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"FORBIDDEN" PLANES FOR RAYLEIGH WAVES

By

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Abstract. Existence of "forbidden" planes, on which Rayleigh waves cannot propagate, is discussed. A mathematical model for anisotropic materials possessing "forbidden" planes is constructed. An example of transversely isotropic material having "forbidden" planes is presented.

1. Introduction. It has been known, beginning from Rayleigh's pioneering work [1], that elastic surface waves "play an important part in earthquakes, and in the collision of elastic solids", since amplitudes of these waves exponentially decay with depth, and most of the associated energy is concentrated in a relatively thin layer beneath the free surface. The latter circumstance, as was pointed out by Rayleigh, allows surface waves to propagate along much greater distances and without considerable attenuation in comparison with bulk waves.

An interesting consequence, flowing out from the theory of Rayleigh waves propagating in isotropic media, is that for any admissible values of the elastic parameters, which satisfy the positive-definite condition for the elasticity tensor, and for any direction, Rayleigh waves can propagate. It should be mentioned that the original Rayleigh theory describes propagation of the surface waves on the free surface of the isotropic media only, while adequate theories capable of analysis of the surface waves in anisotropic media were developed much later. Thus, taking into account isotropic materials, "forbidden" directions along which Rayleigh waves cannot propagate do not exist.

Of course, search of anisotropic materials, which cannot transmit surface wave energy along some specific directions, has played an important role in both theoretical and experimental analyses on surface waves. Had such materials and corresponding directions existed, they could have led to the development of a kind of reflecting barrier on the path of surface waves.

One of the first theoretical works on propagation of Rayleigh waves in anisotropic media is due to Stoneley [2], where Rayleigh's method is extrapolated to the analysis of surface waves propagating on the basal plane and in the direction of elastic symmetry of cubic crystals. Unfortunately, an oversight of the whole class of decaying and

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oscillating-with-depth solutions led Stoneley to the erroneous conclusion that a great variety of cubic crystals possess "forbidden" directions. Later on, the amended Stoneley's approach was extrapolated to analysis of surface waves propagating on the basal planes of orthorhombic and tetragonal crystals [3, 4] in the directions of high symmetries. More general anisotropic classes and more general directions of propagation were analysed by a numerical method [5]–[7], based on the three-dimensional complex formalism. In the course of the studies [3]–[7], an assumption, that "forbidden" directions for Rayleigh waves do not exist, was made.

The final solution for the problem of "forbidden" directions for Rayleigh waves was obtained in a course of comprehensive theoretical analyses [8]–[15], based on the Stroh sextic formalism [16]. Theorems on uniqueness and existence for Rayleigh waves proved in [8]–[15], along with some numerical results [17, 18] based on the sextic formalism, do not allow "forbidden" directions to appear, regardless of elastic anisotropy.

However, as will be shown further, not only separate "forbidden" directions, but the whole "forbidden" planes, on which Rayleigh waves cannot propagate at any direction, exist. Moreover, "forbidden" planes coincide with the basal planes of the specific transversely isotropic media; and, what can be highly important for practical applications, such transversely isotropic media only slightly differ from the isotropic ones.

2. Basic notation. Equations of motion for an anisotropic elastic medium can be written in the form

$$\mathbf{A}(\partial_x, \partial_t)\mathbf{u} \equiv \operatorname{div}_x \mathbf{C} \cdot \nabla_x \mathbf{u} - \rho\ddot{\mathbf{u}} = 0, \tag{2.1}$$

where $\mathbf{u}$ is the displacement field, ρ is the density of a medium, and $\mathbf{C}$ is the fourth-order elasticity tensor assumed to be *positive definite*:

$$\forall_{\mathbf{A} \in \operatorname{sym}(R^3 \otimes R^3), \mathbf{A} \neq 0} \mathbf{A} \quad (\mathbf{A} \cdot \mathbf{C} \cdot \mathbf{A}) \equiv \sum_{i,j,m,n} A_{ij} C^{ijmn} A_{mn} > 0. \tag{2.2}$$

Traditionally, [1]–[18], a Rayleigh wave propagating in a homogeneous iso- or anisotropic medium is constructed in the form

$$\mathbf{u}(\mathbf{x}) = \sum_{k=1}^{3} C_k \mathbf{m}_k e^{ir(\gamma_k \boldsymbol{\nu} \cdot \mathbf{x} + \mathbf{n}' \cdot \mathbf{x} - ct)}, \tag{2.3}$$

where C_k are complex coefficients determined up to a multiplier by the traction-free boundary conditions; $\mathbf{m}_k$ are complex eigenvectors of the Christoffel equation, which will be introduced further; these eigenvectors correspond to complex roots γ_k of the characteristic polynomial; r is the (real) wave number; $\boldsymbol{\nu}$ is an outward normal to the boundary $\Pi_{\boldsymbol{\nu}}$ of the regarded half-space; $\mathbf{n}' \in \Pi_{\boldsymbol{\nu}}$ is the unit vector determining direction of propagation of the surface wave, refer to Fig. 1; and c is the phase speed.

