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FORBIDDEN-MINOR CHARACTERIZATION FOR THE CLASS OF GRAPHIC ELEMENT SPLITTING MATROIDS

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Abstract

This paper is based on the element splitting operation for binary matroids that was introduced by Azadi as a natural generalization of the corresponding operation in graphs. In this paper, we consider the problem of determining precisely which graphic matroids M have the property that the element splitting operation, by every pair of elements on M yields a graphic matroid. This problem is solved by proving that there is exactly one minor-minimal matroid that does not have this property.

Keywords: binary matroid, graphic matroid, minor, splitting operation, element splitting operation.

2000 Mathematics Subject Classification: 05B35.

1. Introduction

Let $M(G)$ and $M^*(G)$ denote the circuit matroid and the cocircuit matriod, respectively of a graph G. A matroid is *Eulerian* if its ground set can be expressed as a union of disjoint circuits of the matroid (see [14]). A matroid is *bipartite* if every circuit of it has an even number of elements. Welsh [14] proved that a binary matroid is Eulerian if and only if its dual is bipartite. As the matroids F_7 and $M(K_5)$ are Eulerian, their dual matroids F_7^* and $M^*(K_5)$ are bipartite. It is easy to see that a binary matroid M is Eulerian iff the sum of column vectors of A is zero where A is a matrix over $GF(2)$ that represents M. For undefined notation and terminology in graphs and matroids, we refer [6] and [8].

Fleischner [3] defined the *splitting operation* for a graph with respect to a pair of adjacent edges as follows: Let G be a connected graph and v be a vertex of degree at least three in G. If $x = uv$ and $y = wv$ are two edges incident at v, then splitting away the pair x, y from v results in a new graph $G_{x,y}$ obtained from G by deleting the edges x and y, and adding a new vertex $v_{x,y}$ adjacent to u and w. The transition from G to $G_{x,y}$ is called the splitting operation on G. For practical purposes, we denote the new edges $v_{x,y}u$ and $v_{x,y}w$ in $G_{x,y}$ by x and y, respectively (See Figure 1). Fleischner [3] characterized Eulerian graphs and developed an algorithm to find all distinct Eulerian trails in an Eulerian graph using the splitting operation.

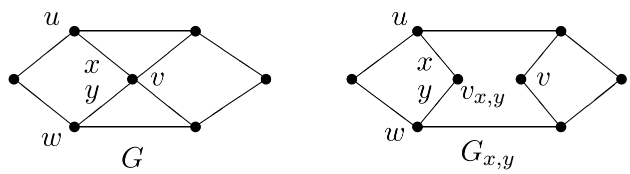

Figure 1

In a similar way, Tutte [13] specified the *point splitting operation* for graphs as follows: Let G be a graph and v be a vertex of degree at least 4 in G. Let H be the graph obtained from G by replacing v by two adjacent vertices v_1, v_2 such that each point formerly joined to v is joined to exactly one of v_1 and v_2 so that in H, deg $v_1 \geq 3$ and deg $v_2 \geq 3$. We say that H arises from G by point-splitting operation. Tutte [13] characterized 3-connected graphs using this operation. Later on, Slater [12] classified 4-connected graphs using n-point splitting operation which is a natural generalization of the point splitting operation.

Azadi [1] defined an operation which, in a sense, combines the splitting operation and the point splitting operation as follows: Let v be a vertex of G and let x, y be distinct edges of G incident at v. Let $G'_{x,y}$ be the graph obtained from G such that $G'_{x,y} = G_{x,y} + v_{x,y}v$, where $G_{x,y}$ is the graph obtained from G by splitting operation with respect to the edges x and y.

Then we say that $G'_{x,y}$ is obtained from G by the *element splitting operation with respect to the pair* of edges x and y (see Figure 2).

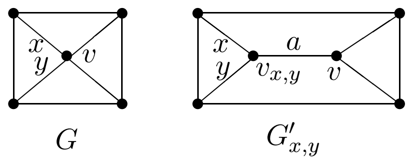

Figure 2

Raghunathan *et al.* [7] extended the definition of Fleischner's splitting operation to binary matroids as follows: Let A be a matrix over $GF(2)$ that represents the matroid M. Consider distinct elements x and y of M. Let $A_{x,y}$ be the matrix that is obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to x and y where it takes the value 1. Suppose $M_{x,y}$ is the matroid represented by the matrix $A_{x,y}$. Then $M_{x,y}$ is said to be obtained from M by *splitting* away the pair x, y. Various properties concerning the splitting matroid have been studied in [2, 7, 9, 10, 11].

Azadi [1] further extended the operation of element splitting with respect to the pair of edges in graphs to binary matroids as follows: Let A be a matrix over $GF(2)$ that represents the matroid M. Suppose that x and y are distinct elements of M. Let $A'_{x,y}$ be the matrix that is obtained by adjoining an extra row to A with this row being zero everywhere except in the columns corresponding to x and y where it takes the value 1 and then adjoining an extra column (corresponding to a) with this column being zero everywhere except in the last row where it takes the value 1. Suppose $M'_{x,y}$ is the matroid represented by the matrix $A'_{x,y}$. Then $M'_{x,y}$ is said to be obtained from M by *element splitting* the pair of elements x and y.

