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## DEPARTMENT OF CIVIL ENGINEERING



# FORCE AT A POINT IN THE INTERIOR OF A SEMI-INFINITE SOLID 

by<br>R. D. MINDLIN

Office of Naval Research Project NR-064-388
Contract Nonr-266(09)
Technical Report-No. 8
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## Introduction

In a paper ${ }^{(1)}$ under the same titie, the solution of the linear equationa of equilibrium of an elastic body was given for the case of a force acting at a point within an isotropic body bounded by a plane. The result was obtained by starting with Kelvin's solution for a force in an infinite body and guessing the nuclei of strain to add outaide of the semi-infinite body so as to annul the tractions on the plane boundary. In the present paper it is shown how these results may be obtained, directly, by means of an application of potential theory.

## Prokoritch Punctions

In an isotropic elastic body in equilibrium, the displacement $u$ is governed by the equation ${ }^{(2)}$

$$
\begin{equation*}
\mu \nabla^{2} u_{n}+\frac{\mu}{1-2 \nu} \nabla \nabla \cdot u_{n}+F=0, \tag{1}
\end{equation*}
$$

where $\mu$ is the shear modulus, $\nu$ is Poisson's ratio and $F$ is the body force per unit of volume.

Por an isotropic body, the stress, $\sigma$, is related to the displacement by

$$
\begin{equation*}
\underset{\sim}{\sigma}=\lambda \nabla \cdot \underset{n}{u} I_{n}+\mu\left(\nabla u_{n}+u_{n} \nabla\right) \tag{2}
\end{equation*}
$$

(1) R. D. Mindlin, Physics, Vol. 7 (1936), pp. 195-202.
(2) For the vector notation used in this paper, see C. E. Weatherburn, Advanced Vector Analysis, G. Bell and Sons, Itd., London, 1928.
where $\lambda=2 \mu \nu /(i-2 \nu)$.
By Helmholtz's theorem ${ }^{(3)}$, $u$ may be resolved into lamellar and solenodal components:

$$
\begin{equation*}
\underset{n}{u}=\underset{n}{\nabla} \varphi+\underset{n}{\nabla} \times \underset{n}{H}, \underset{n}{\nabla} \cdot \underset{n}{H}=0, \tag{3}
\end{equation*}
$$

so that (1) may be written

$$
\begin{equation*}
\mu \nabla^{2}\left(r<\underset{n}{ } \varphi+\nabla_{n} \times \underset{n}{H}\right)+\underset{n}{F}=0 \tag{4}
\end{equation*}
$$

where $\alpha=2(1-\nu) /(1-2 \nu)$.
The quantity in parenthesis in (4) is a vector, say, $B=i_{n} B_{x}+j B_{j}+k B$ i.e.,

$$
\begin{gather*}
\alpha{ }_{i} \cdot \varphi+\nabla_{n} \times \underset{n}{H}=\underset{\sim}{B},  \tag{5}\\
\mu \nabla^{2} \underset{n}{B}=-F \tag{6}
\end{gather*}
$$

Operating on (5) with $\nabla \cdot$, we find

$$
\begin{equation*}
\alpha \nabla^{2} \varphi=\nabla \cdot \underset{n}{B} \tag{7}
\end{equation*}
$$

the complete solution of wish is

$$
\begin{equation*}
2 \alpha \varphi=\underset{n}{r} \cdot \beta+\beta \text {, } \tag{8}
\end{equation*}
$$

where $\beta$ is a scalar function, which satisfies

$$
\begin{equation*}
\mu \nabla^{2} \beta=r \cdot F, \tag{9}
\end{equation*}
$$

and $r_{n}=i x+j y+k z$ is the position vector.
Substituting (8) in (3) and eliminating ${\underset{n}{n}}^{\nabla} \times \underset{n}{ }$ by means of (5), there results

$$
\begin{gather*}
u_{u}=B_{n}-\frac{1}{4(1-\nu)} \nabla_{n}\left(r_{n} \cdot B_{n}+\beta\right)  \tag{10}\\
\mu \nabla^{2}{\underset{n}{B}}_{B}=-\underset{n}{F} \tag{11}
\end{gather*}
$$

(3) Weatherburn, p. 44 .

$$
\begin{equation*}
\mu \nabla^{2} \beta=r \cdot F . \tag{12}
\end{equation*}
$$

Thus the displacement is expressed in terms of the Papkofitch functions, $B$ and $\beta$, whose Laplacians are known if the body force $F$ is known.

