Forced Symmetric Oscillations

Michal Fečkan* Ruyun Ma† Bevan Thompson

Abstract

We show the existence of forced periodic solutions to certain symmetric ordinary differential equations. First and second order systems of ordinary differential equations are investigated with and without damping with periodic and symmetric forcings. We study both resonance and nonresonance cases.

1 Introduction

Let $A : \mathbb{R}^n \to \mathbb{R}^n$ be an orthogonal matrix with respect to a scalar product $(\cdot, \cdot)$ on $\mathbb{R}^n$, i.e. $A^* = A^{-1}$. So $|Ax| = |x|$ for any $x \in \mathbb{R}^n$. We also suppose that $A^p = \mathbb{I}$ for some $p \in \mathbb{N}$. Let $T > 0$ be fixed. Then we consider the Banach spaces

$$\begin{aligned} X &:= \left\{ x \in C^0(\mathbb{R}, \mathbb{R}^n) \mid x(t+T) = Ax(t), \quad \forall t \in \mathbb{R} \right\}, \\ Y &:= X \cap C^1(\mathbb{R}, \mathbb{R}^n)\,. \end{aligned}$$

Clearly, if $x \in X$ then it is pT-periodic. Moreover, $|x(t)|$ is T-periodic, since $|x(t+T)| = |Ax(t)| = |x(t)|$.

Now we consider the differential equation

$$\dot{x} = f(x) + h(t)\,, \tag{1.1}$$

*Support from the VEGA-MS 1/2001/05, and the Raybould Fellowship is gratefully acknowledged

†Support from the NSFC (No. 10271095), GG-110-10736-1003, NWNU-KJCXGC-212, the Foundation of Excellent Young Teacher of the Chinese Education Ministry, and the ARC Centre of Excellence for Mathematics and Statistics of Complex Systems is gratefully acknowledged

Received by the editors November 2005.

Communicated by J. Mawhin.

1991 *Mathematics Subject Classification :* 34C14, 34C15.

Key words and phrases : periodic solution, symmetric systems, topological degree.

for $h \in X$, $f(x) = \nabla F(x)$ and $F : \mathbb{R}^n \to \mathbb{R}$ smooth and A-invariant, i.e. $F(Ax) = F(x)$, $\forall x \in \mathbb{R}^n$.

When $A = -\mathbb{I}$ then we get the anti-periodic case studied in [2]. Our results for $1 \notin \sigma(A)$ of Section 2 are extensions of similar ones for the anti-periodic case $A = -\mathbb{I}$. We call that case a nonresonant one, since we derive existence results without further conditions.

On the other hand, we get for $1 \in \sigma(A)$ a different situation studied in Section 3. We call that case a resonant one, since the solvability of studied equations is reduced to a global bifurcation condition. So we need addition assumptions on such equations in order to solve that bifurcation condition. In both Sections 2 and 3, we also derive similar results for the damped system

$$\ddot{x} + \delta\dot{x} + \nabla F(x) = h(t) \tag{1.2}$$

with $\delta > 0$. There we use topological degree arguments.

The last Section 4 is devoted to undamped symmetric second order equations with gradient nonlinearities, i.e. to (1.2) with $\delta = 0$, which are not studied in previous Sections 2 and 3. That is again a resonant case. Hence we use a Ljapunov-Schmidt method to get a global characterization of forcing terms $h(t)$ for which studied equations are solvable. There we use the Banach fixed point method to solve such equations.

Finally we note that we have already investigated related problems in [3] for symmetric heteroclinic cycles of perturbed symmetric ordinary differential equations, and more recently, in [1] for symmetric evolution partial differential equations, respectively.

2 Nonresonance Systems

We start with the following results

Lemma 1. It holds $f(Ax) = Af(x)$, $\forall x \in \mathbb{R}^n$.

Proof. Differentiating $F(Ax) = F(x)$ we get $DF(Ax)Av = DF(x)v$. Since $DF(x)v = (\nabla F(x), v)$, we have $(\nabla F(Ax), Av) = (\nabla F(x), v)$ and so $A^*\nabla F(Ax) = \nabla F(x)$, and $A^{-1}\nabla F(Ax) = \nabla F(x)$ which implies $\nabla F(Ax) = A\nabla F(x)$. The proof is finished.

