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Forced Convex *n*-Gons in the Plane*

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Abstract. In a seminal paper from 1935, Erdős and Szekeres showed that for each n there exists a least value $g(n)$ such that any subset of $g(n)$ points in the plane in general position must always contain the vertices of a convex n-gon. In particular, they obtained the bounds

$$2^{n-2} + 1 \le g(n) \le \binom{2n-4}{n-2} + 1,$$

which have stood unchanged since then. In this paper we remove the $+1$ from the upper bound for $n \ge 4$.

In 1935, Paul Erdős and George Szekeres published a short paper "A combinatorial problem in geometry" [1] which was destined to have a profound influence on the development of combinatorics (and especially Ramsey theory) during the next 60 years (see [3]). In particular, in this paper, Erdős and Szekeres rediscovered Ramsey's theorem, which had only just appeared (unknown to them) five years earlier. Their investigations arose from a geometrical question of the talented young mathematician Esther Klein (soon to become Mrs. Szekeres). She asked, "Is it true that for every n, there is a least value $g(n)$ such that any set X of $g(n)$ points in the plane in general position always contains the vertices of a convex n-gon?"

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Fig. 1. Caps and cups.

Erdős and Szekeres gave several proofs of the existence of $g(n)$ in [1] and established the following bounds:

$$2^{n-2} + 1 \le g(n) \le \binom{2n-4}{n-2} + 1. \tag{1}$$

They also conjectured that the lower bound in (1) in fact always holds with equality. This is known [2] to be the case for $n \le 5$. Despite repeated attempts over the years, no general improvement on (1) has been found.

In this note, we make a very small improvement on the upper bound of (1). Namely, we show

$$g(n) \le \binom{2n-4}{n-2} \tag{2}$$

for $n \ge 4$.

While this is admittedly rather modest, we hope[1] that it might suggest methods which could give rise to more substantial reductions in the upper bound.

By an *m-cap* we mean a sequence of m points $x_1, x_2, \ldots, x_m$ such that the polygonal path connecting them is concave, i.e., the x_i have increasing x-coordinates and the path from x_1 to x_m turns clockwise at each intermediate vertex. Similarly, an *m-cup* is a set of points $y_1, y_2, \ldots, y_m$ with increasing x-coordinates such that the polygonal path joining them is convex, i.e., the path from y_1 to y_m always turns counterclockwise.

The following result from [1] follows easily by induction:

Lemma 1. *If $X \subset \mathbb{E}^2$ is in general position and $|X| > \binom{a+b-4}{a-2}$, then X contains either an a-cap or a b-cup.*

In fact, as shown in [1], this bound is sharp.

Theorem 1. *If $X \subset \mathbb{E}^2$ is in general position and $|X| \ge \binom{2n-4}{n-2}$ for $n \ge 4$, then X contains the vertices of a convex n-gon.*

Proof. Suppose the contrary. Rotate X if necessary so that no line determined by two points of X is either horizontal or vertical. We can further assume without loss of generality that all lines determined by two points of X have slopes less than 0.1 in absolute value (by uniformly compressing X in the y-direction, if necessary).

Define $A := \{x \in X : x$ is the left-hand endpoint of some $(n-1)$-cap in $X\}$.

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[1] In fact, this is exactly what happened! See *Note added in proof.*

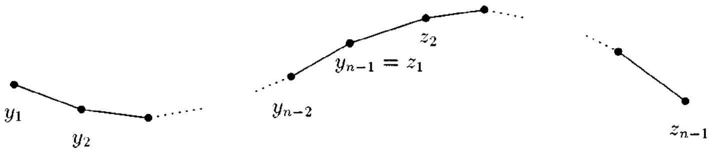

Fig. 2. A cup joining a cap.

Case 1: $|A| > \binom{2n-5}{n-3}$.
Then by Lemma 1, A contains an $(n-1)$-cup, say, $y_1, y_2, \ldots, y_{n-1}$. Since $y_{n-1} \in A$, there exists an $(n-1)$-cap $y_{n-1} = z_1, z_2, \ldots, z_{n-1}$ in X. However, this is impossible since either $y_1, y_2, \ldots y_{n-1}, z_2$ is an n-cup, or $y_{n-2}, z_1, z_2, \ldots, z_{n-1}$ is an n-cap (see Fig. 2).

Case 2: $|A| < \binom{2n-5}{n-3}$.
Then $B := X \backslash A$ satisfies $|B| > \binom{2n-4}{n-2} - \binom{2n-5}{n-3} = \binom{2n-5}{n-3}$. Again, by Lemma 1, B must contain an $(n-1)$-cup. However, this is impossible by the definition of A.

This leaves as the only possibility:

Case 3: $|A| = |B| = \binom{2n-5}{n-3} = \frac{1}{2}\binom{2n-4}{n-2}$.
For any $b \in B$, consider the set $A \cup \{b\}$. Since this set has size greater than $\binom{2n-5}{n-3}$ then by Lemma 1, it contains an $(n-1)$-cup, say with right-hand endpoint y. Now, if $y \in A$, then as in Case 1, we reach a contradiction. Hence we must have $y = b$.

Thus, each $b \in B$ is the right-hand endpoint of an $(n-1)$-cup with left-hand endpoint in A. It follows in a similar way that each $a \in A$ is the left endpoint of an $(n-1)$-cap with right-hand endpoint in B.

