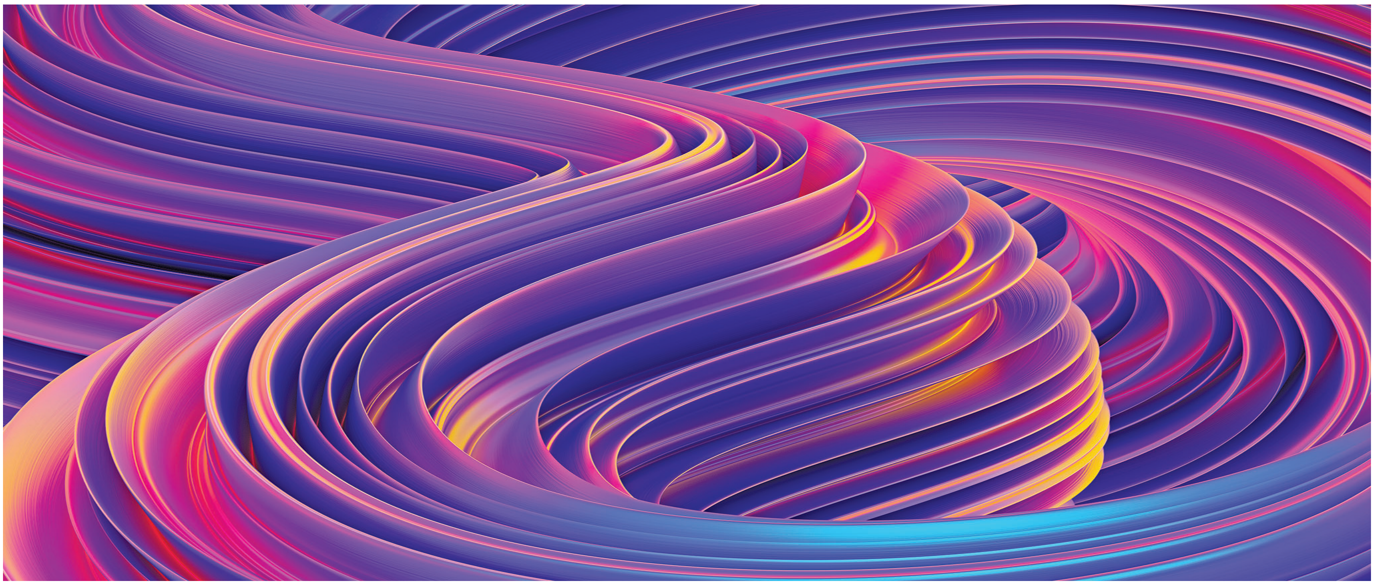
Foliations on Complex Manifolds



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In these notes we survey some aspects of the theory of holomorphic foliations on complex manifolds. The origins of the theory go back to works of Darboux, Poincaré and Painlevé, where it was developed to study solutions of ordinary differential equations on $\mathbb{C}^2$. We briefly discuss some of the early works on this theory, mostly concerned with the local behavior of the leaves near the singularities. We then move the focus from local to global properties. Birational geometry has had a great influence on the development of a global theory of holomorphic foliations. After reviewing the Enriques-Kodaira classification of projective surfaces and explaining the general philosophy of the Mininal Model Program, we explore some of their recent counterparts for foliations.

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Communicated by Notices *Associate Editor Steven Sam.*

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DOI: https://doi.org/10.1090/noti2507

Foliations

A foliation $\mathcal{F}$ of dimension r on a differentiable manifold M^n is a decomposition of M into a disjoint union of immersed submanifolds of dimension r, called *leaves*, which pile up locally like fibers of a submersion. Formally, $\mathcal{F}$ is defined by an atlas $\{\varphi_i : U_i \to M\}$, with $U_i \subset \mathbb{R}^r \times \mathbb{R}^{n-r}$, and differentiable transition functions of the form

$$\varphi_{ij}(x, y) = \big(f(x, y), g(y)\big), \quad (x, y) \in \mathbb{R}^r \times \mathbb{R}^{n-r}.$$

The leaves of $\mathcal{F}$ are locally given by the fibers of the projection $U_i \to \mathbb{R}^{n-r}$. If the transition functions are C^∞, we say that $\mathcal{F}$ is a C^∞ foliation. The integer $n-r$ is called the codimension of the foliation $\mathcal{F}$. In the literature, the integer r is also referred to as the *rank* of the foliation.

Despite their simple local description, the existence of a foliation on a compact manifold M is subject to global topological constraints, and carries relevant information about the geometry of M. For instance, consider the 2-dimensional sphere S^2. If there were a foliation $\mathcal{F}$ of dimension one on S^2, then one could cook up a non-vanishing smooth vector field everywhere tangent to $\mathcal{F}$, contradicting the Poincaré-Hopf theorem. On the other hand, when $M = \mathbb{R}^2/\mathbb{Z}^2$ is a 2-dimensional torus, one can construct plenty of foliations of dimension one on M. For any choice of angle θ, the foliation by lines on $\mathbb{R}^2$ having slope equal to θ induces a foliation $\mathcal{F}_\theta$ of dimension one on the quotient $\mathbb{R}^2/\mathbb{Z}^2$. If θ is rational, then the leaves of $\mathcal{F}_\theta$ are homeomorphic to S^1. If θ is irrational, then the leaves are homeomorphic to $\mathbb{R}$, and dense in M by a theorem of Kronecker.