The terms

$$\mathbf{u}_k(\mathbf{x}) \equiv \mathbf{m}_k e^{ir(\gamma_k \boldsymbol{\nu} \cdot \mathbf{x} + \mathbf{n}' \cdot \mathbf{x} - ct)} \tag{2.4}$$

are called *partial waves*.

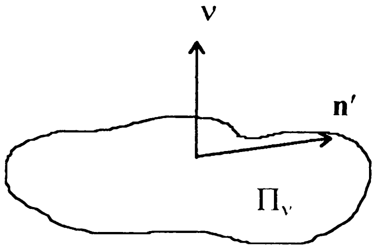

FIG. 1

Substitution of the partial waves (2.4) in Eq. (2.1) produces the Christoffel equation:

$$[(\gamma_k \boldsymbol{\nu} + \mathbf{n}') \cdot \mathbf{C} \cdot (\mathbf{n}' + \gamma_k \boldsymbol{\nu}) - \rho c^2 \mathbf{I}] \cdot \mathbf{m}_k = 0, \tag{2.5}$$

where $\mathbf{I}$ is the unit diagonal matrix. Equation (2.5) can be written in the equivalent form:

$$\det[(\gamma_k \boldsymbol{\nu} + \mathbf{n}') \cdot \mathbf{C} \cdot (\mathbf{n}' + \gamma_k \boldsymbol{\nu}) - \rho c^2 \mathbf{I}] = 0. \tag{2.6}$$

This shows that the left-hand side of the latter equation represents a polynomial of degree 6 with respect to γ_k.

The following definition for the *limiting speed* c_k^{lim}, $k = 1, 2, 3$ (see [12]) is needed for further analysis:

$$c_k^{\text{lim}} \equiv \inf_{\varphi \in [-\frac{\pi}{2}; \frac{\pi}{2}]} (\cos^{-1}(\varphi) \sqrt{\rho^{-1} \lambda_k (\mathbf{w} \cdot \mathbf{C} \cdot \mathbf{w})}), \tag{2.7}$$

where

$$\mathbf{w} = \sin(\varphi) \boldsymbol{\nu} + \cos(\varphi) \mathbf{n}' \tag{2.8}$$

and $\lambda_k(\mathbf{w} \cdot \mathbf{C} \cdot \mathbf{w})$, $k = 1, 2, 3$, are eigenvalues of the matrix $(\mathbf{w} \cdot \mathbf{C} \cdot \mathbf{w})$ arranged in descending mode.

Let c_k, $k = 1, 2, 3$, denote the speed of the bulk waves propagating in the direction $\mathbf{n}'$. The following expression [19] determines c_k:

$$c_k = \sqrt{\rho^{-1} \lambda_k (\mathbf{n}' \cdot \mathbf{C} \cdot \mathbf{n}')}, \quad k = 1, 2, 3. \tag{2.9}$$

The combination of (2.7) and (2.9) yields:

PROPOSITION 2.1. The limiting speed does not exceed corresponding bulk wave speed: $c_k^{\text{lim}} \leq c_k$, $k = 1, 2, 3$.

PROPOSITION 2.2. Let c be the phase speed.

a) If $c > c_1^{\text{lim}}$, then all six roots of the Christoffel equation are real.

b) If $c_1^{\text{lim}} > c > c_2^{\text{lim}}$, then the Christoffel equation has four real and one pair of complex-conjugate roots.

c) If $c_2^{\text{lim}} > c > c_3^{\text{lim}}$, then the Christoffel equation has two real and two pairs of complex-conjugate roots.

d) If $c_3^{\text{lim}} > c > 0$, then the Christoffel equation has three pairs of complex-conjugate roots.

