Foliations and complemented framed structures

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Abstract

On an odd dimensional manifold, we define a structure which generalizes several known structures on almost contact manifolds, namely Sasakian, trans-Sasakian, quasi-Sasakian, Kenmotsu and cosymplectic structures. This structure, hereinafter called a G.Q.S. manifold, is defined on an almost contact metric manifold and satisfies an additional condition (1.5). We then consider a codimension-one distribution on a G.Q.S. manifold. Necessary and sufficient conditions for the normality of the complemented framed structure on the distribution defined on a G.Q.S manifold are studied (Th. 3.2). The existence of the foliation on G.Q.S. manifolds and of bundle-like metrics are also proven. It is shown that under certain circumstances a new foliation arises and its properties are investigated. Some examples illustrating these results are given in the final part of this paper.

Introduction

The geometry of foliation on a Riemannian manifold has been intensively studied in the latest years and many interesting results have been obtained, among others, by Pitiş [11], Pang [10], Libermann [7]. In their book [2], Bejancu and Faran studied the foliations defined on a Riemannian manifold by using only two adapted linear connections. The notion of *bundle-like* metric on a Riemannian manifold was introduced by Reinhart [12] and intensively studied by several authors (see Tondeur [15] and the related references cited therein.) It was subsequently proved that there exists a bundle-like metric on a Riemannian manifold (M, g) endowed with two complementary orthogonal non-integrable distributions (see [2] p.32).

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Blair [3] introduced the notion of *complemented framed structure*, on a Riemannian manifold (M, g). This notion is defined for manifolds with an f-structure and it is the analogue of the Kaehler structure in the almost complex case and of the quasi-Sasakian structure in the almost contact case (see [3] p.155). Several important results in this direction were established by Goldberg and Yano in [6].

The purpose of this paper is to study some properties of a foliated manifold endowed with an almost contact metric structure satisfying an additional condition (1.5). The remaining part of this paper is structured as follows.

In the first section, several general results regarding quasi-Sasakian manifolds are stated for later use. We introduce the notion of a generalised quasi-Sasakian manifold (on short a G.Q.S manifold), defined as a manifold endowed with an almost contact metric structure enjoying property (1.5). Some important results of the G.Q.S manifolds are proven, for later use (Proposition 1.2). In the second section, it is introduced for the first time in literature, after the best of this author's knowledge, the notion of *complemented framed structure* (Definition 2.1) on any codimension-one distribution tangent to the structure vector field of a G.Q.S manifold. Th. 2.1 proves the existence of this structure. The third section is aims at studying the existence of *normal complemented framed structure*. The existence of a normal complemented framed structure is proved by veryfing the necessary and sufficient conditions for the existence of this structure. (Th. 3.2, Th. 3.3). Next, it is shown in Th. 3.6 that the existence of a *normal complemented framed structure* implies the existence of a foliation of dimension three. The existence of a *bundle-like metric* and of a minimal foliation are also studied in Th. 3.7 and 3.8, respectively. The paper is concluded with an example which illustrates the above-mentioned theoretical results.

1 Preliminaries

Throughout this paper, all manifolds and maps are differentiable of class C^∞. Consider M be an $(n+p)$-dimensional paracompact manifold and TM be the tangent bundle of M. $F(M)$ represents the algebra of the differentiable functions on M and $\Gamma(E)$ the $F(M)$-module of the sections of a vector bundle E over M. In the following M is supposed to be a Riemannian manifold with the Riemannian metric g.

Now suppose that there exists a pair of complementary orthogonal distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M, that is, TM has the decomposition

$$TM = \mathcal{D} \oplus \mathcal{D}^\perp, \tag{1.1}$$

with respect to the Riemannian metric g.

We denote by Q and Q' the projection morphisms of TM on $\mathcal{D}$ and $\mathcal{D}^\perp$, respectively. Based on the ideas from [2] p.97 we consider two connections denoted by D and $D^\perp$ on the distributions $\mathcal{D}$ and $\mathcal{D}^\perp$, called *intrinsic linear connections* and defined by

$$a)\ \ D_X QY = Q\tilde{\nabla}_{QX} QY + Q[Q'X, QY],$$
$$b)\ \ D^\perp_X Q'Y = Q'\tilde{\nabla}_{Q'X} Q'Y + Q'[QX, Q'Y], \quad \forall\ X, Y \in \Gamma(TM),$$

where $\tilde{\nabla}$ is the Levi-Civita connection on (M, g). Suppose that the distribution $\mathcal{D}$ is integrable. Then $\mathcal{D}$ defines a foliation on M, which we denote by $\mathcal{F}$. The Riemannian metric g is said to be *bundle-like* for the foliation $\mathcal{F}$ (cf. [12]), if each geodesic in (M, g) that is tangent to the normal distribution to $\mathcal{F}$ at one point remains tangent for its entire length. Bejancu-Faran (see [2] p.110) gave a characterization for a *bundle-like* metric on a Riemannian manifold (M, g) as follows: the Riemannian metric g on M is *bundle-like* for the foliation $\mathcal{F}$ if the Riemannian metric induced by g on $\mathcal{D}^\perp$, denoted by the same symbol g, is parallel with respect to the intrinsic connection $D^\perp$, that is, we have

$$(D^\perp_X g)(Q'Z, Q'Y) = 0, \quad \forall X, Y, Z \in \Gamma(TM).$$

From [2] p.112, we recall the following result.