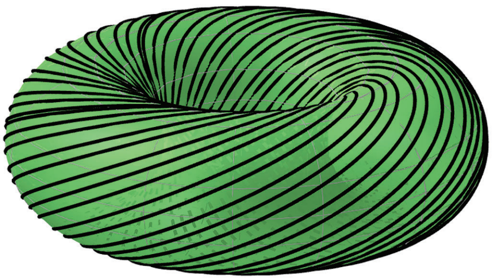

Figure 1. Foliation on the torus.

More generally, we have the following result by Thurston characterizing which closed manifolds admit a C^∞ foliation of codimension one in terms of the topological Euler characteristic.

Theorem 1 ([Thu76]). *Let M be a closed connected smooth manifold. Then M admits a C^∞ foliation of codimension one if and only if $\chi(M) = 0$.*

When X is a *complex manifold*, it is natural to consider *holomorphic foliations* on X. These are defined by atlases $\{\varphi_i : U_i \to X\}$, with $U_i \subset \mathbb{C}^r \times \mathbb{C}^{n-r}$, and biholomorphic transition functions of the form

$$\varphi_{ij}(x, y) = \big(f(x, y), g(y)\big), \ \ (x, y) \in \mathbb{C}^r \times \mathbb{C}^{n-r}.$$

The integer n is the complex dimension of X, r is the dimension of the holomorphic foliation, and $n - r$ its codimension. The geometric theory of holomorphic foliations was first introduced to better understand solutions of complex ordinary differential equations on the plane $\mathbb{C}^2$. In this context, it is natural to allow for singularities. Given a meromorphic function F on $\mathbb{C}^2$, consider the ODE

$$\frac{dy}{dx} = F(x, y). \tag{1}$$

By the theorem of existence and uniqueness of solutions of ordinary differential equations, the solutions of (1) induce a holomorphic foliation of dimension 1 on the complement of a closed subset of $\mathbb{C}^2$. Indeed, if $F(x, y)$ is holomorphic at a point $(x_0, y_0) \in \mathbb{C}^2$, then the ODE (1) admits a unique holomorphic solution $y = y(x)$ satisfying $y(x_0) = y_0$.

More generally, a singular holomorphic foliation $\mathcal{F}$ of dimension r on a complex manifold X is an equivalence class of holomorphic foliations of dimension r defined on the complement of a proper closed subset of codimension at least 2 in X. There is a minimal closed subset $\mathrm{Sing}(\mathcal{F})$ such that $\mathcal{F}$ can be extended to a holomorphic foliation of dimension r on $X \setminus \mathrm{Sing}(\mathcal{F})$. This closed set is called the *singular locus* of $\mathcal{F}$. Outside of $\mathrm{Sing}(\mathcal{F})$, the vectors that are tangent to the leaves of $\mathcal{F}$ form a sub-vector bundle of rank r of the tangent bundle T_X, which can be extended on the whole X to a coherent subsheaf $T_\mathcal{F}$ of T_X, called the *tangent sheaf* of $\mathcal{F}$. The subsheaf $T_\mathcal{F}$ is closed under the Lie bracket: if v and w are two local vector fields on $X \setminus \mathrm{Sing}(\mathcal{F})$ everywhere tangent to the leaves of $\mathcal{F}$, then their Lie bracket $[v, w]$ is also everywhere tangent to the leaves of $\mathcal{F}$. Conversely, we have the following classical theorem:

Theorem 2 (Frobenius). *There is a one to one correspondence between singular holomorphic foliation on X and saturated coherent subsheaves of the tangent bundle T_X that are closed under the Lie bracket.*

Much of the early work on singular holomorphic foliations focused on the behavior of the leaves near the singular locus. Let $\mathcal{F}$ be a singular holomorphic foliation on $\mathbb{C}^2$ with an isolated singularity at the origin, defined by a holomorphic vector field

$$v = p(x, y)\frac{\partial}{\partial x} + q(x, y)\frac{\partial}{\partial y},$$

where p and q are holomorphic functions. Denote by $p_1(x, y) = ax + by$ and $q_1(x, y) = cx + dy$ the linear parts of p and q, respectively, and suppose that they are not both identically zero. Under some genericity assumptions, the local behavior of the leaves near the origin is controlled by the eigenvalues of the nonzero matrix

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Denote by λ_1 and λ_2 the two eigenvalues of A, with $\lambda_2 \neq 0$, and suppose that λ_1/λ_2 is not real, or it is a positive real number that is not an integer nor the inverse of an integer. A fundamental theorem of Poincaré states that v is linearizable. This means that locally around the singularity, after a suitable analytic change of coordinates, the vector field v can be written in the form

$$v = \lambda_1 z\frac{\partial}{\partial z} + \lambda_2 w\frac{\partial}{\partial w}, \tag{2}$$

where z and w are the new coordinates. The vector field (2) can be easily integrated. On a punctured neighborhood of the origin, the leaves of the foliation are parametrised by

$$\gamma(t) = (c_1 e^{\lambda_1 t}, c_2 e^{\lambda_2 t}), \ t \in \mathbb{C}.$$

The ones given by the equations $z = 0$ and $w = 0$ accumulate in the origin.