Lemma 2. *If $1 \notin \sigma(A)$ then there is a constant $M > 0$ such that*

$$M\|x\|_0 \le \|\dot{x}\|_2 \tag{2.1}$$

for any $x \in Y$, where $\|x\|_0 := \max_{t\in\mathbb{R}} |x(t)|$ and $\|x\|_2 := \sqrt{\int_0^T |x(t)|^2\, dt}$.

Proof. We solve the boundary value problem

$$\begin{aligned} \dot{x}(t) &= h(t) \in X \\ x(T) &= Ax(0)\,. \end{aligned}$$

We get $x(t) = x(0) + \int_0^t h(s)\,ds$ and

$$Ax(0) = x(T) = x(0) + \int_0^T h(s)\,ds\,,$$
$$x(0) = (A - \mathbb{I})^{-1} \int_0^T h(s)\,ds\,.$$

So

$$x(t) = (A - \mathbb{I})^{-1} \int_0^T h(s)\,ds + \int_0^t h(s)\,ds\,.$$

The proof is finished.

Remark 1. If $1 \in \sigma(A)$ then (2.1) is not valid, since then $\exists v \in \mathbb{R}^n$ such that $Av = v$ and a constant function $x(t) = v$ belongs to Y, for which (2.1) is not true.

Theorem 1. *For any $h \in X$, equation* (1.1) *has a solution $x \in Y$.*

Proof. By using the Leray-Schauder topological degree method [5], it is enough to prove that there is a constant $K > 0$ such that any solution $x \in Y$ of (1.1) satisfies

$$\|x\|_0 \le K\,.$$

So let $x \in Y$ solve (1.1). Then we get

$$\begin{aligned}\|\dot{x}\|_2^2 = {} & \int_0^T \Big(\nabla F(x(t)), \dot{x}(t)\Big)\,dt + \int_0^T \Big(h(t), \dot{x}(t)\Big)\,dt = \\ & F(x(T)) - F(x(0)) + \int_0^T \Big(h(t), \dot{x}(t)\Big)\,dt = \\ & F(Ax(0)) - F(x(0)) + \int_0^T \Big(h(t), \dot{x}(t)\Big)\,dt = \\ & F(x(0)) - F(x(0)) + \int_0^T \Big(h(t), \dot{x}(t)\Big)\,dt \\ & \le \|h\|_2 \|\dot{x}\|_2\,.\end{aligned}$$

Hence

$$\|\dot{x}\|_2 \le \|h\|_2\,.$$

Then Lemma 2 gives

$$K = \|h\|_2 / M\,.$$

The proof is finished.

Similarly, if we consider (1.2) for $\delta > 0$, then we have

Theorem 2. *For any $h \in X$, equation* (1.2) *has a solution $x \in Z := X \cap C^2(\mathbb{R}, \mathbb{R}^n)$.*

Proof. Again, it is enough to show the existence of a constant $K > 0$ such that any solution of (1.2) satisfies $\|x\|_0 \le K$. So let $x \in Z$ solve (1.2). Then

$$\int_0^T (\ddot{x}(t), \dot{x}(t))\, dt + \delta \int_0^T |\dot{x}(t)|^2\, dt + \int_0^T \Big(\nabla F(x(t)), \dot{x}(t)\Big)\, dt = \int_0^T (h(t), \dot{x}(t))\, dt\,.$$

Hence

$$\frac{|\dot{x}(T)|^2 - |\dot{x}(0)|^2}{2} + \delta \int_0^T |\dot{x}(t)|^2\, dt \le \|h\|_2 \|\dot{x}\|_2\,.$$

But $|\dot{x}(T)| = |A\dot{x}(0)| = |\dot{x}(0)|$. Consequently, we get

$$K = \frac{1}{M\delta}\|h\|_2\,.$$

The proof is finished.