Theorem 1.1. *If $(M, g, \mathcal{F})$ is a foliated Riemannian manifold, then the following assertions are equivalent*

a) g is bundle-like metric for $\mathcal{F}$,

b) QX is a $\mathcal{D}^\perp$-Killing vector field, that is

$$g(\tilde{\nabla}_{Q'Y} QX, Q'Z) + g(\tilde{\nabla}_{Q'Z} QX, Q'Y) = 0, \ \forall \ X, Y, Z \in \Gamma(TM).$$

Now, we denote by ∇ (resp. $\nabla^\perp$) the connection induced by $\tilde{\nabla}$ on $\mathcal{D}$ (resp. $\mathcal{D}^\perp$) and by h, h' the $F(M)$-bilinear mappings

$$h : \Gamma(TM) \times \Gamma(\mathcal{D}) \to \Gamma(\mathcal{D}^\perp), \quad h' : \Gamma(TM) \times \Gamma(\mathcal{D}^\perp) \to \Gamma(\mathcal{D}).$$

The connections ∇, $\nabla^\perp$ and the $F(M)$-bilinear mappings h, h' are given by

$$\nabla_X QY = Q\tilde{\nabla}_X QY, \quad \nabla^\perp_X Q'Y = Q'\tilde{\nabla}_X Q'Y,$$
$$h(X, QY) = Q'\tilde{\nabla}_X QY, \quad h'(X, Q'Y) = Q\tilde{\nabla}_X Q'Y,$$

for any $X, Y \in \Gamma(TM)$. In connection with the decomposition (1.1) we have (see [2] p.27)

$$\begin{aligned} &a) \ \tilde{\nabla}_X QY = \nabla_X QY + h(X, QY), \\ &b) \ \tilde{\nabla}_X Q'Y = \nabla^\perp_X Q'Y + h'(X, Q'Y), \ \forall X, Y \in \Gamma(TM), \end{aligned} \tag{1.2}$$

relations which are called the Gauss formulae for the Riemannian distributions $(\mathcal{D}, g)$ and $(\mathcal{D}^\perp, g)$, respectively. For any $Q'X \in \Gamma(D^\perp)$ and $QX \in \Gamma(D)$, we define two $F(M)$-linear operators (see [2] p.27), namely

$$A_{Q'X} : \Gamma(D) \to \Gamma(D), \quad A_{QX} : \Gamma(D^\perp) \to \Gamma(D^\perp),$$

by

$$A_{Q'X} QY = -h'(QY, Q'X),$$
$$A'_{QX} Q'Y = -h(Q'Y, QX), \ \forall X, Y \in \Gamma(TM).$$

According to the theory of submanifolds, $A_{Q'X}$, and A'_{QX} are called the shape operators of $\mathcal{D}$ and $\mathcal{D}^\perp$ with respect to the normal sections $Q'X$ and QX, respectively.

It is easy to see that (see [2] p.28)

$$\begin{aligned}
&a)\ g(h(QX,QY),Q'Z) = g(A_{Q'Z}QX,QY),\\
&b)\ [QX,QY] \in \Gamma(\mathcal{D}) \Leftrightarrow h(QX,QY) = h(QY,QX),\\
&\quad \forall X,Y,Z \in \Gamma(TM).
\end{aligned} \tag{1.3}$$

Next, let M be a real $(2n+1)$-dimensional differentiable manifold, endowed with an almost contact metric structure (f,ξ,η,g). As a result, the following equalities are satisfied (see [3])

$$\begin{aligned}
&a)\ f^2 = -I + \eta \otimes \xi,\ \ b)\ \eta(\xi) = 1,\ \ c)\ \eta \circ f = 0,\\
&d)\ f(\xi) = 0,\ \ e)\ \eta(X) = g(X,\xi),\\
&f)\ g(fX,Y) + g(X,fY) = 0,\ \ \forall X,Y \in \Gamma(TM),
\end{aligned} \tag{1.4}$$

where I is the identity of the tangent bundle TM of M, f is a tensor field of type (1,1), η is a 1-form, ξ is a vector field tangent to M and g is a metric tensor field on M. The Nijenhuis tensor field with respect to the tensor field f, denoted by N_f, is given by

$$N_f(X,Y) = [fX,fY] + f^2[X,Y] - f[fX,Y] - f[X,fY],$$
$$\forall X,Y \in \Gamma(TM).$$

An almost contact metric manifold $M(f,\xi,\eta,g)$ is said to be normal if the almost complex structure J on the manifold $M \times R$, given by

$$J(X,\lambda\frac{d}{dt}) = (fX - \lambda\xi, \eta(X)\frac{d}{dt}),\ X \in \Gamma(TM),\ t \in R,$$

is integrable, where λ is a real-valued function on $M \times R$. This condition is equivalent to

$$N_f(X,Y) + 2d\eta(X,Y)\xi = 0,\ \forall\ X,Y \in \Gamma(TM).$$

In [4] the author proved that the almost contact metric structure, (f,ξ,η,g) is normal if and only if

$$(\tilde{\nabla}_{fX}f)Y = f(\tilde{\nabla}_X f)Y - g(\tilde{\nabla}_{fX}\xi,Y)\xi,\quad \forall X,Y \in \Gamma(TM).$$

The above result was used by Tanno [13], who cites Nakagawa [9].