If λ_1/λ_2 is a positive integer n or the inverse of a positive integer $1/n$, then Dulac proved that, in suitable analytic coordinates z and w, the vector field v can be written in the form

$$v = z\frac{\partial}{\partial z} + (nw + \mu z^n)\frac{\partial}{\partial w},$$

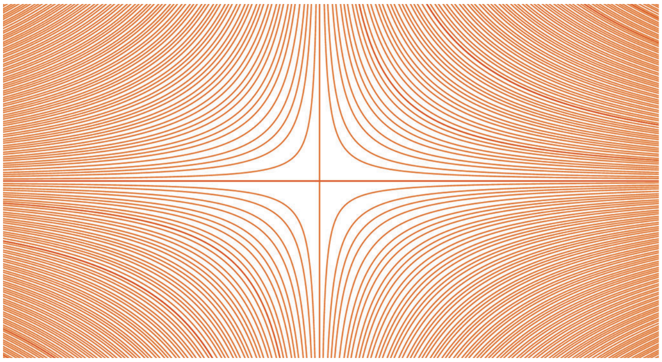

Figure 2. Singularity of a foliation on the plane.

where $\mu \in \mathbb{C}$. Again we see that there are leaves of the foliation on a punctured neighborhood of the origin that accumulate in $(0,0)$. More generally, let $p \in S$ be an isolated singularity for a foliation $\mathcal{F}$ of dimension 1 on a nonsingular complex surface S. A *separatrix* is a complex curve $C \subset S$ through p such that $C \setminus \{p\}$ is a leaf of $\mathcal{F}$ on a punctured neighborhood of p in S. By a theorem of Camacho and Sad [CS82], a separatrix always exists for foliations on surfaces.

In general, we say that a vector field v is in normal form if $v = v_s + v_n$, where $v_s = \lambda_1 z\frac{\partial}{\partial z} + \lambda_2 w\frac{\partial}{\partial w}$ is a semisimple vector field, v_n is a nilpotent vector field (i.e., the matrix associated to the linear part of v_n is nilpotent), and $[v_s, v_n] = 0$. We say that v is *analytically normalizable* if there is an analytic change of coordinates putting it in normal form. So we have seen that if $\lambda_1/\lambda_2 \notin \mathbb{R}_-$, then v is analytically normalizable. There exist examples of vector fields which are not analytically normalizable, such as $v = (z+w)\frac{\partial}{\partial z} + w^2\frac{\partial}{\partial w}$. In this case, after the formal change of coordinates

$$(Z, w) = \Big(z + \sum_{n\geq 1}(n-1)!\, w^n, w\Big),$$

v becomes $Z\frac{\partial}{\partial Z} + w^2\frac{\partial}{\partial w}$, which is in normal form. However, this formal change of coordinates is not holomorphic, since $\sum_{n\geq 1}(n-1)!\, w^n$ diverges. Questions about analytic linearization and normalization of local vector fields on the plane were very much studied in the twentieth century (see for instance [Mar81]).

Global Aspects

Now we move the focus from local to global properties of foliations. One example of a global invariant of a codimension one foliation $\mathcal{F}$ on $\mathbb{CP}^n$ is its *degree*. It is defined as the number of tangencies of a generic line in $\mathbb{CP}^n$ with $\mathcal{F}$. Codimension one foliations on $\mathbb{CP}^n$ of small degree were classified by Jouanolou in [Jou79]. In order to explain this classification, it is convenient to describe foliations using differential forms. The tangent sheaf of a codimension one (singular) foliation $\mathcal{F}$ on $\mathbb{CP}^n$ can be given by a polynomial one-form

$$\Omega = \sum_{i=0}^{n} P_i(x_0, \dots, x_n)dx_i, \tag{3}$$

where the P_i's are homogeneous polynomials of the same degree without common factors. This polynomial form is uniquely determined up to scalar, and the P_i's satisfy $\sum_{i=0}^{n} x_i P_i(x_0, \dots, x_n) = 0$. In the language of differential forms, the Frobenius' integrability condition translates into the condition

$$\Omega \wedge d\Omega = 0. \tag{4}$$

If $\mathcal{F}$ has degree d, then the homogeneous polynomials P_i in (3) have degree $d+1$. Using this description, it is not difficult to deduce that any codimension one foliation $\mathcal{F}$ on $\mathbb{CP}^n$ of degree $d = 0$ is induced by a linear projection $\mathbb{CP}^n \dashrightarrow \mathbb{CP}^1$. We say that $\mathcal{F}$ is induced by a pencil of hyperplanes containing a fixed linear subspace $\mathbb{CP}^{n-2}$, which is the singular locus of $\mathcal{F}$. In particular, the leaves of $\mathcal{F}$ are algebraic submanifolds of $\mathbb{CP}^n$.