Alternatively, the element splitting operation can be defined in terms of circuits of binary matroids [1] as follows:

Let $M = (S, \mathcal{C})$ be a binary matroid, $\{x, y\} \subseteq S$, and $a \notin S$. Let
$\mathcal{C}_0 = \{C \in \mathcal{C} : x, y \in C \text{ or } x, y \notin C\}$,
$\mathcal{C}_1$ = set of minimal members of $\{C_1 \cup C_2 : C_1, C_2 \in \mathcal{C}, C_1 \cap C_2 = \phi$ and $x \in C_1, y \in C_2$ such that $C_1 \cup C_2$ does not contain any member of $\mathcal{C}_0\}$, and
$\mathcal{C}_2 = \{C \cup \{a\} : C \in \mathcal{C}$ and contains exactly one of x and $y\}$.
Let $\mathcal{C}' = \mathcal{C}_0 \cup \mathcal{C}_1 \cup \mathcal{C}_2$. Then $M'_{x,y} = (S \cup \{a\}, \mathcal{C}')$.

If x and y are non-adjacent edges of a graph G, then $M(G)_{x,y}$ may not be graphic. Shikare and Waphare [11] characterized graphic matroids whose splitting matroids are also graphic in the following theorem.

Theorem 1.1 [11]. *The splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if the circuit matroid of the corresponding graph has no minor isomorphic to the circuit matroid of any of the following four graphs.*

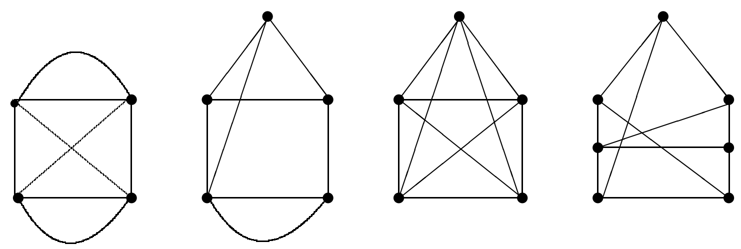

Figure 3 ■

The element splitting operation on a graphic matroid may not yield a graphic matroid. In this paper, we obtain the forbidden-minor characterization for graphic matroids whose element splitting matroid is graphic. The main result in this paper is the following theorem.

Theorem 1.2. *The element splitting operation, by any pair of elements, on a graphic matroid yields a graphic matroid if and only if it has no minor isomorphic to* $M(K_4)$*, where* K_4 *is the complete graph on* 4 *vertices.*

2. The Element Splitting Operation and its Properties

In this section we provide necessary lemmas. We assume that M is a binary matroid and x, y are distinct elements of M.

Lemma 2.1. *Let* x *and* y *be elements of a binary matroid* M *and let* $r(M)$ *denote the rank of* M*. Then, using the notations introduced in Section* 1,

(i) $M_{x,y} = M'_{x,y} \setminus \{a\}$;

(ii) $M = M'_{x,y}/\{a\}$;

(iii) $r(M'_{x,y}) = r(M) + 1$;

(iv) *every cocircuit of* M *is a cocircuit of the matroid* $M'_{x,y}$;

(v) *if* $\{x, y\}$ *is a cocircuit of* M *then* $\{a\}$ *and* $\{x, y\}$ *are cocircuits of* $M'_{x,y}$;

(vi) *if* $\{x, y\}$ *does not contain a cocircuit, then* $\{x, y, a\}$ *is a cocircuit of* $M'_{x,y}$;

(vii) $M'_{x,y} \setminus x/y \cong M \setminus x$;

(viii) *if* M *is graphic and* x, y *are adjacent edges in a corresponding graph, then* $M'_{x,y}$ *is graphic;*

(ix) $M'_{x,y}$ *is not eulerian.*

Proof. (i), (ii), (iii), (v), (vi), (vii) and (viii) are straightforward. The proof of (iv) follows from Lemma 2.4.1 of [4]. If $A'_{x,y}$ represents the matroid $M'_{x,y}$, then the number of one's in the last row of $A'_{x,y}$ is odd. Hence $M'_{x,y}$ is not eulerian. This proves (ix). ∎

The following result is well known.

Lemma 2.2 [6]. *A binary matroid is graphic if and only if it has no minor isomorphic to* $F_7, F_7^*, M^*(K_5)$, *or* $M^*(K_{3,3})$. ∎

Notation. For convenience, let $\mathcal{F} = \{F_7,\ F_7^*,\ M^*(K_5),\ M^*(K_{3,3})\}$.

Lemma 2.3. *Let* M *be a graphic matroid and let* $x, y \in E(M)$ *such that* $M'_{x,y}$ *is not graphic. Then there is a minor* N *of* M *such that no two elements of* N *are in series and* $N'_{x,y} \setminus \{a\}/\{x\} \cong F$ *or* $N'_{x,y} \setminus \{a\}/\{x, y\} \cong F$ *or* $N'_{x,y} \cong F$ *or* $N'_{x,y}/\{x\} \cong F$ *or* $N'_{x,y}/\{y\} \cong F$ *or* $N'_{x,y}/\{x, y\} \cong F$ *for some* $F \in \mathcal{F}$.