We now consider a class of almost contact metric manifolds introduced by Eum [5]. The structure tensor field f of this class of manifolds is assumed to satisfy

$$(\tilde{\nabla}_X f)Y = g(\tilde{\nabla}_{fX}\xi,Y)\xi - \eta(Y)\tilde{\nabla}_{fX}\xi,\quad \forall X,Y \in \Gamma(TM). \tag{1.5}$$

The integrability of invariant hypersurfaces immersed in an almost contact Riemannian manifold complying with condition (1.5) was also studied in [5].
For convenience, we define a tensor field F of type (1,1) by

$$FX = -\tilde{\nabla}_X\xi,\ \ \forall X \in \Gamma(TM). \tag{1.6}$$

Using (1.4a), (1.5) and (1.6), one obtains the following result through direct calculation,

Proposition 1.1. *If M is an almost contact metric manifold enjoying property* (1.5), *the following equalities hold*

$$\begin{aligned} &a)\ M \text{ is normal}, \quad b)\ f \circ F = F \circ f, \quad c)\ F\xi = 0, \\ &d)\ \eta \circ F = 0, \quad e)\ (\tilde{\nabla}_\xi f)X = 0,\ \forall X \in \Gamma(TM). \end{aligned} \tag{1.7}$$

Next we prove the following characterization result for quasi-Sasakian manifold

Proposition 1.2. *Any integral curve of the structure vector field ξ on an almost contact metric manifold M enjoying property* (1.5), *is a geodesic, that is $\tilde{\nabla}_\xi \xi = 0$. Moreover, ξ is a Killing vector field if and only if M is a quasi-Sasakian manifold, that is $d\Phi = 0$, where $\Phi(X,Y) = g(X, fY)$.*

Proof. The first assertion is obviously equivalent to (1.7c). Next, by using (1.5) we infer that

$$\begin{aligned} &3d\Phi(X,Y,Z) = g((\tilde{\nabla}_X f)Z, Y) + g((\tilde{\nabla}_Z f)Y, X) + \\ &g((\tilde{\nabla}_Y f)X, Z) = \eta(X)(g(Y, \tilde{\nabla}_{fZ}\xi) + g(fZ, \tilde{\nabla}_Y \xi)) + \\ &\eta(Y)(g(Z, \tilde{\nabla}_{fX}\xi) + g(fX, \tilde{\nabla}_Z \xi)) + \eta(Z)(g(X, \tilde{\nabla}_{fY}\xi) + \\ &g(fY, \tilde{\nabla}_X \xi)). \end{aligned} \tag{1.8}$$

Now, suppose that ξ is a Killing vector field. Then from (1.8) one deduces that $d\Phi = 0$. Conversely, if $d\Phi = 0$, then we put $X = \xi$ in (1.8), and for $\eta(Y) = \eta(Z) = 0$, we obtain

$$g(Y, \tilde{\nabla}_{fZ}\xi) + g(fZ, \tilde{\nabla}_Y \xi) = 0.$$

Finally, replacing Y by $Y - \eta(Y)\xi$ and Z by fZ in the relation above, we obtain that ξ is a Killing vector field.

Remark 1. It is easy to see that on an almost contact metric manifold M enjoying property (1.5) the structure vector field ξ is not necessarily a Killing vector field. Also, it is interesting to see that:

1) if $FX = -fX$, then M is a Sasakian manifold,
2) if $FX = -X + \eta(X)\xi$, then M becomes a Kenmotsu manifold,
3) if $FX = 0$, then M is a cosymplectic manifold,
4) if ξ is a Killing vector field, then M is a quasi-Sasakian manifold,
5) if $FX = \alpha fX + \beta f^2 X$, $\alpha, \beta \in F(M)$, then M is trans-Sasakian manifold.

This was the reason for which we called M a *generalized quasi-Sasakian manifold* (shortly $G.Q.S$ manifold).

2 Distributions with complemented framed structures

In this paragraph, we first define the notion of complemented framed structure and study its normality. We then prove that a codimension-one distribution tangent to the structure vector field of G.Q.S manifold M admits a complemented framed structure and further on we establish some of its properties.

First, we suppose that a tensor field ϕ of type (1,1) is defined on a n-dimensional Riemannian manifold (M, g). The tensor field ϕ defines on M an f-structure (see [16]) if

$$\phi^3 + \phi = 0.$$

Stong [13] has proved that the rank of ϕ is constant, say r.
If $n = r$ then an f-structure gives an almost complex structure and n is even.
If $n - 1 = r$ and M is orientable, then an f-structure defines an almost contact structure and n is necessarily odd.

Next, suppose that there exists the unit vector fields $\xi_1, ..., \xi_r \in \Gamma(TM)$ and the 1-forms $\eta^1, ..., \eta^r$ on a $(2n + r)$-dimensional manifold M with an f-structure ϕ of rank $2n$.

We say that the f-structure ϕ defines an complemented framed structure on the manifold M, if the following conditions are fulfilled for $i, j \in \{1, ..., r\}$ (see [3])

$$a)\ \phi^2 X = -X + \sum_{i=1}^{r} \eta^i(X)\xi_i, \quad X \in \Gamma(TM),$$
$$b)\ \phi^i(\xi_j) = 0, \quad c)\ \eta^i(\xi_j) = \delta_{ij}; \quad d)\ \eta^i \circ \phi = 0,$$

where δ_{ij} *is the Kronecker delta.*

In the following we suppose there exist an f-structure defined by the tensor field ϕ, a distribution $\mathcal{D}'$ and the 1-dimensional distributions $\mathcal{D}''$ and $\mathcal{D}'''$ on a Riemannian manifold (M, g) so that we have the orthogonal decomposition below

$$TM = \mathcal{D}' \oplus \mathcal{D}'' \oplus \mathcal{D}'''; \quad \phi(\mathcal{D}'') = \phi(\mathcal{D}''') = 0; \quad \phi(\mathcal{D}') = \mathcal{D}'. \tag{2.1}$$