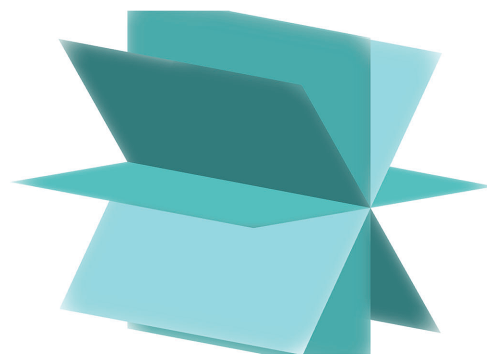

Figure 3. Foliation of degree 0 on $\mathbb{CP}^3$.

In general, the following result characterizes foliations with infinitely many algebraic leaves. It is often referred to as the Darboux-Jouanolou integrability theorem.

Theorem 3 ([Jou79, Théorème 3.3]). *Let $\mathcal{F}$ be a codimension one foliation on $\mathbb{CP}^n$ with infinitely many algebraic leaves. Then $\mathcal{F}$ has a first integral, i.e., it is induced by a rational map $f : \mathbb{CP}^n \dashrightarrow \mathbb{C}$. In particular, all the leaves of $\mathcal{F}$ are algebraic.*

The set $\mathrm{Fol}(d, n)$ of codimension one foliations of degree d on $\mathbb{CP}^n$ has a natural structure of a quasi-projective variety. When $n = 2$, the Frobenius' integrability condition (4) is automatic, and thus $\mathrm{Fol}(d, 2)$ is an open subset of the projective space of polynomial one-forms as in (3), where the P_i's are homogeneous polynomials of degree $d+1$ without common factors. When $d = 0$, we have seen above that $\mathrm{Fol}(0, n)$ can be identified with the Grassmannian parametrizing codimension 2 linear subspaces of $\mathbb{CP}^n$. Jouanolou also classified the next case, $d = 1$. When $n \geq 3$, he showed that $\mathrm{Fol}(1, n)$ has two irreducible components. One of these irreducible components corresponds to foliations given by homogeneous one-forms that depend only on three variables, after suitable projective change of coordinates. Geometrically, they are pullbacks under linear projections of foliations on $\mathbb{CP}^2$ induced by global vector fields. The other irreducible component of $\mathrm{Fol}(1, n)$ corresponds to foliations having a first integral of the form

$$f = \frac{\ell^2}{q} : \mathbb{CP}^n \dashrightarrow \mathbb{C},$$

where ℓ is a linear form and q is a quadratic form. Geometrically, they are induced by pencils of quadric hypersurfaces containing a double hyperplane. Later in [CN96], Cerveau and Lins Neto showed that $\mathrm{Fol}(2, n)$ has six irreducible components when $n \geq 3$, and described each of them. More recently, Da Costa, Lizarbe, and Pereira studied the case $d = 3$, $n \geq 3$ in [dCLP21]. They classify the 18 irreducible components of $\mathrm{Fol}(3, n)$ whose general member does not have a first integral. They also show that there are at least 6 components whose general member has a first integral.

Another global invariant of a foliation is the *algebraic dimension*, or *algebraic rank*. The *algebraic dimension* $r_a(\mathcal{F})$ of a foliation $\mathcal{F}$ on a complex projective manifold X is the maximum dimension of an algebraic subvariety Z through a general point of X that is tangent to $\mathcal{F}$. By this we mean that for every point $z \in Z \setminus \mathrm{Sing}(\mathcal{F})$, we have $T_z Z \subset T_z \mathcal{F}$. A foliation is said to be *purely transcendental* if its algebraic dimension is 0. It is said to be *algebraically integrable* if its algebraic dimension equals the dimension.

From the classification of codimension one foliations on $\mathbb{CP}^n$ of small degree, we observe a lower bound for the algebraic dimension in terms of the degree. Indeed, let $\mathcal{F}$ be a codimension one foliation on $\mathbb{CP}^n$ of degree d. If $d = 0$, then $\mathcal{F}$ is algebraically integrable. If $d = 1$, then $r_a(\mathcal{F}) \geq n - 2$, and this bound is attained when $\mathcal{F}$ is the pullback under a linear projection of a purely transcendental foliation on $\mathbb{CP}^2$ induced by a global vector field. In general, we have that

$$r_a(\mathcal{F}) \geq n - 1 - d.$$

This is a special case of Theorem 4, which gives a lower bound for the algebraic dimension of foliations in a more general context. Notice that the algebraic dimension makes sense for foliations on any complex manifold, while the notion of degree is particular to $\mathbb{CP}^n$ or other varieties covered by lines. As we shall see, the degree can be read off from a more general object that can be attached to any foliation, its *canonical class*. This notion has its origin in connection with birational geometry.

Birational Geometry

A central theme in algebraic geometry is the classification of complex projective varieties up to birational equivalence. Two projective varieties are said to be birationally equivalent if they have isomorphic dense open subsets. Examples of birational invariants of projective manifolds are the *genus* and *irregularity*. Given a complex projective manifold X, we consider the tangent bundle T_X of X, and its dual vector bundle $\Omega^1_X = T_X^\vee$. Let $\omega_X = \det(\Omega^1_X)$ denote the *canonical bundle* of X. It is the line bundle on X whose sections are the top holomorphic differential forms on X. The *genus* $p_g(X)$ of X is the dimension of the space of holomorphic global sections of the canonical bundle ω_X:

$$p_g(X) \coloneqq \dim \Gamma(X, \omega_X).$$

The *irregularity* $q(X)$ of X is the dimension of the space of holomorphic global sections of the cotangent bundle:

$$q(X) \coloneqq \dim \Gamma(X, \Omega^1_X).$$

Moreover, some special arithmetical properties of algebraic varieties turn out to be invariant under birational equivalence, making this notion fundamental also in connection with number theory and arithmetic geometry.