Proof. Since $M'_{x,y}$ is not graphic, $M'_{x,y} \setminus T_1/T_2 \cong F$ for some $T_1, T_2 \subseteq E(M'_{x,y})$. Let $T'_i = T_i - \{a, x, y\}$ for $i = 1, 2$. Then $T'_i \subseteq E(M)$ for each i. Let $N = M \setminus T'_1/T'_2$. Then $N'_{x,y} = M'_{x,y} \setminus T'_1/T'_2$. Let $T''_i = T_i - T'_i$ for $i = 1, 2$. Then $N'_{x,y} \setminus T''_1/T''_2 \cong F$. If $a \in T''_2$, then F is a minor of $M'_{x,y}/a$ and hence, by Lemma 2.1(i), F is a minor of M, which is a contradiction. Suppose $a \in T''_1$. By Lemma 2.1(i), $M_{x,y} = M'_{x,y} \setminus a$. Hence F is a minor of $M_{x,y}$. It follows from Theorem 2.3 of [11] that N does not contain a 2-cocircuit and further, $N_{x,y}/x \cong F$ or $N_{x,y}/\{x, y\} \cong F$. This implies that $N'_{x,y} \setminus \{a\}/x \cong F$ or $N'_{x,y} \setminus \{a\}/\{x, y\} \cong F$. Suppose that $a \notin T''_1 \cup T''_2$. Hence $a \notin T_1 \cup T_2$. If $T''_1 \cup T''_2 = \phi$, then $N'_{x,y} \cong F$. If $T''_2 = \phi$, then $N_{x,y} \setminus x \cong F$ or $N_{x,y} \setminus y \cong F$ or $N'_{x,y} \setminus \{x, y\} \cong F$. In the first case, a forms a 2-cocircuit with x or y whichever is remained, and in later case, a is a coloop. It is a contradiction.

Hence $T_2'' \neq \phi$. If $T_1'' \neq \phi$ then, by Lemma 2.1(vi), F is minor of M, which is a contradiction. Hence $T_1'' = \phi$. Hence $N'_{x,y}/x \cong F$ or $N'_{x,y}/y \cong F$ or $N'_{x,y}/\{x,y\} \cong F$.

Assume that N contains a 2-cocircuit Q. By Lemma 2.1(*iv*), Q is 2-cocircuit in $N'_{x,y}$. Since F is 3-connected, it does not contain a 2-cocircuit. It follows that $N'_{x,y}$ is not isomorphic to F. Hence $N'_{x,y} \setminus \{a\}/x \cong F$ or $N'_{x,y} \setminus \{a\}/\{x,y\} \cong F$ or $N'_{x,y}/\{x\} \cong F$ or $N'_{x,y}/\{y\} \cong F$ or $N'_{x,y}/\{x,y\} \cong F$. If $Q \cap \{x,y\} = \phi$, then it is retained in all these cases and thus F has a 2-cocircuit, which is a contradiction. If $Q = \{x,y\}$, a contradiction follows from Lemma 2.1(v). Hence Q contains exactly one of x,y. Suppose that $x \in Q$. Then $N'_{x,y}/y \ncong F$. Let x_1 be the other element of Q. Let $L = N/x_1$. Then L is a minor of M in which no pair of elements is in series. Further, $L'_{x,y} = N'_{x,y}/x_1 \cong N'_{x,y}/x$. Thus we have $L'_{x,y} \setminus \{a\} \cong F$ or $L'_{x,y} \setminus \{a\}/y \cong F$ or $L'_{x,y} \cong F$ or $L'_{x,y}/y \cong F$. Since $L_{x,y} \cong L'_{x,y} \setminus \{a\}$, and x,y are in series in $L_{x,y}$, it follows that $L'_{x,y} \setminus \{a\} \ncong F$ and also $L'_{x,y} \setminus \{a\}/y \cong L'_{x,y} \setminus \{a\}/x$. If $y \in Q$, then $N'_{x,y}/x \ncong F$. Also, $L'_{x,y} \cong N'_{x,y}/y$. In this case we get $L'_{x,y} \setminus \{a\}/x \cong F$ or $L'_{x,y} \cong F$ or $L'_{x,y}/x \cong F$. ■

Definition 2.4. Let M be a graphic matroid in which no two elements are in series and let $F \in \mathcal{F}$. We say that M is minimal with respect to F if there exist two elements x and y of M such that $M'_{x,y} \setminus \{a\}/\{x\} \cong F$ or $M'_{x,y} \setminus \{a\}/\{x,y\} \cong F$ or $M'_{x,y} \cong F$ or $M'_{x,y}/\{x\} \cong F$ or $M'_{x,y}/\{x,y\} \cong F$.

Corollary 2.5. *Let M be a graphic matroid. For any $x,y \in E(M)$, the matroid $M'_{x,y}$ is graphic if and only if M has no minor isomorphic to a minimal matroid with respect to any $F \in \mathcal{F}$.*

Proof. If $M'_{x,y}$ is not graphic for some x,y, then by Lemma 2.3, M has a minor N in which no two elements are in series and $N'_{x,y} \setminus \{a\}/\{x\} \cong F$ or $N'_{x,y} \setminus \{a\}/\{x,y\} \cong F$ or $N'_{x,y} \cong F$ or $N'_{x,y}/\{x\} \cong F$ or $N'_{x,y}/\{y\} \cong F$ or $N'_{x,y}/\{x,y\} \cong F$ for some $F \in \mathcal{F}$. If $N'_{x,y}/y \cong F$ but $N'_{x,y}/x \ncong F$, then interchange roles of x and y. Conversely, suppose that M has a minor N isomorphic to a minimal matroid with respect to some $F \in \mathcal{F}$. Then $N'_{x,y} \setminus \{a\}$ or $N'_{x,y}/\{x\}$ or $N'_{x,y}/\{x,y\}$ or $N'_{x,y} \cong F$, for some $x,y \in E(M)$. Then $M'_{x,y}$ has a minor isomorphic to F and hence it is not graphic. ■

