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FOLIATIONS OF ASYMPTOTICALLY FLAT 3-MANIFOLDS BY 2-SURFACES OF PRESCRIBED MEAN CURVATURE

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Abstract

We construct 2-surfaces of prescribed mean curvature in 3-manifolds carrying asymptotically flat initial data for an isolated gravitating system with rather general decay conditions. The surfaces in question form a regular foliation of the asymptotic region of such a manifold. We recover physically relevant data, especially the ADM-momentum, from the geometry of the foliation.

For a given set of data (M, g, K), with a three dimensional manifold M, its Riemannian metric g, and the second fundamental form K in the surrounding four dimensional Lorentz space time manifold, the equation we solve is $H+P = \text{const}$ or $H-P = \text{const}$. Here H is the mean curvature, and $P = trK$ is the 2-trace of K along the solution surface. This is a degenerate elliptic equation for the position of the surface. It prescribes the mean curvature anisotropically, since P depends on the direction of the normal.

1. Introduction and Statement of Results

Surfaces with prescribed mean curvature play an important role for example in the field of general relativity. Slicings are frequently used to find canonic objects simplifying the treatment of the four dimensional space-time. A prominent setting is the ADM 3+1 decomposition [**1**] of a four dimensional manifold into three dimensional spacelike slices. Such slices are often chosen by prescribing their mean curvature in the four geometry. In contrast, we consider the subsequent slicing of a three dimensional spacelike slice by two dimensional spheres with prescribed mean curvature in the three geometry.

To be more precise, let (M, g, K) be a set of initial data. That is, (M, g) is a three dimensional Riemannian manifold and K is a symmetric bilinear form on M. This can be interpreted as the extrinsic curvature of M in the surrounding four dimensional space time. We

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consider 2-surfaces Σ satisfying one of the quasilinear degenerate elliptic equations $H \pm P = \text{const}$, where H is the mean curvature of Σ in (M, g) and $P = \operatorname{tr}^\Sigma K$ is the two dimensional trace of K.

In the case where $K \equiv 0$ this equation particularizes to $H = \text{const}$, which is the Euler-Lagrange equation of the isoperimetric problem. This means that surfaces satisfying $H = \text{const}$ are stationary points of the area functional with respect to volume preserving variations. Yau suggested to use such surfaces to describe physical information in terms of geometrically defined objects. Indeed, Huisken and Yau [**9**] have shown that the asymptotic end of an asymptotically flat manifold, with appropriate decay conditions on the metric, is uniquely foliated by such surfaces which are stable with respect to the isoperimetric problem. The Hawking mass

$$m_{\mathrm{H}}(\Sigma) = \frac{|\Sigma|^{1/2}}{(16\pi)^{3/2}}\left(16\pi - \int_\Sigma H^2 \,\mathrm{d}\mu\right)$$

of such a surface Σ is monotone on this foliation and converges to the ADM-mass. This foliation can also be used to define the center of mass of an isolated system since for a growing radius, the surfaces approach Euclidean spheres with a converging center. Therefore, the static physics of an isolated system considered as point mass is contained in the geometry of the $H = \text{const}$ foliation. However, these surfaces are defined independently of K, such that no dynamical physics can be found in their geometry. A different proof of the existence of CMC surfaces is due to Ye [**16**].

The goal of this paper is to generalize the CMC foliations to include the dynamical information into the definition of the foliation. The equation $H \pm P = \text{const}$ was chosen since apparent horizons satisfying $H = 0$ in the case $K \equiv 0$ generalize to surfaces satisfying $H \pm P = 0$ when K does not necessarily vanish. We made this choice with the Penrose inequality [**11**] in mind. This inequality estimates the ADM-mass of an isolated system by the area of a black hole horizon Σ

$$m_{\mathrm{ADM}} \geq \sqrt{\frac{|\Sigma|}{16\pi}}.$$

In the case $K \equiv 0$ this becomes the Riemannian Penrose inequality, which says that if Σ is an outermost minimal surface then the above inequality is valid. It was proved by Huisken and Ilmanen [**8**] and Bray [**3**], both using prescribed mean curvature surfaces. The proof of Bray generalizes the situation in which an outermost minimal surface is part of the stable CMC foliation from [**9**], in that it shows that the Hawking mass is monotone on isoperimetric surfaces when their enclosed volume and area increase even though they may not form a foliation. While a fully general apparent horizon Penrose inequality does not seem to be true [**2**], generalizing this picture is of interest as it may help to investigate whether a replacement is still true.

We consider asymptotically flat data describing isolated gravitating systems. For constants $m > 0$, $\delta \geq 0$, $\sigma \geq 0$, and $\eta \geq 0$ data (M, g, K) will be called $(m, \delta, \sigma, \eta)$-asymptotically flat if there exists a compact set $B \subset M$ and a diffeomorphism $x : M \setminus B \to \mathbf{R}^3 \setminus B_\sigma(0)$ such that in these coordinates g is asymptotic to the conformally flat spatial Schwarzschild metric g^S representing a static black hole of mass m. Here, $g^S = \phi^4 g^e$, where $\phi = 1 + \frac{m}{2r}$, g^e is the Euclidean metric, and r is the Euclidean radius. The asymptotics we require for g and K are

$$\sup_{\mathbf{R}^3 \setminus B_\sigma(0)} \Big(r^{1+\delta} |g - g^S| + r^{2+\delta} |\nabla^g - \nabla^S| + r^{3+\delta} |\mathrm{Rc}^g - \mathrm{Rc}^S| \Big) < \eta, \tag{1.1}$$

$$\sup_{\mathbf{R}^3 \setminus B_\sigma(0)} \Big(r^{2+\delta} |K| + r^{3+\delta} |\nabla^g K| \Big) < \eta. \tag{1.2}$$

Here ∇^g and ∇^S denote the Levi-Civita connections of g and g^S on TM, such that $\nabla^g - \nabla^S$ is a $(1,2)$-tensor. Furthermore, Rc^g and Rc^S denote the respective Ricci tensors of g and g^S. That is, we consider data arising from a perturbation of the Schwarzschild data $(g^S, 0)$.

The main theorem will be proved for $\delta = 0$ and $\eta = \eta(m)$ small compared to $m > 0$. These conditions are optimal in the sense that we only impose conditions on geometric quantities, not on partial derivatives. They include far more general data than similar results. Huisken and Yau [**9**], for example, demand that $g - g^S$ decays like r^{-2} with corresponding conditions on the decay of the derivatives up to fourth order, while we only need derivatives up to second order. Christodoulou and Klainerman [**4**] use asymptotics with $g - g^S$ decaying like $r^{-3/2}$ with decay conditions on the derivatives up to fourth order, and K like $r^{-5/2}$ with decay conditions on derivatives up to third order, whereas our result needs two levels of differentiability less. In addition, we allow data with nonzero ADM-momentum. For such data with $\delta = 0$ we can prove the following:

Theorem 1.1. *Given $m > 0$ there is $\eta_0 = \eta_0(m) > 0$, such that if the data (M, g, K) are $(m, 0, \sigma, \eta_0)$-asymptotically flat for some $\sigma > 0$, there is $h_0 = h_0(m, \sigma)$ and a differentiable map*

$$F : (0, h_0) \times S^2 \to M : (h, p) \mapsto F(h, p)$$

satisfying the following statements.

(i) *The map $F(h, \cdot) : S^2 \to M$ is an embedding. The surface $\Sigma_h = F(h, S^2)$ satisfies $H + P = h$ with respect to (g, K). Each Σ_h is convex, that is $|A|^2 \leq 4 \det A$.*

(ii) *There is a compact set $\bar{B} \subset M$, such that $F((0, h_0), S^2) = M \setminus \bar{B}$.*

(iii) *The surfaces $F(h,S^2)$ form a regular foliation.*
(iv) *There is a constant C such that for all h the surfaces Σ_h satisfy*

$$\|\nabla \mathring{A}\|^2_{L^2(\Sigma_h)} + |\Sigma_h|^{-1}\|\mathring{A}\|^2_{L^2(\Sigma_h)} \leq C\eta^2|\Sigma_h|^{-2},$$

where $\mathring{A}$ is the traceless part of the second fundamental form of Σ_h.
(v) *There are* sup*-estimates for all curvature quantities on Σ_h; in particular, these quatities are asypmtotic to their values on centered spheres in Schwarzschild, cf. Section* 4.
(vi) *If the data is (m,δ,σ',η')-asymptotically flat for some $\delta > 0$, $\sigma' > 0$ and $\eta' > 0$, then for small enough $h \ll h_1(m,\delta,\sigma',\eta')$, then every convex surface Σ with $H+P = h$ contained in $\mathbf{R}^3\setminus B_{h^{-2/3}}(0)$ equals Σ_h. Hence, the foliation is unique in the class of convex foliations.*

An analogous theorem holds for foliations with $H - P = \text{const}$.

This theorem does not need that (M,g,K) satisfy the constraint equations. It can be generalized to give the existence of a foliation satisfying $H + P_0(\nu) = \text{const}$, where $P_0 : SM \to \mathbf{R}^3$ is a function on the sphere bundle of M with the same decay as K.

Our result includes the existence results for constant mean curvature foliations from Huisken and Yau [**9**]. Their uniqueness result for individual surfaces can be proved in a larger class, while the global uniqueness result holds in the general case (cf. Remark 4.2).

By rescaling (g,K), the dependence of η_0 and h_0 on m can be exposed. Consider the map $F_\sigma : x \mapsto \sigma x$, and let $g_\sigma := \sigma^{-2}F^*_\sigma g^S$ and $K_\sigma := \sigma^{-1}F^*_\sigma K$. If (g,K) is $(m,0,\sigma,\eta)$-asymptotically flat, then (g_m,K_m) is $(1,0,m\sigma,m^{-1}\eta)$-asymptotically flat. Therefore $\eta_0(m) = m\eta_0(1)$, and $h_0(\sigma,m) = mh_0(m\sigma,1)$.

Section 2 introduces some notation. In Sections 3 and 4 we carry out the a priori estimates for the geometric quantities, first in Sobolev norms and then in the sup-norm. Using these estimates we examine the linearization of the operator $H \pm P$ in Section 5 and show that it is invertible. This is used in Section 6 to prove Theorem 1.1. Finally, in Section 7 we use special asymptotics of (g,K) to investigate the connection between the foliation and the linear momentum of these data.

2. Preliminaries

2.1. Notation. Let M be a three dimensional manifold. We will denote a Riemannian metric on M by g, or in coordinates by g_{ij}. Its inverse is written as $g^{-1} = \{g^{ij}\}$. The Levi-Civita connection of g is denoted by ∇, the Riemannian curvature tensor by R, the Ricci tensor by Rc, and the scalar curvature by Sc.

Let Σ be a hypersurface in M. Let γ^g denote the metric on Σ induced by g, and let ν^g denote its normal. The second fundamental form of Σ

is denoted by A^g, its mean curvature by H^g, and the traceless part of A^g by $\mathring{A}^g = A^g - \frac{1}{2}H^g\gamma^g$.

We follow the Einstein summation convention and sum over Latin indices from 1 to 3 and over Greek indices from 1 to 2.

We use the usual function spaces on compact surfaces with their usual norms. The L^p-norm of an (s,t)-tensor T with respect to the metric γ on Σ is denoted by

$$\|T\|^p_{L^p_{(s,t)}(\Sigma,\gamma)} = \int_\Sigma |T|^p_\gamma \, \mathrm{d}\mu^\gamma.$$

The space $L^p_{(s,t)}(\Sigma)$ of (s,t)-tensors is the completion of the space of smooth (s,t)-tensors with respect to this norm. In the sequel we will drop the subscripts (s,t), since norms will be used unambiguously. The Sobolev-norm $W^{k,p}(\Sigma)$ is defined as

$$\|T\|^p_{W^{k,p}(\Sigma)} = \|T\|^p_{L^p(\Sigma)} + \ldots + \|\nabla^k T\|^p_{L^p(\Sigma)},$$

where $\nabla^K T$ is the k-th covariant derivative of T. Again, the space $W^{k,p}(\Sigma)$ is the completion of the smooth tensors with respect to this norm.

For a smooth tensor T, define the Hölder semi-norm by

$$[T]_{p,\alpha} := \sup_{p\neq q} \frac{|P_q T(q) - T(p)|}{\mathrm{dist}(p,q)^\alpha},$$

where P_q denotes parallel translation along the shortest geodesic from p to q, and the supremum is taken over all $p \neq q$ with $\mathrm{dist}(p,q)$ less than the injectivity radius of Σ. Define the Hölder norm $\|T\|_{C^{k,\alpha}(\Sigma)}$ as

$$\|T\|_{C^{k,\alpha}_{(s,t)}(\Sigma)} := \sup_\Sigma |T| + \sup_\Sigma |\nabla T| + \cdots + \sup_\Sigma |\nabla^k T| + \sup_{p\in\Sigma} [\nabla^k T]_{p,\alpha}.$$

We assume in the following that (M,g,K) and all hypersurfaces are smooth, i.e., C^∞. However, to prove Theorem 1.1 it is obviously enough to assume g to be C^2 and K to be C^1. The a priori estimates from Sections 3 and 4 are valid for surfaces of class $W^{3,p}$, when p is large enough.

2.2. Extrinsic Geometry. Let $\Sigma \subset (M,g)$ be a hypersurface. The second fundamental form $A_{\alpha\beta}$ and the Riemannian curvature tensor $\mathcal{R}_{\alpha\beta\gamma\delta}$ of Σ are connected to the curvature Rm_{ijkl} of M via the Gauss and Codazzi equations

$$\mathcal{R}_{\alpha\beta\delta\varepsilon} = \mathrm{Rm}_{\alpha\beta\delta\varepsilon} + A_{\alpha\delta}A_{\beta\varepsilon} - A_{\alpha\varepsilon}A_{\beta\delta} \tag{2.1}$$

$$\mathrm{Rm}_{i\alpha\beta\delta}\nu^i = \nabla_\delta A_{\alpha\beta} - \nabla_\beta A_{\alpha\delta}. \tag{2.2}$$

Together, these imply the Simons identity [**14**, **12**]

$$\begin{aligned}\Delta^\Sigma A_{\alpha\beta} = {} & \nabla^\Sigma_\alpha \nabla^\Sigma_\beta H + H A^\delta_\alpha A_{\delta\beta} - |A|^2 A_{\alpha\beta} + A^\delta_\alpha \mathrm{Rm}_{\varepsilon\beta\varepsilon\delta} \\ & + A^{\delta\varepsilon}\mathrm{Rm}_{\delta\alpha\beta\varepsilon} + \nabla^\Sigma_\beta(\mathrm{Rc}_{\alpha k}\nu^k) + \nabla^{\Sigma\delta}(\mathrm{Rm}_{k\alpha\beta\delta}\nu^k).\end{aligned} \tag{2.3}$$

Note that the last two terms were not differentiated with the Leibniz rule. Equation (2.3), therefore, differs slightly from how the Simons identity is usually stated.

2.3. Round surfaces in Euclidean space. The key tool in obtaining a priori estimates for the surfaces in question is the following theorem by DeLellis and Müller [**6**, Theorem 1].

Theorem 2.1. *There exists a universal constant C such that for each compact connected surface without boundary $\Sigma \subset \mathbf{R}^3$, with area $|\Sigma| = 4\pi$, the following estimate holds:*

$$\|A - \gamma\|_{L^2(\Sigma)} \leq C\|\mathring{A}\|_{L^2(\Sigma)}.$$

If in addition $\|\mathring{A}\|_{L^2(\Sigma)} \leq 8\pi$, then Σ is a sphere, and there exists a conformal map $\psi : S^2 \to \Sigma \subset \mathbf{R}^3$ such that

$$\|\psi - (a + \mathrm{id}_{S^2})\|_{W^{2,2}(S^2)} \leq C\|\mathring{A}\|_{L^2(\Sigma)},$$

where id_{S^2} is the standard embedding of S^2 onto the sphere $S_1(0)$ in $\mathbf{R}^3$, and $a = |\Sigma|^{-1}\int_\Sigma \mathrm{id}_\Sigma \,\mathrm{d}\mu$ is the center of gravity of Σ.