Example 1. Consider $n = 2$ and

$$A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

Then (1.1) has the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + h_1(t) \\ \dot{x}_2 &= f_2(x_1, x_2) + h_2(t)\,, \end{aligned} \tag{2.2}$$

where $f_i(x_1, x_2) = \frac{\partial F}{\partial x_i}(x_1, x_2)$, $F \in C^1(\mathbb{R}^2, \mathbb{R})$ with $F(-x_2, x_1) = F(x_1, x_2)$. This gives

$$\begin{aligned} &f_1(x_1, x_2) = f_2(-x_2, x_1), \quad f_2(x_1, x_2) = -f_1(-x_2, x_1) \\ &h_1(t + T) = -h_2(t), \quad h_2(t + T) = h_1(t)\,. \end{aligned} \tag{2.3}$$

Consequently, Theorem 1 can be applied to a forced gradient system (2.2) under symmetry conditions (2.3).

3 Resonance Systems

Now we study the case when $1 \in \sigma(A)$. Let

$$R_1 := \ker(\mathbb{I} - A)\,, \quad R_2 := R_1^{\perp}\,.$$

We consider the Banach spaces

$$\begin{aligned} &X_i := \Big\{x \in X \mid x(t) \in R_i\Big\}\,, \\ &Y_i := X_i \cap C^1(\mathbb{R}, \mathbb{R})\,, \\ &Z_i := X_i \cap C^2(\mathbb{R}, \mathbb{R})\,, \quad i = 1, 2\,. \end{aligned}$$

We note that X_1, Y_1 and Z_1 are just Banach spaces of T-periodic functions. We need the following result.

Lemma 3. *There is a constant $M > 0$ such that inequality (2.1) holds for any $x \in Y_2$.*

Proof. We again solve the boundary value problem

$$\begin{aligned} &\dot{x}(t) = h(t) \in X_2 \\ &x(T) = Ax(0), \quad x \in Y_2\,. \end{aligned}$$

We get $x(t) = x(0) + \int_0^t h(s)\,ds$ and

$$(A - \mathbb{I})x(0) = \int_0^T h(s)\,ds\,.$$

Now we have

$$h(t) \in X_2 \Leftrightarrow h(t) \perp \ker(\mathbb{I} - A) \Leftrightarrow h(t) \in \mathrm{Im}\,(\mathbb{I} - A^*)\,.$$

But since

$$\mathbb{I} - A^* = \mathbb{I} - A^{-1} = A^{-1} \circ (A - \mathbb{I}) = (A - \mathbb{I}) \circ A^{-1}\,,$$

we obtain

$$\mathrm{Im}\,(\mathbb{I} - A) = \mathrm{Im}\,(\mathbb{I} - A^*) = \ker(\mathbb{I} - A)^{\perp}\,.$$

Consequently, $R_2 = \mathrm{Im}\,(\mathbb{I} - A)$ and hence

$$h(t) \in \mathrm{Im}\,(\mathbb{I} - A)$$

for any $t \in \mathbb{R}$. Hence $\int_0^T h(s)\,ds \in \mathrm{Im}\,(\mathbb{I} - A)$, and then

$$x(0) = (A - \mathbb{I})^{-1} \int_0^T h(s)\,ds\,,$$

where $(A - \mathbb{I})^{-1} : R_2 \to R_2$. Thus

$$x(t) = (A - \mathbb{I})^{-1} \int_0^T h(s)\,ds + \int_0^t h(s)\,ds \in R_2$$

for any $t \in \mathbb{R}$. The proof is finished.

Let $P : \mathbb{R}^n \to \mathbb{R}^n$ be the orthogonal projection onto R_2, and $Q := \mathbb{I} - P$. We split any $x \in X$ by

$$x(t) = u(t) + v(t), \quad v(t) \in R_1, \quad u(t) \in R_2, \quad \forall t \in \mathbb{R}\,, \tag{3.1}$$

and then we decompose (1.1) as follows

$$\dot{u}(t) = P\nabla F\Big(u(t)+v(t)\Big) + h_2(t) \tag{3.2}$$

and

$$\dot{v}(t) = Q\nabla F\Big(u(t)+v(t)\Big) + h_1(t) \tag{3.3}$$

for $h_i(t) \in X_i$, $i = 1, 2$.

Lemma 4. *Decomposition* (3.1) *holds if and only if* $v \in X_1$ *and* $u \in X_2$.