Lemma 2.6. *Let M be a graphic matroid corresponding to a graph G. If M is minimal with respect to some $F \in \mathcal{F}$, then*

(i) *M has neither loops nor coloops;*

(ii) *x and y are non-adjacent edges of G and the minimum degree of G is at least* 3;

(iii) *x and y cannot be parallel in G*;

(iv) *every pair of parallel edges of G must contain either x or y*;

(v) *if $M'_{x,y}$ or $M'_{x,y}/\{x\} \cong F_7^*$ or $M^*(K_5)$, then G is simple;*

(vi) *if $M'_{x,y}/\{x\} \cong F_7$ or $M^*(K_{3,3})$, then G is simple or has exactly one pair of parallel edges and one of these two edges must be y, and further there is no* 3*-circuit in G containing both x and y*;

(vii) *if $M'_{x,y}/\{x,y\} \cong F$ then G is simple and there is no* 3*-circuit or* 4*-circuit in G containing both x and y.*

Proof. (i) On the contrary, suppose M has a loop, say z. If z is different from x and y, then it is a loop in $M'_{x,y}$ and hence in F, a contradiction. If z is one of the two elements x and y, say x, then $M'_{x,y} \setminus \{a\}/\{x\} \cong M \setminus \{x\}$ and $M'_{x,y} \setminus \{a\}/\{x,y\} \cong M \setminus \{x\}/\{y\}$. This implies that F is a minor of M, a contradiction. Also, $M'_{x,y}$ contains a 2-circuit, so it cannot be isomorphic to F. Further $M'_{x,y}/\{x\}$ and $M'_{x,y}/\{x,y\}$ contains a loop, a contradiction. Thus, M cannot have loops.

Suppose that M has a coloop, say w. If w is different from x and y then it is preserved in $M'_{x,y}$ and hence in F, a contradiction. If w is one of the two elements x and y, say x, then $\{y,a\}$ is a 2-cocircuit or $\{y\}$ is a coloop of $M'_{x,y}$. Now, in $M'_{x,y} \setminus \{a\}/\{x\}$, y becomes a coloop, a contradiction. Also, $M'_{x,y} \setminus \{a\}/\{x,y\} \cong M/\{x\} \setminus \{y\}$. This means that F is a minor of M, a contradiction. In $M'_{x,y}$, $\{x\}$ remains a coloop and hence $M'_{x,y}$ cannot be isomorphic to F. Moreover, in $M'_{x,y}/\{x\}$, $\{y,a\}$ remains a 2-cocircuit or $\{y\}$ remains a cocircuit and in F, a contradiction. Also, $M'_{x,y}/\{x,y\} \cong M'_{x,y} \setminus \{x\}/\{y\} \cong M \setminus \{x\}$, that is F is a minor of M, a contradiction. Hence M cannot have coloops.

(ii) Follows from Lemma 2.1(viii) and Lemma 2.3.

(iii) If x and y are parallel in G, then x and y remain parallel in $M'_{x,y}$. So, we get a loop in $M'_{x,y} \setminus \{a\}/\{x\}$, $M'_{x,y}/\{x\}$ and a 2-circuit in $M'_{x,y}$, a contradiction. Also, $M'_{x,y} \setminus \{a\}/\{x,y\} = M_{x,y}/\{x,y\} = M \setminus \{x,y\}$, a contradiction. Now, $M'_{x,y}/\{x,y\} = M'_{x,y}/y \setminus x \cong M \setminus x$, a contradiction to Lemma 2.1(vii). Hence these matroids are not isomorphic to F, a contradiction.

(iv) Suppose that the edges x_1 and x_2 are in a parallel class of G that does not contain x or y, then x_1 and x_2 remain in parallel in each of the matroids $M'_{x,y} \setminus \{a\}/\{x\}$, $M'_{x,y} \setminus \{a\}/\{x,y\}$, $M'_{x,y}$, $M'_{x,y}/\{x\}$ and $M'_{x,y}/\{x,y\}$, a contradiction. If x_1 and x_2 are in a parallel class containing x or y, then we get a loop in $M'_{x,y} \setminus \{a\}/\{x\}$, $M'_{x,y} \setminus \{a\}/\{x,y\}$, $M'_{x,y}/\{x\}$, $M'_{x,y}/\{x,y\}$ and a 2-circuit in $M'_{x,y}$. Hence these matroids are not isomorphic to F, a contradiction.

(v) As F_7^* and $M^*(K_5)$ are bipartite, if G contains a pair of parallel edges then by (iv) above, it must contain x or y. So, we get a 3-circuit in $M'_{x,y}$ containing a. Therefore $M'_{x,y} \ncong F_7^*$ or $M^*(K_5)$. Also, we get a 2-circuit in $M'_{x,y}/\{x\}$ and $M'_{x,y}/\{x,y\}$, a contradiction.

(vi) Suppose that G is not simple. Then by (iv) above, each pair of parallel edges must contain x or y. If $\{x, x_1\}$ is a 2-circuit for some edge x_1 of G, then $\{x, x_1, a\}$ is a 3-circuit in $M'_{x,y}$ and hence, $\{x_1, a\}$ is a 2-circuit in $M'_{x,y}/\{x\}$, a contradiction. Hence G has exactly one pair of parallel edges and one of these two edges must be y.