DeLellis and Müller [**6**, 3,6.1,6.3] also prove the following useful estimates for the normal ν and the conformal factor h^2 of such surfaces:

$$C^{-1} \leq h \leq C,$$
$$\|h - 1\|_{W^{1,2}(S^2)} \leq C\|\mathring{A}\|_{L^2(\Sigma)},$$
$$\|N - \nu \circ \psi\|_{W^{1,2}(S^2)} \leq C\|\mathring{A}\|_{L^2(\Sigma)}.$$

Here, N is the normal of $S_1(a)$, and h^2 is the conformal factor of ψ, such that if γ^S denotes the metric on $S_1(a)$ and γ the metric on Σ, then we have $\psi^*\gamma = h^2\gamma^S$.

To translate these inequalities into a scale invariant form for surfaces which do not necessarily have area $|\Sigma| = 4\pi$, we introduce the Euclidean area radius $R_e = \sqrt{|\Sigma|/4\pi}$. The first part of Theorem 2.1 implies that for a general surface Σ,

$$\|A - R_e^{-1}\gamma\|_{L^2(\Sigma)} \leq C\|\mathring{A}\|_{L^2(\Sigma)}. \tag{2.4}$$

In the case $\|\mathring{A}\|_{L^2(\Sigma)} \leq 8\pi$, the second part of Theorem 2.1 gives that there exist $a_e := |\Sigma|^{-1}\int_\Sigma \mathrm{id}_\Sigma \,\mathrm{d}\mu \in \mathbf{R}^3$ and a conformal parameterization $\psi : S_{R_e}(a_e) \to \Sigma$. By the Sobolev embedding on S^2 we obtain the following estimates for ψ, its conformal factor h^2, and the difference of the normal N of $S_{R_e}(a_e)$ and the normal ν of Σ:

$$\sup_{S_{R_e}(a_e)} \left|\psi - \mathrm{id}_{S_{R_e}(a_e)}\right| \leq CR_e\|\mathring{A}\|_{L^2(\Sigma)}, \tag{2.5}$$

$$\|h^2 - 1\|_{L^2(S_{R_e}(a_e))} \leq CR_e\|\mathring{A}\|_{L^2(\Sigma)}, \tag{2.6}$$

$$\|N \circ \mathrm{id}_{S_{R_e}(a_e)} - \nu \circ \psi\|_{L^2(S_{R_e}(a_e))} \leq CR_e\|\mathring{A}\|_{L^2(\Sigma)}. \tag{2.7}$$

2.4. Asymptotically flat metrics. Let g^S be the spatial, conformally flat Schwarzschild metric on $\mathbf{R}^3 \setminus \{0\}$. Namely, let $g^S_{ij} = \phi^4 g^e_{ij}$ with $\phi = 1 + \frac{m}{2r}$, where $g^e_{ij} = \delta_{ij}$ is the Euclidean metric, and r the Euclidean radius on $\mathbf{R}^3$. Here and in the sequel we will suppress the dependence of g^S on the mass parameter m. However, we will restrict ourselves to the case $m > 0$. The Ricci curvature of g^S is given by

$$\mathrm{Rc}^S_{ij} = \frac{m}{r^3}\phi^{-2}\left(\delta_{ij} - 3\rho_i\rho_j\right), \tag{2.8}$$

where $\rho = x/r$ is the radial vector field on $\mathbf{R}^3$, whence $\mathrm{Sc}^S = 0$.

Omiting K and saying that data (M, g) are $(m, \delta, \sigma, \eta)$-asymptotically flat, we mean that $K \equiv 0$ and $(M, g, K \equiv 0)$ is $(m, \delta, \sigma, \eta)$-asymptotically flat. Recall that then there exists a compact set $B \subset M$ and a diffeomorphism $x : M \setminus B \to \mathbf{R}^3 \setminus B_\sigma(0)$, such that in these coordinates the following "norm"

$$\begin{aligned} &\|g - g^S\|_{C^2_{-1-\delta}(\mathbf{R}^3\setminus B_\sigma(0))} \\ &:= \sup_{\mathbf{R}^3\setminus B_\sigma(0)} \left(r^{1+\delta}|g - g^S| + r^{2+\delta}|\nabla^g - \nabla^S| + r^{3+\delta}|\mathrm{Rc}^g - \mathrm{Rc}^S|\right) \end{aligned} \tag{2.9}$$

satisfies $\|g - g^S\|_{C^2_{-1-\delta}} < \eta$. We let $O(\eta)$ denote a constant for which $O(\eta) \le C\eta$ if $\eta < \eta_0$ is bounded.

The volume element $\mathrm{d}V$ of g is a scalar multiple of the volume element $\mathrm{d}V^S$ of g^S, that is $\mathrm{d}V = h\mathrm{d}V^S$. The asymptotics (2.9) imply that $|h| \le O(\eta)r^{-1-\delta}$. In addition, the scalar curvature Sc of g satisfies $|\mathrm{Sc}| \le O(\eta)r^{-3-\delta}$.

Consider a surface $\Sigma \subset \mathbf{R}^3 \setminus B_\sigma(0)$. Let γ^e, γ^S, and γ be the first fundamental forms of Σ induced by g^e, g^S, and g, respectively, A^e, A^S, and A the corresponding second fundamental forms, H^e, H^S, and H the mean curvatures and $\mathring{A}^e$, $\mathring{A}^S$, and $\mathring{A}$ the respective trace free parts of the second fundamental form.

From the well known transformation behavior for the following geometric quantities under conformal transformations, and the asymptotics (2.9), we see:

Lemma 2.2. *The normals ν^e, ν^S, and ν of Σ in the metrics g^e, g^S, and g satisfy*

$$\begin{aligned} \nu^S &= \phi^{-2}\nu^e, \\ |\nu^S - \nu| &\le O(\eta)r^{-1-\delta}, \\ |\nabla^g\nu^S - \nabla^g\nu| &\le O(\eta)r^{-2-\delta}. \end{aligned}$$

The area elements $\mathrm{d}\mu^e$, $\mathrm{d}\mu^S$, and $\mathrm{d}\mu$ satisfy

$$\begin{aligned} \mathrm{d}\mu^S &= \phi^4\mathrm{d}\mu^e, \\ \mathrm{d}\mu - \mathrm{d}\mu^S &= h\mathrm{d}\mu \quad \textit{with} \quad |h| \le O(\eta)r^{-1-\delta}. \end{aligned}$$

The trace free parts $\mathring{A}^e, \mathring{A}^S$*, and* $\mathring{A}$ *of the second fundamental forms satisfy*

$$\mathring{A}^S = \phi^{-2}\mathring{A}^e,$$
$$|\mathring{A} - \mathring{A}^S| \leq O(\eta)r^{-2-\delta} + O(\eta)r^{1-\delta}|A|.$$

The mean curvatures H^e, H^S*, and* H *are related via*

$$H^S = \phi^{-2}H^e + 4\phi^{-3}\partial_\nu\phi,$$
$$|H - H^S| \leq O(\eta)r^{-2-\delta} + O(\eta)r^{1-\delta}|A|.$$

To obtain integral estimates for asymptotically decaying quantities, we cite the following lemma from [**9**, Lemma 5.2].

Lemma 2.3. *Let* (M, g) *be* $(m, 0, \sigma, \eta)$*-asymptotically flat, and let* $p_0 > 2$ *be fixed. Then there exists* $c(p_0)$*, and* $r_0 = r_0(m, \eta, \sigma)$*, such that for every hypersurface* $\Sigma \subset \mathbf{R}^3 \setminus B_{r_{\min}}(0)$*, and every* $p > p_0$*, the following estimate holds:*

$$\int_\Sigma r^{-p}\,\mathrm{d}\mu \leq c(p_0)r_{\min}^{2-p}\int_\Sigma H^2\,\mathrm{d}\mu.$$

Integration and mean curvature refer to g*, and* r *is the Euclidean radius.*

Using Lemma 2.2 to compare the L^2-norms of $\mathring{A}$ in the g-metric and $\mathring{A}^S$ in the g^S-metric, and using the conformal invariance of $\|\mathring{A}^S\|_{L^2(\Sigma, g^S)}$, we obtain

Lemma 2.4. *Let* (M, g) *be* $(m, \delta, \sigma, \eta)$*-asymptotically flat. Then there exists* $r_1 = r_1(\eta, \sigma)$*, such that for every surface* $\Sigma \subset \mathbf{R}^3 \setminus B_{r_{\min}}(0)$ *with* $r_{\min} > r_1$*, we have*

$$\begin{aligned}&\left|\|\mathring{A}^e\|^2_{L^2(\Sigma, g^e)} - \|\mathring{A}\|^2_{L^2(\Sigma, g)}\right|\\ &\leq O(\eta)r_{\min}^{-1-\delta}\left(\|\mathring{A}\|^2_{L^2(\Sigma, g)} + \|H\|_{L^2(\Sigma)}\|\mathring{A}\|_{L^2(\Sigma)} + r_{\min}^{-1-\delta}\|H\|^2_{L^2(\Sigma)}\right).\end{aligned}$$

Corollary 2.5. *Let* M*,* g*,* r_1 *and* Σ *be as in the previous lemma. Assume in addition that* $\|H\|_{L^2(\Sigma)} \leq C'$*; then*

$$\|\mathring{A}^e\|_{L^2(\Sigma)} \leq C(r_1)\|\mathring{A}\|_{L^2(\Sigma, g)} + C(r_1, C_0, C')O(\eta)r_{\min}^{-1-\delta}.$$

Next, we quote a Sobolev-inequality for surfaces contained in asymptotically flat manifolds. It can be found in [**9**, Proposition 5.4]. The proof uses the well known Michael-Simon-Sobolev inequality in Euclidean space [**10**].

Proposition 2.6. *Let* (M, g) *be* $(m, 0, \sigma, \eta)$*-asymptotically flat. Then there is* $r_0 = r_0(m, \eta, \sigma)$*, and an absolute constant* C_{sob}*, such that each* *surface $\Sigma \subset M \setminus B_{r_0}(0)$ and each Lipschitz function f on Σ satisfy*

$$\left(\int_\Sigma |f|^2 \,\mathrm{d}\mu\right)^{1/2} \leq C_{\mathrm{sob}} \int_\Sigma |\nabla f| + |Hf| \,\mathrm{d}\mu.$$

Using Hölder's inequality, this implies that for all $q \geq 2$

$$\int_\Sigma |f|^q \,\mathrm{d}\mu \leq C_{\mathrm{sob}} \left(\int_\Sigma |\nabla f|^{\frac{2q}{2+q}} + |fH|^{\frac{2q}{2+q}}\right)^{\frac{2+q}{2}}, \tag{2.10}$$

and for all $p \geq 1$

$$\left(\int_\Sigma |f|^{2p} \,\mathrm{d}\mu\right)^{1/p} \leq C_{\mathrm{sob}}\, p^2 \,|\mathrm{supp} f|^{1/p} \int_\Sigma |\nabla f|^2 + H^2 f^2 \,\mathrm{d}\mu. \tag{2.11}$$

3. A priori estimates I

We begin by stating rather general a priori estimates for the geometry of surfaces. For this, let $\Sigma \subset \mathbf{R}^3 \setminus B_\sigma(0)$ be a surface, and let g be $(m, 0, \sigma, \eta)$-asymptotically flat. Let $r_{\min} := \min_\Sigma r$ be the minimum of the Euclidean radius on Σ. Assume that on Σ the following two conditions are satisfied:

$$\int_\Sigma |\nabla H|^2 \,\mathrm{d}\mu \leq C^K \int_\Sigma r^{-4}|A|^2 + r^{-6} \,\mathrm{d}\mu, \tag{3.1}$$

$$\int_\Sigma u\,|A|^2 \,\mathrm{d}\mu \leq C_0^B \int_\Sigma u \det A \,\mathrm{d}\mu \quad \text{for all} \quad 0 \leq u \in C^\infty(\Sigma). \tag{3.2}$$

Remark 3.1.

(i) The first condition states that in a certain sense the mean curvature is nearly constant. This condition will later be implied by the equation by which the mean curvature is prescribed.

(ii) The second condition means that the surfaces are convex. Indeed, on smooth surfaces (3.2) implies that $|A|^2 \leq C_0^B \det A$ pointwise. However, we will need that condition (3.2) is preserved under $W^{2,p}$-convergence of surfaces. Huisken and Yau [**9**] are able to replace this condition by requiring stability of their CMC surfaces. In the present case similar reasoning would work; however, stability is not a natural condition for our surfaces.

Condition (3.2) implies topological restrictions, and an estimate on the L^2-norm of the mean curvature.

Lemma 3.2. *There is $r_0 = r_0(m, \eta, \sigma, C_0^B)$, such that every compact closed surface Σ satisfying (3.2) and $r_{\min} > r_0$ is diffeomorphic to S^2 and satisfies*

$$\int_\Sigma H^2 \,\mathrm{d}\mu \leq C(m, \eta, C_0^B). \tag{3.3}$$

Proof. The Gauss equation (2.1) implies that the Gauss curvature G of Σ is given by $G = \det A + \mathrm{Rc}(\nu,\nu) - \frac{1}{2}\mathrm{Sc}$. Inserting $u \equiv 1$ into (3.2) and applying Lemmas 2.2 and 2.3, we obtain

$$\begin{aligned}\int_\Sigma H^2 \,\mathrm{d}\mu &\le C_0^B \int_\Sigma G \,\mathrm{d}\mu + C_0^B \int_\Sigma |\mathrm{Rc}| \,\mathrm{d}\mu \\ &\le C_0^B \chi(\Sigma) + C_0^B r_{\min}^{-1} \int_\Sigma H^2 \,\mathrm{d}\mu.\end{aligned}$$

Here $\chi(\Sigma)$ is the Euler characteristic of Σ. If $r_{\min}$ is large enough, this gives $0 \le \|H\|_{L^2(\Sigma)} \le C_0^B \chi(\Sigma)$, which implies $\chi(\Sigma) \ge 0$. If $\chi(\Sigma) = 0$, i.e., Σ is a torus, then $\int_\Sigma H^2 \,\mathrm{d}\mu = 0$, whence $\|\mathring{A}\|_{L^2(\Sigma)} = 0$. Using Corollary 2.5 and Theorem 2.1, we obtain that Σ is a sphere, a contradiction. q.e.d.