Proof. Let (3.1) hold. Then from $Ax(t) = x(t+T)$ we get

$$Au(t) - u(t+T) = v(t+T) - v(t)\,,$$

and so

$$R_2 \ni (A - \mathbb{I})u(t) + u(t) - u(t+T) = v(t+T) - v(t) \in R_1\,.$$

Hence

$$v(t) = v(t+T), \quad Au(t) = u(t+T)$$

which gives $v \in X_1$ and $u \in X_2$. Reversely, if $v \in X_1$ and $u \in X_2$ then clearly $v(t) \in R_1$ and $u(t) \in R_2$, $\forall t \in \mathbb{R}$. The proof is finished.

Lemma 5. *If* $v \in X_1$ *and* $u \in X_2$ *then*

$$Q\nabla F(u+v) \in X_1, \quad P\nabla F(u+v) \in X_2\,.$$

Proof. We must show

$$Q\nabla F\Big(u(t+T) + v(t+T)\Big) = Q\nabla F\Big(u(t)+v(t)\Big)\,, \tag{3.4}$$

and

$$P\nabla F\Big(u(t+T) + v(t+T)\Big) = AP\nabla F\Big(u(t)+v(t)\Big)\,. \tag{3.5}$$

But since $Q \circ (\mathbb{I} - A) = 0$, $u(t+T) = Au(t)$ and $v(t+T) = v(t) = Av(t)$, we get

$$\begin{aligned} &Q\nabla F\Big(u(t+T) + v(t+T)\Big) = Q\nabla F\Big(Au(t) + Av(t)\Big) = \\ &QA\nabla F\Big(u(t)+v(t)\Big) = Q\nabla F\Big(u(t)+v(t)\Big)\,. \end{aligned}$$

This gives (3.4). By using $A \circ P = P \circ A$, which follows from $(\mathbb{I} - A) \circ Q = 0$ and $Q \circ (\mathbb{I} - A) = 0$, we similarly derive (3.5). The proof is finished.

Now we split

$$v(t) = z(t) + w, \quad \int_0^T z(s)\,ds = 0, \quad w \in R_1\,,$$

and decompose (3.2)-(3.3) as follows

$$\dot{u}(t) = P\nabla F\Big(u(t) + z(t) + w\Big) + h_2(t) \tag{3.6}$$

and

$$\dot{z}(t) = Q\nabla F\Big(u(t) + z(t) + w\Big) + h_{11}(t) \\ -\frac{1}{T}\int_0^T Q\nabla F\Big(u(t) + z(t) + w\Big)\,dt \tag{3.7}$$

and

$$0 = \frac{1}{T}\int_0^T Q\nabla F\Big(u(t) + z(t) + w\Big)\,dt + \bar{h}_1 \tag{3.8}$$

with $h_1(t) = h_{11}(t) + \bar{h}_1$, $\bar{h}_1 := \frac{1}{T}\int_0^T h_1(s)\,ds$. Now it is known [2] that

$$\|z\|_0 \le \sqrt{T}\|\dot{z}\|_2\,. \tag{3.9}$$

Furthermore, from (3.6)-(3.7) we have

$$\|\dot{u}\|_2^2 + \|\dot{z}\|_2^2 = \int_0^T \Big(\nabla F(u(t) + z(t) + w), \dot{u}(t) + \dot{z}(t)\Big)\,dt + \\ \int_0^T (h_2(t), \dot{u}(t))\,dt + \int_0^T (h_{11}(t), \dot{z}(t))\,dt \le \\ F(u(T) + z(T) + w) - F(u(0) + z(0) + w) + \|h_2\|_2\|\dot{u}\|_2 + \|h_{11}\|_2\|\dot{z}\|_2 = \\ F(A(u(0) + z(0) + w)) - F(u(0) + z(0) + w) + \|h_2\|_2\|\dot{u}\|_2 + \|h_{11}\|_2\|\dot{z}\|_2 = \\ \|h_2\|_2\|\dot{u}\|_2 + \|h_{11}\|_2\|\dot{z}\|_2 \le \sqrt{\|h_2\|_2^2 + \|h_{11}\|_2^2}\sqrt{\|\dot{u}\|_2^2 + \|\dot{z}\|_2^2}\,.$$