(vii) If G contains a pair of parallel edges, it must contain x or y, say x. Then $M'_{x,y}$ contains a 3-circuit containing x and a. Consequently, $M'_{x,y}/\{x,y\}$ contains a 2-circuit and hence it is in F, a contradiction. Now, if G contains 3 or a 4-circuit containing both x and y, then $M'_{x,y}/\{x,y\}$ contains a loop or 2-circuit respectively and hence it is in F, a contradiction. ■

3. The Element Splitting operation on Graphic Matroids

In this section, we obtain the minimal matroids corresponding to each of the four matroids $F_7, F_7^*, M^*(K_{3,3})$ and $M^*(K_5)$ and use them to give a proof of Theorem 1.2.

In the following lemma, we characterize minimal matroids corresponding to the matroid F_7.

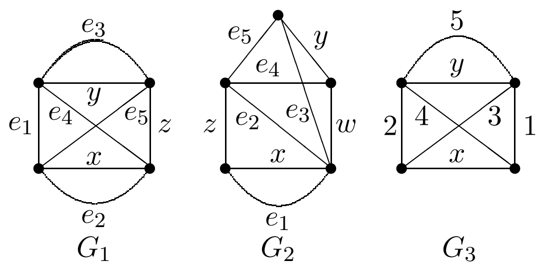

Figure 4

Lemma 3.1. *Let M be a graphic matroid. Then M is minimal with respect to the matroid F_7 if and only if M is isomorphic to one of the three circuit matroids $M(G_1)$, $M(G_2)$ and $M(G_3)$, where G_1, G_2 and G_3 are the graphs of Figure* 4.

Proof. Firstly, we consider the graph G_3 and prove that $M'(G_3)_{x,y}/\{x\} \cong F_7$.

Let matrices A and $A'_{x,y}$ represent the matroids $M(G_3)$ and $M'(G_3)_{x,y}$ respectively. Then

$$A = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & x & y \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

So, we have

$$A'_{x,y} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & x & y & a \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \end{bmatrix} \end{array}.$$

Therefore

$$A'_{x,y}/\{x\} = \begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & y & a \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 1 \end{bmatrix} \end{array}.$$

Hence $M'(G_3)_{x,y}/\{x\} \cong F_7$.

One can check similarly that $M'(G_1)_{x,y} \setminus \{a\}/\{x\} \cong F_7$; $M'(G_2)_{x,y} \setminus \{a\}/\{x,y\} \cong F_7$. Thus the matroids $M(G_1)$, $M(G_2)$ and $M(G_3)$ are minimal with respect to F_7.

Conversely, let M be a minimal matroid with respect to F_7. Let G be a graph corresponding to M. Let the edges x and y of G are such that $M'(G)_{x,y} \setminus \{a\}/\{x\} \cong F_7$ or $M'(G)_{x,y} \setminus \{a\}/\{x,y\} \cong F_7$ or $M'(G)_{x,y} \cong F_7$ or $M'(G)_{x,y}/\{x\} \cong F_7$ or $M'(G)_{x,y}/\{x,y\} \cong F_7$.

By Lemma 2.1(i), $M(G)_{x,y}/\{x\} \cong F_7$. If $M'(G)_{x,y} \setminus \{a\}/\{x\} \cong F_7$, then by Lemma 3.1 of [11], G is isomorphic to the graph G_1 of Figure 4. Similarly, if $M'(G)_{x,y} \setminus \{a\}/\{x,y\} \cong F_7$, then by Lemma 3.1 of [11], G is isomorphic to the graph G_2 of Figure 4. Further, $M'(G)_{x,y} \ncong F_7$ because $M'_{x,y}$ is not eulerian by Lemma 2.1(x).

Suppose that $M'(G)_{x,y}/\{x\} \cong F_7$. Since $r(F_7) = 3$, $r(M'(G)_{x,y}) = 4$. Further $|E(M'(G)_{x,y})| = 8$. Consequently, $r(M(G)) = 3$ and $|E(M(G))| = 7$. Thus, G is a graph with 4 vertices and 7 edges. This implies that G is non-simple. Also, by Lemma 2.6(vi), G has exactly one pair of parallel edges. Hence G can be obtained from a simple graph with 4 vertices and 6 edges by adding an edge in parallel. Since the complete graph K_4 is the only simple graph with 4 vertices and 6 edges (see [5]), G must be isomorphic to the graph G_3 of Figure 4.

Suppose that $M'(G)_{x,y}/\{x, y\} \cong F_7$. Then $r(M'(G)_{x,y}) = 5$ and $|E(M'(G)_{x,y})| = 9$. This implies that $r(M(G)) = 4$ and $|E(M(G))| = 8$. Thus, G is a graph with 5 vertices and 8 edges. Hence, by Lemma 2.6(ii), G has degree sequence (4,3,3,3,3). By Lemma 2.6(vii), G is simple and does not have a 3-circuit or a 4-circuit containing both x and y. There is only one simple graph with 5 vertices and 8 edges (see [5]) as shown in Figure 5. In this graph, any two edges are either in a 3-circuit or in a 4-circuit. Hence we discard this graph. ■

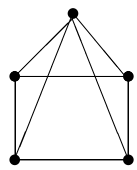

Figure 5

We characterize minimal matroids corresponding to the matroid F_7^* in the following lemma.

Lemma 3.2. *Let M be a graphic matroid. Then M is minimal with respect to the matroid F_7^* if and only if M is isomorphic to one of the two circuit matroids $M(G_4)$ and $M(G_5)$, where G_4 and G_5 are the graphs of Figure* 6.