Definition 2.1. *We say that a complemented framed structure is defined on a Riemannian manifold* (M, g) *if there exist a tensor field* ϕ *of type* (1,1)*, the unit vector fields* $U_1 \in \Gamma(\mathcal{D}'')$, $U_2 \in \Gamma(\mathcal{D}''')$, *the 1-forms* η^1, η^2 *so that* (2.1) *and*

$$a)\ \phi^2 X = -X + \eta^1(X)U_1 + \eta^2(X)U_2, \quad X \in \Gamma(TM),$$
$$b)\ \phi(U_a) = 0, \quad c)\ \eta^a(U_b) = \delta_{ab}; \quad d)\ \eta^a \circ \phi = 0$$

are fulfilled, where δ_{ab} *is the Kronecker delta,* $a, b \in \{1, 2\}$.

The Riemannian manifold $M(\phi, U_1, U_2, \eta^1, \eta^2, g)$ is called a *complemented framed metric manifold* if (see [6])

$$\eta^a(X) = g(X, U_a),\ a = 1; 2 \ \text{and}\ g(\phi X, Y) + g(X, \phi Y) = 0.$$

Definition 2.2. *We say that the Riemannian manifold* (M, g) *with complemented framed structure is normal if*

$$S(X, Y) = N_\phi(X, Y) + 2d\eta^1(X, Y)U_1 + 2d\eta^2(X, Y)U_2 = 0, \quad \forall\, X, Y \in \Gamma(TM). \tag{2.2}$$

Next, we consider a G.Q.S manifold M with the almost contact metric structure (f, ξ, η, g). Suppose that on M there exist a distribution $\mathcal{D}_1$ of codimension 1, tangent to the structure vector field ξ, and let $\mathcal{D}_2$ be the orthogonal complementary distribution to $\mathcal{D}_1$ in TM, that is

$$TM = \mathcal{D}_1 \oplus \mathcal{D}_2,\ \dim \mathcal{D}_2 = 1, \quad \xi \in \Gamma(\mathcal{D}_1). \tag{2.3}$$

Because $\dim \mathcal{D}_2 = 1$, one deduces by using (1.4f) that $f\mathcal{D}_2 \subseteq \mathcal{D}_1$. Next, let $\mathcal{D}$ be the complement orthogonal distribution of $f\mathcal{D}_2 \oplus \{\xi\}$ in $\mathcal{D}_1$. Subsequently, we have the following orthogonal decomposition

$$\mathcal{D}_1 = \mathcal{D} \oplus \{\xi\} \oplus \mathcal{D}^{\perp};\ \ \mathcal{D}^{\perp} = f\mathcal{D}_2. \tag{2.4}$$

Now, we consider a locally-defined unit vector field $N \in \Gamma(\mathcal{D}_2)$ and $U = fN \in \Gamma(\mathcal{D}^{\perp})$. From (1.4a) and (1.4f) one deduces that U is also a unit vector field.

Next, let ∇ be the connection induced by $\tilde{\nabla}$ on $\mathcal{D}_1$. Relative to the decomposition (2.3), the formulae (1.2) have the following expression (see [1])

$$\begin{aligned} &a)\ \ \tilde{\nabla}_X Y = \nabla_X Y + B(X,Y)N,\\ &b)\ \ \tilde{\nabla}_X N = -AX,\ \ \forall X,Y \in \Gamma(\mathcal{D}_1),\ \ N \in \Gamma(\mathcal{D}_2), \end{aligned} \tag{2.5}$$

where A is the shape operator with respect to the section N and B is given by $B(X,Y) = g(\tilde{\nabla}_X Y, N)$, $X,\ Y \in \Gamma(\mathcal{D}_1)$.

Remark 2. Because $\tilde{\nabla}$ is the Levi-Civita connection, one obtains from (1.3b) that $B(X,Y) = B(Y,X) \ \Leftrightarrow\ g(AX,Y) = g(X,AY),\ \ X,Y \in \Gamma(\mathcal{D}_1)$ if and only if the distribution $\mathcal{D}_1$ is involutive. In this case, we denote by $\mathcal{F}_1$ the foliation on M defined by the involutive distribution $\mathcal{D}_1$.

Next, let us denote by P the projection morphism of TM on $\mathcal{D}$. Taking into account the decompositions (2.3) and (2.4), we may express X in the following way

$$X = PX + a(X)U + \eta(X)\xi,\ \ \forall X \in \Gamma(\mathcal{D}_1), \tag{2.6}$$

where a is a 1-form defined by $a(X) = g(X,U)$. From (1.4d) and (2.6), we see that

$$fX = tX - a(X)N,\ \ \forall X \in \Gamma(\mathcal{D}_1), \tag{2.7}$$

where t is a tensor field of type (1,1) given by

$$tX = fPX,\ \ X \in \Gamma(\mathcal{D}_1).$$

Also, from (1.6) and (2.5b), the vector field FX can be expressed in the following way

$$FX = \alpha X - \eta(AX)N,\ \ \forall X \in \Gamma(\mathcal{D}_1), \tag{2.8}$$

where α is a tensor field of type (1,1) so that $\alpha X \in \Gamma(\mathcal{D}_1),\ X \in \Gamma(\mathcal{D}_1)$.