Consequently, we obtain

$$\sqrt{\|\dot{u}\|_2^2 + \|\dot{z}\|_2^2} \le \sqrt{\|h_2\|_2^2 + \|h_{11}\|_2^2}\,.$$

By using Lemma 3 and (3.9), we get

$$\|u\|_0 + \|z\|_0 \le (M^{-1} + \sqrt{T})\sqrt{\|h_2\|_2^2 + \|h_{11}\|_2^2}\,. \tag{3.10}$$

Inequality (3.10) gives a uniform estimate of all possible solutions of (3.6)-(3.7) for any $w \in R_1$. But then the set

$$S_w := \Big\{(u,z) \in X_2 \times X_1 \mid (u,z) \quad \text{solves} \quad (3.6) - (3.7)\Big\}$$

is a nonempty and compact subset. So we can consider the map $w \to B(w)$, $B : R_1 \to 2^{R_1}$ given by

$$B(w) := \left\{\frac{1}{T}\int_0^T Q\nabla F(u(t) + z(t) + w)\,dt \mid (u,z) \in S_w\right\}\,.$$

Clearly, the sets $B(w)$ are nonempty and compact. Moreover, the mapping B is upper-semicontinuous. Summarizing, we have the following result.

Theorem 3. *If* $1 \in \sigma(A)$ *then* (1.1) *is solvable if and only if* $-\bar{h}_1 \in \operatorname{Im} B(\cdot)$.

To get more reasonable results than Theorem 3, we have to consider additional conditions for F.

Theorem 4. *If there is a nonsingular matrix $C : R_1 \to R_1$, a nondecreasing function $\phi : [0,\infty) \to [0,\infty)$ and positive constants $\alpha \geq 1$, c_1, c_2 such that*

i) $(\nabla F(w), Cw) \geq c_1|w|^{\alpha+1} - c_2$, $\forall w \in R_1$,

ii) $(\nabla F(u+w) - \nabla F(w), Cw) \geq -|w|\phi(|u|)(|w|^{\alpha-1}+1)$, $\forall u \in \mathbb{R}^n$, $\forall w \in R_1$,

then (1.1) *has a solution $x \in Y$ for any $h \in X$.*

Proof. For any $u \in \mathbb{R}^n$, $|u| \leq r$, $w \in R_1$, we have

$$\begin{aligned} &(\nabla F(u+w), Cw) = (\nabla F(w), Cw) + (\nabla F(u+w) - \nabla F(w), Cw) \\ &\geq c_1|w|^{\alpha+1} - c_2 - |w|\phi(r)(|w|^{\alpha-1}+1)\,. \end{aligned} \tag{3.11}$$

Let r be the right-hand side of (3.10). Now we put the right-hand side of (3.8) into the homotopy

$$\frac{1}{T}\int_0^T Q\nabla F\Big(\lambda(u(t)+z(t))+w\Big)\,dt + \lambda\bar{h}_1 = 0 \tag{3.12}$$

for $\lambda \in [0,1]$. So we solve (3.6), (3.7) and (3.12). Then the estimate (3.11) implies that for $|w| = R$ with R sufficiently large, we have to compute the Brouwer degree

$$\deg\Big(Q\nabla F, B_R, 0\Big)\,, \tag{3.13}$$

which is just the topological degree to the system (3.6), (3.7) and (3.12). But from i) for $0 \leq \lambda \leq 1$, $u \in R_1$, $|u| = R \gg 1$, we get

$$\begin{aligned} &(\lambda Q\nabla F(u) + (1-\lambda)Cu, Cu) \geq \lambda c_1|u|^{\alpha+1} - \lambda c_2 + \\ &(1-\lambda)\frac{1}{\|C^{-1}\|^2}|u|^2 = \lambda\Big(c_1R^{\alpha+1} - c_2\Big) + \frac{1-\lambda}{\|C^{-1}\|^2}R^2 > 0\,. \end{aligned}$$

So the Brouwer degree (3.13) is equal to $\operatorname{sgn}\det C \neq 0$. Consequently, (1.1) is solvable. The proof is finished.

Similarly we have the next result.