The *Minimal Model Program* (MMP for short) is an algorithmic surgery process designed to transform a given projective variety X into a simplest representative X' in its birational equivalence class. In this way, if one is interested in understanding a birational property of X, one can investigate it in the simpler model X'.

We start by reviewing the classical MMP for surfaces. It was established by the Italian school of algebraic geometry by the beginning of the 20th century, and reviewed in modern language in the 1960's, most notably by Kodaira, Zariski and Shafarevich. Given a smooth projective surface S, the blowup of a point $P \in S$ is a morphism $\pi : \tilde{S} \to S$ from a smooth projective surface $\tilde{S}$ that replaces the point $P \in S$ with the *exceptional curve* $C = \pi^{-1}(P) \cong \mathbb{CP}^1$, and restricts to an isomorphism between $\tilde{S} \setminus C$ and $S \setminus \{P\}$. Viewed as an element of $\mathrm{H}^2(\tilde{S}, \mathbb{Z})$, the exceptional curve C has self-intersection $C^2 = -1$. Conversely, Castelnuovo's contractibility theorem asserts that any curve C on a smooth surface S such that $C \cong \mathbb{CP}^1$ and $C^2 = -1$ is the exceptional curve of a blowup. Such a curve is called a (-1)-*curve*. It turns out that any smooth projective surface can be obtained from a distinguished representative of its birational equivalence class by a sequence of blowups. Such distinguished representatives are characterized by the property that they do not contain any (-1)-curve, and are classically called *minimal surfaces*. Figure 4 summarizes the classical MMP for surfaces. The MMP terminates after a finite number of steps because the second Betti number $b_2(S)$ drops by one every time we blow down a (-1)-curve.

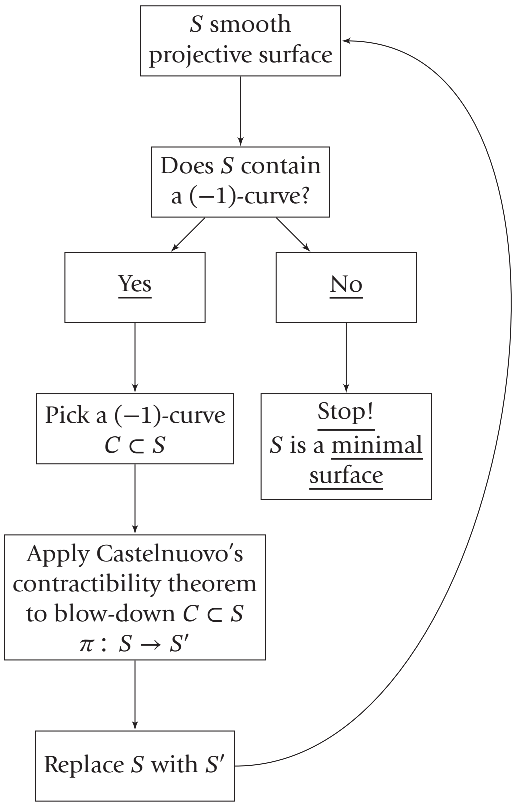

Figure 4. The classical MMP for surfaces.

In order to generalize this to higher dimensions, the question

$$\text{“Does } S \text{ contain a } (-1)\text{-curve?”} \tag{5}$$

must be rephrased in a way that it makes sense in any dimension. This is done with the aid of the *canonical class*.

Definition 1. Let X be a complex projective manifold X. The *canonical class* K_X of X is the first Chern class of the canonical line bundle of X:

$$K_X := c_1(\omega_X) \in \mathrm{H}^2(X, \mathbb{Z}).$$

For any compact complex curve $C \subset X$, the intersection number $-K_X \cdot C$ measures the Ricci curvature of X along C. Usually, the sign of $-K_X \cdot C$ varies with the curve $C \subset X$. Varieties whose canonical class has a definite sign are very special, and play a distinguished role in algebraic geometry. Particularly important are *Fano varieties*. A *Fano variety* is a projective variety X with $-K_X > 0$ (i.e., $-K_X$ is ample).

Let X be a complex projective manifold. In order to generalize to higher dimensions the classical MMP for surfaces, question (5) is replaced with

$$\text{“Is the canonical class } K_X \text{ nef?”}$$

We say that K_X is *nef* if $K_X \cdot C \geq 0$ for every curve $C \subset X$. The goal of the MMP is to produce a finite sequence of elementary birational maps

$$X = X_0 \dashrightarrow X_1 \dashrightarrow X_2 \dashrightarrow \cdots \dashrightarrow X_n = X',$$

ending with a variety X' in the same birational equivalence class of X, and satisfying exactly one of the following two conditions.