The prof of completeness of the Papkovitch functions, given above, is an extension, to include the body force, of one given in a previous paper ${ }^{(4)}$, where $B$ and $\beta$ were called Papkovitch ${ }^{(5)}$ functions after the originator of the solution (10) of the elasticity equations. Recent writsrs associate these functions with the name of Boussinesq ${ }^{(6)}$, who introduced $B_{z}$ and $\beta$, but employed functions of a different type where $B_{n}$ and $B_{y}$ could have been used.

## Green's Formula

The value of a function $V$, at any point in a region, may be expressed in terms of its values at the boundary, its Laplacian and Green's function, $\mathcal{G}$, for the region, by means of Green's formula ${ }^{(7)}$

$$
\begin{equation*}
-4 \pi V=\int V n_{n} \cdot \nabla G d S+\int G \nabla^{2} V d v \tag{13}
\end{equation*}
$$

For the region $z>0$, Green's function is

$$
\begin{equation*}
G=r_{1}^{-1}-r_{2}^{-1} \tag{4}
\end{equation*}
$$

where
(4) R. D. Mindlin, Bull. Am. Math. Soc., Vol. 42 (1936), pp. 373-376.
(5) P. F. Papkovitch, Comptes Rendus, Acad. des Sciences, Paris, Vol. 195 (1922), pp. 513-515 and 754-756.
(6) J. Boussinesq, "Appliçation des Potentiels à l'étude de l'Équilibre et du Mouvement des Solides Elastiques," Gauthier-Villars, Faris, 1885, pp. 63 and 72.
(7) Weatherbirn, p. 34 .

$$
\begin{align*}
& r_{1}^{2}=(x-\xi)^{2}+(y-\eta)^{2}+(z-\xi)^{2}  \tag{15}\\
& r_{2}^{2}=(x-\xi)^{2}+(y-\eta)^{2}+(z+\xi)^{2} \tag{16}
\end{align*}
$$

in which ( $x, y, z$ ) are the coordinates of a point $P(x, y, z)$ in the region and $(\xi, \eta, \xi),(\xi, \eta,-\xi)$ are the coordinates of a source point $Q(\xi, \eta, \xi)$ and its image $Q^{\prime}(5, \eta,-5)$, respectively.

## Porce_at_ Point

Kelvin's definition of a force at a point takes the following form, in the present case. Consider a distribution of body forces $F$ in a closed region $T$ within $z>0$, with $F=0$ outside $T$ but within $z>0$. diminish $T$ indefinitely, always enclosing the point $C(0,0, c$.$) , bot let$

$$
\begin{equation*}
\lim _{T \rightarrow 0} \int_{T} F d v=P \tag{17}
\end{equation*}
$$

where $P$ is a constant force at $C$.
For later use, we note that the limit, as $T$ approaches zero, of

$$
\begin{gather*}
Q(\xi, \eta, \xi)=c(0,0, c),  \tag{18}\\
Q^{\prime}(\xi, \eta,-\xi)=c^{\prime}(0,0,-c),  \tag{19}\\
r_{1}^{2}=x^{2}+y^{2}+(z-c)^{2}=R_{1}^{2},  \tag{20}\\
r_{2}^{2}=x^{2}+y^{2}+(z+c)^{2}=R_{2}^{2},  \tag{21}\\
G=\frac{1}{R_{1}}-\frac{1}{R_{2}},  \tag{22}\\
\frac{\partial G}{\partial \xi}=-\frac{\partial}{\partial z}\left(\frac{1}{R}+\frac{1}{R_{2}}\right),  \tag{23}\\
\frac{\partial^{2} G}{\partial \xi^{2}}=\frac{\partial^{2}}{\partial z^{2}}\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right), \tag{24}
\end{gather*}
$$

$$
\begin{equation*}
\frac{\partial G}{\partial \xi}=-\frac{\partial}{\partial x}\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right) \tag{25}
\end{equation*}
$$