Proposition 3.3. *Let (M, g) be $(m, 0, \sigma, \eta)$-asymptotically flat. Then there exists $r_0 = r_0(m, \eta, \sigma, C_0^B, C^K)$, such that each closed surface Σ satisfying* (3.1), (3.2), *and $r_{\min} > r_0$ also satisfies*

$$\int_\Sigma \left|\nabla|\mathring{A}|\right|^2 + H^2|\mathring{A}|^2 \,\mathrm{d}\mu \le C(m, \eta, C_0^B, C^K) r_{\min}^{-4}.$$

Proof. We begin by computing

$$\mathring{A}^{\alpha\beta}\Delta\mathring{A}_{\alpha\beta} = \mathring{A}^{\alpha\beta}(\Delta A_{\alpha\beta} + \gamma_{\alpha\beta}\Delta H) = \mathring{A}^{\alpha\beta}\Delta A_{\alpha\beta},$$

since $\mathring{A}$ is trace free. By

$$2|\mathring{A}|\Delta|\mathring{A}| + 2|\nabla|\mathring{A}||^2 = \Delta|\mathring{A}|^2 = 2\mathring{A}^{\alpha\beta}\Delta\mathring{A}_{\alpha\beta} + 2|\nabla\mathring{A}|^2,$$

and

$$|\nabla\mathring{A}|^2 - |\nabla|\mathring{A}||^2 \ge 0, \tag{3.4}$$

we obtain, using the Simons identity (2.3),

$$\begin{aligned}|\mathring{A}|\Delta|\mathring{A}| \ge{}& \mathring{A}^{\alpha\beta}\nabla_\alpha\nabla_\beta H + H\mathring{A}^{\alpha\beta}A_\alpha^\delta A_{\beta\delta} - |A|^2\mathring{A}^{\alpha\beta}A_{\alpha\beta} \\ &+ \mathring{A}^{\alpha\beta}A_\alpha^\delta \mathrm{Rm}_{\varepsilon\beta\varepsilon\delta} + \mathring{A}^{\alpha\beta}A^{\delta\varepsilon}\mathrm{Rm}_{\delta\alpha\beta\varepsilon} \\ &+ \mathring{A}^{\alpha\beta}\nabla_\beta\left(\mathrm{Rc}_{\alpha k}\nu^k\right) + \mathring{A}^{\alpha\beta}\nabla^\delta\left(\mathrm{Rm}_{k\alpha\beta\delta}\nu^k\right).\end{aligned} \tag{3.5}$$

Integration, and partial integration of $|\mathring{A}|\Delta|\mathring{A}|$, renders

$$\begin{aligned}\int_\Sigma |\nabla|\mathring{A}||^2 \,\mathrm{d}\mu \le{}& \int_\Sigma -\mathring{A}^{\alpha\beta}\nabla_\alpha\nabla_\beta H + |A|^2\mathring{A}^{\alpha\beta}A_{\alpha\beta} - H\mathring{A}^{\alpha\beta}A_\alpha^\delta A_{\beta\delta} \,\mathrm{d}\mu \\ &- \int_\Sigma \mathring{A}^{\alpha\beta}A_\alpha^\delta\mathrm{Rm}_{\varepsilon\beta\varepsilon\delta} + \mathring{A}^{\alpha\beta}A^{\delta\varepsilon}\mathrm{Rm}_{\delta\alpha\beta\varepsilon} \,\mathrm{d}\mu \\ &- \int_\Sigma \mathring{A}^{\alpha\beta}\left(\nabla_\beta\mathrm{Rc}_{\alpha k}\nu^k\right) + \mathring{A}^{\alpha\beta}\nabla^\delta\left(\mathrm{Rm}_{k\alpha\beta\delta}\nu^k\right) \mathrm{d}\mu.\end{aligned} \tag{3.6}$$

In the first line one computes as follows, and estimates, using convexity (3.2)

$$\begin{aligned}&\int_\Sigma |A|^2 \mathring{A}^{\alpha\beta} A_{\alpha\beta} - H\mathring{A}^{\alpha\beta} A_\alpha^\gamma A_{\beta\gamma}\,\mathrm{d}\mu\\ &= -2\int_\Sigma |\mathring{A}|^2 \det A\,\mathrm{d}\mu \le -\frac{2}{C_0^B}\int_\Sigma |\mathring{A}|^2|A|^2\,\mathrm{d}\mu.\end{aligned}$$

To recast the second line of (3.6), recall that in three dimensions, the Ricci tensor determines the Riemann tensor:

(3.7)

$$\mathrm{Rm}_{ijkl} = \mathrm{Rc}_{ik}g_{jl} - \mathrm{Rc}_{il}g_{jk} - \mathrm{Rc}_{jk}g_{il} + \mathrm{Rc}_{jl}g_{ik} - \frac{1}{2}\mathrm{Sc}\left(g_{ik}g_{jl} - g_{il}g_{jk}\right).$$

This implies that the second line of (3.6) can be expressed as

$$\mathring{A}^{\alpha\beta}A_\alpha^\delta \mathrm{Rm}_{\varepsilon\beta\varepsilon\delta} + \mathring{A}^{\alpha\beta}A^{\delta\varepsilon}\mathrm{Rm}_{\delta\alpha\beta\varepsilon} = 2\mathring{A}^{\alpha\beta}\mathring{A}_\alpha^\delta \mathrm{Rc}_{\beta\delta} - |\mathring{A}|^2\mathrm{Rc}(\nu,\nu).$$

Let $\omega = \mathrm{Rc}(\cdot,\nu)^T$ be the tangential projection of $\mathrm{Rc}(\cdot,\nu)$ to Σ. Then partial integration, the Codazzi-equations (2.2) and (3.7) give for the first term of (3.6)

$$-\int_\Sigma \mathring{A}^{\alpha\beta}\nabla_\alpha\nabla_\beta H\,\mathrm{d}\mu = \int_\Sigma \left(\tfrac{1}{2}|\nabla H|^2 + \omega(\nabla H)\right)\mathrm{d}\mu.$$

In the last line of (3.6) we compute, using partial integration, (3.7), and the Codazzi equation (2.2)

$$\int_\Sigma \mathring{A}^{\alpha\beta}\left(\nabla_\beta(\mathrm{Rc}_{\alpha k}\nu^k) + \nabla^\delta(\mathrm{Rm}_{k\alpha\beta\delta}\nu^k)\right)\mathrm{d}\mu = -\int_\Sigma 2|\omega|^2 + \omega(\nabla H)\,\mathrm{d}\mu.$$

Combining these estimates with (3.6) and $2|\omega(\nabla H)| \le |\omega|^2 + |\nabla H|^2$, we infer

$$\begin{aligned}&\int_\Sigma |\nabla|\mathring{A}||^2 + \tfrac{2}{C_0^B}|A|^2|\mathring{A}|^2\,\mathrm{d}\mu\\ &\le \int_\Sigma \tfrac{3}{2}|\nabla H|^2 + 3|\omega|^2 + |\mathring{A}|^2\mathrm{Rc}(\nu,\nu) - \mathring{A}^{\alpha\beta}\mathring{A}_\alpha^\delta\mathrm{Rc}_{\beta\delta}\,\mathrm{d}\mu.\end{aligned} \tag{3.8}$$

The asymptotics of g imply that $|\mathrm{Rc}| + |\omega| \le C(m,\eta)r^{-3}$. Inserting the estimate (3.1) for $\int_\Sigma |\nabla H|^2\,\mathrm{d}\mu$ into the previous estimate, we arrive at

$$\begin{aligned}&\int_\Sigma |\nabla|\mathring{A}||^2 + \tfrac{2}{C_0^B}|A|^2|\mathring{A}|^2\,\mathrm{d}\mu\\ &\le C(m,\eta,C^K)\int_\Sigma \left(r^{-4}|A|^2 + r^{-6} + r^{-3}|\mathring{A}|^2\right)\mathrm{d}\mu.\end{aligned} \tag{3.9}$$

For the first term on the right we use $|A|^2 = |\mathring{A}|^2 + \frac{1}{2}H^2$ and (3.3) to obtain

$$\int_\Sigma r^{-4}|A|^2\,\mathrm{d}\mu \le C(m,\eta,C_0^B)r_{\min}^{-4} + \int_\Sigma r^{-4}|\mathring{A}|^2\,\mathrm{d}\mu.$$

The third integrand of the right of (3.9) can be estimated together with the last term of this equation by combining the Schwarz inequality with Lemma 2.3:

$$\int_\Sigma r^{-3}|\mathring{A}|^2\,\mathrm{d}\mu \le \int_\Sigma \tfrac{2}{C_0^B}|\mathring{A}|^4+\tfrac{C_0^B}{4r^6}\,\mathrm{d}\mu \le \frac{2}{C_0^B}\int_\Sigma |\mathring{A}|^4\,\mathrm{d}\mu + C(m,\eta,C_0^B)r_{\min}^{-4}.$$

Inserting these estimates into (3.9), and absorbing the first term of this equation on the left hand side, we obtain the assertion of the proposition. q.e.d.

From now on we use the notation $R = R(\Sigma) = \sqrt{\frac{|\Sigma|}{4\pi}}$, where we compute the area with respect to the metric g.

Corollary 3.4. *Under the additional assumption that*

$$(C_1^B)^{-1}R(\Sigma)^{-1} \le |H|,$$

the previous proposition gives an estimate for the L^2-norm of $\mathring{A}$,

$$\|\mathring{A}\|_{L^2(\Sigma)} \le C(m,\eta,C_0^B,C_1^B,C^K)R(\Sigma)r_{\min}^{-2}.$$

Corollary 3.5. *Under the assumptions of Proposition* 3.3*, in fact*

$$\|\nabla\mathring{A}\|_{L^2(\Sigma)} + \|H\mathring{A}\|_{L^2(\Sigma)} \le C(m,\eta,C_0^B,C^K)r_{\min}^{-2}.$$

Proof. The proof works by replacing equation (3.4) in the proof of Proposition 3.3 by

$$|\nabla\mathring{A}|^2 - |\nabla|\mathring{A}||^2 \ge \tfrac{1}{17}|\nabla\mathring{A}|^2 - \tfrac{16}{17}(|\omega|^2 + |\nabla H|^2).$$

This inequality is proved in the same way as a similar inequality for ∇A, which is recorded in [**13**, Section 2]. The right hand side introduces the desired term $|\nabla\mathring{A}|^2$, and the remaining terms are treated as in the proof of Proposition 3.3. q.e.d.

Corollary 3.6. *Under the assumptions of Proposition* 3.3 *and Corollary* 3.4 *there are uniform estimates for the second fundamental form:*

$$\|A\|_{L^2(\Sigma)} \le C(m,\eta,C_0^B,C_1^B,C^K)\left(1 + r_{\min}^{-2}R(\Sigma)\right) \quad \text{and}$$
$$\|\nabla A\|_{L^2(\Sigma)} \le C(m,\eta,C_0^B,C^K)r_{\min}^{-2}.$$

We have now established that surfaces with small ∇H have very small trace free second fundamental form. In the next section we will use this estimate to get estimates about how close we are to centered spheres in Schwarzschild.

4. A priori estimates II

This section specializes on surfaces which satisfy the equation

$$H \pm P = \text{const.} \tag{4.1}$$

We will use Theorem 2.1 to derive estimates on the position of such a surface by using the curvature estimates of the previous section.

As described in the introduction, $P = \operatorname{tr}^\Sigma K = \operatorname{tr}^M K - K(\nu,\nu)$ is the trace of an extra tensor field K along Σ. We will consider data (M,g,K) which are (m,δ,σ,η)-asymptotically flat. That is, in addition to (2.9), we have that the weighted norm of K satisfies

$$\|K\|_{C^1_{-2-\delta}(\mathbf{R}^3\setminus B_\sigma(0))} := \sup_{\mathbf{R}^3\setminus B_\sigma(0)} (r^{2+\delta}|K| + r^{3+\delta}|\nabla^g K|) < \eta.$$

In the sequel, we will consider either (m,δ,σ,η)-asymptotically flat data with $\delta > 0$ and arbitrary $\eta < \infty$, or $(m,0,\sigma,\eta)$- asymptotically flat data with small $\eta \ll 1$.

Remark 4.1. If (M,g,K) are (m,δ,σ,η)-asymptotically flat, equation (4.1) implies condition (3.1). Indeed, $|\nabla H|^2 = |\nabla P|^2$ and

$$\nabla^\Sigma P = \nabla^\Sigma \operatorname{tr}^M K - (\nabla^M_{\cdot} K)(\nu,\nu) - 2K(A(\cdot),\nu),$$

such that $|\nabla P|^2 \le |\nabla K|^2 + |A|^2|K|^2$. Then

$$\begin{aligned}\int_\Sigma |\nabla H|^2 \, d\mu &= \int_\Sigma |\nabla P|^2 \, d\mu \\ &\le \|K\|^2_{C^1_{-2-\delta}} \int_\Sigma r^{-4-2\delta}|A|^2 + r^{-6-2\delta} \, d\mu.\end{aligned}$$

The results of this section require some additional conditions on the surfaces:

(A1) $\quad R(\Sigma) \le C^A_1 r^q_{\min} \quad q < \frac{3}{2}$ for $\delta > 0$ or $q = 1$ for $\delta = 0$,

(A2) $\quad (C^A_2)^{-1} R(\Sigma)^{-1} \le H \pm P$,

(A3) $\quad \displaystyle\int_\Sigma u\,|A|^2 \, d\mu \le C^A_3 \int_\Sigma u \det A \, d\mu$ for all $0 \le u \in C^\infty(\Sigma)$,

(A4) $\quad \displaystyle\frac{1}{4\pi R_e^2}\left|\int_\Sigma \mathrm{id}_\Sigma \, d\mu^e\right| \le R_e$.

Here R_e denotes the geometric radius of Σ computed with respect to the Euclidean metric, as in the previous section. In the sequel, C^A will denote constants which depend only on C^A_1, C^A_2 and C^A_3. If (M,g,K) is (m,δ,σ,η)-asymptotically flat with $\delta > 0$, with $o(1)$ we denote constants depending on m, C^A, δ and η, such that $o(1) \to 0$ for $\sigma < r_{\min} \to \infty$. If (M,g,K) is $(m,0,\sigma,\eta)$-asymptotically flat, $o(1)$ is such that, for each $\varepsilon > 0$, there is η_0 and r_0 such that $|o(1)| < \varepsilon$, provided $\eta < \eta_0$ and

$r_{\min} > r_0$. For fixed m and bounded C^A, both r_0 and η_0 can be chosen independent of C^A.

Remark 4.2.

(i) Note that condition (A3) is the same as condition (3.2), from the previous section. We restated it here for convenience.

(ii) Conditions (A1) and (A2) allow to compare different radius expressions, namely the Euclidean radius r, the geometric radius $R(\Sigma)$ and the curvature radius given by $2/H$. This is necessary, since the curvature estimates of the previous section improve with growing $r_{\min}$, while the estimates of DeLellis and Müller include the geometric radius $R(\Sigma)$. To balance these two radii we use (A1). Condition (A2) will be used to apply Corollary 3.4 to obtain L^2-estimates on $\mathring{A}$.

(iii) Condition (A4) means that the surface is not far off center. We will use this to conclude that the origin is contained in the approximating sphere of Theorem 2.1.

(iv) The distinction of the cases $\delta > 0$ and $\delta = 0$ in condition (A1) is due to the fact that in the proof of Proposition 4.3, we can use Lemma 2.3 only for $\delta > 0$.

(v) To prove the uniqueness result of Huisken and Yau [**9**, 5.1] we do not need conditions (A2) and (A3). Instead, if we impose stability of the CMC surfaces, the estimates $\|\mathring{A}\|_{L^2(\Sigma)} \leq Cr_{\min}^{-1/2}$ and (3.3) can be derived as in [**9**, 5.3]. Condition (A1) is slightly stronger than what Huisken and Yau need; they only require $q < 2$. Using stability and (A1) and (A4), only we can prove all subsequent estimates.

The folowing proposition formulates the position estimates.

Proposition 4.3. *Let (M, g, K) be $(m, \delta, \sigma, \eta)$-asymptotically flat and $m > 0$, and let Σ be a surface which satisfies* (4.1) *and* (A1)–(A4)*. Let $R_e = \sqrt{|\Sigma/4\pi}$ denote the geometric radius of Σ and a its center of gravity, both computed with respect to the Euclidean metric. Let $S := S_{R_e}(a)$ denote the Euclidean sphere with center a and radius R_e. Then there exists a conformal parameterization $\psi : S \to (\Sigma, \gamma^e)$, such that*

$$\sup_S|\psi - \mathrm{id}_S| \leq C(m, C^A)R(\Sigma)^2 r_{\min}^{-2}, \tag{4.2}$$

$$\|h^2 - 1\|_{L^2(S)} \leq C(m, C^A)R(\Sigma)^2 r_{\min}^{-2} \quad \textit{and} \tag{4.3}$$

$$\|N \circ \mathrm{id}_S - \nu \circ \psi\|_{L^2(S)} \leq C(m, C^A)R(\Sigma)^2 r_{\min}^{-2}. \tag{4.4}$$

In addition, the center satisfies the estimate

$$|a|/R_e \leq o(1), \tag{4.5}$$

where $o(1)$ is as described at the beginning of this section.