Theorem 5. *If there is an open bounded subset $\Omega \subset R_1$ such that $0 \notin Q\nabla F(\partial\Omega)$ and*

$$\deg\Big(Q\nabla F, \Omega, 0\Big) \neq 0\,,$$

then for any $\varepsilon \neq 0$ small, the equation

$$\dot{x} = f(x) + \varepsilon h(t)$$

has a solution $x \in Y$.

We note that the above approach is valid also for (1.2). Consequently, similar results like Theorems 3, 4, 5 hold for (1.2) as well.

Example 2. Consider $n = 2$ and

$$A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} .$$

Then (1.1) has the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) + h_1(t) \\ \dot{x}_2 &= f_2(x_1, x_2) + h_2(t) \,, \end{aligned}$$

where again $f_i(x_1, x_2) = \frac{\partial F}{\partial x_i}(x_1, x_2)$, $F \in C^1(\mathbb{R}^2, \mathbb{R})$ with $F(x_1, -x_2) = F(x_1, x_2)$. This gives

$$\begin{aligned} &f_1(x_1, -x_2) = f_1(x_1, x_2), \quad f_2(x_1, -x_2) = -f_2(x_1, x_2) \\ &h_1(t + T) = h_1(t), \quad h_2(t + T) = -h_2(t) \,. \end{aligned}$$

Now

$$\begin{aligned} R_1 &= \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_2 = 0\Big\} \\ R_2 &= \Big\{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 = 0\Big\} \\ Q\nabla F(x_1, x_2) &= f_1(x_1, x_2) \,. \end{aligned}$$

Then Theorem 4 can be applied, for instance, if

$$f_1(x_1, x_2) = x_1^{2n+1} + g(x_1, x_2)$$

for g bounded on $\mathbb{R}^2$. To be more concrete, we can take

$$F(x_1, x_2) = \frac{x_1^{2n+2}}{2n+2} + x_1 \cos x_2 \,.$$

Then $g(x_1, x_2) = \cos x_2$.

4 Undamped Second Order Equations

In this section, we study (1.2) for $\delta = 0$ of the form

$$\ddot{x} + \nabla F(x) = h(t) \,. \tag{4.1}$$

Then Theorem 2 is not valid for (4.1). In order to solve (4.1), we use Theorem 1 of [4], which we quote here for the reader's convenience:

Theorem A. *Let H be a Hilbert space. Assume that L is selfadjoint and N is a continuous gradient operator. Let $\widetilde{A}, \widetilde{B} : H \to H$ be two continuous linear and selfadjoint operators such that the following hold:*

i) *$N - \widetilde{A}$ and $\widetilde{B} - N$ are monotone,*

ii) $L-(1-\lambda)\widetilde{A}-\lambda\widetilde{B}$ *has a bounded inverse for every* $\lambda \in [0,1]$,

then $L-N$ *is a bijection from* $D(L)$ *to* H.

In our case we take

$$\begin{aligned} &H = L^2((0,T),\mathbb{R}^n), \quad Lx = \ddot{x}, \quad N(x)(t) = -\nabla F(x(t))\,, \\ &D(L) = \Big\{x \in W^{2,2}((0,T),\mathbb{R}^n) \mid Ax(0) = x(T), \quad A\dot{x}(0) = \dot{x}(T)\Big\}\,. \end{aligned}$$

It is not difficult to verify that L is really selfadjoint with a point spectrum $\sigma(L)$. So in order to find $\sigma(L)$, we solve

$$\ddot{x} = \lambda x, \quad x \in D(L)\,. \tag{4.2}$$

Elementary calculation shows that (4.2) has the following solutions:

i) $\lambda = 0$ if and only if $1 \in \sigma(A)$ and then $x(t) = v \in \ker(\mathbb{I}-A)$,

ii) $\lambda = -\omega^2 < 0$ with $\cos\omega T \in \sigma\Big(\frac{A^*+A}{2}\Big)$.

