

FOLIATIONS AND GROUPS OF DIFFEOMORPHISMS

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John Mather has described a close relation between framed codimension-one Haefliger structures (these form a class of singular foliations), and the group of compactly supported diffeomorphisms of \mathbf{R}^1 , with discrete topology [11], [12], [14]. In this announcement I will describe generalizations of his ideas to higher codimension Haefliger structures and groups of diffeomorphisms of arbitrary manifolds. See Haefliger [7] for a development of Haefliger structures and their classifying spaces.

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Let $\text{Diff}^r(M^p)$ denote the group of C^r diffeomorphisms of M^p , a closed manifold. Let $\text{Diff}_0^r(M^p)$ denote the connected component of the identity.

THEOREM 1. $\text{Diff}_0^\infty(M^p)$ is a simple group.

The proof makes use of both the theorem of Epstein [4] that the commutator subgroup of $\text{Diff}_0(M^p)$ is simple, and of the result of M. Herman [9] which gives the case M^p is a p -torus.

THEOREM 2. $B\bar{\Gamma}_p^\infty$ is $(p+1)$ -connected, where $B\bar{\Gamma}_p^\infty$ is the classifying space for framed, codimension p , C^∞ , Haefliger structures.

The more usual notation is $F\Gamma_p^\infty = B\bar{\Gamma}_p^\infty$. Haefliger proved [6] that $B\bar{\Gamma}_p^r$ is p -connected for $1 \leq r \leq \infty$; Mather proved that $B\bar{\Gamma}_1^\infty$ is 2-connected.

Theorem 2 means that two C^∞ foliations of a manifold coming from nonsingular vector fields are homotopic as Haefliger structures if and only if the normal bundles are isomorphic.

Theorems 1 and 2 are proven by showing they are related; cf. Theorem 4 for a statement of a relationship.

COROLLARY. $P_1^{[p/2]}$ is nontrivial in $H^*(B\bar{\Gamma}_p^\infty; \mathbf{R})$ where P_1 is the first real Pontrjagin class of the normal bundle to the canonical Haefliger structure.

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Thus, Bott's vanishing theorem [1], which says real Pontrjagin classes in $B\Gamma_p^r$ ($r \geq 2$) vanish above dimension $2p$, gives a sharp bound on dimensions.

This corollary in the case $p=2$ follows easily from Theorem 2.

For higher codimensions, product foliations then yield examples.

THEOREM 3. $B\bar{\Gamma}_p^0$ is contractible.

Again, Mather proved this when the codimension is one.

This means topological Haefliger structures are completely determined up to homotopy by their normal micro-bundles.

Theorem 3 implies that Bott's vanishing theorem is quite false in the topological case—any normal micro-bundle is the normal micro-bundle for a topological foliation. In fact, if the micro-bundle is differentiable, it even admits a Haefliger structure Lipschitz close to being differentiable.

A little background and notation is necessary before the statement of the more general relationships. Let G be a topological group. Let G_δ be G with discrete topology. Then the map $G_\delta \rightarrow G$ is a continuous map which has a homotopy-theoretic fiber \bar{G} . \bar{G} is also a topological group: the explicit construction for \bar{G} is the space of paths α in G ending at the identity $e = \alpha(1)$, with discrete topology on $\alpha(0)$. Then multiplication is pointwise. There are maps, now,

$$\bar{G} \rightarrow G_\delta \rightarrow G \rightarrow B\bar{G} \rightarrow BG_\delta \rightarrow BG,$$

and any two consecutive arrows define a fibration.

BG is the classifying space for G -bundles. BG_δ classifies flat G -bundles: for instance, $B\text{Diff}^\infty(M^n)_\delta$ has an associated M -bundle, with discrete structure group: i.e., a C^∞ foliation transverse to the fibers of the bundle. Thus, $B\text{Diff}^\infty(M^p)_\delta$ classifies "foliated M^p -bundles". Finally, $B\bar{G}$ classifies G -bundles with a flat structure, together with a global trivialization defined (up to homotopy); e.g. $B\bar{\text{Diff}}^\infty(M^p)$ classifies "foliated M^p -products".