By straightforward calculations, using (1.4a), (1.4d), (1.4f) we obtain the following result

Proposition 2.1. *The next equalities hold on a G.Q.S manifold M*

$$\begin{aligned} &a)\ \ tU = 0,\quad b)\ \ t\xi = 0,\quad c)\ \ t^2X = -X + a(X)U + \eta(X)\xi,\\ &d)\ \ g(tX,Y) + g(X,tY) = 0,\quad \forall X,Y \in \Gamma(\mathcal{D}_1). \end{aligned} \tag{2.9}$$

According to Definition 2.1 and Proposition 2.1, we can state the following theorem

Theorem 2.1. *On a* $2n$*-dimensional distribution* $\mathcal{D}_1$ *of a G.Q.S manifold M, tangent to the structure vector field* ξ*, there exists a complemented metric framed structure given by* (t, ξ, U, η, a, g).

Using (1.5), (1.6), (2.5a), (2.5b), (2.7) and (2.8), we infer the next result through direct calculations,

Propositon 2.2. *The covariant derivatives of* t *and* a *on a* $2n$*-dimensional distribution* $\mathcal{D}_1$ *of a G.Q.S manifold M are given by*

$$\begin{aligned} &a)\ (\nabla_X t)Y = \eta(Y)(t\alpha(X) - \eta(AX)U) - a(Y)AX + \\ &\qquad\qquad g(FX, fY)\xi + g(AX, Y)U, \\ &b)\ (\nabla_X a)Y = g(AX, tY) + \eta(Y)\eta(AtX),\ \ \forall X, Y \in \Gamma(\mathcal{D}_1). \end{aligned} \tag{2.10}$$

We also establish the following proposition for later use

Proposition 2.3. *Let M be a G.Q.S manifold and let* $\mathcal{D}_1$ *be a* $2n$*-dimensional distribution tangent to the structure vector field* ξ*. The following equalities hold*

$$\begin{aligned} &a)\ \nabla_X U = -tAX + (\eta(AtX) + a(X)a(FU))\xi, \\ &b)\ \nabla_\xi U = -tA\xi,\ \ c)\ [U, \xi] = f[N, \xi], \\ &d)\ a(FX) = \eta(AtX) + a(X)a(FU), \\ &e)\ g(X, FU) = -g(tX, A\xi) - g(X, [U, \xi]),\ \ \forall\, X \in \Gamma(\mathcal{D}_1). \end{aligned} \tag{2.11}$$

Proof. Taking $Y = U$ in (2.10a), one deduces the assertion a) by using (2.9a) - (2.9c) and (2.7). The assertion b) comes from (2.9a), (2.9b) and (2.11a). The statement c) is a consequence of (1.7b) and (1.7e). Finally the assertions d) and e) are obtained through direct calculation using (2.7), (2.5b) and the fact that $\tilde{\nabla}$ is a torsion free connection.

3 Normality for a complemented framed structure on a G.Q.S manifold M

The purpose of this section is to study the normality of complemented framed structure on the distribution $\mathcal{D}_1$ of a G.Q.S manifold M and to establish a necessary and sufficient condition for the existence of normal complemented framed structure.

First, for the complemented framed structure (t, ξ, U, η, a) on the distribution $\mathcal{D}_1$, the normality tensor S defined in (2.2) can be expressed as follows

$$S(X, Y) = N_t(X, Y) + 2da(X, Y)U + 2d\eta(X, Y)\xi, \forall X, Y \in \Gamma(\mathcal{D}_1). \tag{3.1}$$

Thus the complemented framed structure (t, ξ, U, η, a) is normal if $S(X, Y) = 0$, $\forall\, X, Y \in \Gamma(\mathcal{D}_1)$. Next, we state the following result

Theorem 3.1. *On a $2n$-dimensional distribution $\mathcal{D}_1$ of a G. Q. S manifold, tangent to the structure vector field ξ, the tensor field S is expressed by*

$$\begin{aligned} &S(X,Y) = a(X)(AtY - tAY + \eta(Y)\alpha U - g(Y,FU)\xi) - \\ &a(Y)(AtX - tAX + \eta(X)\alpha U - g(X,FU)\xi) + [g(AtX,Y) - \\ &g(tX,AY) + g(AX,tY) - g(AtY,X)]U + \\ &[B(tX,tY) - B(tY,tX)]N, \ \ \forall X,Y \in \Gamma(\mathcal{D}_1). \end{aligned} \tag{3.2}$$

Proof. Because $\tilde{\nabla}$ is a torsion free connection, using (2.5a), (2.10a) we infer that

$$\begin{aligned} &N_t(X,Y) = (\nabla_{tX}t)Y - (\nabla_{tY}t)X + t((\nabla_Y t)X - (\nabla_X t)Y) \\ &= \eta(Y)(t\alpha tX - t^2\alpha X) - \eta(X)(t\alpha tY - t^2\alpha Y) + a(X)(AtY - \\ &tAY) - a(Y)(AtX - tAX) + (g(fY,FtX) - g(fX,FtY)\xi + \\ &[B(tX,Y) - B(X,tY) + \eta(X)\eta(AtY) - \eta(Y)\eta(AtX)]U + \\ &[B(tX,tY) - B(tY,tX)]N, \ \ \forall X,Y \in \Gamma(\mathcal{D}_1). \end{aligned} \tag{3.3}$$