## Force Normal to the Plane Boundary

In this case we take $F_{x}=F_{y}=0$ and $B_{x}=B_{y}=0$. The remaining Papkoritch (in this case Boussinesq) functions, $B_{z}$ and $\beta$, must satisfy the condition of vanishing traction on $z=0$. Thus, we have, from (2) and (10), on $z=0$

$$
\begin{align*}
& \sigma_{z z}=\frac{\mu}{2(1-\nu)}\left[2(1-\nu) \frac{\partial B_{x}}{\partial z}-\frac{\partial^{2} \beta}{\partial z^{2}}\right]=0,  \tag{26}\\
& \sigma_{z x}=\frac{\mu}{2(1-\nu)}\left[(1-2 \nu) \frac{\partial B_{x}}{\partial x}-\frac{\partial^{2} \beta}{\partial x \partial z}\right]=0,  \tag{27}\\
& \sigma_{z y}=\frac{\mu}{2(1-\nu)}\left[(1-2 \nu) \frac{\partial B_{z}}{\partial y}-\frac{\partial^{2} \beta}{\partial y \partial z}\right]=0 . \tag{28}
\end{align*}
$$

The function in brackets in (26) is one whose Laplacian

$$
\begin{equation*}
\nabla^{2}\left[2(1-\nu) \frac{\partial B_{z}}{\partial z}-\frac{\partial^{2} \beta}{\partial z^{2}}\right]=-\frac{2(1-\nu)}{\mu} \frac{\partial F_{x}}{\partial z}-\frac{1}{\mu} \frac{\partial^{2}\left(z F_{x}\right)}{\partial z^{2}} \tag{29}
\end{equation*}
$$

is known throughout $z \rightarrow 0$ and whose boundary value is zero. Hence, by (13),

$$
2(1-\nu) \frac{\partial B_{z}}{\partial z}-\frac{\partial^{2} \beta}{\partial z^{2}}=\frac{1}{4 \pi \mu} \int G\left[2(1-\nu) \frac{\partial F_{x}}{\partial \xi}+\frac{\partial^{2}\left(\xi F_{2}\right)}{\partial \xi^{2}}\right] d v
$$

Now, integrating the first term in the volume integral by parts,

$$
\iiint G \frac{\partial F_{2}}{\partial \xi} d \xi d \eta d \xi=\iint G F_{x} d \xi d \eta-\iiint F_{z} \frac{\partial G}{\partial \zeta} d \xi d \eta d \xi . \text { (31) }
$$

The surface integral in (31) vanishes because $F_{2}=G=0$ on the boundary of the body. Then, try (17) and (23),

$$
\begin{equation*}
\lim _{T \rightarrow 0} \int G \frac{\partial F_{z}}{\partial \zeta} d v=P_{z} \frac{\partial}{\partial z}\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right) \tag{32}
\end{equation*}
$$

STmilarly, integrating by parts twice and using (17), (18) and (24),

$$
\begin{equation*}
\lim _{r \rightarrow 0} \int G \frac{\partial^{2} F_{1}}{\partial \xi^{2}} d v=c P_{2} \frac{\partial^{2}}{\partial z^{2}}\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right) . \tag{33}
\end{equation*}
$$

Hence

$$
2(1-\nu) B_{2}-\frac{\partial \beta}{\partial z}=\frac{P_{2}}{4 \pi \mu}\left[2(1-\nu)\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right)+c \frac{\partial}{\delta z}\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right)\right],(34)
$$

where one integration with respect to $Z$ has been performed. (The arbitrary function of $x$ and $y$, thereby introduced, must vanish since $2(1-\nu) B_{z}-\partial \beta / \partial z$ must vanish as $Z \rightarrow \infty$ ).