Proof. Using (4.1), Remark 4.1, and condition (A3), Corollaries 3.4 and 2.5 imply the following roundness estimates with respect to the Euclidean metric

$$\|\overset{\circ}{A}{}^e\|_{L^2(\Sigma,g^e)} \leq C(m, C^A) R_e r_{\min}^{-2}. \tag{4.6}$$

Therefore Theorem 2.1 and the subsequent remarks, as well as Lemma 2.2, imply $R_e \leq 2R(\Sigma)$ and (4.2)–(4.4). Condition (A1) then implies that $R_e r_{\min}^{-1} \leq 2C_1^A r_{\min}^{q-1}$, from which by (4.2)

$$|\mathrm{id}_S| \geq |\psi| - C(m, C^A) r_{\min}^{2q-2} \geq r - \frac{1}{2} r_{\min} \geq \frac{1}{2} r \geq \frac{1}{2} r_{\min},$$

if $r_{\min}$ is large enough. Every convex combination with $0 \leq \lambda \leq 1$ also satisfies

$$|\lambda \mathrm{id}_{S_{R_e}(a)} + (1-\lambda)\psi| \geq \frac{1}{2} r. \tag{4.7}$$

Similar to Huisken and Yau [**9**], we compute for a fixed vector $b \in \mathbf{R}^3$ with $|b|^e = 1$

$$0 = (H \pm P) \int_\Sigma g^e(b, \nu^e) \mathrm{d}\mu^e = \int_\Sigma H g^e(b, \nu^e) \mathrm{d}\mu^e \pm \int_\Sigma P g^e(b, \nu^e) \mathrm{d}\mu^e. \tag{4.8}$$

We estimate using (A1),

$$\left| \int_\Sigma P g^e(b, \nu^e) \mathrm{d}\mu^e \right| \leq o(1) \int_\Sigma r^{-2-\delta} g^e(b, \nu^e) \mathrm{d}\mu^e \leq o(1). \tag{4.9}$$

This follows from Lemma 2.3 in the case $\delta > 0$, and by brute force and (A1) in the case $\delta = 0$. In the first term we express H by H^e. Using Lemma 2.2 we obtain that the error is of the order $o(1)$, such that

$$\left| \int_\Sigma \left(H^e \phi^{-2} + 4\phi^{-3} \partial_{\nu^e} \phi \right) g^e(b, \nu^e) \, \mathrm{d}\mu^e \right| \leq o(1). \tag{4.10}$$

The first variation formula with respect to the Euclidean metric gives

$$\int_\Sigma H^e \phi^{-2} g^e(b, \nu^e) \mathrm{d}\mu^e = \int_\Sigma \mathrm{div}_\Sigma^e(\phi^{-2} b) \, \mathrm{d}\mu^e = -2 \int_\Sigma \phi^{-3} g^e(b, \nabla^e \phi) \, \mathrm{d}\mu^e.$$

Using $g^e(\nabla^e \phi, b) = g^e(D\phi, b) - g^e(b, \nu^e)\partial_{\nu^e}\phi$ and $|D\phi| \leq C(m) r^{-2}$ gives

$$\left| \int_\Sigma 6 g^e(b, \nu^e) \partial_{\nu^e} \phi \, \mathrm{d}\mu^e - \int_\Sigma 2 g^e(b, D\phi) \, \mathrm{d}\mu^e \right| \leq o(1). \tag{4.11}$$

Now we will use that Σ is approximated by the sphere S as described by (4.2)–(4.4), and replace the integrals of (4.11) by integrals over S. For the first term estimate

$$\begin{aligned} &\left| \int_\Sigma g^e(b, D\phi) \, \mathrm{d}\mu^e - \int_S g^e(b, D\phi) \, \mathrm{d}\mu^e \right| \\ &\leq \left| \int_S (h^2 - 1) g^e(b, D\phi \circ \psi) \, \mathrm{d}\mu^e \right| + \left| \int_S g^e(b, (D\phi)\circ\psi - D\phi) \, \mathrm{d}\mu^e \right|. \end{aligned}$$

Using (4.2), (4.3), (4.7), $|D\phi| \le C(m)r^{-2}$, $|D^2\phi| \le C(m)r^{-3}$, and Lemma 2.3 we can estimate the error terms

$$\begin{aligned}&\left|\int_S (h^2-1)g^e(b, D\phi\circ\psi)\,\mathrm{d}\mu^e\right|\\ &\le \|h^2-1\|_{L^2(S)}\,\|g^e(b,D\phi\circ\psi)\|_{L^2(S)} \le C\|\mathring{A}^e\|_{L^2(\Sigma)}R_e r_{\min}^{-1},\end{aligned}$$

and

$$\begin{aligned}&\left|\int_S g^e(b,(D\phi)\circ\psi - D\phi)\,\mathrm{d}\mu^e\right|\\ &\le \sup_S|\psi-\mathrm{id}|\int_S\big(\max_{\lambda\in[0,1]}\left|D^2\phi\,(\lambda\mathrm{id}+(1-\lambda)\psi)\right|\big)\,\mathrm{d}\mu\\ &\le C\|\mathring{A}^e\|_{L^2(\Sigma)}R_e r_{\min}^{-1}.\end{aligned}$$

The second term in (4.11) can be replaced similarly, with analogous treatment of the error terms, additionally using (4.4). In the end the error is also controlled by $C\|\mathring{A}^e\|_{L^2(\Sigma)}R_e r_{\min}^{-1}$. Therefore, both error terms can be estimated by $C(m,C^A)R_e^2 r_{\min}^{-3}$. Using (A1) gives $R_e^2 \le C^A r_{\min}^{-q}$, and finally (4.11) implies that

$$\left|\int_S 6g^e(b,N)\partial_N\phi\,\mathrm{d}\mu^e - \int_S 2g^e(b,D\phi)\,\mathrm{d}\mu^e\right| \le o(1). \tag{4.12}$$

Set $b = \frac{a}{|a|}$, and choose coordinates φ and ϑ on S such that $g^e(b,N) = \cos\varphi$. Compute $D\phi = -\frac{m}{2r^2}\rho$, $N = R_e^{-1}(x-a)$, and $g^e(N,\rho) = R_e r^{-1} + r^{-1}|a|\cos\varphi$, where again $\rho = x/r$ is the radial direction of $\mathbf{R}^3$. Inserting this into (4.12) gives

$$\left|m\int_S 3|a|r^{-3}\cos^2\varphi + 2R_e r^{-3}\cos\varphi - |a|r^{-3}\,\mathrm{d}\mu^e\right| \le o(1). \tag{4.13}$$

From condition (A4) we conclude that $|a| \le R_e$. Using the integration formula

$$\int_{S_{R_e}(a)} r^{-k}\cos^l\varphi\,\mathrm{d}\mu^e = \frac{\pi R_e}{|a|}(2R_e|a|)^{-l}\int_{R_e-|a|}^{R_e+|a|} r^{1-k}(r^2-R_e^2-|a|^2)^l\,\mathrm{d}r,$$

we compute the terms in (4.13) and obtain

$$8\pi m|a|/R_e \le o(1). \tag{4.14}$$

Since $m > 0$ this implies the last assertion of the proposition. q.e.d.

Corollary 4.4. *For each $\varepsilon > 0$ we can choose $o(1)$ sufficiently small such that instead of* (A1) *we have the stronger estimate*

$$(1+\varepsilon)^{-1}R(\Sigma) \le r_{\min} \le (1+\varepsilon)R(\Sigma). \tag{4.15}$$

In addition, we have strengthened (A4) *to*

$$\frac{1}{4\pi R_e^2}\left|\int_\Sigma \mathrm{id}_\Sigma \,\mathrm{d}\mu^e\right| \le \varepsilon R_e, \tag{4.16}$$

provided $r_{\min} > r_0$*, and* $r_0 = r_0(\varepsilon, m, \sigma, C^A)$ *is large enough.*

Proof. From the position estimates equations (4.2) and (4.5) we obtain for every $p \in S$

$$(1 - o(1))R_e \le R_e - |a| \le |\mathrm{id}_S(p)| \le |\psi(p)| + C(m, C^A)R_e^2 r_{\min}^{-2}.$$

The left hand side is independent of p, so we can take the minimum over p to get $r_{\min} = \min_\Sigma |\psi(p)|$ on the right. By arranging that $|o(1)| < \varepsilon$ we obtain

$$(1-\varepsilon)R_e \le r_{\min} + C(m, C^A)R_e^2 r_{\min}^{-2},$$

which implies the corollary in view of (A1). q.e.d.

Corollary 4.5. *Condition* (A2) *holds with improved constants. In addition the following upper bound is also true:*

$$(1+\varepsilon)^{-1}R(\Sigma)^{-1} \le \frac{H}{2} \le (1+\varepsilon)R(\Sigma)^{-1}, \tag{4.17}$$

provided $\eta < \eta_0$ *is small enough and* $r_{\min} > r_0$ *is large enough.*

Proof. Using the first part of Theorem 2.1, the roundness estimates (4.6), and $|H - H^e| \le Cr^{-2} + Cr^{-1}|A|$ from Lemma 2.2, we obtain the following estimate for H:

$$\|H - 2/R_e\|^2_{L^2(\Sigma)} \le C(m, C^A)r_{\min}^{-2}.$$

By equation (4.1), the mean curvature H is nearly constant, whence we derive

$$\begin{aligned}(H \pm P - 2/R_e)^2|\Sigma| &\le 2\|H - 2/R_e\|^2_{L^2(\Sigma)} + 2\|P\|^2_{L^2(\Sigma)}\\ &\le C(m, C^A)r_{\min}^{-2}.\end{aligned}$$

In view of $|P| \le C(\|K\|_{C^1_{-2}})r^{-2}$, this implies

$$|H - 2/R_e| \le C(m, C^A)r_{\min}^{-2},$$

which gives the assertion of the corollary. q.e.d.

We now take a closer look at those terms in the proof of Proposition 3.3 that came from the geometry of M. The Ricci tensor of the Schwarzschild metric, when restricted to a centered coordinate sphere for example splits orthogonally into a positive tangential part $(\mathrm{Rc}^S)^T = mr^{-3}\phi^{-6}\gamma^S \ge 0$ and a negative normal part $\mathrm{Rc}^S(\nu, \nu) = -2mr^{-3}\phi^{-6} \le 0$, the mixed term ω^S vanishes. We now combine the estimates of Proposition 4.3 to estimate the analogous terms on Σ.

Proposition 4.6. *Let Σ be as in Proposition 4.3; then for $r_{\min} > r_0(m,\sigma,C^A)$ large enough, we have*

$$\begin{aligned}\|\nu-\phi^{-2}\rho\|^2_{L^2(\Sigma,g)} &\leq o(1)r^2_{\min}+C(m,C^A),\\ \|\mathrm{Rc}(\nu,\nu)-\phi^{-4}\mathrm{Rc}^S(\rho,\rho)\|^2_{L^2(\Sigma,g)} &\leq o(1)r^{-4}_{\min}+C(m,C^A)r^{-6}_{\min},\\ \|\omega\|^2_{L^2(\Sigma,g)} &\leq o(1)r^{-4}_{\min}+C(m,C^A)r^{-6}_{\min},\\ \|\mathrm{Rc}^T-P^S_{\phi^{-2}\rho}\mathrm{Rc}^S\|^2_{L^2(\Sigma,g)} &\leq o(1)r^{-4}_{\min}+C(m,C^A)r^{-6}_{\min},\end{aligned}$$

where $\rho = x/r$ is the radial direction of $\mathbf{R}^3$, $P^S_{\phi^{-2}\rho}\mathrm{Rc}^S$ is the orthogonal projection with respect to g^S of the Ricci tensors of g^S onto the subspace of the tangential space of M which is g^S-orthogonal to $\phi^{-2}\rho$, and $o(1)$ is as described at the beginning of the section.

Proof. From Lemma 2.2 and Corollary 4.4 we derive

$$\|\nu-\phi^{-2}\rho\|^2_{L^2(\Sigma,g)} \leq c\|\nu^e-\rho\|^2_{L^2(\Sigma,g^e)}+o(1). \tag{4.18}$$

Now we use Proposition 4.3 to obtain a sphere $S = S_{R_e}(a)$ and a conformal parameterization $\psi : S \to \Sigma$ satisfying the estimates (4.2)–(4.5). From the estimate on the center a, we compute for the difference of the Euclidean normal $N = (x-a)/R_e$ and the radial direction $\rho = x/r$ that

$$|N-\rho|_{g^e} \leq (|R^e-r|_{g^e}+|a|)/R^e \leq 2|a|/R^e \leq o(1).$$

Using (4.2)–(4.4), we estimate

$$\begin{aligned}|\rho\circ\psi(x)-\rho(x)|_{g^e} &\leq \big(\sup_{\lambda\in[0,1]}|D\rho(\lambda x-(1-\lambda)\psi(x))|_{g^e}\big)|\psi(x)-x|_{g^e}\\ &\leq C^A\|\mathring{A}^e\|_{L^2(\Sigma,g^e)},\end{aligned}$$

and

$$\int_\Sigma|\nu^e-\rho|^2_{g^e}\,\mathrm{d}\mu^e = \int_S h^{-2}|\nu^e\circ\psi-\rho\circ\psi|^2_{g^e}\mathrm{d}\mu^e \leq C\int_S|\nu^e\circ\psi-\rho\circ\psi|^2_{g^e}\mathrm{d}\mu^e.$$

By the triangle inequality and the previous inequalities, we obtain

$$\begin{aligned}&\|\nu^e\circ\psi-\rho\circ\psi\|_{L^2(S,g^e)}\\ &\leq \|\nu^e\circ\psi-N\|_{L^2(S)}+\|N-\rho\|_{L^2(S)}+\|\rho-\rho\circ\psi\|_{L^2(S)}\\ &\leq o(1)r_{\min}+C(m,C^A).\end{aligned}$$

This implies the first inequality of the proposition in view of (4.18). The second inequality now easily follows, since

$$\begin{aligned}&\|\mathrm{Rc}(\nu,\nu)-\phi^{-4}\mathrm{Rc}^S(\rho,\rho)\|^2_{L^2(\Sigma)}\\ &\leq \|\mathrm{Rc}^S-\mathrm{Rc}\|^2_{L^2(\Sigma)}+\sup_\Sigma|\mathrm{Rc}^S|^2\|\nu-\phi^{-2}\rho\|^2_{L^2(\Sigma)}\\ &\leq C(m,C^A)r^{-6}_{\min}\big(1+\|\nu-\phi^{-2}\rho\|^2_{L^2(\Sigma)}\big).\end{aligned}$$

For the third inequality, observe that by a similar computation

$$\|\mathrm{Rc}(\nu,\cdot) - \mathrm{Rc}^S(\phi^{-2}\rho,\cdot)\|^2_{L^2(\Sigma)} \le C(m, C^A) r_{\min}^{-6}\big(1 + \|\nu - \phi^{-2}\rho\|^2_{L^2(\Sigma)}\big),$$

such that only the difference of the projections of $\mathrm{Rc}^S(\cdot,\phi^{-2}\rho)$ to the subspaces g-orthogonal to ν and g^S-orthogonal to $\phi^{-2}\rho$ have to be estimated. Note that the latter projection is zero. To estimate the difference, write

$$P^g_\nu \mathrm{Rc}^S(\cdot,\phi^{-2}\rho) = \mathrm{Rc}^S(\cdot,\phi^{-2}\rho) - g(\cdot,\nu)\mathrm{Rc}^S(\nu,\phi^{-2}\rho),$$

where P^g_ν is the g-orthogonal projection on the g orthogonal complement of ν, and

$$P^S_{\phi^{-2}\rho} \mathrm{Rc}^S(\cdot,\phi^{-2}\rho) = \mathrm{Rc}^S(\cdot,\phi^{-2}\rho) - g^S(\cdot,\phi^{-2}\rho)\mathrm{Rc}^S(\phi^{-2}\rho,\phi^{-2}\rho).$$

Therefore the third estimate of the proposition follows as before. The last estimate can be obtained using a similar computation. q.e.d.