We note

$$\Big(\frac{A^*+A}{2}x, x\Big) = (Ax, x)\,,$$

which implies

$$-|x|^2 \le \Big(\frac{A^*+A}{2}x, x\Big) \le |x|^2\,.$$

So

$$\sigma\Big(\frac{A^*+A}{2}\Big) \subset [-1,1]\,.$$

We remark that $1 \in \sigma\Big(\frac{A^*+A}{2}\Big)$ if and only if $1 \in \sigma(A)$. Summarizing we obtain the following result

Lemma 6. *It holds* $\sigma(L) = \mathcal{M}$ *for*

$$\mathcal{M} := \Big\{-\Big(\frac{\omega+2k\pi}{T}\Big)^2 \mid \omega \in [0,\pi], \quad \cos\omega \in \sigma\Big(\frac{A^*+A}{2}\Big), \quad k \in \mathbb{Z}\Big\}\,.$$

Now we can apply Theorem A to derive the next result

Theorem 6. *If there are two constants* $\alpha < \beta$ *such that*

i) $[\alpha,\beta] \cap \mathcal{M} = \emptyset$,

ii) $\sigma(\operatorname{Hess} F(u)) \subset [-\beta,-\alpha]$, $\forall u \in \mathbb{R}^n$.

Then (4.1) *has a unique solution* $x \in Z$ *for any* $h \in X$.

Proof. We apply Theorem A to (4.1) in the above framework with $\widetilde{A} = \alpha\mathbb{I}$ and $\widetilde{B} = \beta\mathbb{I}$. The proof is finished.

When h is only pT-periodic continuous in (4.1) then we can apply the above procedure, but then we get

$$\mathcal{M}_p = \left\{ -\frac{4k^2\pi^2}{p^2T^2} \mid k \in \mathbb{Z} \right\}$$

instead of $\mathcal{M}$. Indeed, we have here the case $A = \mathbb{I}$ and T is replaced by pT. The set $\mathcal{M}_p$ for large p is denser than the set $\mathcal{M}$. This gives that for some F we can show the existence of pT-periodic solution of (4.1) with $h \in Y$ but not for general pT-periodic continuous h. For instance, if $1 \notin \sigma(A)$ then $0 \notin \mathcal{M}$ while $0 \in \mathcal{M}_p$. If F satisfies i), ii) of Theorem 6 for $\alpha < 0 < \beta$ then we have a unique solution $x \in Z$ for any $h \in Y$, while for a general continuous pT-periodic h we do not know the existence of pT-periodic solution of (4.1). More precisely, assuming conditions i), ii) along with $[\alpha, \beta] \cap \mathcal{M}_p = \{0\}$ and then by using the Ljapunov-Schmidt reduction procedure and Theorem A like in [6] to (4.1), we obtain the bifurcation equation

$$\frac{1}{pT} \int_0^{pT} \nabla F(\psi(c, h_2)(t) + c)\, dt = h_1 \tag{4.3}$$

for a mapping ψ and

$$h(t) = h_2(t) + h_1, \quad \int_0^{pT} h_2(t)\, dt = 0, \quad \int_0^{pT} \psi(c, h_2)(t)\, dt = 0, \quad c \in \mathbb{R}^n .$$

Hence the existence of a pT-periodic solution of (4.1) for pT-periodic continuous h is reduced to (4.3). Summarizing we arrive at the particular following result.

Theorem 7. *Let* $1 \notin \sigma(A)$. *If there are two constants* $\alpha < 0 < \beta$ *such that*

a) $[\alpha, \beta] \cap \mathcal{M} = \emptyset$,

b) $[\alpha, \beta] \cap \mathcal{M}_p = \{0\}$,

c) $\sigma(\operatorname{Hess} F(u)) \subset [-\beta, -\alpha]$, $\forall u \in \mathbb{R}^n$,

d) ∇F *is uniformly bounded on* $\mathbb{R}^n$, *i.e.* $\sup_{u \in \mathbb{R}^n} |\nabla F(u)| < \infty$,

then for any $h \in X$, (4.1) *has a unique solution* $x \in Z$, *which is of course* pT*-periodic. While there is a huge set* $\mathcal{S}$ *of* pT*-periodic continuous functions* $h \in C_{pT}(\mathbb{R}, \mathbb{R}^n)$ *for which* (4.1) *has no* pT*-periodic solution.*

Proof. Let $M_1 := \sup_{u \in \mathbb{R}^n} |\nabla F(u)|$. Then the mentioned set $\mathcal{S}$ is given by

$$\mathcal{S} := \left\{ h \in C_{pT}(\mathbb{R}, \mathbb{R}^n) \mid |h_1| > M_1, \quad h(t) = h_2(t) + h_1, \quad \int_0^{pT} h_2(t)\, dt = 0 \right\} .$$

This follows directly from (4.3). The proof is finished.