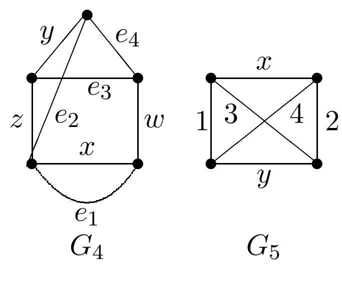

Figure 6

Proof. Observe that $M'(G_4)_{x,y} \setminus \{a\}/\{x\} \cong F_7^*$ and $M'(G_5)_{x,y} \cong F_7^*$. Therefore $M(G_4)$ and $M(G_5)$ are minimal with respect to F_7^*.

Conversely, let $M(G)$ be a minimal graph with respect to F_7^* and let x and y be edges of G such that $M'(G)_{x,y} \setminus \{a\}/\{x\} \cong F_7^*$ or $M'(G)_{x,y} \setminus \{a\}/\{x,y\} \cong F_7^*$ or $M'(G)_{x,y} \cong F_7^*$ or $M'(G)_{x,y}/\{x\} \cong F_7^*$ or $M'(G)_{x,y}/\{x,y\} \cong F_7^*$. If $M'(G)_{x,y} \setminus \{a\}/\{x\} \cong F_7^*$, then by Lemma 2.1(i), $M(G)_{x,y}/\{x\} \cong F_7^*$. Hence, by Lemma 3.2 of [11], G is isomorphic to the graph G_4 of Figure 6. Similarly, if $M'(G)_{x,y} \setminus \{a\}/\{x,y\} \cong F_7^*$, then $M(G)_{x,y}/\{x,y\} \cong F_7^*$. By Lemma 3.2 of [11], there is no minimal graphic matroid such that $M(G)_{x,y}/\{x,y\} \cong F_7^*$. In each of the remaining three cases, G is simple by Lemma 2.6(v).

Suppose that $M'(G)_{x,y} \cong F_7^*$. Then $r(M(G)) = r(M'(G)_{x,y}) - 1 = 3$. Further, $|E(M)| = 7$. Since $r(F_7^*) = 4$, $r(M(G)) = 3$. Consequently, G is a simple graph with 4 vertices and 6 edges. Hence G is isomorphic to K_4, which is the graph G_5 of Figure 6. Suppose that $M'(G)_{x,y}/\{x\} \cong F_7^*$. Then $r(M(G)) = 4$ and $|E(M(G))| = 7$. Hence G is a graph with 5 vertices and 7 edges and has a vertex of degree less than 3, a contradiction to Lemma 2.6(ii). Finally, if $M'(G)_{x,y}/\{x,y\} \cong F_7^*$, then G has 6 vertices and 8 edges and hence a vertex of degree less than 3, a contradiction. ■

The minimal matroids corresponding to the matroid $M^*(K_{3,3})$ are characterized as follows.

Lemma 3.3. *Let M be a graphic matroid. Then M is minimal with respect to the matroid $M^*(K_{3,3})$ if and only if M is isomorphic to one of the five circuit matroids $M(G_6), M(G_7), M(G_8), M(G_9)$ and $M(G_{10})$, where G_6, G_7, G_8, G_9 and G_{10} are the graphs of Figure* 7.

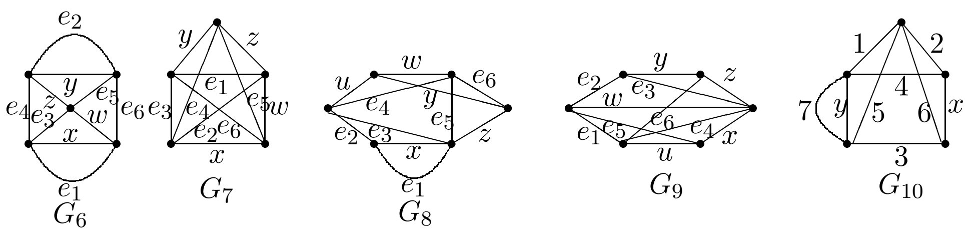

Figure 7

Proof. Observe that $M'(G_6)_{x,y} \setminus \{a\}/\{x\} \cong M^*(K_{3,3})$; $M'(G_7)_{x,y} \setminus \{a\}/\{x\} \cong M^*(K_{3,3})$; $M'(G_8)_{x,y} \setminus \{a\}/\{x,y\} \cong M^*(K_{3,3})$; $M'(G_9)_{x,y} \setminus \{a\}/\{x,y\} \cong$

$M^*(K_{3,3})$ and $M'(G_{10})_{x,y}/\{x\} \cong M^*(K_{3,3})$. This implies that $M(G_6)$, $M(G_7)$, $M(G_8)$, $M(G_9)$ and $M(G_{10})$ are minimal matroids with respect to the matroid $M^*(K_{3,3})$.