We can now improve the roundness estimates of Proposition 3.3.

Proposition 4.7. *Let (M, g, K) be $(m, \delta, \sigma, \eta)$-asymptotically flat. Then there exists a constant $C(m, C^A)$ and $r_0 = r_0(m, \sigma, C^A)$, such that for all surfaces Σ satisfying* (4.1)*, conditions* (A1)–(A4)*, and $r_{\min} > r_0$, the following estimate holds*

$$\int_\Sigma |\nabla|\mathring{A}||^2 + H^2|\mathring{A}|^2 \,\mathrm{d}\mu \le o(1) r_{\min}^{-4} + C(m, C^A) r_{\min}^{-6}.$$

Proof. We use the Simons identity as in the proof of Proposition 3.3

$$\begin{aligned}&\int_\Sigma |\nabla|\mathring{A}||^2 + \tfrac{2}{C_0^A} H^2|\mathring{A}|^2 \,\mathrm{d}\mu \\ &\le \int_\Sigma \big(\tfrac{3}{2}|\nabla H|^2 + 3|\omega|^2 + |\mathring{A}|^2 \mathrm{Rc}(\nu,\nu) - \mathring{A}^{\alpha\beta}\mathring{A}^\delta_\alpha \mathrm{Rc}_{\beta\delta}\big) \,\mathrm{d}\mu.\end{aligned}$$

By Remark 4.1 we have $|\nabla H|^2 \le o(1)(r^{-4}|A|^2 + r^{-6})$. We further proceed as in the proof of Proposition 3.3, but now estimate the resulting terms using Proposition 4.6. For example, with $\mathrm{Rc}^S(\rho,\rho) \le 0$ and the Schwarz inequality we derive

$$\begin{aligned}\int_\Sigma |\mathring{A}|^2 \mathrm{Rc}(\nu,\nu)\,\mathrm{d}\mu &\le \|\mathring{A}\|^2_{L^4(\Sigma)} \|\mathrm{Rc}(\nu,\nu) - \phi^{-4}\mathrm{Rc}^S(\rho,\rho)\|_{L^2(\Sigma)} \\ &\le o(1) r_{\min}^{-5} + C(m, C^A) r_{\min}^{-6}.\end{aligned}$$

Here we used the Sobolev inequality from Proposition 2.6 together with Proposition 3.3 and Corollary 4.4, to estimate the L^4-norm of $\mathring{A}$:

$$\|\mathring{A}\|^4_{L^4(\Sigma)} \le C(m, C^A)|\Sigma| r_{\min}^{-8} \le C(m, C^A) r_{\min}^{-6}.$$

The estimates for the other terms are obvious. q.e.d.

Our next step is to prove sup-estimates for $\mathring{A}$ using the Stampaccia iteration technique.

Proposition 4.8. *Let Σ be as in Proposition 3.3; then for each $\varepsilon > 0$ there exists $r_0 = r_0(m, \sigma, C^A)$ and a constant $C(\varepsilon, m, C^A)$, such that if $r_{\min} \geq r_0$*

$$\sup_\Sigma |\mathring{A}| \leq C(\varepsilon, m, C^A)\big(o(1)r_{\min}^{-2} + r_{\min}^{-3+\varepsilon}\big).$$

Proof. Let $u := |\mathring{A}|$, and $u_k := \max(u - k, 0)$ for all $k \geq 0$. Let $A(k) := \{x \in \Sigma : u_k > 0\}$. Let $p > 1$, and multiply equation (3.5) with u_k^p and integrate. Partial integration, proceeding as in Proposition 3.3, and using the Schwarz inequality to absorb all gradient terms on the left hand side gives

$$\begin{aligned}&\int_{A(k)} p u_k^{p-1} u|\nabla u|^2 + u_k^p|\nabla u|^2 + C_3^A u_k^p u^2 H^2 \mathrm{d}\mu \\ &\leq c(p)\int_{A(k)} (u_k^{p-1}u + u_k^p)|\nabla H|^2 + u_k^p|\omega|^2 + u_k^p u^2|\mathrm{Rc}| + u_k^{p-1}u|\omega|^2 \mathrm{d}\mu.\end{aligned} \tag{4.19}$$

We have the bounds $|\mathrm{Rc}| + |\omega| \leq C(m)r^{-3}$, and Remark 4.1 and Corollary 4.5 imply that $|\nabla H|^2 \leq o(1)(r^{-6} + r^{-4}u^2)$. Equation (4.19) therefore gives

$$\begin{aligned}&\int_{A(k)} p u_k^{p-1} u|\nabla u|^2 + u_k^p|\nabla u|^2 + C_3^A u_k^p u^2 H^2 \,\mathrm{d}\mu \\ &\leq C(m, C^A)\int_{A(k)} u_k^p r^{-6} + u_k^{p-1}ur^{-6} + u_k^p u^2 r^{-3}\,\mathrm{d}\mu.\end{aligned} \tag{4.20}$$

Using the Sobolev inequality (2.11), Proposition 3.3, $u_k \leq u$, and $\nabla u_k = \nabla u$ on $A(k)$, we infer that for all $1 < q < \infty$

$$\begin{aligned}\int_{A(k)} u_k^p \mathrm{d}\mu &\leq C(q, m, C^A)|A(k)|(o(1)r_{\min}^{-2p} + r_{\min}^{-3p}) \\ &\leq C(q, m, C^A)|A(k)|^{1-1/q}|\Sigma|^{1/q}(o(1)r_{\min}^{-2p} + r_{\min}^{-3p}).\end{aligned}$$

We proceed to estimate the second term on the right hand side of (4.20). We use the Sobolev inequality (2.10) to conclude that

$$\begin{aligned}&\int_{A(k)} u_k^{p-1}u\,\mathrm{d}\mu = \int_{A(k)} (u_k u^{\frac{1}{p-1}})^{p-1} \\ &\leq C(p)\left(\int_{A(k)} \big|\nabla(u_k u^{\frac{1}{p-1}})\big|^{\frac{2(p-1)}{p+1}} + |(u_k u^{\frac{1}{p-1}})H|^{\frac{2(p-1)}{p+1}}\,\mathrm{d}\mu\right)^{\frac{p+1}{2}}.\end{aligned}$$

Since $|\nabla(u_k u^{\frac{1}{p-1}})| \le c(p)u^{\frac{1}{p-1}}|\nabla u|$, we estimate the first term, using Hölder,

$$\int_{A(k)} \left|\nabla(u_k u^{\frac{1}{p-1}})\right|^{\frac{2(p-1)}{p+1}} \mathrm{d}\mu \le c(p)\int_{A(k)} |\nabla u|^{\frac{2(p-1)}{p+1}} u^{\frac{2}{p+1}} \mathrm{d}\mu$$
$$\le c(p)\left(\int_{A(k)} |\nabla u|^2 \,\mathrm{d}\mu\right)^{\frac{p-1}{p+1}} \left(\int_{A(k)} u \,\mathrm{d}\mu\right)^{\frac{2}{p+1}}.$$

Similary, for the second term

$$\int_{A(k)} |(u_k u^{\frac{1}{p-1}})H|^{\frac{2(p-1)}{p+1}} \mathrm{d}\mu \le \left(\int_{A(k)} u^2H^2 \,\mathrm{d}\mu\right)^{\frac{p-1}{p+1}} \left(\int_{A(k)} u \,\mathrm{d}\mu\right)^{\frac{2}{p+1}}.$$

Combining these, we get that

$$\int_{A(k)} u_k^{p-1} u \,\mathrm{d}\mu \le C(p)\left(\int_{A(k)} u \,\mathrm{d}\mu\right)\left(\int_{A(k)} |\nabla u|^2 + H^2u^2 \,\mathrm{d}\mu\right)^{\frac{p-1}{2}}.$$

Observe that for any $0 < q < \infty$, by an application of the Hölder inequality and the Soboloev inequality (2.11),

$$\int_{A(k)} u \,\mathrm{d}\mu \le \left(\int_\Sigma u^q \,\mathrm{d}\mu\right)^{\frac{1}{q}} |A(k)|^{\frac{q-1}{q}}$$
$$\le C(q)|A(k)|^{\frac{q-1}{q}}|\Sigma|^{\frac{1}{q}}\left(\int_\Sigma |\nabla u|^2 + H^2u^2 \,\mathrm{d}\mu\right)^{\frac{1}{2}}.$$

In view of Proposition 4.7, this yields that for all $1 < q < \infty$

$$\int_{A(k)} u_k^{p-1} u \,\mathrm{d}\mu \le C(p,q,m,C^A)|A(k)|^{\frac{q-1}{q}}|\Sigma|^{\frac{1}{q}}(o(1)r_{\min}^{-2p} + r_{\min}^{-3p}).$$

A similar treatment of the last term in equation (4.20) gives that

$$\int_{A(k)} u_k^p u^2 \le C(p,q,m,C^A)|A(k)|^{\frac{q-1}{q}}|\Sigma|^{\frac{1}{q}}(o(1)r_{\min}^{-2p-2} + r_{\min}^{-3p-4}).$$

Thus, we infer that (4.20) yields

$$\int_{A(k)} pu_k^{p-1}u|\nabla u|^2 + u_k^p|\nabla u|^2 + C_3^A u_k^p u^2H^2 \,\mathrm{d}\mu$$
$$\le C(p,q,m,C^A)|A(k)|^{\frac{q-1}{q}}|\Sigma|^{\frac{1}{q}}(o(1)r_{\min}^{-2p-6} + r_{\min}^{-3p-6}).$$

Let $f := u_k^{p/2+1}$; then the above etimate is equivalent to

$$\int_{A(k)} |\nabla f|^2 + H^2f^2 \,\mathrm{d}\mu$$
$$\le C(p,q,m,C^A)|A(k)|^{\frac{q-1}{q}}|\Sigma|^{\frac{1}{q}}(o(1)r_{\min}^{-2p-6} + r_{\min}^{-3p-6}).$$

Using the Sobolev inequality (2.11) to estimate $\int_\Sigma f^2\,\mathrm{d}\mu$, and reexpressing this in terms of $f^2 = u_k^{p+2}$, we obtain the iteration inequality

$$\begin{aligned}|h-k|^{p+2}|A(h)| &\leq \int_{A(h)} u_k^{p+2}\,\mathrm{d}\mu \leq \int_{A(k)} u_k^{p+2}\,\mathrm{d}\mu \\ &\leq C(p,q,m,C^A)|A(k)|^{2-\frac{1}{q}}|\Sigma|^{\frac{1}{q}}(o(1)r_{\min}^{-2p-6} + r_{\min}^{-3p-6}).\end{aligned}$$

By [**15**, Lemma 4.1], this iteration inequality implies that $|A(d)| = 0$ for $d \geq d_0$ with

$$d_0^{p+2} \leq C(p,q,m,C^A)(o(1)r_{\min}^{-2p-6} + r_{\min}^{-3p-6})|\Sigma|^{\frac{1}{q}}|A(0)|^{1-\frac{1}{q}}.$$

As $|A(0)| \leq |\Sigma|$ we see that we can fix any $1 < q < \infty$. Corollary 4.4 implies that $|\Sigma| \leq C(m,C^A)r_{\min}^2$. Therefore, we can estimate

$$d_0 \leq C(p,m,C^A)\big(o(1)r_{\min}^{-2} + r_{\min}^{-3+2/(p+2)}\big).$$

This yields

$$\sup_\Sigma |\mathring{A}| \leq C(p,m,C^A)\big(o(1)r_{\min}^{-2} + r_{\min}^{-3+2/(p+2)}\big).$$

Thus the claimed estimate follows provided p is large enough. q.e.d.

We now have a sup-estimate for $A = \nabla\nu$. This can be combined with the L^2-estimates for $|\nu - \phi^{-2}\rho|$ to prove a sup-estimate for this expression.

Proposition 4.9. *Let Σ be as in Proposition 4.3 such that in particular Σ satisfies $\|\nu - \phi^{-2}\rho\|_{L^2(\Sigma)} \leq o(1)r_{\min} + C(m,C^A)$ and $|A| \leq C(m,C^A)r_{\min}^{-1}$. Then there exists $r_0 = r_0(m,\sigma,C^A)$ such that*

$$\sup|\nu - \phi^{-2}\rho| \leq o(1) + C(m,C^A)r_{\min}^{-2/3}$$

provided $o(1)$ is small enough, and $r_{\min} > r_0$.

Proof. From the above assumptions, $|\nabla(\nu - \phi^{-2}\rho)| \leq C(m,C^A)r_{\min}^{-1}$. Therefore $f := |\nu - \phi^{-2}\rho|^2$ satisfies

$$|\nabla f| = \big|g(\nabla(\nu - \phi^{-2}\rho), \nu - \phi^{-2}\rho)\big| \leq C(m,C^A)r_{\min}^{-1},$$

provided r_0 is large enough. Assume there exists $p_0 \in \Sigma$ such that for $M > 0$ the inequality $f(p_0) \geq 2M(o(1) + r_{\min}^{-1})^{2/3}$ holds. Let $B := \{p \in \Sigma : |p - p_0| \leq M(o(1) + r_{\min}^{-1})^{2/3}C(m,C^A)^{-1}r_{\min}\}$. Then for all $p \in B$ we have that $f(p) \geq M(o(1) + r_{\min}^{-1})^{2/3}$, which implies that

$$\int_\Sigma f\,\mathrm{d}\mu \geq \int_B f\mathrm{d}\mu \geq C\frac{M^3}{C(m,C^A)^2}(o(1)r_{\min} + 1)^2,$$

where we used that $|B| \geq CM^2(\varepsilon + r_{\min}^{-1})^{4/3}C(m,C^A)^{-2}r_{\min}^2$. This follows from the estimate on the conformal factor of $\psi : S \to \Sigma$ from Theorem 2.1, if ε and $r_{\min}^{-1}$ are small enough. If M is large enough, this is a contradiction. q.e.d.

Corollary 4.10. *In the same way we obtain an estimate*

$$\sup_\Sigma |\nu^e - \rho| \le o(1) + C(m, C^A) r_{\min}^{-2/3},$$

and therefore

$$\int_\Sigma g^e(\nu^e, \rho) \ge \frac{1}{2},$$

if $o(1)$ *is small enough. Hence* Σ *is globally a graph over* S^2*, i.e., there is a function* $u \in C^\infty(S^2)$ *such that*

$$\Sigma = \{u(p)p : p \in S^2 \subset \mathbf{R}^3\}.$$

Corollary 4.11. *Surfaces* Σ *as in Proposition* 4.3 *satisfy*

$$|\mathrm{Rc}(\nu,\nu) + 2mr^{-3}| \le o(1) r_{\min}^{-3} + C(m, C^A) r_{\min}^{-3-2/3}.$$

This enables us to precisely compute the curvature of Σ taken with respect to g.

Theorem 4.12. *Let* Σ *be as in Proposition* 3.3*. Let* $R_e = \sqrt{|\Sigma|^e/4\pi}$ *be its Euclidean geometric radius, and define* $\bar\phi = 1 + \frac{m}{2R_e}$ *and* $\bar H = \frac{2}{\bar\phi^2 R_e} - \frac{2m}{\bar\phi^3 R_e^2}$*. Then there exist* $r_0 = r_0(m, \sigma, C^A)$ *and* $C(m, C^A)$*, such that if* $r_{\min} > r_0$ *the following estimates hold:*

$$\sup_\Sigma |H - \bar H| \le o(1) r_{\min}^{-2} + C(m, C^A) r_{\min}^{-2-2/3},$$

$$\sup_\Sigma \left|\det A - \bar H^2/4\right| \le o(1) r_{\min}^{-3} + C(m, C^A) r_{\min}^{-3-2/3},$$

$$\sup_\Sigma \left|G - \bar H^2/4 - 2m/R_e^3\right| \le o(1) r_{\min}^{-3} + C(m, C^A) r_{\min}^{-3-2/3}.$$

Here $G = \det A - \mathrm{Rc}(\nu,\nu) + \frac{1}{2}\mathrm{Sc}$ *is the Gauss-curvature of* Σ.