We can easily check that conditions a) and b) of Theorem 7 hold for an arbitrary $\beta > 0$ and α such that

$$0 > \alpha > \max\left\{ -\frac{\arccos^2 \lambda_{max}}{T^2}, -\frac{4\pi^2}{p^2T^2} \right\},$$

where λ_{max} is the largest eigenvalue of matrix $\frac{A+A^*}{2}$.

Moreover, we note that if $h \in X \subset C_{pT}(\mathbb{R}, \mathbb{R}^n)$ then

$$pTh_1 = \int_0^{pT} h(t)\,dt = \sum_{i=0}^{p-1} \int_{iT}^{(i+1)T} h(t)\,dt = \sum_{i=0}^{p-1} \int_0^T A^i h(t)\,dt =$$
$$\left(\sum_{i=0}^{p-1} A^i\right) \int_0^T h(t)\,dt = (\mathbb{I} - A^p) \circ (\mathbb{I} - A)^{-1} \int_0^T h(t)\,dt = 0\,.$$

Hence $h_1 = 0$ for any $h \in X$. Consequently, $X \cap \mathcal{S} = \emptyset$ which of course follows also from the statement of Theorem 7.

Finally, it follows from (4.3) that for any $h \in C_{pT}(\mathbb{R}, \mathbb{R}^n)$ with

$$h(t) = h_2(t) + \frac{1}{pT} \int_0^{pT} \nabla F(\psi(c, h_2)(t) + c)\,dt$$

for $c \in \mathbb{R}$ and $h_2 \in C_{pT}(\mathbb{R}, \mathbb{R}^n) \setminus X$ with $\int_0^{pT} h_2(t)\,dt = 0$, there is an pT-periodic solution of (4.1) which does not belong to Z.

Summarizing, under conditions of Theorem 7, there are many $h \in C_{pT}(\mathbb{R}, \mathbb{R}^n)$ for which there is no a pT-periodic solution of (4.1), while there are many $h \in C_{pT}(\mathbb{R}, \mathbb{R}^n)$ for which there is a pT-periodic solution of (4.1) not belonging to Z, but for any $h \in X$, there is a unique pT-periodic solution of (4.1) belonging to Z.

References

[1] AIZICOVICI, S. and FEČKAN, M. Forced symmetric oscillations of evolution equations, *Nonlinear Analysis* **64** (2006), 1621-1640.

[2] AIZICOVICI, S. and PAVEL, N. H. Anti-periodic solutions to a class of nonlinear differential equations in Hilbert space, *J. Funct. Anal.* **99** (1991), 387-408.

[3] BATTELLI, F. and FEČKAN, M. Heteroclinic period blow–up in certain symmetric ordinary differential equations, *Z. angew. Math. Phys. (ZAMP)* **47** (1996), 385-399.

[4] FONDA, A. and MAWHIN, J. Iterative and variational methods for the solvability of some semilinear equations in Hilbert spaces, *J. Differential Equations* **98** (1994), 355-375.

[5] MAWHIN, J. Topological Degree Methods in Nonlinear Boundary Value Problems, *Reg. Conf. Ser. Math.* **40**, Amer. Math. Soc., Providence RI 1979.

[6] MAWHIN, J. and WALTER, W. A general symmetry principle and some implications, *J. Math. Anal. Appl.* **186** (1994), 778-798.

Michal Fečkan
Department of Mathematical Analysis
and Numerical Mathematics,
Comenius University,
Mlynská dolina, 842 48 Bratislava - Slovakia
e-mail: michal.feckan@fmph.uniba.sk

Ruyun Ma
Department of Mathematics
Northwest Normal University
Lanzhou 730070 - P. R. China
e-mail: mary@nwnu.edu.cn

Bevan Thompson
Department of Mathematics
The University of Queensland
Brisbane, Qld 4072a - Australia
e-mail: hbt@maths.uq.edu.au