Returning to the boundary conditions, we note that (27) and (28) can be integrated with respect to $x$ and $y$, respectively, so that, on $z=0$,

$$
\begin{equation*}
(1-2 \nu) B_{z}-\frac{\partial \beta}{\partial z}=0 \tag{35}
\end{equation*}
$$

The Laplacian

$$
\begin{equation*}
\nabla^{2}\left[(1-2 v) B_{z}-\frac{\partial \beta}{\partial z}\right]=-\frac{1-2 v}{\mu} F_{z}-\frac{1}{\mu} \frac{\partial\left(z F_{z}\right)}{\partial z} \tag{36}
\end{equation*}
$$

is known throughout $z>0$ and hence, by (13),

$$
(1-2 \nu) B_{z}-\frac{\partial \beta}{\partial z}=\frac{1}{4 \pi \mu} \int G\left[(1-2 \nu) F_{z}+\frac{\partial}{\partial \xi}\left(\zeta F_{z}\right)\right] d v
$$

Es the same process as before, using (17), (18), (22) and (23), we find

$$
\begin{equation*}
(1-2 \nu) B_{z}-\frac{\partial \beta}{\partial z}=\frac{P_{z}}{4 \pi \mu}\left[(1-2 \nu)\left(\frac{1}{R}-\frac{1}{R_{2}}\right)+c \frac{\partial}{\delta z}\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right)\right] . \tag{38}
\end{equation*}
$$

Finally, from (34) and (38),
$B_{z}=-\frac{P_{x}}{4 \pi \mu}\left[\frac{1}{R_{1}}+\frac{3-4 \nu}{R_{2}}+\frac{2 c(z+c)}{R_{2}^{3}}\right]$
$\beta=\frac{P_{3}}{4 \pi \mu}\left[4(1-2)(1-2 \nu) \log \left(R_{1}+z+c\right)-\frac{c}{R_{1}}-\frac{(3-4 \nu) c}{R_{2}}\right]$

These two functions constitute the solution for the case of the force at ( $0,0, C$ ) normal to the plane boundary.

## Force Parallel to the Plane Boundary

In this case we take $F_{y}=F_{z}=0$ and $B_{y}=0$. The boundary conditions then become, on $z=0$,

$$
\begin{align*}
& \sigma_{z z}=\frac{\mu}{2(1-\nu)}\left[2(1-\nu) \frac{\partial B_{x}}{\partial z}+2 \nu \frac{\partial B_{x}}{\partial x}-x \frac{\partial^{2} B_{x}}{\partial z^{2}}-\frac{\partial^{2} \beta}{\partial z^{2}}\right]=0  \tag{47}\\
& \sigma_{z x}=\frac{\mu}{2(1-\nu)}\left[(1-2 \nu)\left(\frac{\partial B_{z}}{\partial x}+\frac{\partial B_{x}}{\partial z}\right)-x \frac{\partial^{2} B_{x}}{\partial x \partial z}-\frac{\partial^{2} \beta}{\partial x \partial z}\right]=0  \tag{42}\\
& \sigma_{z y}=\frac{\mu}{2(1-\nu)}\left[(1-2 \nu) \frac{\partial B_{z}}{\partial y}-x \frac{\partial^{2} B_{x}}{\partial y \partial z}-\frac{\partial^{2} \beta}{\partial y \partial z}\right]=0 \tag{43}
\end{align*}
$$

Differentiating. (42) with respect to $y$ and (43) with respect to $x$ and subtracing, we find, on $z=0$,

$$
\frac{\partial^{2} B_{k}}{\partial y \partial z}=0
$$

Hence, on $z=0$,

$$
\begin{equation*}
\frac{\partial B_{x}}{\partial z}=0 \tag{44}
\end{equation*}
$$

Also, in $z=0$,

$$
\nabla^{2} \frac{\partial B_{X}}{\partial z}=-\frac{1}{\mu} \frac{\partial F_{x}}{\partial z}
$$