Proof. The polynomial coefficients in (2.6) are all real. This ensures that if γ_k is a complex root, then $\overline{\gamma}_k$ is a root also. The assumption that γ_k is a real root allows us to introduce a new parameter $\varphi \in (-\pi/2; \pi/2)$ such that $\gamma_k = \tan\varphi$. In terms of the parameter φ, Eq. (2.6) can be represented in the form

$$\det(\cos^{-2}\varphi(\mathbf{w}\cdot\mathbf{C}\cdot\mathbf{w}) - \rho c^2\mathbf{I}) = 0, \tag{2.10}$$

where $\mathbf{w}$ is the real-valued vector defined by (2.8). Reduction of the matrix $(\mathbf{w}\cdot\mathbf{C}\cdot\mathbf{w})$ in (2.10) to the diagonal form gives

$$c = \cos^{-1}\varphi\sqrt{\rho^{-1}\lambda_k(\mathbf{w}\cdot\mathbf{C}\cdot\mathbf{w})}, \quad k = 1, 2, 3. \tag{2.11}$$

This along with definitions (2.7), (2.9) completes the proof. □

REMARK 2.1. a) Attenuation with depth in the "lower" half-space $(\boldsymbol{\nu}\cdot\mathbf{x}) < 0$ requires only complex roots γ_k with negative imaginary part $\gamma_k = \beta_k + i\alpha_k$, $\alpha_k < 0$, to appear in (2.3). This will be assumed throughout in the text.

b) Generally, there is no objection against considering a wider speed interval: $c < c_1^{\text{lim}}$, where *at least* one pair of complex-conjugate roots exists; so attenuation with depth in (2.3) can be achieved by substituting corresponding partial wave(s) with $\alpha_k < 0$. Still, the following analysis is confined to the interval $0 < c < c_3^{\text{lim}}$.

c) It will be shown further that under assumption $0 < c < c_3^{\text{lim}}$, representation (2.3) is valid for both (i) non-multiple roots of the polynomial (2.6); and (ii) an exceptional case relevant to isotropic materials, when two linearly independent eigenvectors correspond to two multiple roots.

3. Six-dimensional formalism. To analyse the case when multiple complex roots of Eq. (2.6) arise, differential operator (2.1) should be decomposed in terms of the spatial variables $\mathbf{x}' = \text{Pr}_{\Pi_\nu}\mathbf{x}$ and $x'' = \boldsymbol{\nu}\cdot\mathbf{x}$ (thus, $\mathbf{x}'$ belongs to the $\Pi_{\boldsymbol{\nu}}$-plane):

$$\mathbf{A}(\partial_x, \partial_t) \equiv (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})\partial_{x''}^2 + (\boldsymbol{\nu}\cdot\mathbf{C}\cdot\nabla_{\mathbf{x}'} + \nabla_{\mathbf{x}'}\cdot\mathbf{C}\cdot\boldsymbol{\nu})\partial_{x''} + \nabla_{\mathbf{x}'}\cdot\mathbf{C}\cdot\nabla_{\mathbf{x}'} - \rho\mathbf{I}\partial_t^2. \tag{3.1}$$

At this stage it is not clear what is the appropriate representation for the partial wave corresponding to multiple roots. In such a situation a more general representation for the partial wave will be adopted:

$$\mathbf{v}(x'')e^{ir(\mathbf{n}'\cdot\mathbf{x}-ct)}, \tag{3.2}$$

where $\mathbf{v}(x'')$ is an unknown vector function, while the exponential multiplier in (3.2) corresponds to propagation of the plane wave along the direction $\mathbf{n}'$ with the phase speed c. Substituting representation (3.2) into (3.1) produces the following system of ordinary differential equations:

$$((\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})\partial_{x''}^2 + ir(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n}' + \mathbf{n}'\cdot\mathbf{C}\cdot\boldsymbol{\nu})\partial_{x''} - r^2(\mathbf{n}'\cdot\mathbf{C}\cdot\mathbf{n}' - \rho c^2\mathbf{I}))\mathbf{v}(x'') = 0. \tag{3.3}$$

Direct analysis of system (3.3) is rather difficult, and reduction to the first-order system can simplify it, allowing us to exploit the available technique [20].

Introduction of a new function $\mathbf{w} = \partial_{x''}\mathbf{v}$ allows us to reduced the second-order system (3.3) in C^3 to the first-order one in C^6:

$$\partial_{x''}\mathbf{I}_6\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix} = \mathbf{R}_6\begin{pmatrix}\mathbf{v}\\ \mathbf{w}\end{pmatrix}. \tag{3.4}$$

In (3.4), $\mathbf{I}_6$ denotes the unit (diagonal) matrix in C^6, and the complex six-dimensional matrix $\mathbf{R}_6$ has the form

$$\mathbf{R}_6 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ r^2(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})^{-1}\cdot(\mathbf{n}'\cdot\mathbf{C}\cdot\mathbf{n}' - \rho c^2\mathbf{I}) & -ir(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})^{-1}\cdot(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n}' + \mathbf{n}'\cdot\mathbf{C}\cdot\boldsymbol{\nu}) \end{pmatrix}. \tag{3.5}$$

In (3.5), $\mathbf{I}$ stands for the unit (diagonal) matrix in the three-dimensional space.