On the other hand, from (2.10b) we deduce that

$$\begin{aligned} &2da(X,Y) = (\nabla_X a)Y - (\nabla_Y a)X = B(tY,X) + \\ &\eta(Y)a(FX) - B(tX,Y) - \eta(X)a(FY), \forall X,Y \in \Gamma(\mathcal{D}_1). \end{aligned} \tag{3.4}$$

From (1.7b), (2.9c), (2.9d) and (2.11d), we infer that

$$\begin{aligned} &g(FtX,fY) - g(FtY,fX) = g(Ft^2Y,X) - g(Ft^2X,Y) \\ &= g(FX,Y) - g(X,FY) + a(Y)g(X,FU) - a(X)g(FU,Y) \\ &= -2d\eta(X,Y) + a(Y)g(X,FU) - a(X)g(Y,FU). \end{aligned} \tag{3.5}$$

Next, by using (1.7b), (2.9c), (2.9d), (2.7), (2.8) and (2.11d) we obtain that

$$\begin{aligned} &t\alpha tX = t(FtX + \eta(AtX)N) = tF(fX + a(X)N) + \eta(AtX)U = \\ &t(f\alpha X\eta(AX)U) + a(X)(fFN + a(FN)N) + \eta(AtX) = t^2\alpha X - \\ &a(\alpha X)U + a(X)\alpha U + \eta(AtX)U = t^2\alpha X + a(X)(\alpha U - a(FU)U) \\ &\ \forall X \in \Gamma(\mathcal{D}_1). \end{aligned} \tag{3.6}$$

Finally, the relation (3.2) comes from (3.1), (3.3) - (3.6) and the proof is complete. We now give a characterization for the normality of complemented metric framed structures defined on the distribution D_1 of a G.Q.S manifold M.

Theorem 3.2. *The complemented metric framed structure on a $2n$-dimensional distribution $\mathcal{D}_1$ of a G.Q.S manifold M, tangent to ξ, is normal if and only if the following conditions are fulfilled*

$$\begin{aligned} &a)\ \mathcal{D} \oplus \{\xi\} \ \text{ is integrable}, \\ &b)\ AtX = tAX + a(AtX)U, \ \forall\ X \in \Gamma(\mathcal{D}), \\ &c)\ \ [U,\xi] = -g([U,\xi],N)N, \quad d)\ \ tAU = 0 = tA\xi. \end{aligned} \tag{3.7}$$

Proof. First, for $X, Y \in \Gamma(\mathcal{D})$, the relation (3.2) becomes

$$S(X,Y) = [g(AtX,Y) - g(tX,AY) + g(AX,tY) - g(X,AtY)]U + [g(AtX,tY) - g(AtY,tX)]N. \tag{3.8}$$

Also, from the same relation (3.2) we obtain

$$S(X,\xi) = [\eta(AtX) - g(tX,A\xi)]U, \quad \forall X \in \Gamma(\mathcal{D}). \tag{3.9}$$

$$S(X,U) = tAX - AtX + g(X,FU)\xi + [a(AtX) - g(AU,tX)]U, \ \forall\, X \in \Gamma(\mathcal{D}). \tag{3.10}$$

$$S(U,\xi) = \alpha U - tA\xi = -[U,\xi] - g([U,\xi],N)N. \tag{3.11}$$

Now, suppose that $S(X,Y) = 0, \ \forall X, Y \in \Gamma(\mathcal{D}_1)$. Then, the relations (3.8) and (3.9) imply

$$a)\ g(AtX,tY) = g(AtY,tX), \qquad b)\ \eta(AtX) = g(tX,A\xi), \ \forall X \in \Gamma(\mathcal{D}). \tag{3.12}$$

The relations (1.3b), (3.12a) and (3.12b) yield the assertion (3.7a). Using the fact that $S(X,U) = 0$, from (3.10) we obtain $g(AU,tX) = 0, \ X \in \Gamma(\mathcal{D})$, that is, $tAU = 0$. The same relation (3.10) implies the fact that $\eta(AtX) = g(X,FU)$, which together with (3.12b) and (2.11e) prove that $\eta(AtX) = g(X,FU) = 0 \Rightarrow tA\xi = 0, \ X \in \Gamma(\mathcal{D})$. Next, it is easy to see that (3.7c) comes from (3.11). The relations (3.7) are proved. Conversely, if the relations (3.7) are true, then it is easy to see that $S(X,Y) = 0, \ \forall\, X, Y \in \Gamma(\mathcal{D})$. Taking into account that $tA\xi = 0$, we infer that $S(X,\xi) = 0$. Finally, from (3.10), (3.7a) - (3.7e) and (2.11e) one deduces that $S(X,U) = 0, \ \forall X \in \Gamma(\mathcal{D})$. This completes the proof.

Remark 3. The result proved above justifies the fact that the existence of the normal complemented metric framed structure implies the existence of the involutive distribution $\mathcal{D} \oplus \{\xi\}$ and consequently the existence of a foliation which we shall denote by $\mathcal{F}_1$. We shall also say that the foliation $\mathcal{F}_1$ is normal provided that Theorem 3.2 is true.