Hence, from (13),

$$
\begin{aligned}
\frac{\partial B_{x}}{\partial z} & =\frac{1}{4 \pi \mu} \int G \frac{\partial F_{x}}{\partial \zeta} d v \\
& =\frac{P_{x}}{4 \pi \mu} \frac{\partial}{\partial z}\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right)
\end{aligned}
$$

by (32). Thus

$$
\begin{equation*}
B_{x}=\frac{P_{x}}{4 \pi \mu}\left(\frac{1}{R_{1}}+\frac{1}{R_{2}}\right) \tag{45}
\end{equation*}
$$

From (43), on $z=0$,

$$
(1-2 \nu) B_{z}-\frac{\partial \beta}{\partial z}=x \frac{\partial B_{x}}{\partial z}=0
$$

by (44). Also, in $z=0$

$$
\nabla^{2}\left[(1-2 \nu) B_{z}-\frac{\partial \beta}{\partial z}\right]=-\frac{x}{\mu} \frac{\partial F_{x}}{\partial z}
$$

Hence, by (13)

$$
\begin{equation*}
(!-2 \nu) B_{z}-\frac{\partial \beta}{\partial z}=\frac{1}{4 \pi \mu} \int G \xi \frac{\partial F_{x}}{\partial \xi} d v . \tag{46}
\end{equation*}
$$

But the right hand side of (46) vanishes since $\xi \rightarrow 0$ as $T \rightarrow 0$. Hence

$$
\begin{equation*}
(1-2 \nu) B_{z}-\frac{\partial \beta}{\partial z}=0 \tag{47}
\end{equation*}
$$

throughout the region $z>0$.

$$
\text { From }(41) \text {, on } \geq=0 \text {, }
$$

$$
\begin{equation*}
2(1-\nu) \frac{\partial B_{z}}{\partial z}-\frac{\partial^{2} \beta}{\partial z^{2}}+2 \nu \frac{\partial B_{x}}{\partial x}-x \frac{\partial^{2} B_{x}}{\partial z^{2}}=0 \tag{48}
\end{equation*}
$$

Now, on $z=0$, we have, from (45),

$$
2 \nu \frac{\partial B_{x}}{\partial x}-x \frac{\partial^{2} B_{x}}{\partial z^{2}}=\frac{(1-2 \nu) P_{x} x}{2 \pi \mu R_{0}}-\frac{3 P_{x} c^{2} x}{2 \pi \mu R_{0}^{3}}
$$

where $R_{0}^{2}=x^{2}+y^{2}+c^{2}$. But, on $z=0$,

$$
-(1-2 v) \frac{\partial B_{x}}{\partial x}=\frac{(1-2 \nu) P_{x} x}{2 \pi \mu R_{x}}
$$

and

$$
\frac{P_{x} c}{2 \pi \mu} \frac{\partial^{2}}{\partial x \partial z}\left(\frac{1}{R_{z}}\right)=\frac{3 P_{x} c^{2} x}{2 \pi \mu R_{0}}
$$

Hence, (48) may be rewritten as

$$
\begin{equation*}
\chi=2(1-\nu) \frac{\partial B_{X}}{\partial z}-\frac{\partial^{2} \beta}{\partial z^{2}}-(1-2 \nu) \frac{\partial B_{x}}{\partial x}-\frac{P_{x} C}{2 \pi \mu} \frac{\partial^{2}}{\partial \times \partial z}\left(\frac{1}{R_{2}}\right)=0 \tag{4}
\end{equation*}
$$

on z-0. The Laplacian of the left side of (49) is, in the region $z \geqslant 0$,

$$
\nabla^{2} \chi=-\frac{x}{\mu} \frac{\partial^{2} F_{x}}{\partial z^{x}}+\frac{1-2 \nu}{\mu} \frac{\partial F_{a}}{\partial x}
$$

Hence, from (13),

$$
\begin{equation*}
X=\frac{1}{4 \pi \mu} \int G\left[\xi \frac{\partial^{2} F_{i n}}{\partial \xi^{2}}-(1-2 \nu) \frac{\partial F_{B}}{\partial \xi}\right] d v . \tag{50}
\end{equation*}
$$