Let $\text{Diff}_K(\mathbf{R}^p)$ be the group of diffeomorphisms of \mathbf{R}^p with compact support. Then again, $B\bar{\text{Diff}}_K^r(\mathbf{R}^p) \times \mathbf{R}^p$ has a foliation of codimension p transverse to the \mathbf{R}^p -factors. Thus, there is a classifying map

$$B\bar{\text{Diff}}_K^r(\mathbf{R}^p) \times \mathbf{R}^p \rightarrow B\bar{\Gamma}_p^r.$$

(The image is in $B\bar{\Gamma}_p^r$ since there is a natural trivialization of the normal bundle to the foliation.)

The foliation agrees with the trivial, product foliation in a neighborhood of ∞ in the \mathbf{R}^p factors. Thus, one obtains a map of the p -fold suspension of $B\bar{\text{Diff}}_K^r \mathbf{R}^p$,

$$S^p(B\bar{\text{Diff}}_K^r \mathbf{R}^p) \rightarrow B\bar{\Gamma}_p^r.$$

This defines an adjoint map $B \overline{\text{Diff}}_K^r(\mathbb{R}^p) \rightarrow \Omega^p(B\overline{\Gamma}_p^r)$ to the p -fold loop space of $B\overline{\Gamma}_p^r$.

THEOREM 4. *The map $B \overline{\text{Diff}}_K^r(\mathbb{R}^p) \rightarrow \Omega^p(B\overline{\Gamma}_p^r)$ induces an isomorphism on homology.*

This theorem is due to Mather in the case $p=1$.

The map is certainly not a homotopy equivalence since $\pi_1(B \overline{\text{Diff}}_K^r \mathbb{R}^p)$ is highly nonabelian while $\pi_1(\Omega^p B\overline{\Gamma}_p^r) = \pi_{p+1}(B\overline{\Gamma}_p^r)$ is abelian.

Similarly, there is a map $B \overline{\text{Diff}}^r(M^p) \times M^p \rightarrow B\overline{\Gamma}_p^r$ which is a lifting of the classifying map for the tangent bundle of M^p , so there is a commutative diagram

$$\begin{array}{ccc}
 & & B\overline{\Gamma}_p^r \\
 & \nearrow & \downarrow \\
 B \overline{\text{Diff}}^r(M^p) \times M^p & \rightarrow & B\overline{O}_p
 \end{array}$$

Let X be the space of liftings of the classifying map for $T(M^p)$ in $B\overline{O}_p$ to $B\overline{\Gamma}_p^r$. Then we have a map $B \overline{\text{Diff}}^r(M^p) \rightarrow X$.

THEOREM 5. *The map*

$$B \overline{\text{Diff}}^r(M^p) \rightarrow X$$

induces an isomorphism on homology.

Again, this is not a homotopy equivalence since $\pi_1(X)$ is abelian.

For the case $r=0$, we assume M^p is a differentiable manifold.

COROLLARY. (a) $B \text{Homeo}(M^p)$ is acyclic, where $\text{Homeo}(M^p) = \text{Diff}^0(M^p)$ is the group of homeomorphisms of M^p .

(b) *The map $B \text{Homeo}(M^p)_\delta \rightarrow B \text{Homeo}(M^p)$ induces an isomorphism on homology.*

This corollary is implied by Theorems 3 and 5. Cf. Mather [13], who showed $B \text{Homeo}_K(\mathbb{R}^p)_\delta$ is acyclic.

COROLLARY. *The following groups are isomorphic, where k is the first positive integer such that one of them is nontrivial:*

- (i) $H_k(B \overline{\text{Diff}}^r(M^p); \mathbb{Z})$,
- (ii) $H_k(B \overline{\text{Diff}}_K^r(\mathbb{R}^p); \mathbb{Z})$,
- (iii) $H_{k+p}(B\overline{\Gamma}_p^r; \mathbb{Z})$.

CONJECTURE. *This first k is $p+1$, for $r = \infty$.*

Mather's theorem [11] shows this for $p=1$. Bott and Haefliger showed

that all differentiable characteristic classes (in some sense) vanish below this dimension, $H_{2p+1}(B\bar{\Gamma}_p^r; \mathbf{Z})$ [2], [3].

In [16] I sketched examples showing there is a surjective homomorphism

$$H_3(B\bar{\Gamma}_1^\infty; \mathbf{Z}) \twoheadrightarrow \mathbf{R},$$

using the Godbillon-Vey invariant gv [5]. Recently I have extended this to arbitrary codimension, so there is a surjective homomorphism

$$H_{2p+1}(B\bar{\Gamma}_p^\infty; \mathbf{Z}) \twoheadrightarrow \mathbf{R}.$$

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