Proof. From Proposition 4.3 we obtain an approximating sphere $S = S_{R_e}(a)$ and a conformal map $\psi : S \to \Sigma$ which satisfies (4.2)–(4.5). We compare Σ with the *centered* sphere $\bar S = S_{R_e}(0)$ and consider the map $\xi : \bar S \to \Sigma : x \mapsto \psi(x + a)$. From (4.2) and (4.5) we obtain that

$$\begin{aligned}\sup_\Sigma |r - R^e| &= \sup_{\bar S} \big|\, |\xi(x)| - |x|\, \big| \\ &\le o(1) r_{\min} + C(m, C^A),\end{aligned}$$

which in particular implies that $|r_{\min} - R^e| \le o(1) r_{\min} + C(m, C^A)$. In addition, $\sup_\Sigma |\bar\phi - \phi| \le o(1) r_{\min}^{-1} + C(m, C^A) r_{\min}^{-2}$ as well as $\sup_\Sigma |\bar\phi^{-2} - \phi^{-2}| + \sup_\Sigma |\bar\phi^{-3} - \phi^{-3}| \le o(1) r_{\min}^{-1} + C(m, C^A) r_{\min}^{-2}$. Take a point $x \in \bar S$, and let ν^e be the Euclidean normal to Σ. Estimate

$$\begin{aligned}&|D_\rho(x)\phi(x) - D_{\nu^e(\xi(x))}\phi(\xi(x))| \\ &\le |D_{\rho(x)}\phi(x) - D_{\rho(x)}\phi(\xi(x))| + |D_{\rho(x)}\phi(\xi(x)) - D_{\rho(\xi(x))}\phi(\xi(x))| \\ &\quad + |D_{\rho(\xi(x))}\phi(\xi(x)) - D_{\nu^e(\xi(x))}\phi(\xi(x))| \\ &\le o(1) r_{\min}^{-2} + C(m, C^A) r_{\min}^{-2-2/3}.\end{aligned}$$

The L^2-norm of $H - \bar{H}$ can then be estimated by using Lemma 2.2 to replace H by H^e, and estimating $\|H^e - 2/R_e^2\|_{L^2(\Sigma)}$ by taking the trace of (2.4).

$$\begin{aligned}\int_\Sigma |H-\bar{H}|^2 \,\mathrm{d}\mu^e &\leq \int_\Sigma \left| \frac{H^e}{\phi^2} + 4\frac{D_{\nu^e}\phi}{\phi^3} - \frac{2}{\bar{\phi}^2 R_e} + \frac{2m}{\bar{\phi}^3 R_e^2}\right|^2 \mathrm{d}\mu^e + o(1) r_{\min}^{-3} \\ &\leq o(1) r_{\min}^{-2} + C(m, C^A) r_{\min}^{-2-4/3}.\end{aligned}$$

Proceeding as in the proof of Corollary 4.5 we obtain the asserted sup-estimate on $H - \bar{H}$ by using (4.1) and $|P| \leq o(1) r_{\min}^{-2}$.

The sup-estimates on $\mathring{A}$ of Proposition 4.8 imply that A on Σ satisfies

$$\begin{aligned}\left|A - \frac{1}{2}\bar{H}\mathrm{Id}\right| &\leq \left|A - \frac{1}{2}H\mathrm{Id}\right| + |H - \bar{H}| \\ &\leq o(1) r_{\min}^{-2} + C(m, C^A) r_{\min}^{-2-2/3},\end{aligned} \tag{4.21}$$

which implies the second assertion of the theorem. Corollary 4.11 gives that

$$\left|\mathrm{Rc}(\nu,\nu) + 2m/R_e^3\right| \leq o(1) r_{\min}^{-3} + C(m, C^A) r_{\min}^{-3-2/3},$$

which, in view of $|\mathrm{Sc}| \leq o(1) r_{\min}^{-3}$, equation (4.21), and the Gauss equation $G = \det A - \mathrm{Rc}(\nu,\nu) + \frac{1}{2}\mathrm{Sc}$, implies the last assertion. q.e.d.

In summary, we have established that given the assumptions (A1)–(A4), solutions to (4.1) are very close to centered spheres. We have shown that the key geometric objects are very close to what they are on centered spheres in Schwarzschild, and have improved the constants in (A1)–(A4). This will give us some room for deformations of the surfaces, while retaining the estimates. In particular, we will start with the $H = \mathit{const}$. foliation of the centered spheres in Schwarzschild and deform it in $C^{2,\alpha}$ to retain $H + P = \text{const}$, while changing the data to (g, K). This is made precise in Section 6, while the next section uses the estimates to show that we can apply the inverse function theorem.

5. The linearization of the operator $\mathcal{H} \pm \mathcal{P}$

In this section we will examine the linearization of the operator $\mathcal{H} \pm \mathcal{P}$ which assigns the function $H \pm P$ to a surface. We will prove that this linearization is invertible, whence we can apply the implicit function theorem in Section 6 to find surfaces with $H \pm P = \text{const}$. We begin by computing the linearization. For this let $\Sigma \subset M$ be a closed surface. In a neighborhood of Σ we introduce Gaussian normal coordinates $y : \Sigma \times (-\varepsilon, \varepsilon) \to M$, such that $y(\cdot, 0) = \mathrm{id}_\Sigma$, and $\partial y/\partial t = \nu_{\Sigma_t}$, with $\Sigma_t = y(\Sigma, t)$. For a function $f \in C^\infty(\Sigma)$ with $|f| \leq \varepsilon$ define the graph of f over Σ as

$$\mathrm{graph}(f) := \{y(p, f(p)) : p \in \Sigma\}.$$

Let $\mathcal{H} : C^\infty(\Sigma) \to C^\infty(\Sigma)$ be the operator, which assigns to a function f the mean curvature $\mathcal{H}(f)$ of graph(f), and let $\mathcal{P} : C^\infty(\Sigma) \to C^\infty(\Sigma)$ be the operator which assigns to a function f the function $P = \mathrm{tr}^{\mathrm{graph}(f)} K$ evaluated on graph(f). To compute the linearization of $\mathcal{H} \pm \mathcal{P}$ at $f = 0$, we need the following lemma:

Lemma 5.1. *Let $\Sigma \subset M$ be a surface, and $F : \Sigma \times (-\varepsilon, \varepsilon) \to M$ a variation of Σ, with $F(\cdot, 0) = \mathrm{id}_\Sigma$. If F is normal to Σ, i.e., $\frac{\partial F}{\partial t}\big|_{t=0} = f\nu$ for $f \in C^\infty(\Sigma)$, then*

$$\left.\frac{dH}{dt}\right|_{t=0} = -\Delta^\Sigma f - f\left(|A|^2 + \mathrm{Rc}(\nu,\nu)\right),$$
$$\left.\frac{dP}{dt}\right|_{t=0} = f\left(\nabla^M_\nu \mathrm{tr}^M K - \nabla^M_\nu K(\nu,\nu)\right) + 2K(\nabla^\Sigma f, \nu).$$

Here A is the second fundamental form, H the mean curvature, and ν the normal of Σ. The covariant derivative of M is denoted by ∇^M and that of Σ by ∇^Σ.

Proof. The first equation is well known. It can be found in [**3**, Appendix A]. The second immediately follows from $P = \mathrm{tr}^M K - K(\nu,\nu)$ and $\frac{d\nu}{dt}\big|_{t=0} = -\nabla^\Sigma f$. q.e.d.

Lemma 5.1 implies that the linearization $L^{\mathcal{H}\pm\mathcal{P}}$ of $\mathcal{H} \pm \mathcal{P}$ is given by

$$\begin{aligned} L^{\mathcal{H}\pm\mathcal{P}} f = &-\Delta f \pm 2K(\nabla^\Sigma f, \nu) \\ &- f\left(|A|^2 + \mathrm{Rc}(\nu,\nu) \pm \nabla^M_\nu K(\nu,\nu) \mp \nabla^M_\nu \mathrm{tr} K\right). \end{aligned} \tag{5.1}$$

To obtain a form which is easier to handle, we multiply this by f and integrate by parts.

Proposition 5.2. *Let $f \in C^\infty(\Sigma)$. Then*

$$\begin{aligned} &\int_\Sigma f L^{\mathcal{H}\pm\mathcal{P}} f \,\mathrm{d}\mu \\ &= \int_\Sigma |\nabla f|^2 - f^2\big(8\pi\left(\mu \mp J(\nu)\right) + \tfrac{1}{2}\big|(K^T)^\circ \pm \mathring{A}\big|^2 + |\theta|^2\big) \\ &\qquad - \tfrac{1}{2} f^2\big(\tfrac{1}{2}(H \pm P)^2 + (H \mp K(\nu,\nu))^2 - (\mathrm{tr}K)^2 - 2G\big)\mathrm{d}\mu. \end{aligned}$$

Here μ and J are given by the constraint equations $16\pi\mu = \mathrm{Sc} - |K|^2_g + (\mathrm{tr}K)^2$ and $8\pi J = \nabla^M \mathrm{tr}K - \mathrm{div}^M K$, and $(K^T)^\circ$ denotes the trace free part of the tangential projection of K onto Σ, i.e., $(K^T)^\circ_{\alpha\beta} = K_{\alpha\beta} - \frac{1}{2}\gamma^{\varepsilon\delta} K_{\varepsilon\delta}\gamma_{\alpha\beta}$. Moreover, $\theta = K(\cdot,\nu)^T$, and G denotes the Gaussian curvature of Σ.

Proof. Multiply (5.1) with f and integrate to obtain

$$\int_\Sigma f L^{\mathcal{H}\pm\mathcal{P}} f \,\mathrm{d}\mu = \int_\Sigma |\nabla f|^2 \pm 2fK(\nabla^\Sigma f, \nu) - f^2\big(|A|^2 + \mathrm{Rc}(\nu,\nu) \mp \nabla_\nu \mathrm{tr}K \pm \nabla_\nu K(\nu,\nu)\big)\,\mathrm{d}\mu.$$

By the Gauss equation and the constraint equation we compute

$$|A|^2 + \mathrm{Rc}(\nu,\nu) = 8\pi\mu + \frac{1}{2}\left(|K|^2 - (\mathrm{tr}K)^2 + H^2 + |A|^2\right) - G.$$

Considering the term $2\int_\Sigma fK(\nabla^\Sigma f,\nu)\,\mathrm{d}\mu$, we obtain by partial integration that

$$2\int_\Sigma fK(\nabla^\Sigma f,\nu)\,\mathrm{d}\mu = \int_\Sigma f^2(8\pi J(\nu) - \nabla^M_\nu \mathrm{tr}K + \nabla^M_\nu K(\nu,\nu) + HK(\nu,\nu) - K^T\cdot A)\,\mathrm{d}\mu.$$

This gives the asserted identity in view of $|K|^2 = |K^T|^2 + 2|\theta|^2 + K(\nu,\nu)^2$. q.e.d.

This expression can be used to prove positivity. In the sequel we will restrict ourselves to data (M,g,K) which are $(m,0,\sigma,\eta)$-asymptotically flat. By eventually increasing σ, every set of $(m,\delta,\sigma,\bar\eta)$-asymptotically flat data can be made $(m,0,\sigma,\eta)$-asymptotically flat for any choice of $\eta > 0$.

Proposition 5.3. *For $m > 0$ and constants C_1^A, C_2^A, and C_3^A, there are $\eta_0 = \eta_0(m, C^A)$ and $r_0 = r_0(m,\sigma,C^A)$ such that if the data (g,K) are $(m,0,\sigma,\eta_0)$-asymptotically flat and Σ satisfies* (4.1)*, conditions* (A1)–(A4) *as well as $r_{\min} > r_0$, then there is μ_1 with*

$$\mu_1 \geq 6mR_e^{-3} - o(1)R_e^{-3} + C(m,C^A)R_e^{-3-2/3},$$

such that for all functions $f \in C^\infty(\Sigma)$ with $\int_\Sigma f\,\mathrm{d}\mu = 0$ the following inequality holds

$$\mu_1 \int_\Sigma f^2\,\mathrm{d}\mu \leq \int_\Sigma f L^{\mathcal{H}\pm\mathcal{P}} f\,\mathrm{d}\mu.$$

Here $o(1)$ is as described at the beginning of Section 4.

Proof. It is a well-known fact that a lower bound on the Gauss curvature $G \geq \kappa$ of a surface gives a lower bound $\lambda_1 \geq 2\kappa$ on the first eigenvalue of its Laplace-Beltrami operator. This bound is provided by Theorem 4.12, such that for all f with $\int_\Sigma f\,\mathrm{d}\mu = 0$ we obtain

$$\left(\tfrac{1}{2}\bar H^2 + \tfrac{4m}{R_e^3} - o(1)R_e^{-3} + C(m,C^A)R_e^{-3-2/3}\right)\int_\Sigma f^2\,\mathrm{d}\mu \leq \int_\Sigma |\nabla f|^2\,\mathrm{d}\mu.$$

From Proposition 5.2, the asymptotics of K, the sup-estimates for $\mathring{A}$ from Proposition 4.8, and the expression for G in Theorem 4.12, we obtain

$$\begin{aligned}\int_\Sigma f L^{\mathcal{H}\pm\mathcal{P}} f \,\mathrm{d}\mu &\geq \int_\Sigma (|\nabla f|^2 - |\mathring{A}|^2 - \tfrac{3}{4}H^2 + G - o(1)R_e^{-3}) f^2 \,\mathrm{d}\mu \\ &\geq \left(6mR_e^{-3} - o(1)R_e^{-3} - C(m, C^A)R_e^{-3-2/3}\right)\int_\Sigma f^2 \,\mathrm{d}\mu.\end{aligned}$$

If $o(1)$ is small enough, the factor on the right hand side is positive, and this gives the assertion. q.e.d.

We are now able to show that solutions u of $L^{\mathcal{H}\pm\mathcal{P}}u = \text{const}$ are almost constant.

Proposition 5.4. *Let (M, g, K) and Σ be as in Proposition 5.3. Consider a solution u of $L^{\mathcal{H}\pm\mathcal{P}}u = f$ with $\int_\Sigma (f-\bar{f})^2 \,\mathrm{d}\mu \leq \mu_1^2/4\bar{u}^2$ where μ_1 is as in Proposition 5.3, $\bar{f} = |\Sigma|^{-1}\int_\Sigma f \,\mathrm{d}\mu$ is the mean value of f, and $\bar{u}$ is the mean value of u. Then*

$$\sup_\Sigma |u - \bar{u}| \leq \left(o(1) + C(m, C^A)R_e^{-2/3}\right)\bar{u}.$$

Proof. We can assume that u is normalized such that $\bar{u} = 1$. Then

$$L^{\mathcal{H}\pm\mathcal{P}}(u-1) = f + \left(|A|^2 + \mathrm{Rc}(\nu,\nu) \pm \nabla^M_\nu K(\nu,\nu) \mp \nabla^M_\nu \mathrm{tr}K\right).$$

Multiplying by $(u-1)$, integrating, and using Proposition 5.3, we obtain

$$\begin{aligned}&\mu_1 \int_\Sigma (u-1)^2 \,\mathrm{d}\mu \\ &\leq \int_\Sigma (u-1)\left(f + |A|^2 + \mathrm{Rc}(\nu,\nu) \pm \nabla^M_\nu K(\nu,\nu) \mp \nabla^M_\nu \mathrm{tr}K\right)\mathrm{d}\mu.\end{aligned}$$

Using the Schwarz inequality and the assumption on f, we estimate

$$\int_\Sigma (u-1) f \,\mathrm{d}\mu = \int_\Sigma (u-1)(f-\bar{f})\,\mathrm{d}\mu \leq \frac{\mu_1}{2}\left(\int_\Sigma (u-1)^2 \,\mathrm{d}\mu\right)^{1/2}. \tag{5.2}$$

Define R_e and $\bar{H}$ as in Theorem 4.12; then

$$|\mathring{A}|^2 + \tfrac{1}{2}|H^2 - \bar{H}^2| + |\mathrm{Rc}(\nu,\nu) + 2mR_e^{-3}| \leq o(1)R_e^{-3} + C(m, C^A)R_e^{-3-2/3}.$$

Combining $\int_\Sigma (u-1)\bar{H}^2 \,\mathrm{d}\mu = 0$ with the Schwarz inequality gives

$$\begin{aligned}&\left|\int_\Sigma (u-1)\left(|A|^2 + \mathrm{Rc}(\nu,\nu) \pm \nabla^M_\nu K(\nu,\nu) \mp \nabla^M_\nu \mathrm{tr}K\right)\mathrm{d}\mu\right| \\ &\leq \left(o(1)R_e^{-2} + C(m, C^A)R_e^{-2-2/3}\right)\|u-1\|_{L^2(\Sigma)}.\end{aligned}$$

Inserting this into (5.2), we obtain the L^2-estimate

$$\|u-1\|^2_{L^2(\Sigma)} \leq \mu_1^{-2}\left(o(1)R_e^{-4} + C(m, C^g)R_e^{-4-4/3}\right).$$

By standard estimates from the theory of linear elliptic partial differential equations of second order [**7**] we can obtain a sup-estimate from this

$$\sup_\Sigma |u-1| \leq \mu_1^{-1}\left(o(1)R_e^{-3} + C(m,C^A)R_e^{-3-2/3}\right),$$

which implies the assertion, in view of the estimate for μ_1 from Proposition 5.3. q.e.d.