Conversely, let $M(G)$ be a minimal matroid with respect to $M^*(K_{3,3})$. Let x and y be edges of G such that $M'(G)_{x,y} \setminus \{a\}/\{x\} \cong M^*(K_{3,3})$ or $M'(G)_{x,y} \setminus \{a\}/\{x, y\} \cong M^*(K_{3,3})$ or $M'(G)_{x,y} \cong M^*(K_{3,3})$ or $M'(G)_{x,y}/\{x\} \cong M^*(K_{3,3})$ or $M'(G)_{x,y}/\{x, y\} \cong M^*(K_{3,3})$. If $M'(G)_{x,y} \setminus \{a\}/\{x\} \cong M^*(K_{3,3})$, then by Lemma 2.1(i), $M(G)_{x,y}/\{x\} \cong M^*(K_{3,3})$. Hence, by Lemma 3.3 of [11], G is isomorphic to one of the two graphs G_6 and G_7 of Figure 7. If $M'(G)_{x,y} \setminus \{a\}/\{x, y\} \cong M^*(K_{3,3})$, then by Lemma 2.1(i), $M(G)_{x,y}/\{x, y\} \cong M^*(K_{3,3})$. Hence by Lemma 3.3 of [11], G is isomorphic to one of the two graphs G_8 and G_9 of Figure 7. $M'_{x,y}$ is not eulerian, by Lemma 2.1(x). Therefore $M'(G)_{x,y} \ncong M^*(K_{3,3})$.

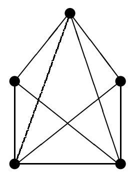

Figure 8

Suppose that $M'(G)_{x,y}/\{x\} \cong M^*(K_{3,3})$. Then $r(M(G)) = 4$ and $|E(M(G))| = 9$. Consequently, G is a graph with 5 vertices and 9 edges. Suppose that G is simple. By [5], any simple graph with 5 vertices and 9 edges is isomorphic to the graph of Figure 8. Suppose G is isomorphic to this graph. Then G has two edge-disjoint 3-cocircuits. Out of which, by Lemma 2.1(iv), at least one 3-cocircuit is preserved in $M'(G)_{x,y}/\{x\}$ and hence it is preserved in $M^*(K_{3,3})$, a contradiction. Thus G is non-simple. By Lemma 2.6(vi), G has exactly one pair of parallel edges. Since the degree of a vertex in G is at least 3, the degree sequence of G is (6,3,3,3,3), (5,4,3,3,3) or (4,4,4,3,3). Therefore G can be obtained from a simple graph with 5 vertices and 8 edges by adding an edge in parallel. There are in all 2 non-isomorphic simple graphs with 5 vertices and 8 edges (see [5]). So, there are in all 3 possibilities for G as shown in Figure 9.

If G is isomorphic to one of the two graphs (i) and (ii) of Figure 9, then it has two edge-disjoint 3-cocircuits, and hence at least one of them is survived in $M'(G)_{x,y}/\{x\} \cong M^*(K_{3,3})$, a contradiction. Hence G is isomorphic to third graph which is nothing but the graph G_{10} of Figure 7.

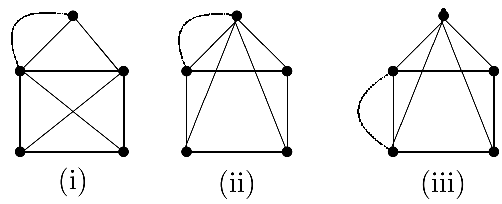

Figure 9

Finally, suppose that $M'(G)_{x,y}/\{x,y\} \cong M^*(K_{3,3})$. Since $r(M^*(K_{3,3})) = 4$, $r(M'(G)_{x,y}) = 6$. This shows that $r(M(G)) = 5$ and $|E(M(G))| = 10$. Consequently, G is a graph with 6 vertices and 10 edges with minimum degree at least 3. So, the degree sequence of G is (4,4,3,3,3,3) or (5,3,3,3,3,3). By Lemma 2.6(vii), G is simple. There are in all 4 non-isomorphic simple graphs with 6 vertices and 10 edges having the said degree sequences as shown in Figure 10 (see [5]). By Lemma 2.6(vii), G does not have a 3-circuit or a 4-circuit containing both x and y. As there are no 3-cocircuits and 5-cocircuits in $M^*(K_{3,3})$, every such cocircuit in G contains x or y. Suppose G is the graph (i) or graph (ii) of Figure 10. Then there is only one choice for x, y, as shown in the figure. For these choices $M'(G)_{x,y}/\{x,y\}$ is not Eulerian, a contradiction. If G is the graph (iii) or graph (iv) of Figure 10, then we get a 3-cocircuit or a 5-cocircuit in $M'(G)_{x,y}/\{x,y\}$ and hence it is in $M^*(K_{3,3})$, a contradiction. ■

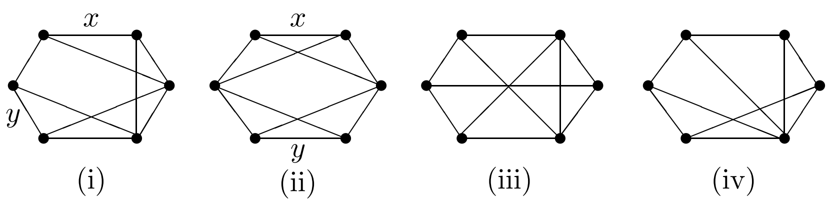

Figure 10

Finally, we characterize minimal matroids corresponding to the matroid $M^*(K_5)$ in the following lemma.

Lemma 3.4. *Let M be a graphic matroid. Then M is minimal with respect to the matroid $M^*(K_5)$ if and only if M is isomorphic to one of the three circuit matroids $M(G_{11})$, $M(G_{12})$ and $M(G_{13})$, where G_{11}, G_{12} and G_{13} are the graphs of Figure* 11.