1. $K_{X'}$ is *nef*. Such a variety X' is called a *minimal model*.
2. There is a morphism $f : X' \to Y$ onto a lower dimensional variety whose fibers are Fano varieties. Such a morphism $f : X' \to Y$ is called a *Mori fiber space*.

The MMP involves making some choices, and the outcome is not unique in general. However, whether the MMP for X ends with a minimal model or a Mori fiber space depends only on the birational equivalence class of X. Next we introduce a fundamental birational invariant for projective manifolds, the *Kodaira dimension*, defined in terms of the rate of growth of holomorphic sections of the line bundle $\omega_X^{\otimes m}$ as m increases.

Definition 2. Let $\mathcal{L}$ be a line bundle on a complex projective manifold X. The *Iitaka dimension* $\kappa(\mathcal{L})$ of $\mathcal{L}$ is defined as follows. Consider the semigroup $\mathbb{N}(\mathcal{L})$ of nonnegative integers m for which $\mathcal{L}^{\otimes m}$ admits nonzero holomorphic sections. If $\mathbb{N}(\mathcal{L}) = \{0\}$, then we set $\kappa(\mathcal{L}) = -\infty$. If $\mathbb{N}(\mathcal{L}) \neq \{0\}$, then there is an integer κ such that, for $m \in \mathbb{N}(\mathcal{L})$ sufficiently large,

$$c_1 \cdot m^{\kappa} \ \leq \dim \Gamma(X, \mathcal{L}^{\otimes m}) \ \leq c_2 \cdot m^{\kappa},$$

for suitable positive constants c_1 and c_2. We set $\kappa(\mathcal{L}) = \kappa$.

The *Kodaira dimension* of X is

$$\kappa(X) := \kappa(\omega_X) \in \{-\infty, 0, \ldots, \dim(X)\}.$$

It is a birational invariant for complex projective manifolds.

For surfaces, the elementary birational maps in the MMP are blowups of points. Surfaces that are birational to products $\mathbb{CP}^1 \times C$ are called *ruled surfaces*, and are characterized by the condition that $\kappa(S) = -\infty$. Surfaces with non-negative Kodaira dimension have a unique minimal model in their birational class. They can be divided into the following classes. This is known as the Enriques-Kodaira classification.

- $\kappa(S) = 0$. There are 4 classes.
  - *Enriques' surfaces*: $p_g(S) = q(S) = 0$.
  - *Bielliptic surfaces*: $p_g(S) = 0$, $q(S) = 1$.
  - *K3 surfaces*: $p_g(S) = 1$, $q(S) = 0$.
  - *Abelian surfaces*: $p_g(S) = 1$, $q(S) = 2$.
- $\kappa(S) = 1$. These surfaces are not ruled, and admit a fibration $f : S \to B$ onto a smooth curve whose generic fiber is an elliptic curve.
- $\kappa(S) = 2$. Most surfaces lie in this class. These are called *surfaces of general type*.

In higher dimensions, there are two types of elementary birational maps in the MMP, called *divisorial contractions* and *flips*. Divisorial contractions can be viewed as generalizations of the blowup of a point on a surface. Flips are birational maps $X \dashrightarrow X'$ that restrict to isomorphisms between the complements of *small* subsets of X and X', i.e., subsets of codimension ≥ 2. They have no parallel in surface theory. Another source of complication in higher dimensions is that the birational models X_i's may not be smooth, although they only have very mild singularities, called *terminal singularities*. In dimension three, the MMP was completed in [Mor88]. In higher dimensions a major breakthrough was achieved in [BCHM10], where it was proved that the flip surgery can always be performed. The major open problem in general is to show that there is no infinite sequence of flips. However, [BCHM10] establishes a weaker version of termination of flips, which allows them to prove the MMP in the following cases:

1. $\kappa(X) = \dim(X)$, in which case the MMP ends with a minimal model; and
2. X is *uniruled* (i.e., X is covered by rational curves), in which case the MMP ends with a Mori fiber space.

The *abundance conjecture* predicts that X is uniruled if and only if $\kappa(X) = -\infty$. What is currently known is that the condition that X is uniruled is equivalent to the condition that K_X is not *pseudo-effective*, i.e., K_X is not a limit of classes of effective divisors (with rational coefficients). We refer to [KM98] for an introduction to the MMP.

Canonical Class of Foliations

Similarly, there are relevant properties of holomorphic foliations $\mathcal{F}$ that depend only on the birational equivalence class of $\mathcal{F}$. The algebraic rank is an example of birational invariant for foliations. In recent years, techniques from birational geometry and the MMP have been successfully applied to the study of global properties of holomorphic foliations. This led, for instance, to a birational classification of codimension one foliations on surfaces similar to the Enriques-Kodaira classification, which we review below.

Starting with the tangent sheaf $T_{\mathcal{F}}$ of a foliation $\mathcal{F}$ on a complex projective manifold X, we can define its canonical class $K_{\mathcal{F}} \in \mathrm{H}^2(X, \mathbb{Z})$ and Kodaira dimension $\kappa(\mathcal{F}) \in \{-\infty, 0, \dots, \dim(X)\}$. This is analogous to the definition of the canonical class and Kodaira dimension of X starting with T_X. The Kodaira dimension of codimension one foliations on projective surfaces was first considered in [Men00]. Under restrictions on the singularities of $\mathcal{F}$, the Kodaira dimension $\kappa(\mathcal{F})$ is a birational invariant for foliations.