Corollary 3.1. *If $\mathcal{D}_1$ is an involutive distribution of a* G.Q.S *manifold, tangent to the structure vector field ξ, then the complemented framed structure is normal if and only if*

$$a)\ AtX = tAX, \ \forall\, X \in \Gamma(\mathcal{D}_1), \quad b)\ [U,\xi] = 0.$$

We consider the distribution $\mathcal{D}'_1 = \mathcal{D} \oplus \mathcal{D}_2 \oplus \{\xi\}$ on a G.Q.S manifold M and therefore we have the following decomposition $TM = f\mathcal{D}_2 \oplus \mathcal{D}'_1$. Next, we denote by A' the shape operator with respect to the section U. Through direct calculation, we obtain the next result

Proposition 3.1. *The next equivalences hold on a G.Q.S manifold M*

$$AtX = tAX + a(AtX)U \Leftrightarrow A'tX = tA'X + g(A'tX,N)N, \ \ X \in \Gamma(\mathcal{D}),$$

$$[U,\xi] = -g([U,\xi],N)N, \quad \Leftrightarrow \quad [N,\xi] = -g([N,\xi],U)U.$$

We denote by $\mathcal{F}'_1$ the foliation corresponding to the distribution $\mathcal{D}'_1$. The result of Proposition 3.1 justifies the next result

Theorem 3.3. *On a G.Q.S manifold, the foliation $\mathcal{F}_1$ is normal if and only if the foliation $\mathcal{F}'_1$ is normal, too.*

Next we obtain a new characterisation of the normality of foliation $\mathcal{F}_1$.

Theorem 3.4. *The foliation $\mathcal{F}_1$ on a G.Q.S manifold M is normal if and only if the following conditions are fulfilled*

$$a)\ \mathcal{D}\oplus\{\xi\}\ \text{ is involutive,}$$
$$b)\ U\ \text{ is a } \mathcal{D}_1\text{-Killing vector field,}$$
$$c)\ [U,\xi] = -g([U,\xi],N)N.$$

Proof. Using (2.11a), we obtain through direct calculation that

$$\begin{aligned} &g(\nabla_X U, Y) + g(\nabla_Y U, X) = g(AY, tX) + \\ &g(AX, tY) + \eta(AtX)\eta(Y) + \eta(AtY)\eta(X) + \\ &(\eta(X)a(Y) + \eta(Y)a(X))a(FU), \ \ \forall X, Y \in \Gamma(\mathcal{D}_1). \end{aligned} \tag{3.13}$$

Suppose that the foliation $\mathcal{F}_1$ is normal. Then from (2.9d), (3.7a) - (3.7d) and (3.13) we deduce that

$$g(\nabla_X U, Y) + g(\nabla_Y U, X) = 0, \ \ \forall X \in \Gamma(\mathcal{D}), \ \ Y \in \Gamma(\mathcal{D}_1).$$

Now, if we consider $X = U$ in (3.13), using (3.7d) we get

$$g(\nabla_U U, Y) = -g(tAU, Y) = 0, \ \forall\, Y \in \Gamma(\mathcal{D}_1). \tag{3.14}$$

Finally, the relation (3.13) for $X = \xi$ is expressed as follows

$$g(\nabla_Y U, \xi) + g(\nabla_\xi U, Y) = \eta(AtY) - g(tA\xi, Y) = 0, \ \ \ \forall\, Y\ \in \Gamma(\mathcal{D}_1).$$

Consequently, U is a $\mathcal{D}_1$-Killing vector field. Conversely, suppose that U is a $\mathcal{D}_1$-Killing vector field and $[U,\xi] = -g([U,\xi],N)N$. It follows from (3.13) that

$$g(AtX - tAX + \eta(AtX)\xi, Y) = 0, \ \ \forall X \in \Gamma(\mathcal{D}\oplus\{\xi\}), \ Y\ \in \Gamma(\mathcal{D}_1). \tag{3.15}$$

In particular, from the relation above for $Y = \xi$, we get that $\eta(AtX) = 0, \ \ X \in \Gamma(\mathcal{D})$ and by substituting it in (3.15) the proof of the Theorem is finalized.

From (3.14) and Theorem 3.3, we deduce the following consequence

Corollary 3.2. *Let M be a G.Q.S manifold with normal foliation $\mathcal{F}_1$ (or equivalently normal foliation $\mathcal{F}'_1$, respectively). Then*

a) *any integral curve of a $\mathcal{D}_1$-Killing vector field U is a geodesic,*

b) *any integral curve of a $\mathcal{D}'_1$-Killing vector field N is a geodesic.*

If M is a Kenmotsu manifold, then it can be proved by direct calculation that $[U,\xi] = -U + tA\xi - a(A\xi)N$, and consequently

Corollary 3.3. *In a Kenmotsu manifold, there exists no normal foliation tangent to the structure vector filed ξ.*

Remark 4. The above result is also true in case of trans-Sasakian manifolds.

If M is a quasi-Sasakian manifold, we can state the following result

Theorem 3.5. *The next assertions are equivalent on a quasi-Sasakian manifold M*

a) *The foliations $\mathcal{F}_1$ and $\mathcal{F}'_1$ are normal,*

b) $AtX = tAX, \quad \forall X \in \Gamma(\mathcal{D} \oplus \{\xi\}),$ *$\mathcal{D} \oplus \{\xi\}$ is involutive and* $[U,\xi] = -g([U,\xi],N)N,$

c) *U is a $\mathcal{D}_1$-Killing vector field, $\mathcal{D} \oplus \{\xi\}$ is involutive and* $[U,\xi] = -g([U,\xi],N)N,$

d) *N is a $\mathcal{D}'_1$-Killing vector field, $\mathcal{D} \oplus \{\xi\}$ is involutive and* $[N,\xi] = -g([N,\xi],U)U.$

Next, denote by $\mathcal{A} = \mathcal{D}^{\perp} \oplus f\mathcal{D}^{\perp} \oplus \{\xi\}$. We shall now prove our next important result.