The first term in the integrand vanishes since $\xi \rightarrow 0$ as $T \rightarrow 0$. The second term is integrated ky parts and the surface integral vanishes, leaving

$$
\chi=\frac{1-2 \nu}{4 \pi \mu} \int F_{x} \frac{\partial G}{\partial \xi} d v
$$

which, by (17) and (25), is

$$
\begin{equation*}
X=-\frac{(1-2 \nu) R_{1}}{4 \pi \mu} \frac{\partial}{\partial x}\left(\frac{1}{R_{1}}-\frac{1}{R_{2}}\right) \tag{51}
\end{equation*}
$$

Then, from (51), (49) and (45),

$$
\begin{equation*}
2(1-\nu) \frac{\partial B_{x}}{\partial z}-\frac{\partial^{2} \beta}{\partial z^{2}}=-\frac{(1-2 \nu) P_{x} x}{2 \pi \mu R_{2}^{3}}+\frac{3 P_{x} c \times(z+c)}{2 \pi \mu R_{2}^{5}} \tag{52}
\end{equation*}
$$

or

$$
\begin{equation*}
2(1-\nu) B_{z}-\frac{\partial \beta}{\partial z}=\frac{(1-2 \nu) P_{x} x}{2 \pi \mu R_{2}\left(R_{2}+z+c\right)}-\frac{P_{1} c x}{i \pi \mu R_{2}^{3}} \tag{53}
\end{equation*}
$$

Mnally, from (53) and (47), we have

$$
\begin{align*}
& B_{z}=\frac{(1-2 \nu) P_{x} x}{2 \pi \mu R_{2}\left(R_{2}+z+c\right)}-\frac{P_{x} c x}{2 \pi \mu R_{2}^{3}}  \tag{54}\\
& \beta=-\frac{(1-2 \nu)^{2} P_{x} x}{2 \pi \mu\left(R_{2}+z+c\right)}+\frac{(1-2 \nu) P_{x} c x}{2 \pi \mu R_{2}\left(R_{2}+z+c\right)} \tag{55}
\end{align*}
$$

These two functions, in addition to (45), comprese the solution for the case of the force at ( $0,0, \mathrm{c}$ ) parallel to the plane boundary.

## Comparison rith Previous Re日ults

The previous solution, mentioned in the Introduction, was given in terms of the Calerkin vector ${ }_{n}$ (not to be confused with the body force $F$ in the present paper).

For the case of a force normal to the plane boundary the solution obtcined was

$$
\begin{align*}
F=\frac{P_{2} R}{8 \pi(1-\nu)}\left\{R_{1}\right. & +[8 \nu(1-\nu)-1] R_{2}-2 c z / R_{2}  \tag{56}\\
& \left.+4(1-2 \nu)[(1-\nu) z-\nu c] \log \left(R_{2}+z+c\right)\right\}
\end{align*}
$$

and, for the case of a force parallel to the plane boundary,

$$
\begin{aligned}
F_{n}= & \frac{P_{x}^{i}}{8 T_{i}(i-\nu)}\left\{\begin{array}{r}
R_{1}+R_{2}-2 c^{2} / R_{2} \\
\\
\left.+4(1-\nu)(1-2 \nu)\left[(z+c) \log \left(R_{2}+z+c\right)-R_{2}\right]\right\}
\end{array}\right. \\
& +\frac{P_{x} R}{8 \pi(1-\nu)}\left\{2 c x / R_{2}+2(1-2 \nu) \times \log \left(R_{2}+z+c\right)\right\}
\end{aligned}
$$

The relation between the Gaierkin and Papkoritch functions has been shown to be 4

$$
\begin{align*}
& \mu B=(1-\nu) \nabla^{2} F  \tag{58}\\
& \mu \beta=(1-\nu)\left(2 \nabla \cdot F-r \cdot \nabla^{2} F\right) \tag{59}
\end{align*}
$$

By inserting (56) and (57) in (58) and (59), it may be verified that the previous and present solutions are identical.