REMARK 3.1. a) The right-hand side of (3.5) ensures that

$$\det \mathbf{R}_6 = -r^6 \det((\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu})^{-1}(\mathbf{n}'\cdot\mathbf{C}\cdot\mathbf{n}' - \rho c^2\mathbf{I})), \tag{3.6}$$

and, since the tensor $\mathbf{C}$ is positive definite, the kernel space of the matrix $\mathbf{R}_6$ is trivial, with the only exceptions when $\rho c^2 = \lambda_k(\mathbf{n}'\cdot\mathbf{C}\cdot\mathbf{n}')$, $k = 1, 2, 3$. For these exceptional cases, the phase speed c coincides with the speed of bulk waves propagating in the direction $\mathbf{n}'$. Now, taking into account Proposition 2.1, it becomes clear that $\ker \mathbf{R}_6 = 0$ for the considered speed interval $0 < c < c_3^{\mathrm{lim}}$.

PROPOSITION 3.1. Let $c \in (0; c_3^{\mathrm{lim}})$.

a) If λ is an eigenvalue of the matrix $\mathbf{R}_6$, then $\mathrm{Re}\,\lambda \neq 0$.

b) If $\lambda = \alpha + i\beta$ is an eigenvalue and $\mathbf{m} = (\mathbf{m}', \mathbf{m}'')$ is the corresponding eigenvector of the matrix $\mathbf{R}_6$, where $\mathbf{m}', \mathbf{m}'' \in C^3$, then $\lambda' = -\alpha + i\beta$ is also an eigenvalue with corresponding eigenvector $\overline{\mathbf{m}} = (\overline{\mathbf{m}'}, \overline{\mathbf{m}''})$.

Proof. If λ is an eigenvalue with corresponding eigenvector $\mathbf{m} = (\mathbf{m}', \mathbf{m}'')$, then due to (3.5), action of the matrix $\mathbf{R}_6$ on this eigenvector gives:

$$\mathbf{m}', \mathbf{m}'' \in \ker(\lambda^2(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + \lambda ir(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n}' + \mathbf{n}'\cdot\mathbf{C}\cdot\boldsymbol{\nu}) - r^2(\mathbf{n}'\cdot\mathbf{C}\cdot\mathbf{n}' - \rho c^2\mathbf{I})). \tag{3.7}$$

The latter expression is equivalent to

$$\mathbf{m}', \mathbf{m}'' \in \ker(\gamma^2(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + \gamma(\boldsymbol{\nu}\cdot\mathbf{C}\cdot\mathbf{n}' + \mathbf{n}'\cdot\mathbf{C}\cdot\boldsymbol{\nu}) + (\mathbf{n}'\cdot\mathbf{C}\cdot\mathbf{n}' - \rho c^2\mathbf{I})), \tag{3.8}$$

where γ is defined to be

$$\gamma = -ir^{-1}\lambda. \tag{3.9}$$

Now, it is clear that the matrix in the right-hand side of (3.8) coincides with the one in the Christoffel equation (2.6). To complete the proof it remains to note that the polynomial in (2.6) has real coefficients. □

REMARK 3.2. a) Actually, the proof of Proposition 3.1 establishes the equivalence between the spectral properties of the matrix $\mathbf{R}_6$ and the roots and corresponding kernel spaces of the Christoffel equation (2.6).

b) According to the general theory of ordinary differential equations [20], the structure of the eigensolutions of the system (3.4) depends on the Jordan form of the matrix $\mathbf{R}_6$. Directly from Proposition 3.1 it follows that, for $c \in (0; c_3^{\mathrm{lim}})$, there can be only three kinds of Jordan normal forms for the matrix $\mathbf{R}_6$:

$$\mathbf{J}_6^{(I)} = ir \begin{pmatrix} \gamma_1 &&&&& \\ & \overline{\gamma}_1 &&&& \\ && \gamma_2 &&& \\ &&& \overline{\gamma}_2 && \\ &&&& \gamma_3 & \\ &&&&& \overline{\gamma}_3 \end{pmatrix}, \quad \mathbf{J}_6^{(II)} = ir \begin{pmatrix} \begin{pmatrix} \gamma_1 & 1 \\ 0 & \gamma_1 \end{pmatrix} &&& \\ & \begin{pmatrix} \overline{\gamma}_1 & 1 \\ 0 & \overline{\gamma}_1 \end{pmatrix} && \\ && \gamma_3 & \\ &&& \overline{\gamma}_3 \end{pmatrix},$$

$$\mathbf{J}_6^{(III)} = ir \begin{pmatrix} \begin{pmatrix} \gamma_1 & 1 & 0 \\ 0 & \gamma_1 & 1 \\ 0 & 0 & \gamma_1 \end{pmatrix} & \\ & \begin{pmatrix} \overline{\gamma}_1 & 1 & 0 \\ 0 & \overline{\gamma}_1 & 1 \\ 0 & 0 & \overline{\gamma}_1 \end{pmatrix} \end{pmatrix}. \tag{3.10}$$