Motivated by the special role of Fano varieties in birational geometry, we introduce the following class of foliations.

Definition 3. A foliation $\mathcal{F}$ on a complex projective manifold is called a *Fano foliation* if $-K_{\mathcal{F}} > 0$ (i.e., $-K_{\mathcal{F}}$ is ample).

Example 1. Recall that $\mathrm{H}^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$, where we choose the positive generator to be the cohomology class of a hyperplane section. Under this identification, $K_{\mathbb{CP}^n} = -(n+1) \in \mathbb{Z}$. Let $\mathcal{F}$ be a codimension one foliation of degree d on $\mathbb{CP}^n$. An easy computation shows that $K_{\mathcal{F}} = d - n + 1 \in \mathbb{Z}$. In particular, Fano foliations on $\mathbb{CP}^n$ are those with small degree, $d < n - 1$.

In a series of papers started in [AD13], the first named author and S. Druel have developed the theory of Fano foliations. They showed that the positivity of $-K_{\mathcal{F}}$ has an effect on the algebraicity of the leaves of $\mathcal{F}$. In order to measure the positivity of $-K_{\mathcal{F}}$, define the index $\iota(\mathcal{F})$ of a Fano foliation $\mathcal{F}$ on a complex projective manifold X to be the largest integer dividing $-K_{\mathcal{F}}$ in $H^2(X, \mathbb{Z})$. This is analogous to the index $\iota(X)$ of a Fano manifold X. The following result is a lower bound for the algebraic dimension $r_a(\mathcal{F})$ of Fano foliations in terms of the index, and a classification of the cases when this bound is attained.

Theorem 4 ([AD19, Corollary 1.6.]). *Let $\mathcal{F}$ be a Fano foliation of index $\iota(\mathcal{F})$ on a complex projective manifold X. Then $r_a(\mathcal{F}) \geq \iota(\mathcal{F})$, and equality holds if and only if $X \cong \mathbb{CP}^n$ and $\mathcal{F}$ is the pullback under a linear projection of a purely transcendental foliation on $\mathbb{CP}^{n-r_a(\mathcal{F})}$ with zero canonical class.*

Fano manifolds with large index have been classified. By a theorem of Kobayachi and Ochiai, the index $\iota(X)$ of a Fano manifold X satisfies $\iota(X) \leq \dim(X)+1$, equality holds if and only if X is a projective space, and $\iota(X) = \dim(X)$ if and only if X is a quadric hypersurface. Fano manifolds with $\iota(X) = \dim(X) - 1$ are called *del Pezzo manifolds*, and were classified by Fujita. Those with $\iota(X) = \dim(X) - 2$ were later classified by Mukai. We refer to [AC13] for a survey on the classification of Fano manifolds with large index, with many references. As a corollary of Theorem 4 above, we have a version of the Kobayachi-Ochiai's theorem for foliations: the index $\iota(\mathcal{F})$ of a Fano foliation $\mathcal{F}$ is bounded by the dimension, $\iota(\mathcal{F}) \leq r$. Moreover, equality holds if and only if $X \cong \mathbb{CP}^n$ and $\mathcal{F}$ is induced by a linear projection $\varphi : \mathbb{CP}^n \dashrightarrow \mathbb{CP}^{n-r}$. This means that, away from the center of the projection, where φ is not defined, the leaves of $\mathcal{F}$ are the fibers of φ. These foliations are precisely the ones of degree 0. The next cases have also been investigated. In analogy with the theory of Fano manifolds, Fano foliations $\mathcal{F}$ with index $\iota(\mathcal{F}) = r - 1$ are called *del Pezzo foliations*. Under restrictions on the singularities, there is a classification of del Pezzo foliations of rank $r \geq 3$, parallel to Fujita's classification of del Pezzo manifolds. We refer to [Fig22] for an account on the classification of del Pezzo foliations.

The birational classification of codimension one foliations on surfaces can be summarized as follows (see [Bru15] and [McQ08]). First of all, by a theorem of Seidenberg [Sei68], after a sequence of blowups, it is always possible to reduce the singularities of a foliation $\mathcal{F}$ on a surface. A point $p \in \text{Sing}(\mathcal{F})$ is a *reduced singularity* if, locally around p, $\mathcal{F}$ is given by a vector field whose linear part has eigenvalues λ_1 and λ_2, with $\lambda_2 \neq 0$, and $\lambda_1/\lambda_2 \notin \mathbb{Q}_+$. This condition implies that there are only finitely many separatices through p. For foliations with reduced singularities, the Kodaira dimension $\kappa(\mathcal{F})$ is a birational invariant. We have the following birational classification of foliations with reduced singularities and Kodaira dimension $\kappa(\mathcal{F}) < 2$.