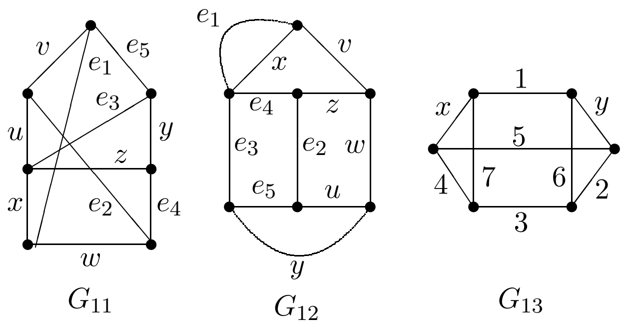

Figure 11

Proof. Observe that $M'(G_{11})_{x,y} \setminus \{a\}/\{x\} \cong M^*(K_5)$; $M'(G_{12})_{x,y} \setminus \{a\}/\{x\} \cong M^*(K_5)$ and $M'(G_{13})_{x,y} \cong M^*(K_5)$. Therefore $M(G_{11})$, $M(G_{12})$ and $M(G_{13})$ are minimal matroids with respect to the matroid $M^*(K_5)$.

Conversely, let $M(G)$ be a minimal matroid with respect to $M^*(K_5)$ and let x and y be edges of G such that $M'(G)_{x,y} \setminus \{a\}/\{x\} \cong M^*(K_5)$ or $M'(G)_{x,y} \setminus \{a\}/\{x, y\} \cong M^*(K_5)$ or $M'(G)_{x,y} \cong M^*(K_5)$ or $M'(G)_{x,y}/\{x\} \cong M^*(K_5)$ or $M'(G)_{x,y}/\{x, y\} \cong M^*(K_5)$. If $M'(G)_{x,y} \setminus \{a\}/\{x\} \cong M^*(K_5)$, then by Lemma 2.1(i), $M(G)_{x,y}/\{x\} \cong M^*(K_5)$. Therefore, by Lemma 3.4 of [11], G is isomorphic to one of the two graphs G_{11} and G_{12} of Figure 11. If $M'(G)_{x,y} \setminus \{a\}/\{x, y\} \cong M^*(K_5)$, then $M(G)_{x,y}/\{x, y\} \cong M^*(K_5)$. By Lemma 3.4 of [11], there is no minimal graphic matroid $M(G)$ such that $M(G)_{x,y}/\{x, y\} \cong M^*(K_5)$. By Lemma 2.6(v), G is simple in the remaining three cases. Suppose that $M'(G)_{x,y} \cong M^*(K_5)$. Then $r(M(G)) = 5$ and $|E(M(G))| = 9$. Hence, G is a graph with 6 vertices and 9 edges having degree sequence (3,3,3,3,3,3). There are only two such non-isomorphic simple graphs, (see [5]) as shown in Figure 12. In graph (i) of Figure 12,

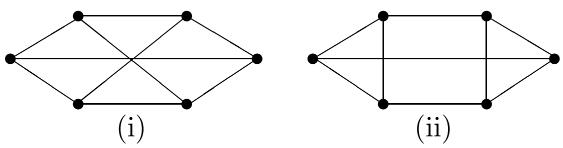

Figure 12

for every choice of non-adjacent edges x, y, there is a 4-circuit containing exactly one of x and y. Such circuit becomes a 5-circuit in $M'_{x,y}$, a contradiction. Hence, the circuit matroid of this graph is not minimal with respect to $M^*(K_5)$. Hence G is isomorphic to graph (ii) of Figure 12 which is in fact the graph G_{13} of Figure 11.

Suppose that $M'(G)_{x,y}/\{x\} \cong M^*(K_5)$. Then G has 7 vertices and 10 edges. Hence G has a vertex of degree less than 3, a contradiction. Suppose that $M'(G)_{x,y}/\{x, y\} \cong M^*(K_5)$. Then G is a graph with 8 vertices and 11 edges. Hence G has a vertex of degree less than 3, a contradiction. ∎

Now, we prove Theorem 1.2.

***Proof of Theorem* 1.2.** Let M be a graphic matroid and let G be a graph such that $M = M(G)$. On combining Corollary 2.5 and Lemmas 3.1, 3.2, 3.3 and 3.4, it follows that $M'(G)_{x,y}$ is graphic for every pair $\{x, y\}$ of edges of G if and only if $M(G)$ has no minor isomorphic to any of the matroids $M(G_i)$, $i = 1, 2, \ldots, 13$, where the graphs G_i are as shown in Figures 4, 6, 7 and 11. However, we have $M(G_5) \cong M(G_1) \setminus \{e_2, e_3\} \cong M(G_2)/\{z\} \setminus \{e_1, e_2\} \cong M(G_3) \setminus \{5\} \cong M(G_4)/\{w\} \setminus \{e_1\} \cong M(G_6)/\{e_6\} \setminus \{w, e_1, e_2\} \cong M(G_7)/\{z\} \setminus \{y, e_4, e_5\} \cong M(G_8)/\{z, e_2\} \setminus \{e_1, e_3, e_5\} \cong M(G_9)/\{x, e_1\} \setminus \{w, e_4, e_5\} \cong M(G_{10})/\{3\} \setminus \{5, 7\} \cong M(G_{11})/\{x, e_4, e_5\} \setminus \{w, e_1\} \cong M(G_{12})/\{v, z, e_3\} \setminus \{x, e_1\} \cong M(G_{13})/\{1, 5\} \setminus \{x\}$. Thus, $M'(G)_{x,y}$ is graphic if and only if $M(G)$ has no minor isomorphic to the matroid $M(G_5)$. Observe that the graph G_5 is isomorphic to the complete graph K_4. This completes the proof of the theorem. ∎

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