde{\nabla}}_X Y, L^*) = G(Y, \tilde{\tilde{\nabla}}_X L^*)$ and $G(\tilde{\tilde{\nabla}}_Y X, L^*) = G(X, \tilde{\tilde{\nabla}}_Y L^*)$. Therefore (4.38) is equivalent to

$$G(\tilde{\tilde{\nabla}}_X L^*, Y) + G(\tilde{\tilde{\nabla}}_Y L^*, X) = 0, \quad \forall X, Y \in \Gamma(\mathcal{L}' \oplus \mathcal{L}''), \tag{4.39}$$

where $\tilde{\tilde{\nabla}}$ is the Levi-Civita connection on $(IM(c), G)$. Since $\mathcal{L}$ is the normal bundle to $IM(c)$, then (4.39) is equivalent to

$$G(\tilde{\tilde{\nabla}}_X L^*, Y) + G(\tilde{\tilde{\nabla}}_Y L^*, X) = 0, \quad \forall X, Y \in \Gamma(\mathcal{L}' \oplus \mathcal{L}''), \tag{4.40}$$

where $\tilde{\tilde{\nabla}}$ is the Levi-Civita connection on (TM°, G).

Now, we consider three cases to analyze (4.40). In the first case, let X and Y belong to $\Gamma(\mathcal{L}')$. Then by using (3.20), we conclude that $\tilde{\tilde{\nabla}}_X L^*$ and $\tilde{\tilde{\nabla}}_Y L^*$ belong to $\Gamma(HTM^\circ)$. Thus we have $G(\tilde{\tilde{\nabla}}_X L^*, Y) = G(\tilde{\tilde{\nabla}}_Y L^*, X) = 0$, because $\mathcal{L}'$ and HTM° are orthogonal vector bundles with respect to G. Consequently, in this case (4.40) is identically satisfied. In the second case, we let X and Y belong to $\Gamma(\mathcal{L}'')$. Then by using (3.20), we conclude that $\tilde{\nabla}_X L^*$ and $\tilde{\nabla}_Y L^*$ belong to $\Gamma(VTM^\circ)$. Similar to the previous case, we can deduce that (4.40) is again identically satisfied. In the third case, we let $X = X^i\dot{\partial}_i \in \Gamma(\mathcal{L}')$ and $Y = Y^i\delta_i \in \Gamma(\mathcal{L}'')$. Since $\mathcal{L}''$ is the complementary orthogonal distribution to $\mathcal{L}^*$ in HTM°, then we have

$$0 = G(Y, L^*) = Y^i y^j G(\delta_i, \delta_j) = Y^i y^j (ug_{ij} + vy_i y_j) = (u + 2tv)Y^i y_i. \tag{4.41}$$

Also, (3.31) gives us

$$X^i y_i = 0. \tag{4.42}$$

According to (3.20), we get

$$G(\widetilde{\nabla}_X L^*, Y) = (u + tv)X^k Y^r g_{kr} - \frac{1}{2u} X^i Y^r R_{ir}. \tag{4.43}$$

Similarly, we obtain

$$G(\widetilde{\nabla}_Y L^*, X) = -tvX^r Y^i g_{ir} - \frac{1}{2u} Y^i R_{ri} X^r. \tag{4.44}$$

Using (4.43) and (4.44), we obtain the following expression of (4.40):

$$\Big(ug_{ij} - \frac{1}{u}R_{ij}\Big)X^i Y^j = 0. \tag{4.45}$$

On other hand, (4.42) implies

$$h_{ij}X^i Y^j = g_{ij}X^i Y^j. \tag{4.46}$$

By using (4.34), (4.45) and (4.46) we obtain

$$\begin{aligned}\Lambda_{ij}X^i Y^j &= R_{ij}X^i Y^j - h_{ij}X^i Y^j = R_{ij}X^i Y^j - g_{ij}X^i Y^j \\ &= R_{ij}X^i Y^j - \frac{1}{u^2}R_{ij}X^i Y^j = \Big(1 - \frac{1}{u^2}\Big)R_{ij}X^i Y^j.\end{aligned} \tag{4.47}$$

Now, we consider the isomorphism of vector bundles $\Phi : \mathcal{L}' \to \mathcal{L}''$ defined by $\Phi(X^i\dot{\partial}_i) = X^i\delta_i = X^*$. Then (4.47) is equivalent to

$$\Lambda(X^*, Y) = (1 - \frac{1}{u^2})R(X^*, Y), \quad \forall X^*, Y \in \Gamma(\mathcal{L}''). \tag{4.48}$$