In (3.10), $\mathbf{J}_6$ denotes the Jordan normal form of the matrix $\mathbf{R}_6$ and the parameters γ_k are associated with the eigenvalues λ_k by (3.9).

c) In view of the preceding remark and the theory of ordinary differential equations [20], the following three types of representations for surface waves can occur:

(i) for the Jordan normal form $\mathbf{J}_6^{(I)}$, the corresponding representation is given by (2.3);

(ii) for the Jordan normal form $\mathbf{J}_6^{(II)}$, the representation is as follows:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) = (C_1 + irC_2\boldsymbol{\nu}\cdot\mathbf{x})\mathbf{m}_1' e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}'\cdot\mathbf{x}-ct)} + C_2\mathbf{m}_2' e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}'\cdot\mathbf{x}-ct)} \\ + C_3\mathbf{m}_3' e^{ir(\gamma_3\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}'\cdot\mathbf{x}-ct)}, \end{aligned} \tag{3.11}$$

where $\mathbf{m}_1' \in C^3$ corresponds to the eigenvalue $ir\gamma_1$ of $\mathbf{J}_6^{(II)}$, $\mathbf{m}_2' \in C^3$ corresponds to the generalized eigenvector associated with $\mathbf{m}_1'$, and $\mathbf{m}_3' \in C^3$ corresponds to the eigenvalue $ir\gamma_3$;

(iii) for the Jordan normal form $\mathbf{J}_6^{(III)}$, the representation is as follows:

$$\begin{aligned} \mathbf{u}(\mathbf{x}) = {} & (C_1 + irC_2\boldsymbol{\nu}\cdot\mathbf{x} + C_3(ir\boldsymbol{\nu}\cdot\mathbf{x})^2)\mathbf{m}_1' e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}'\cdot\mathbf{x}-ct)} \\ & + (C_2 + 2irC_3\boldsymbol{\nu}\cdot\mathbf{x})\mathbf{m}_2' e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}'\cdot\mathbf{x}-ct)} \\ & + 2C_3\mathbf{m}_3' e^{ir(\gamma_1\boldsymbol{\nu}\cdot\mathbf{x}+\mathbf{n}'\cdot\mathbf{x}-ct)}, \end{aligned} \tag{3.12}$$

where $\mathbf{m}_1' \in C^3$ corresponds to the eigenvalue $ir\gamma_1$ of $\mathbf{J}_6^{(III)}$, and $\mathbf{m}_2', \mathbf{m}_3'$ are the generalized eigenvectors.

Thus, expressions (3.11), (3.12) determine the unknown vector function $\mathbf{v}$ in (3.2). It should be noted that the coefficients C_k in the representations (3.11), (3.12) generally are not independent, in contrast to the arbitrary coefficients in the representation (2.3); see [20].

4. Boundary conditions. The traction-free boundary conditions on the surface $\Pi_{\boldsymbol{\nu}}$ can be written in the form

$$\mathbf{t}_{\boldsymbol{\nu}} \equiv \boldsymbol{\nu}\cdot\mathbf{C}\cdot\nabla\mathbf{u}|_{\mathbf{x}\in\Pi_{\boldsymbol{\nu}}} = 0. \tag{4.1}$$

The following two cases can occur when substituting the corresponding representations for the surface wave into the boundary conditions (4.1):

(i) for the Jordan normal form $\mathbf{J}_6^{(I)}$ and the representation (2.3),

$$\sum_{k=1}^{3} C_k(\gamma_k \boldsymbol{\nu} \cdot \mathbf{C} \cdot \boldsymbol{\nu} + \boldsymbol{\nu} \cdot \mathbf{C} \cdot \mathbf{n}') \cdot \mathbf{m}'_k = 0; \tag{4.2}$$

(ii) for the Jordan normal form $\mathbf{J}_6^{(II)}$ and the representation (3.11),

$$(\boldsymbol{\nu} \cdot \mathbf{C} \cdot \boldsymbol{\nu}) \cdot ((\gamma_1 C_1 + C'_2)\mathbf{m}'_1 + \gamma_1 C_2 \mathbf{m}'_2 + \gamma_3 C_3 \mathbf{m}'_3) + (\boldsymbol{\nu} \cdot \mathbf{C} \cdot \mathbf{n}') \cdot \left(\sum_{k=1}^{3} C_k \mathbf{m}'_k \right) = 0, \tag{4.3}$$

where $C'_2 = (ir)^{-1} C_2$.