Theorem 3.6. *If the foliation $\mathcal{F}_1$ on a G.Q.S manifold M is normal, then the distribution $\mathcal{A}$ defines a foliation $\mathcal{F}$ of dimension 3 on M.*

Proof. It is enough to prove that the distribution $\mathcal{A}$ is involutive. As $\mathcal{F}_1$ is normal, we have $[U,\xi], \ [N,\xi] \in \Gamma(\mathcal{A})$. Using the fact that $\tilde{\nabla}$ is a torsion free connection and Theorem 3.4, we obtain through direct calculation that

$$g(\tilde{\nabla}_U N, tX) = -g(AU, tX) = -g(U, AtX) = 0, \quad g(\tilde{\nabla}_N U, X) =$$

$$g((\tilde{\nabla}_N f)N + f(\tilde{\nabla}_N N, X) = -g(\tilde{\nabla}_N N, fX) = 0, \quad \forall X \in \Gamma(\mathcal{D}).$$

Therefore $[N,U] \in \Gamma(\mathcal{A})$. Also, in the case of quasi-Sasakian manifolds, a new interesting result holds.

Theorem 3.7. *Let $(M,g,\mathcal{F})$ be a 3-foliated G.Q.S manifold with the normal foliation $\mathcal{F}_1$. Then the metric tensor g is bundle-like for $\mathcal{F}$ if and only if M is quasi Sasaki manifold.*

Prof. Suppose that the metric tensor g is bundle-like. By using Theorem 1.1, it follows that U, N, ξ are $\mathcal{D}$-Killing vector fields. Then, we obtain through direct calculation that

$$g(\tilde{\nabla}_N \xi, N) = (\tilde{\nabla}_U \xi, U) = a(FU) = 0;$$
$$g(\tilde{\nabla}_U \xi, N) = -g(\tilde{\nabla}_N \xi, U),$$

and hence ξ is a Killing vector field. Thus, from Proposition 1.2 we deduce that M is a quasi-Sasaki manifold. The converse is trivial and the proof is complete.

Next, by using the result from [2] p.138, we define the mean curvature vector field of the foliation $\mathcal{F}$, denoted by H, as follows

$$H = \frac{1}{3}(h(N,N) + h(U,U) + h(\xi,\xi)), \tag{3.17}$$

where h is the second fundamental form of the distribution $\mathcal{A}$. In order to evaluate H, we suppose that the foliation $\mathcal{F}_1$ is normal. Then $h(X,Y) \in \Gamma(\mathcal{D}), \ \forall X, Y \in \Gamma(\mathcal{A})$ since $\mathcal{D}$ is the orthogonal complement of $\mathcal{A}$. Now, from Theorem 3.5 and Propositon 1.2, it is easy to see that $H = 0$. Therefore, we obtain the following

Theorem 3.8. *Let M be a G.Q.S manifold. If the foliation $\mathcal{F}_1$ is normal then the foliation $\mathcal{F}$ is minimal, that is, the mean curvature vector field H vanishes.*

At the end of this paper, we would like to offer some examples of foliations in order to illustrate the results obtained in this paper. Let us suppose that R^5

is endowed with a quasi-Sasakian structure of rank 3 given by (f,ξ,η,g). The 1-form is $\eta = dz - 2y^1dx^1$ where $(x^1,x^2,x^3,x^4,x^5) = (x^1,x^2,y^1,y^2,z)$, and the structure vector field is $\xi = (0,0,0,0,1)$. The matrix of the tensor f, F and g are given by

$$[f_i^h] = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 2y^1 & 0 & 0 \end{pmatrix}, \ [F_i^h] = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2y^1 & 0 & 0 \end{pmatrix},$$

and

$$[g_{ij}] = \begin{pmatrix} 1+4(y^1)^2 & 0 & 0 & 0 & -2y^1 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ -2y^1 & 0 & 0 & 0 & 1 \end{pmatrix},$$

respectively. On this manifold, we consider the following distributions

$$\mathcal{D} = span\{E_1 = (1,0,0,0,2x^4);\ \ E_2 = (0,0,1,0,0)\} = f\mathcal{D};$$
$$\mathcal{D}^\perp = span\{U = (0,2x^4,0,1,0)\};\ \ \mathcal{D}_2 = f\mathcal{D}^\perp = span\{N = (0,1,0,-2x^4,0)\};$$
$$\xi = \frac{\partial}{\partial x^5}.$$

First, we see that the distribution $\mathcal{D}_1 = \mathcal{D} \oplus \mathcal{D}^\perp \oplus \{\xi\}$ is involutive and therefore on R^5 one may define a foliation denoted by $\mathcal{F}_1$. Moreover, we may deduce the following through direct calculation

Theorem 3.9. *On the quasi-Sasakian manifold R^5, endowed with the given metric structure (f,ξ,η,g) the next assertions are true*

a) *the foliation $\mathcal{F}_1$ is normal,*

b) *g is a bundle-like metric for the foliation $\mathcal{F}_1$,*

c) *the distribution $\mathcal{A} = \mathcal{D}^\perp \oplus f\mathcal{D}^\perp \oplus \{\xi\}$ defines a 3-dimensional foliation on R^5, and it is minimal.*

After the best of this author's knowledge, this minimal 3-dimensional foliation appears here for the first time.

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