Finally, from (4.35) and (4.48) we deduce that (4.40) is equivalent to $\Lambda = (1 - \frac{1}{u^2})R$ on $IM(c)$. □

Taking into account that $\mathcal{L}^\perp$ is orthogonal to the vertical Liouville distribution $\mathcal{L}$ we deduce that L^* is a Killing vector field on $IM(c)$ if and only if (see [11])

$$G(\widetilde{\nabla}_X L^*, Y) + G(\widetilde{\nabla}_Y L^*, X) = 0, \quad \forall X, Y \in \Gamma(\mathcal{L}^\perp). \tag{4.49}$$

Now, we can prove the following theorem.

Theorem 5 *Let (M, F) be a Finsler manifold and $IM(c)$ be a c-indicatrix bundle over M. Then the horizontal Liouville vector field L^* is a Killing vector field on $IM(c)$ if and only if $\Lambda = (1 - \frac{1}{u^2})R$ on $IM(c)$.*

Proof If L^* is a Killing vector field on $IM(c)$, then according to (4.49), the relation (4.40) is held and consequently from Theorem 4 we infer that $\Lambda = (1-\frac{1}{u^2})R$ on $IM(c)$. Conversely let $\Lambda = (1-\frac{1}{u^2})R$ on $IM(c)$. Then (4.40) gives us (4.49), only for any X, Y belong to $\Gamma(\mathcal{L}'\oplus\mathcal{L}'')$. Also, if $X = Y = L^*$ then (3.16) implies (4.49). Hence in order to complete the proof we need to show that (4.49) is held for $X = L^*$ and $Y \in \Gamma(\mathcal{L}'\oplus\mathcal{L}'')$. According to (3.16), since $\widetilde{\nabla}_{L^*}L^* = 2t(2\alpha_4+\alpha_6+2t\alpha_5)L$ then we deduce that we should prove that

$$G(\widetilde{\nabla}_Y L^*, L^*) = 0, \quad \forall Y \in \Gamma(\mathcal{L}'\oplus\mathcal{L}''). \tag{4.50}$$

We consider two cases to analyze (4.50).

Case 1. $Y \in \Gamma(\mathcal{L}'')$. Then from (3.20) we infer that $\widetilde{\nabla}_Y L^* \in \Gamma(VTM^\circ)$, and consequently (4.50) is held in this case.

Case 2. $Y \in \Gamma(\mathcal{L}')$. In this case we have $Y = Y^i\dot{\partial}_i$, where Y^i satisfy (4.42). Then by using (3.20), we obtain

$$\widetilde{\nabla}_Y L^* = Y^i\Big((1+2t\alpha_9)\delta_i^k + \frac{1}{2u^2}y^jR_{isj}g^{ks}\Big)\delta_k.$$

Hence we get

$$G(\widetilde{\nabla}_Y L^*, L^*) = Y^i\Big((1+2t\alpha_9)(u+2tv)y_i + \frac{1}{2u}y^ry^jR_{irj}\Big). \tag{4.51}$$

But by using (ii) of (2.2), (ii) of (2.3) and (4.42) we have $Y^ky_k = 0$ and $R_{irj}y^jy^r = 0$. Hence $G(\widetilde{\nabla}_Y L^*, L^*) = 0$, where $Y \in \Gamma(\mathcal{L}')$.

By using the above cases, we deduce that (4.49) is identically satisfied, and therefore L^* is a Killing vector field on $IM(c)$. □

Theorem 6 *A Finsler manifold (M,F) is of positive constant curvature k if and only if $\Lambda = (1-\frac{1}{u^2})R$ on the indicatrix bundle $IM(c)$ where $c = \frac{u}{\sqrt{k}}$.*

Proof Let (M,F) be a Finsler manifold of constant curvature k. Then by Theorem 1, we have

$$R_{ij} = kF^2h_{ij}. \tag{4.52}$$

But on $IM(c)$ we have $F(x,y) = c = \frac{u}{\sqrt{k}}$. Hence we obtain $F^2 = \frac{u^2}{k}$ or equivalently

$$kF^2 = u^2. \tag{4.53}$$

Substituting the above equation into (4.52), we obtain

$$h_{ij} = \frac{1}{u^2}R_{ij}. \tag{4.54}$$

Substituting (4.54) into (4.34), we get

$$\Lambda_{ij} = R_{ij} - \frac{1}{u^2}R_{ij} = (1-\frac{1}{u^2})R_{ij}. \tag{4.55}$$