Expressions (4.2), (4.3) can be regarded as linear equations with respect to the unknown coefficients C_k. The existence of nontrivial solutions of (4.2), (4.3) is equivalent to the vanishing of the corresponding determinants, and this also provides a necessary and sufficient condition for the existence of a Rayleigh wave.

5. Rayleigh waves on the basal planes of transversely isotropic media. Let the unit vectors $\mathbf{e}_k$, $k = 1, 2, 3$, form an orthogonal basis in R^3, where vector $\mathbf{e}_1$ is normal to the $\Pi_{\boldsymbol{\nu}}$-basal plane of a transversely isotropic medium. Thus vectors $\mathbf{e}_1$ and $\boldsymbol{\nu}$ coincide. The corresponding elasticity tensor has the following components:

$$\begin{matrix} c_{11} & c_{12} & c_{12} & 0 & 0 & 0 \\ & c_{22} & c_{23} & 0 & 0 & 0 \\ & & c_{22} & 0 & 0 & 0 \\ & & & c_{44} & 0 & 0 \\ & & & & c_{55} & 0 \\ & & & & & c_{55} \end{matrix} \tag{5.1}$$

where $c_{44} = \frac{1}{2}(c_{22} - c_{23})$ and, as before, the elasticity tensor is assumed to be positive-definite.

Substituting tensor (5.1) in (3.5) yields

$$\mathbf{R}_6 = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{M} & -i\mathbf{N} \end{pmatrix}, \tag{5.2}$$

where the matrices $\mathbf{M}$ and $\mathbf{N}$, having the same structure as in (3.13), are of the form

$$\begin{aligned} \mathbf{M} &= r^2 \left(\frac{c_{55} - \rho c^2}{c_{11}} \boldsymbol{\nu} \otimes \boldsymbol{\nu} + \frac{c_{22} - \rho c^2}{c_{55}} \mathbf{n}' \otimes \mathbf{n}' + \frac{c_{22} - c_{23} - \rho c^2}{2c_{55}} \mathbf{w} \otimes \mathbf{w} \right), \\ \mathbf{N} &= r \left(\frac{c_{12} + c_{55}}{c_{11}} \boldsymbol{\nu} \otimes \mathbf{n}' + \frac{c_{12} + c_{55}}{c_{55}} \mathbf{n}' \otimes \boldsymbol{\nu} \right). \end{aligned} \tag{5.3}$$

The following proposition flows out directly from the analysis of the eigenproblem for the matrix $\mathbf{R}_6$:

PROPOSITION 5.1. a) The relation between the elastic constants, density, and the phase speed,

$$\rho c^2 = 2\frac{|c_{12}+c_{55}|}{(c_{11}-c_{55})^2}\sqrt{c_{11}c_{55}(c_{11}c_{55}+c_{22}c_{55}+2c_{12}c_{55}-c_{11}c_{22}+c_{12}^2)} \\ -\frac{2c_{12}c_{55}^2+2c_{11}c_{12}c_{55}-c_{11}^2c_{22}+c_{11}c_{22}c_{55}+c_{11}c_{12}^2+c_{12}^2c_{55}+2c_{11}c_{55}^2}{(c_{11}-c_{55})^2} \tag{5.4}$$

is necessary and sufficient for the rise of the Jordan normal form $\mathbf{J}_6^{(II)}$.

b) The different roots γ_k of the Christoffel equation (2.4) with $\operatorname{Im}\gamma_k \equiv \alpha_k < 0$ corresponding to (5.4) are as follows:

$$\gamma_1 = -i\left(\frac{c_{11}c_{22}-(c_{11}+c_{55})\rho c^2-2c_{12}c_{55}-c_{12}^2}{2c_{11}c_{55}}\right)^{1/2}, \\ \gamma_3 = -i\left(\frac{c_{22}-c_{23}-2\rho c^2}{2c_{55}}\right)^{1/2}, \tag{5.5}$$

where γ_1 is the multiple root, and ρc^2 is determined by (5.4).

c) The complex amplitudes $\mathbf{m}_1'$ and $\mathbf{m}_3'$ in representation (3.11) have the form:

$$\mathbf{m}_1' = ip\boldsymbol{\nu}\left(\frac{c_{22}-\rho c^2}{c_{11}}\right)^{1/4} + p\mathbf{n}'\left(1-\frac{\rho c^2}{c_{55}}\right)^{1/4}, \qquad \mathbf{m}_3' = \boldsymbol{\nu}\times\mathbf{n}', \tag{5.6}$$

where p is the normalisation factor:

$$p = \left(\left(\frac{c_{22}-\rho c^2}{c_{11}}\right)^{1/2} + \left(1-\frac{\rho c^2}{c_{55}}\right)^{1/2}\right)^{-1/2}. \tag{5.7}$$

REMARK 5.1. Analysis of expression (5.4) shows that a (necessary and sufficient) condition for parameter ρc^2 to be real and positive is as follows:

$$c_{11}c_{22} < (c_{12}+2c_{55})^2. \tag{5.8}$$

In obtaining (5.8), the positive-definite condition for the elasticity tensor is taken into account.