We now form a directed bipartite graph G with vertex sets A and B, and edge set E consisting of all pairs (u, v), where either $u \in A$ is the left-hand endpoint and $v \in B$ is the right-hand endpoint of some $(n-1)$-cap in X, or $v \in A$ is the left-hand endpoint and $u \in B$ is the right-hand endpoint of some $(n-1)$-cup in X.

By the preceding remarks, it follows that all vertices of G have outdegree at least 1. This implies G has some (directed) cycle $C = a_{i_1} b_{i_1} \cdots a_{i_r} b_{i_r}$.

Now consider an edge $(a, b) \in E$. Let $L^+(a)$ denote the half-line starting at a and going down with slope 0.1, and let $R^-(b)$ denote the half-line starting at b and going

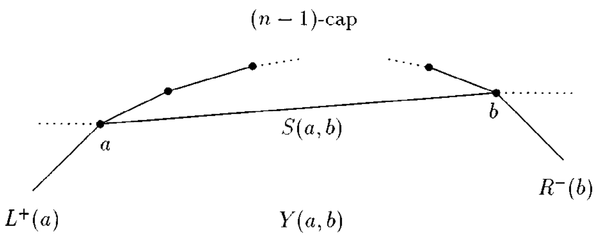

Fig. 3

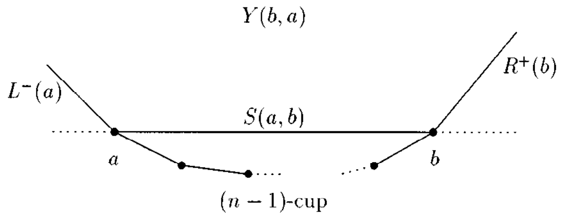

Fig. 4

down with slope -0.1. Also, let $S(a, b)$ denote the line segment joining a and b. Finally, let $Y(a, b)$ denote the region of $\mathbb{E}^2$ (strictly) below the path $L^+(a)S(a, b)R^-(b)$ (see Fig. 3). $\square$

Claim 1. *X has no point in $Y(a, b)$.*

Otherwise, if $x \in X \cap Y(a, b)$, then the $(n-1)$-cap spanned by (a, b) together with x forms a convex n-gon in X, which is a contradiction.

By an analogous argument for $(b, a) \in E$, with $L^-(a)$, $R^+(b)$, $Y(b, a)$ defined accordingly (see Fig. 4), we also see that $Y(b, a)$ can contain no point of X.

Next, consider two connected edges (a, b) and (b, a') in E. We cannot have $a = a'$, since if we did, then X would contain a convex $(2n-4)$-*gon* (formed by the $(n-1)$-cap and $(n-1)$-cup spanned by a and b), which is impossible.

Claim 2. *a' must lie* above *the line through a and b.*

Proof. Suppose not. Then from the geometry of the situation (see Fig. 5), either $a' \in Y(a, b)$ or $a \in Y(b, a')$, a contradiction. A similar argument shows if $(b, a) \in E$ and $(a, b') \in E$ then b' must lie below the line through b and a. $\square$

Finally, consider the cycle $C = a_{i_1} b_{i_1} \cdots a_{i_r} b_{i_r}$ occurring in G. If $r = 1$, then we find a convex $(2n-4)$-gon, which is impossible. So, we may assume $r \geq 2$. By

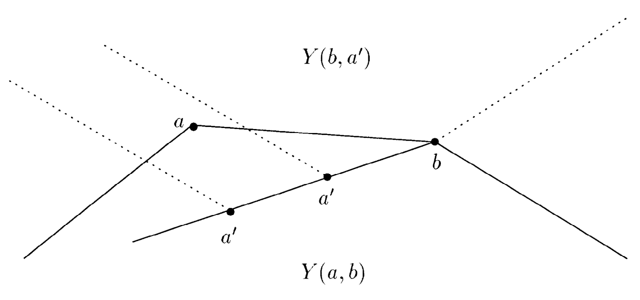

Fig. 5

Claim 2, each of the angles between adjacent edges, $a_{i_1}b_{i_1}, b_{i_1}a_{i_2}, a_{i_2}b_{i_2} \cdots a_{i_r}b_{i_r}, b_{i_r}a_{i_1}$ must turn in a counterclockwise direction. Hence, the lines through the consecutive edges $a_{i_1}b_{i_1}, b_{i_1}a_{i_2}, a_{i_2}b_{i_2} \cdots$, have decreasing slopes. However, since C is a cycle, we reach a contradiction. □

References

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2. P. Erdős and G. Szekeres, On some extremum problems in elementary geometry, *Ann. Univ. Sci. Budapest. Eötvös Sect. Math.*, **3–4** (1961), 53–62.
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Note added in proof. We were pleased to learn that D. Kleitman and L. Pachter, by cleverly analyzing the preceding situation more carefully, have managed to lower the upper bound on $g(n)$ to $\binom{2n-4}{n-2} + 7 - 2n$. We also wish to thank them for pointing out a simplification of our earlier argument. Very shortly after this improvement, G. Tóth and P. Valtr further reduced the upper bound on $g(n)$ to $\binom{2n-5}{n-2} + 2$, which is the current record.

We are inclined to believe (as did Erdős and Szekeres) that the lower bound $2^{n-2} + 1$ is the true value of $g(n)$. However, we admit that there is little real evidence yet for this belief. A first step would be to show that $g(n) = O((4 - c)^n)$ for some $c > 0$, a result for which the authors gladly offer \$100 for the first proof.