Corollary 5.5. *Provided $o(1)$ is small enough, and f is as in the previous proposition, a solution of $Lu = f$ does not change sign.*

Corollary 5.6. *Let u be a solution of $L^{\mathcal{H}\pm\mathcal{P}}u = f$. If*

$$\int_\Sigma (u-\bar{u})f\,\mathrm{d}\mu \leq \frac{\mu_1}{2}\left(\int_\Sigma (u-\bar{u})^2\,\mathrm{d}\mu\right)^{1/2},$$

with μ_1 from Proposition 5.3, then

$$\sup_\Sigma |u-\bar{u}| \leq o(1) + C(m,C^A)R_e^{-2/3}\bar{u}.$$

This corollary implies that $L^{\mathcal{H}\pm\mathcal{P}}$ is invertible in suitable Banach spaces.

Theorem 5.7. *Under the assumptions of the previous proposition, $L^{\mathcal{H}\pm\mathcal{P}}$ is invertible as operator $L^{\mathcal{H}\pm\mathcal{P}} : C^{2,\alpha}(\Sigma) \to C^{0,\alpha}(\Sigma)$ for each $0<\alpha<1$. Its inverse $L^{\mathcal{H}\pm\mathcal{P}}_{inv} : C^{0,\alpha}(\Sigma) \to C^{2,\alpha}(\Sigma)$ exists and is continuous. It satisfies $\|L^{\mathcal{H}\pm\mathcal{P}}_{inv} f\|_{L^2(\Sigma)} \leq R_e^3/3m\|f\|_{L^2(\Sigma)}$ and the Hölder norm estimate*

$$\|L^{\mathcal{H}\pm\mathcal{P}}_{inv} f\|_{C^{2,\alpha}(\Sigma)} \leq C(\alpha,\Sigma)\tfrac{R_e^3}{3m}\|f\|_{C^{2,\alpha}(\Sigma)}.$$

Proof. Assume that there exists a function u with $\|u\|_{L^2(\Sigma)} = 1$ and

$$\sup_{\|v\|_{L^2(\Sigma)}=1}\left|\int_\Sigma vL^{\mathcal{H}\pm\mathcal{P}}u\,\mathrm{d}\mu\right| \leq \frac{3m}{R_e^3}. \tag{5.3}$$

From Proposition 5.3 we have that $\bar{u}\neq 0$. Without loss of generality, $\bar{u}>0$. Choosing $v = u-\bar{u}$ in (5.3) implies that the assumptions of Corollary 5.6 are satisfied. If $o(1)$ is small enough, we obtain that $\bar{u}/2 \leq u \leq 2\bar{u}$. From $\|u\|_{L^2(\Sigma)} = 1$ we obtain that $\bar{u} \geq \frac{1}{2}|\Sigma|^{-1/2}$, and from Hölder's inequality $\bar{u} \leq |\Sigma|^{-1/2}$. Using $v=1$ in (5.3) gives

$$\left|\int_\Sigma L^{\mathcal{H}\pm\mathcal{P}}u\,\mathrm{d}\mu\right| \leq \frac{3m}{R_e^3}|\Sigma| \leq C(m,C^A)R_e^{-1}. \tag{5.4}$$

On the other hand, we compute from (5.1), using partial integration, that

$$\int_\Sigma L^{\mathcal{H}\pm\mathcal{P}}u\,\mathrm{d}\mu = -\int_\Sigma u\big(|A|^2 + \mathrm{Rc}(\nu,\nu) \pm \nabla^M_\nu K(\nu,\nu) \mp \nabla^M_\nu \mathrm{tr}K$$
$$\pm \nabla^M_{e_\alpha}K(e_\alpha,\nu) \mp HK(\nu,\nu) \pm K^T\cdot A\big)\,\mathrm{d}\mu.$$

Inserting this into the previous estimate, we infer using (5.4) that

$$\left|\int_\Sigma u|A|^2\,\mathrm{d}\mu\right| \le \left|\int_\Sigma L^{\mathcal{H}\pm\mathcal{P}}u\,\mathrm{d}\mu\right| + C(m)R_e^{-3}|\Sigma|\bar{u} \le C(m)R_e^{-1}\bar{u}.$$

From $\bar{u} \le 2u$ we obtain that $\int_\Sigma H^2\,\mathrm{d}\mu \le C(m)R_e^{-1}$, which contradicts (A2) for large R_e. This implies that $L^{\mathcal{H}\pm\mathcal{P}}$ is injective, and since it is a linear elliptic operator, the Fredholm alternative consequently implies its surjectivity. The existence of a continuous inverse $L^{\mathcal{H}\pm\mathcal{P}}_{\mathrm{inv}}$ with the asserted bounds follows [**7**, Chapter 5]. Note that by the a priori estimates of Theorem 4.12 the Gauss curvature, and therefore the injectivity radius, are controlled. q.e.d.

Remark 5.8. The constant $C(\alpha,\Sigma)$ can be chosen uniformly by using Schauder estimates in harmonic coordinate patches on Σ. Analogous estimates in the spaces $W^{2,p}(\Sigma)$ can be found in [**4**, Chapter 2]

$$\|L^{\mathcal{H}\pm\mathcal{P}}_{\mathrm{inv}}f\|_{W^{2,p}(\Sigma)} \le C(2,p)\tfrac{R_e^3}{3m}\|f\|_{L^p(\Sigma)}.$$

The constants $C(2,p)$ therein can be chosen uniformly since they only depend on $k_{\min} := |\Sigma|^{-1}\min_\Sigma G$, and $k_{\max} := |\Sigma|^{-1}\max_\Sigma G$, which are controlled in our case.

6. The foliation

To prove the existence of surfaces satisfying $H \pm P = \mathrm{const}$, we use the following strategy. Let (g,K) be $(m,0,\sigma,\eta)$-asymptotically flat with $m>0$. Let $g_\tau := (1-\tau)g^S + \tau g$, and $K_\tau := \tau K$. Then the data (g_τ, K_τ) is also $(m,0,\sigma,\eta)$-asymptotically flat. For the initial reference data $(g^S,0)$ we know a lot of solutions to the equation $H = \mathrm{const}$, namely the centered spheres (note that if $K\equiv 0$ then $P\equiv 0$). The mean curvature of a centered sphere of radius r with respect to g^S can be computed using 2.2 and equals

$$H^S(r) = \left(1+\frac{m}{2r}\right)^{-3}\left(\frac{2}{r}-\frac{m}{r^2}\right).$$

This function is invertible for $r > r_1(m)$. The inverse function satisfies $|r - 2/h| \le C$ on $(0,h_1)$, for any $C>0$ provided $h_1>0$ is chosen small enough. Let $h>0$ be a constant. Then we can solve $H^S(r) = h$ with $r > r_1(m)$, provided $h < h_1(m)$. Therefore the equation

$$H \pm P = h$$

is satisfied on a sphere of radius $r(h)$ for $\tau = 0$. To deform this solution for $\tau = 0$ to a family of solutions for $\tau\in[0,1]$, we introduce two classes of surfaces. For this consider the following conditions related to (A1)–(A4) by appropriately choosing the constants

(B1) $R(\Sigma) \leq 8 r_{\min}$,

(B2) $R(\Sigma)^{-1} \leq 8(H \pm P)$,

(B3) $\int_\Sigma u|A|^2 \,\mathrm{d}\mu \leq 8 \int_\Sigma u \det A \,\mathrm{d}\mu \quad \text{for all} \quad 0 \leq u \in C^\infty(\Sigma)$,

(B4) $|\Sigma|_e^{-1} \left| \int_\Sigma \mathrm{id}_\Sigma \,\mathrm{d}\mu^e \right| \leq \frac{3}{4} R_e$.

Choose η_0 so small, and r_0 so large, that Corollaries 4.4, 4.5, and Theorem 4.12 imply that these conditions hold with better constants on surfaces Σ with $r_{\min} > r_0$

(C1) $R(\Sigma) \leq 4 r_{\min}$,

(C2) $R(\Sigma)^{-1} \leq 4(H \pm P)$,

(C3) $\int_\Sigma u|A|^2 \,\mathrm{d}\mu \leq 4 \int_\Sigma u \det A \,\mathrm{d}\mu \quad \text{for all} \quad 0 \leq u \in C^\infty(\Sigma)$,

(C4) $|\Sigma|_e^{-1} \left| \int_\Sigma \mathrm{id}_\Sigma \,\mathrm{d}\mu^e \right| \leq \frac{7}{8} R_e$.

By eventually decreasing η_0 and increasing r_0, we can assume that (C1) – (C4) imply that the linearized operator $L^{\mathcal{H}\pm\mathcal{P}}$ from the previous section is invertible, Corollary 4.10 guarantees that Σ is globally a graph over S^2, and $g^e(\nu^e, \rho) > 1/2$. Moreover, from Theorem 4.12 we can assume that for all surfaces satisfying (B1) – (B4), also

$$\frac{1}{4} r_{\min} \leq (H \pm P)^{-1} \leq 4\, r_{\min}. \tag{6.1}$$

Let (g, K) be data such that for fixed $m > 0$ the data (g_τ, K_τ) as before all are $(m, 0, \sigma, \eta_0)$-asymptotically flat. Define the following nested sets of surfaces:

$\mathcal{S}_1(\tau) = \{\Sigma \subset M : \Sigma \text{ satisfies } r_{\min} > r_0 \text{ and (B1)–(B4) w.r.t. } (g_\tau, K_\tau)\}$,

$\mathcal{S}_2(\tau) = \{\Sigma \subset M : \Sigma \text{ satisfies } r_{\min} > 2r_0 \text{ and (C1)-(C4) w.r.t. } (g_\tau, K_\tau)\}$.

Choose $0 < h_2 \leq h_1$ such that the centered spheres $S_r(0)$ with mean curvature $H < h_2$ are in $\mathcal{S}_2(0)$. Choose $h_0 < \min\{h_1, h_2, \frac{1}{8} r_0^{-1}\}$. Let

$$\kappa : [0,1] \to (0, h_0) \times [0,1] : t \mapsto (h(t), \tau(t))$$

be a continuous, piecewise smooth curve with $\tau(0) = 0$. Denote by $(H \pm P)_\tau$ the function $H \pm P$ evaluated with respect to (g_τ, K_τ). Let $I_\kappa \subset [0,1]$ be the set

$$I_\kappa := \left\{ t \in [0,1] : \exists\, \Sigma(t) \in \mathcal{S}_2(\tau(t)) \text{ with } (H \pm P)_{\tau(t)} = h(t) \right\}.$$

Proposition 6.1. *Under the assumptions of this section, $I_\kappa = [0,1]$.*

Proof. We can assume that κ is smooth. By choice of h_0, $0 \in I_\kappa$, so I_κ is nonempty.

For proving that I_κ is open, let $t_0 \in I_\kappa$, and $\Sigma \in \mathcal{S}_2(\tau(t_0))$ the surface with $(H \pm P)_{\tau(t_0)} = h(t_0)$. Consider Gaussian normal coordinates $y : \Sigma \times (-\varepsilon, \varepsilon) \to M$, and let $B := \{f \in C^{2,\alpha}(\Sigma) : \sup |f| < \varepsilon\}$. Define the operator

$$\mathcal{L} : B \times [0,1] \to C^{0,\alpha}(\Sigma) : (f,t) \mapsto (\mathcal{H} \pm \mathcal{P})_t(f) - h(t),$$

where $(\mathcal{H} \pm \mathcal{P})_t(f)$ is the function $(H \pm P)_t$ on graph(f). This operator is differentiable, and we have $\mathcal{L}(0, t_0) = 0$.

The differential of $\mathcal{L}$ with respect to the first variable is the operator $L^{\mathcal{H}\pm\mathcal{P}}$ from Section 5, and is invertible since $\Sigma \in \mathcal{S}_2(\tau(t))$. By the implicit function theorem there exists $\delta > 0$, and a differentiable function $\xi : (t_0 - \delta, t_0 + \delta) \to B$, such that $\mathcal{L}(\xi(t), t) = 0$ for all t with $|t - t_0| < \delta$.

Hence, for each such t there is a surface $\Sigma(t)$ with $(H \pm P)_{\tau(t)} =$ const. By continuity, and by eventually decreasing δ, we can assume that $\Sigma(t) \in \mathcal{S}_1(\tau(t))$. By choice of r_0 and η_0 conditions (B1)–(B4) imply (C1)–(C4). By choice of h_0 we obtain $r_{\min} > 2r_0$ whence $\Sigma(t) \in \mathcal{S}_2(\tau(t))$. That is, I_κ contains a small neighborhood of t_0.

To show that I_k is closed, assume that $\{t_n\} \subset I_\kappa$ is a convergent series with $\lim_{n\to\infty} t_n \to t$. Let $\Sigma(t_n) \in \mathcal{S}_2(\tau(t_n))$ be the surface with $(H \pm P)_{\tau(t_n)} = h(t_n)$. By Corollary 4.10 all $\Sigma(t_n) = \text{graph}(u_n)$ are graphs over S^2 as described in Section 5.

From the position estimates in Proposition 4.3, the uniform estimates for the angle $g^e(\nu^e, \rho)$, and the uniform curvature estimates from Corollary 4.5 and Proposition 4.8 we obtain uniform $C^2(S^2)$-estimates for the sequence (u_n). In addition, the $W^{1,2}$-estimates on the curvature imply uniform $W^{3,2}$-estimates for (u_n).

We can assume that (u_n) converges in $W^{2,p}(S^2)$ to $u \in W^{2,p}(S^2)$ for a $1 < p < \infty$. Furthermore, we can assume that $(u_n) \to u$ in $C^{1,\alpha}(S^2)$ for a fixed $0 < \alpha < 1$.

On graph(u) a weak version of the quasilinear equation $(H \pm P)_{\tau(t)} = h(t)$ is satisfied. By fixing coefficients, this can be interpreted as a linear equation. Since $u \in C^{1,\alpha}$, the coefficients of this equation are $C^{0,\alpha}$. Regularity theory for such equations [**7**, Chapter 8] implies that u, and therefore Σ, are smooth. By $C^{1,\alpha}$-convergence Σ satisfies (C1), (C4), and $r_{\min} > 2r_0$. By $W^{2,p}$-convergence (C2) and (C3) are satisfied, provided p is large enough. Therefore $t \in I_\kappa$, and I_κ is closed. q.e.d.