Assume that at $c_0 \in (0; c_3^{\lim})$ the Jordan normal form $\mathbf{J}_6^{(II)}$ arises. Substitution of the amplitudes (5.6) in the boundary conditions (4.3) requires $C_3 = 0$ and

$$c_{12} = -\sqrt{c_{11}c_{55}}\sqrt{\frac{c_{22}-\rho c^2}{c_{55}-\rho c^2}}, \tag{5.9}$$

where ρc^2 is determined by (5.4). Expression (5.9) shows that satisfaction of the boundary conditions (4.3) can be achieved by negative c_{12} only. The latter is physically unlikely, yet can be compatible with the positive-definite condition. Still, further analysis is confined to the case when $c_{12} > 0$. This allows us to formulate

PROPOSITION 5.2. No Rayleigh wave propagates at $c_0 \in (0; c_3^{\lim})$ when (i) the phase speed c_0 is determined by (5.4); and (ii) $c_{12} > 0$.

Still, at some other value of the same speed interval, the Rayleigh wave can propagate. Cases of non-multiple roots, which lead to the Jordan normal form $\mathbf{J}_6^{(I)}$, were studied analytically in [4]. The following proposition is, in principle, due to Royer and Dieulesaint [4]:

PROPOSITION 5.3. Suppose that: (i) parameter ρc_R^2 is a positive root of the third-order polynomial:

$$c_{11}(c_{11}-c_{55})x^3 + c_{11}(c_{22}c_{55}+2c_{12}^2-c_{11}c_{55}-2c_{11}c_{22})x^2 + (c_{11}c_{22}-c_{12}^2)(c_{11}c_{22}+2c_{11}c_{55}-c_{12}^2)x - c_{55}(c_{11}c_{22}-c_{12}^2)^2 = 0 \quad (5.10)$$

and (ii) relation (5.4) does not hold at this value of ρc_R. Then c_R determines the speed of the Rayleigh wave.

Suppose now that two values of the phase speed defined by (5.4) and (5.10) are equal. Then, for such a situation, no Rayleigh wave can propagate, which is due to Proposition 5.2. This phenomenon admits an easy explanation: when both of these values of the phase speed coincide, representation (2.3) is not valid, and the corresponding representation (3.11) does not satisfy the boundary conditions. For the regarded case, a combination of expressions (5.4) and (5.10) produces the following polynomial equation (with respect to an unknown parameter c_{12}):

$$(c_{55}-9c_{11})Z^4 + 2c_{55}(c_{55}-17c_{11})Z^3 + (c_{55}^3-45c_{11}c_{55}^2-5c_{11}c_{22}c_{55}-8c_{11}^2c_{55}+9c_{11}^2c_{22})Z^2 + 2c_{11}c_{55}(5c_{11}c_{22}-12c_{55}^2-4c_{11}c_{55}-5c_{22}c_{55})Z + c_{11}c_{22}c_{55}(4c_{11}c_{22}-9c_{55}^2-3c_{11}c_{55}) = 0. \quad (5.11)$$

As will be shown further, for some combinations of the elastic parameters c_{11}, c_{22}, c_{55}, Eq. (5.11) has positive root(s), which determines c_{12}, and such combinations of elastic parameters are compatible with the positive-definite condition. This allows us to formulate the principle result of the paper.

SCHOLIUM 5.1. "Forbidden" planes exist, and they can coincide with the basal planes of transversely isotropic media. For these media, "forbidden" planes arise at the elastic parameter c_{12}, it being the real positive root(s) of Eq. (5.11).

6. Example. Let $c_{11} = c_{22} = 1$. Then for such a transversely isotropic medium, Eq. (5.11) reduces to a single parametric one:

$$(c_{55}-9)Z^4 + 2c_{55}(c_{55}-17)Z^3 + (c_{55}^3-45c_{55}^2-13c_{55}+9)Z^2 + 2c_{55}(5-12c_{55}^2-9c_{55})Z + c_{55}(4-9c_{55}^2-3c_{55}) = 0, \quad (6.1)$$

where, as before, the positive root(s) determines the value of c_{12} at which no Rayleigh wave can propagate.