- $\kappa(\mathcal{F}) = 1$. Then $\mathcal{F}$ is birational to one of the following.
    - *Riccati foliation*: there is a rational fibration $S \to C$ with general fiber everywhere transverse to $\mathcal{F}$.
    - *Turbulent foliation*: there is an elliptic fibration $S \to C$ with general fiber everywhere transverse to $\mathcal{F}$.
    - Foliations induced by non-isotrivial elliptic fibrations.
    - Foliations induced by isotrivial fibrations of genus ≥ 2.
- $\kappa(\mathcal{F}) = 0$. After a ramified covering, $\mathcal{F}$ is birational to a foliation induced by a global holomorphic vector field with isolated zeroes.
- $\kappa(\mathcal{F}) = -\infty$. Either $\mathcal{F}$ is birational to a foliation on a ruled surface induced by a fibration by rational curves, or to a *Hilbert modular foliation*.

A Hilbert modular surface S is the minimal desingularization of the Baily-Borel compactification of the quotient of $\mathbb{H}^2 = \mathbb{H} \times \mathbb{H}$ by an irreducible lattice in $\text{PSL}(2, \mathbb{R})^2$. The two projections $\mathbb{H}^2 \to \mathbb{H}$ induce two foliations on S with dense leaves, called *Hilbert modular foliations*. Hilbert modular foliations provide examples of a completely new behavior which is not seen in classical geometry. Namely, its canonical class $K_{\mathcal{F}}$ is pseudo-effective, while $\kappa(\mathcal{F}) = -\infty$. On the other hand, it is a remarkable theorem due to Miyaoka that foliations with non-pseudo-effective canonical class are uniruled (i.e., their leaves are covered by rational curves).

Much progress has been made toward the birational classification of foliations on higher dimensional varieties. Reduction theorems for foliations on threefolds have been established by Cano in codimension one, and by McQuillan and Panazzolo in dimension one. Building on works of McQuillan, versions of the MMP for foliations on threefolds were recently established by Cascini and Spicer. We refer to [CS21] for details and references about the MMP for codimension one foliations, and [CS20] for foliations of dimension one on threefolds.

Regular Foliations

We end these notes by discussing the classification problem for *regular foliations* on complex projective manifolds. These are holomorphic foliations $\mathcal{F}$ with $\text{Sing}(\mathcal{F}) = \emptyset$. In view of Theorem 1, it is natural to ask for a characterization of projective manifolds that admit regular foliations. For surfaces, the first result in this direction is the following classification of regular foliations on minimal ruled surfaces by Gomez-Mont.

Theorem 5 ([GM89]). *Let $\pi : S \to C$ be a $\mathbb{CP}^1$-fibration, with $g(C) \neq 1$, and let $\mathcal{F}$ be a regular foliation of rank 1 on S. Then*

- *either $\mathcal{F}$ is induced by the fibration $\pi : S \to C$; or*
- *$\mathcal{F}$ is transverse to the fibration $\pi : S \to C$, and constructed by suspension of a representation $\rho : \pi_1(C) \to \text{PSL}(2, \mathbb{C})$.*

Brunella extended this result, and completely classified regular foliations on complex projective surfaces S with Kodaira dimension $\kappa(S) < 2$. It is still not known whether there exist purely transcendental regular foliations on surfaces of general type. The following is a corollary of this classification.

Corollary 1. *Let $\mathcal{F}$ be a regular foliation of rank 1 on a smooth projective rational surface S. Then $\mathcal{F}$ is induced by a $\mathbb{CP}^1$-fibration $S \to \mathbb{CP}^1$.*

In higher dimensions, there is no classification of regular foliations. However, Touzet has conjectured the following generalization of Corollary 1 (see [Dru17, Conjecture 1.2]). From the point of view of birational geometry, a natural higher dimensional analog of the class of rational surfaces is that of *rationally connected varieties*. A complex projective manifold X is *rationally connected* if any two points of X can be connected by a rational curve. We refer to [Kol01] for an introductory discussion on this topic.

Conjecture 1 (Touzet). *Let $\mathcal{F}$ be a regular foliation on a rationally connected projective manifold X. Then $\mathcal{F}$ is induced by a fibration $X \to Y$.*

The class of rationally connected varieties includes that of Fano manifolds, for which the conjecture was verified in [Dru17]. While Touzet's conjecture is open already in dimension 3, for a regular codimension one foliation $\mathcal{F}$ on a projective threefold X, the MMP can be used to greatly reduce the problem. After a sequence of smooth blowdowns centered at smooth curves,

$$X \to X_1 \to X_2 \to \cdots \to X_n,$$

one reaches a smooth threefold X_n, together with a regular codimension one foliation $\mathcal{F}_n$ on X_n, such that either $K_{\mathcal{F}_n}$ is nef, or $X_n \to Y_n$ is a Mori fiber space with fibers tangent to $\mathcal{F}_n$. In the latter case, the second named author showed in [Fig21] that $\mathcal{F}_n$ is induced by a fibration by rational surfaces, and hence so is $\mathcal{F}$. This proves Touzet's conjecture for codimension one foliations on threefolds in some special cases, and reduces the problem to understanding regular foliations with nef canonical class.

ACKNOWLEDGMENT. We thank Alex Massarenti, Jorge Vitório Pereira, and the referee for their valuable comments and suggestions. We thank CAPES/COFECUB, CNPq, and FAPERJ for the financial support.

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