This gives the following:

Theorem 6.2. *Let $m > 0$ be fixed. Then there exist constants $h_0 = h_0(m, \sigma)$ and $\eta_0 = \eta_0(m)$ such that for every $(m, 0, \sigma, \eta_0)$-asymptotically flat data set (g, K) and every curve $\kappa : [0,1] \to (0, h_0) \times [0,1] : t \mapsto$*

$(h(t), \tau(t))$ there exists a smooth family of surfaces $\Sigma_\kappa(t) \in \mathcal{S}_2(\tau(t))$ satisfying $H \pm P = h(t)$ with respect to the $\tau(t)$-data.

Remark 6.3. At first glance, the resulting $H \pm P =$ const-surface could depend on the choice of the curve κ from $\kappa(0)$ to $\kappa(1)$. However, since the range of κ is simply connected, and the solutions obtained from the implicit function theorem are locally unique, a standard argument using the homotopy of two curves with common endpoints shows that the surfaces in fact only depend on the endpoints of κ.

We are now ready to prove the existence part of Theorem 1.1.

Theorem 6.4. *Let $m > 0$ be fixed and η_0 and h_0 be as in Theorem 6.2. By possibly decreasing η_0 and h_0 we assure that Corollary 5.5 is valid. Then the surfaces satisfying $H \pm P =$ const constructed in Theorem 6.2 form a foliation. For small $H \pm P$ these surfaces have arbitrarily large radius. In addition, there is a differentiable map*

$$\mathcal{F} : S^2 \times (0, h_0) \times [0, 1] \to M$$

such that the surfaces $\mathcal{F}(S^2, h, \tau)$ satisfy $H \pm P = h$ with respect to the data (g_τ, K_τ). This foliation can therefore be obtained by deforming a piece of the $H =$ const foliation of $(\mathbf{R}^3, g^S)$ by centered spheres.

Proof. Choose $0 < h < h_0$, and define the curve $\kappa_h(t) = (h, t)$ for $t \in [0, 1]$. Using Theorem 6.2 we obtain a family of surfaces $\Sigma_{h,\tau}$ with $H \pm P = h$ by deforming the centered sphere which has $H^S = h$ with respect to g^S along κ. The position estimates and (6.1) imply $h^{-1} \leq 4r_{\min}(\Sigma_{h,t})$, such that by choosing h small, we can make $r_{\min}$ of $\Sigma_1(h)$ large.

The map $\mathcal{F}$ can be constructed by setting $\mathcal{F}(S^2, h, \tau) = \Sigma_{h,\tau}$ and defining the parametrization of $\Sigma_{h,\tau}$ by the fact that $\Sigma_{h,\tau}$ is a graph over S^2. This implies the differentiability of $\mathcal{F}$ with respect to $p \in S^2$ and $\tau \in [0, 1]$.

To show that these surfaces form a foliation, choose another curve. Let $h_1 \in (0, h_0)$ be fixed. The curve κ_{h_1} gives a fixed reference surface $\Sigma_{h_1,1}$. For $h_2 < h_1$ consider the curves $\lambda_{h_2}(t) = ((1 - t)h_1 + th_2, 1)$. Concatenating κ_{h_1} and λ_{h_2} gives a family of surfaces $\Sigma'_{h,1}$ with $h \in [h_2, h_1]$, as well as a differentiable map $F : S^2 \times [h_2, h_1] \to M$ such that $F(S^2, h) = \Sigma'_{h,1}$. Remark 6.3 implies that $\Sigma'_{h,1} = \Sigma_{h,1} =: \Sigma_h$. Therefore $\mathcal{F}$ is differentiable with respect to $h \in (0, h_0)$.

Let ν_h denote the normal to Σ_h. Then the lapse α_h of the family F is defined as $\alpha_h := g\left(\nu_h, \frac{dF}{dh}\right)$. Since $H \pm P =$ const along Σ_h, and therefore the tangential part of $\frac{dF}{dh}$ is irrelevant for the evolution of $H \pm P$, we have

$$h_1 - h_2 = \frac{d}{dh}(H \pm P) = L^{\mathcal{H} \pm \mathcal{P}} \alpha_h$$

with the operator $L^{\mathcal{H}\pm\mathcal{P}}$ from Section 5. By Corollary 5.5, α_h does not change sign. Therefore the family of the Σ_h is a foliation. q.e.d.

We can also prove the uniqueness of $H \pm P = \text{const}$ surfaces.

Theorem 6.5. *For $m > 0$ there are constants $\eta_0(m, C^A) > 0$ and $h_0(m, \sigma, C^A) > 0$ such that if (g, K) is $(m, 0, \sigma, \eta_0)$-asymptotically flat, then two surfaces Σ_1 and Σ_2 satisfying* (A1)–(A4) *and $H \pm P = h = \text{const}$ with $h \in (0, h_0)$ coincide.*

Proof. We prove this by reversing the process we used in the proof of the existence result. That is, we start for the data (g_τ, K_τ) at $\tau = 1$ with Σ_1 and Σ_2 and obtain surfaces Σ_1' and Σ_2' with $H = h = \text{const}$ with respect to the Schwarzschild metric g^S at $\tau = 0$. Here η_0 and h_0 have to be adjusted as in the beginning of this section, such that this process works.

By the uniqueness of constant mean curvature surfaces satisfying (A1)–(A4) in the Schwarzschild metric, as follows for example from Huisken and Yau [**9**, Section 5], we infer that Σ_1' coincides with Σ_2'. Then by the local uniqueness of the implicit function theorem, also Σ_1 and Σ_2 coincide. q.e.d.

Corollary 6.6. *The $H \pm P = \text{const}$ foliations from Theorem 1.1 consisting of surfaces satisfying* (A1)–(A4) *are unique at infinity.*

7. Special data

We want to interpret the foliation of $H \pm P = \text{const}$ surfaces in a physical manner. A foliation of surfaces satisfying $H = \text{const}$ was interpreted in [**9**] as the center of mass of an isolated system. The definition of this foliation does not refer to the extrinsic curvature K and therefore cannot contain information on dynamical physics. In contrast, Proposition 7.1 shows that the $H \pm P = \text{const}$ foliation allows an interpretation as linear momentum.

We restrict ourselves to data (g, K) with $\|g - g^S\|_{C^2_{-1-\delta}} < \infty$ with $\delta > 0$ and

$$K = \frac{3}{2r^2}\big(\rho \otimes p + p \otimes \rho - 2\langle p, \rho\rangle(g^e - \rho \otimes \rho)\big) + O(r^{-2-\delta})$$

where $p \in \mathbf{R}^3$ is a fixed vector, $\rho = x/r$ is the radial direction, and the derivatives of $O(r^{-2-\delta})$ are of order $O(r^{-3-\delta})$. This structure of K was proposed by York [**17**] and represents a trace free extrinsic curvature tensor with ADM-momentum p. There exist initial data satisfying the constraint equations with these asymptotics. Using this representation of K, we can refine the estimates from Proposition 4.3 and obtain

Proposition 7.1. *Let (g, K) be as described above. If $|p| < m$ is small enough, and Σ satisfies $H \pm P = \text{const}$, assumptions* (A1)–(A4), *and $r_{\min} > r_0$, then there exist a vector $a \in \mathbf{R}^3$, a sphere $S = S_{R_e}(a)$, and a parameterization $\psi : S \to \Sigma$ such that*

$$|a/R_e \mp \tau(v)\bar{p}| \leq CR_e^{-\delta},$$
$$\sup_S |\phi - \mathrm{id}_S| \leq CR_e^{-\delta},$$
$$\sup_\Sigma |\nu^e - R_e^{-1}(r\rho - a)| \leq CR_e^{-\delta}$$

with $\bar{p} = \frac{p}{|p|}$, $v = \frac{|p|}{m}$, and $\tau(v) = \frac{1-\sqrt{1-v^2}}{v}$. If $0 \leq v \leq 1$ then $0 \leq \tau(v) \leq 1$ and $\tau(v) = \frac{1}{2}v + O(v^3)$ for $v \to 0$.

Proof. This proof is similar to the proof of Proposition 4.3. However, instead of estimating like (4.9), we compute more carefully using the asymptotics of K. For the test vector $b = \bar{a} := \frac{a}{|a|}$ we obtain

$$\left| -8\pi m \frac{|a|}{R_e} \mp 4\pi \left\langle p, \frac{a}{|a|} \right\rangle \frac{|a|^2 + R_e^2}{R_e^2} \right| \leq CR_e^{-\delta}. \tag{7.1}$$

Now we split $p = g^e(\bar{a}, p)\bar{a} + g^e(\bar{q}, p)\bar{q}$ with $\bar{q}$ orthogonal to a and $|\bar{q}| = 1$. Then we use $\bar{q}$ as an additional test vector. This gives the second estimate

$$|\langle p, \bar{q} \rangle| \left| \frac{4\pi}{5} \frac{5R_e^3 - 2|a|^2 R_e - 3|a|^2}{R_e^3} \right| \leq CR_e^{-\delta}. \tag{7.2}$$

Proposition 4.3 gives $\tau := |a|/R_e < 1$ if p is small. Then (7.2) implies that $|g^e(p, \bar{q})| \leq CR_e^{-\delta}$, and therefore $|g^e(p, \bar{a}) - |p|| = |g^e(p, \bar{q})| \leq CR_e^{-\delta}$. Using (7.1) we infer that

$$\left| -2m\tau \mp |p|(1 + \tau^2) \right| \leq CR^{-\delta},$$

which implies the proposition. q.e.d.

Remark 7.2.

(i) This means that surfaces $\Sigma(h)$ satisfying $H \pm P = h = \mathrm{const}$ are not only increasing in size for $h \to 0$, but that they also translate. The magnitude of this translation can be used to compute p. The asymptotic translation τ from the previous proposition can be found by comparing the Euclidean center of gravity to the center of gravity computed using the g-metric. In particular,

$$\lim_{h \to 0} \left(|\Sigma^e|^{-1} \int_{\Sigma(h)} x \, \mathrm{d}\mu^e - |\Sigma|^{-1} \int_{\Sigma(h)} x \, \mathrm{d}\mu \right) = \frac{2}{3} m\tau\bar{p}.$$

Here $\tau = \lim_{h \to 0} |a|/R_e(\Sigma(h))$ is the limit of the magnitude of the translation vector and $\bar{p}$ the unit vector pointing into its direction. Then p can be computed from

$$\pm p = \frac{2m\tau}{1 + \tau^2} \, \bar{p}.$$

(ii) Corvino and Schoen [**5**] also propose a standard form of the extrinsic curvature tensor, namely

$$K^{\mathrm{CS}} = \frac{2}{r^2}\left(p \otimes \rho + \rho \otimes p - \langle p, \rho \rangle g^e\right) + O(r^{-3}).$$

Contrary to the York-form this is not trace free in the terms of highest order. Corvino and Schoen prove that data satisfying this asymptotic condition for K and $g = g^S + O(r^{-2})$ are dense with respect to suitable, weighted Sobolev norms in the set of data $(\bar{g}, \bar{K})$ satisfying the constraint equations and

$$\bar{g} = g^e + O(r^{-1}) \qquad \text{and} \qquad \bar{K} = O(r^{-2}).$$

Therefore these asymptotics possess a certain universality.

For these asymptotics we can also compute the asymptotic translation. It satisfies $\tilde{\tau}(v) = \frac{1-\sqrt{1-\frac{16}{15}v^2}}{\frac{8}{5}v}$. Here $\tilde{\tau}(v) = \frac{1}{3}v + O(v^3)$ for $v \to 0$.

This is not satisfactory for two reasons. Firstly, this asymptotic translation and the associated linear momentum formula do not coincide with the formula obtained from the York asymptotics. Secondly, $0 \leq \tilde{\tau}(v) \leq 1$ only for $0 < v < \frac{15}{16}$, while from physical reasons at least the interval $v \in [0, 1]$ should be admitted.

On the other hand, we cannot expect to obtain a valid formula independent of the slicing condition. For the $H \pm P =$ const foliation, therefore, the slicing condition $\mathrm{tr} K = 0$ seems to be appropriate.

(iii) Both the asymptotics of York and the asymptotics of Corvino and Schoen allow examples of initial data satisfying the vacuum constraint equations. This implies that the Sobolev norm used by Corvino and Schoen to prove density of data with their asymptotics is strong enough to preserve ADM-mass and -momentum, but not strong enough to reproduce the fine structure of the $H \pm P =$ const foliation.

References

[1] R. Arnowitt, S. Deser, & C.W. Misner, *Coordinate invariance and energy expressions in general relativity*, Phys. Rev. (2) **122** (1961) 997–1006. MR 0127946, Zbl 0094.23003.

[2] I. Ben-Dov, *Penrose inequality and apparent horizons*, Phys. Rev. D (3) **70**(**12**) (2004) 124031, 11, MR 2124204.

[3] H. Bray, *The Penrose inequality in General Relativity and volume comparison theorems involving scalar curvature*, Ph.D. thesis, Stanford University, 1997.

[4] D. Christodoulou & S. Klainerman, *The global nonlinear stability of the Minkowski space*, Princeton Mathematical Series, **41**, Princeton University Press, Princeton, NJ, 1993, MR 1316662, Zbl 0827.53055.

[5] J. Corvino & R.M. Schoen, *On the asymptotics for the vacuum Einstein constraint equations*, J. Differential Geom. **73**(**2**) (2006) 185–217, MR 2225517.

[6] C. De Lellis & S. Müller, *Optimal rigidity estimates for nearly umbilical surfaces*, J. Differential Geom. **69**(**1**) (2005) 75–110, MR 2169583.

[7] D. Gilbarg & N.S. Trudinger, *Elliptic partial differential equations of second order*, second ed., Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], **224**, Springer-Verlag, Berlin, 1983, MR 737190.

[8] G. Huisken & T. Ilmanen, *The inverse mean curvature flow and the Riemannian Penrose inequality*, J. Differential Geom. **59**(**3**) (2001), 353–437, MR 1916951, Zbl 1055.53052.

[9] G. Huisken & S.-T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature*, Invent. Math. **124**(**1-3**) (1996) 281–311, MR 1369419, Zbl 0858.53071.

[10] J.H. Michael & L.M. Simon, *Sobolev and mean-value inequalities on generalized submanifolds of* R^n, Comm. Pure Appl. Math. **26** (1973), 361–379, MR 0344978, Zbl 0256.53006.

[11] R. Penrose, *Naked Singularities*, Ann. New York Acad. Sci. **224** (1973), 125–134.

[12] R. Schoen, L. Simon, & S.-T. Yau, *Curvature estimates for minimal hypersurfaces*, Acta Math. **134**(**3-4**) (1975) 275–288, MR 0423263, Zbl 0323.53039.

[13] R. Schoen & S.-T. Yau, *Proof of the positive mass theorem,* II, Comm. Math. Phys. **79**(**2**) (1981) 231–260, MR 612249, Zbl 0494.53028.

[14] J. Simons, *Minimal varieties in riemannian manifolds*, Ann. of Math. (2) **88** (1968) 62–105, MR 0233295, Zbl 0181.49702.

[15] G. Stampacchia, *Èquations elliptiques du second ordre à coefficients discontinus*, Séminaire de Mathématiques Supérieures, No. 16 (Été, 1965), Les Presses de l'Université de Montréal, Montreal, Que., 1966, MR 0251373.

[16] R. Ye, *Foliation by constant mean curvature spheres on asymptotically flat manifolds*, Geometric analysis and the calculus of variations, Int. Press, Cambridge, MA, 1996, 369–383, MR 1449417, Zbl 0932.53026.

[17] J.W. York, *Kinematics and dynamics of general relativity*, Sources of Gravitational Radiation (L. Smarr, ed.), Seattle, Cambridge University Press, 1978, 83–126.

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