Numerical analysis of the solutions to Eq. (6.1) allows us to express the dependencies of c_{12} upon c_{55} by the following (approximate) formula:

$$c_{12} = (1-2c_{55}) + 0.2025c_{55}(1-c_{55}^2)^3. \quad (6.2)$$

The first term in the right-hand side of (6.2),

$$c_{12} = (1 - 2c_{55}), \tag{6.3}$$

defines c_{12}, which corresponds to the isotropic medium.

REMARK 6.1. a) Direct verification shows that the elasticity tensor with $c_{11} = c_{22} = 1$, $c_{55} \in (0; 1/2)$, and c_{12} defined by (6.2), satisfies the positive-definite condition.

b) Comparison of numerical data obtained directly from (6.1) and by the approximate expression (6.2) reveals that the latter determines c_{12} with the (absolute) error not exceeding 5.5×10^{-4} at $c_{55} \in (0; 1/2)$.

c) Apparently, the most interesting result flowing out from the analysis of the numerical solution of Eq. (6.1) consists in a weak anisotropy of the transversely isotropic media, which have "forbidden" planes. Thus, deviation c_{12}, obtained by (6.1) from corresponding values for isotropic media obtained by (6.3), does not exceed 4.8×10^{-2} at $c_{55} \in (0; 1/2)$. Such a weak anisotropy of the regarded media possessing "forbidden" planes can be important for developing structural materials that do not transmit Rayleigh waves.

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REFERENCES

[1] J. W. Strutt (Lord Rayleigh), *On wave propagating along the plane surface of an elastic solid*, Proc. London Math. Soc. **17**, 4–11 (1885)

[2] R. Stoneley, *The propagation of surface elastic waves in a cubic crystal*, Proc. Roy. Soc. **A232**, 447–458 (1955)

[3] E. Dieulesaint and D. Royer, *Elastic Waves in Solids*, John Wiley and Sons, New York, 1980

[4] D. Royer and E. Dieulesaint, *Rayleigh wave velocity and displacement in orthorhombic, tetragonal, and cubic crystals*, J. Acoust. Soc. America **76**, 1438–1444 (1985)

[5] T. C. Lim and G. W. Farnell, *Search for forbidden directions of elastic surface-wave propagation in anisotropic crystals*, J. Appl. Phys. **39**, 4319–4325 (1968)

[6] T. C. Lim and G. W. Farnell, *Character of pseudo surface waves on anisotropic crystals*, J. Acoust. Soc. America **45**, 845–851 (1969)

[7] G. W. Farnell, *Properties of elastic surface waves*, Phys. Acoust. **6**, 109–166 (1970)

[8] D. M. Barnett and J. Lothe, *Synthesis of the sextic and the integral formalism for dislocations, Greens functions, and surface waves in anisotropic elastic solids*, Phys. Norv. **7**, 13–19 (1973)

[9] D. M. Barnett and J. Lothe, *Consideration of the existence of surface wave (Rayleigh wave) solutions in anisotropic elastic crystals*, J. Phys., F: Metal Phys. **4**, 671–686 (1974)

[10] J. Lothe and D. M. Barnett, *On the existence of surface wave solutions for anisotropic elastic half-spaces with free surface*, J. Appl. Phys. **47**, 428–433 (1976)

[11] P. Chadwick and G. D. Smith, *Foundations of the theory of surface waves in anisotropic elastic materials, In: Advances in Applied Mechanics* **17**, Academic Press, New York, 1977, pp. 303–376

[12] P. Chadwick and T. C. T. Ting, *On the structure and invariance of the Barnett-Lothe tensors*, Quart. Appl. Math. **45**, 419–427 (1987)

[13] S. A. Gunderson, D. M. Barnett, and J. Lothe, *Rayleigh wave existence theory: A supplementary remark*, Wave Motion **9**, 319–321 (1987)

[14] P. Chadwick, *Some remarks on the existence of one-component surface waves in elastic materials with symmetry*, Physica Scripta **44**, 94–97 (1992)

[15] P. Chadwick, *The application of the Stroh formalism to prestressed elastic media*, Mathematics and Mechanics of Solids **2**, 379–403 (1997)

[16] A. N. Stroh, *Steady state problems in anisotropic elasticity*, J. Math. Phys. **41**, 77–103 (1962)

[17] G. T. Mase, *Rayleigh wave speeds in transversely isotropic materials*, J. Acoust. Soc. America **81**, 1441–1446 (1987)
[18] G. T. Mase and G. C. Johnson, *An acoustic theory for surface waves in anisotropic media*, J. Appl. Mech. **54**, 127–135 (1987)
[19] M. E. Gurtin, *The Linear Theory of Elasticity.* In: *Handbuch der Physik*, **VIa/2**, Springer-Verlag, 1973, pp. 1–295
[20] P. Hartman, *Ordinary Differential Equations*, John Wiley and Sons, New York, 1964