Conversely, let $\Lambda = (1 - \frac{1}{u^2})R$ on $IM(c)$. Then it follows from (4.53) and (4.34) that

$$R_{ij}(x,y) = u^2 h_{ij}(x,y) = kF^2(x,y)h_{ij}(x,y), \quad \forall (x,y) \in IM(c). \tag{4.56}$$

Now, we take a point $(x,y) \in TM^\circ \backslash IM(c)$. Since TM° admits the fundamental foliation $\mathcal{F}_F$, there exist $c^* > 0$ such that $(x,y) \in IM(c^*)$, that is, $F(x,y) = c^*$. Since F is positively homogeneous of degree one, we have $F(x, \frac{c}{c^*}y) = \frac{c}{c^*}F(x,y) = c$, i.e., $(x, \frac{c}{c^*}y) \in IM(c)$. Hence by (4.56), we obtain

$$R_{ij}(x, \frac{c}{c^*}y) - h_{ij}(x, \frac{c}{c^*}y) = (1 - \frac{1}{u^2})R_{ij}(x, \frac{c}{c^*}y), \tag{4.57}$$

or equivalently

$$R_{ij}(x, \frac{c}{c^*}y) = u^2 h_{ij}(x, \frac{c}{c^*}y). \tag{4.58}$$

Since h_{ij} and R_{ij} are positively homogeneous of degree zero and two, respectively, equation (4.58) implies

$$R_{ij}(x,y) = u^2 \frac{c^{*2}}{c^2} h_{ij}(x,y). \tag{4.59}$$

Since $c = \frac{u}{\sqrt{k}}$ and $F(x,y) = c^*$, it follows from (4.59) that

$$R_{ij}(x,y) = kF^2(x,y)h_{ij}(x,y), \quad \forall (x,y) \in TM^\circ \backslash IM(c). \tag{4.60}$$

Thus it follows from (4.56), (4.60) and Theorem 1 that (M,F) is a Finsler manifold of positive constant curvature k. □

Theorem 7 *Let (M,F) be a Finsler manifold, and k, c two positive numbers such that $c = \frac{u}{\sqrt{k}}$. Then the following assertions are equivalent:*
(i) (M,F) is a Finsler manifold of constant curvature k.
(ii) The Sasaki-Finsler metric G on the indicatrix bundle $IM(c)$ is bundle-like for the horizontal Liouville foliation $IM(c)$.
(iii) The horizontal Liouville vector field is a Killing vector field on $(IM(c), G)$.
(iv) The curvature-angular form Λ of (M,F) satisfy $\Lambda = (1 - \frac{1}{u^2})R$ on $IM(c)$.

References

[1] Antonelli, P. L., Ingarden, R. S. and Matsumoto, M.: The Theory of Sprays and Finsler Spaces with Applications in Physics and Biology, Kluwer Acad. Publish, Dordrecht 1993.

[2] Bao, D., Chern, S. S. and Shen, Z.: An Introduction to Riemannian-Finsler Geometry, Spinger-Verlag, New York 2000.

[3] Bejancu, A.: Finsler Geometry and Applications, Ellis Horwood, New York 1990.

[4] Bejancu, A. and Farran, H. R.: Finsler geometry and natural foliations on the tangent bundle, Rep. Math. Phys. 58, 131–146, (2006).

[5] He, Q., Yang, W. and Zhao, W.: On totally umbilical submanifolds of Finsler spaces, Ann. Polon. Math. 100, 147–157, (2011).

[6] Li, J.: Stable harmonic maps between Finsler manifolds and Riemannian manifolds with positive Ricci curvature, Ann. Polon. Math. 99, 67–77, (2010).

[7] Matsumoto, M.: Foundations of Finsler geometry and special Finsler spaces, Kaiseisha, Japan, 1986.

[8] Oproiu, V. and Papaghiuc, N.: A Kaehler structure on the nonzero tangent bundle of a space form, Diff. Geom. Appl. 11, 1–12, (1999).

[9] Shen, Z.: Lectures on Finsler Geometry, Word Scientific, Singapore 2001.

[10] Tayebi, A. and Peyghan, E.: On a class of Riemannian metrics arising from Finsler structures, C. R. Acad. Sci. Paris, Ser. I. 349, 319–322, (2011).

[11] Yano, K. and Kon, M.: Structures on Manifolds, World Scientific, Singapore 1984.