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Research Article

Forced Vibration of a Timoshenko Beam Subjected to Stationary and Moving Loads Using the Modal Analysis Method

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The modal analysis method (MAM) is very useful for obtaining the dynamic responses of a structure in analytical closed forms. In order to use the MAM, accurate information is needed on the natural frequencies, mode shapes, and orthogonality of the mode shapes a priori. A thorough literature survey reveals that the necessary information reported in the existing literature is sometimes very limited or incomplete, even for simple beam models such as Timoshenko beams. Thus, we present complete information on the natural frequencies, three types of mode shapes, and the orthogonality of the mode shapes for simply supported Timoshenko beams. Based on this information, we use the MAM to derive the forced vibration responses of a simply supported Timoshenko beam subjected to arbitrary initial conditions and to stationary or moving loads (a point transverse force and a point bending moment) in analytical closed form. We then conduct numerical studies to investigate the effects of each type of mode shape on the long-term dynamic responses (vibrations), the short-term dynamic responses (waves), and the deformed shapes of an example Timoshenko beam subjected to stationary or moving point loads.

1. Introduction

The dynamic analysis of elastic structures subjected to moving loads (or masses) has been an interesting research topic in structural engineering. When moving loads are applied to a structure, dynamic deflections and stresses may become considerably higher than those induced by static loads. Because of these characteristics of moving load problems, various structures subjected to moving loads have been investigated including beams, bridges, railroads, highway structures, pavement, and overhead cranes. The discussion in this study will be limited to the flexural one-dimensional (1D) beam structures.

To examine the transverse vibrations of a 1D beam structure, the Timoshenko beam model has been widely adopted to take into account the effects of shear deformation and rotatory inertia on the dynamic responses. In transverse vibration analysis, various solution techniques have been described in the literature including MAM or eigenfunction expansion methods [1, 2], mode summation methods or assumed mode methods [3–5], semianalytical methods [6–11], integral transform methods (Laplace-Carson transform and Fourier transform) [12–14], transfer matrix method [15], Lagrange multiplier methods [16, 17], Galerkin methods [18, 19], finite element methods [20, 21], finite difference method [22], time-domain spectral element method [23], and frequency-domain spectral element method [24].

In order to obtain analytical closed-form solutions for a moving load problem by using the MAM, information is needed regarding the eigensolutions (natural frequencies and mode shapes) and the orthogonality properties of the mode shapes. To obtain the eigensolutions for a Timoshenko beam subjected to specific boundary conditions, we begin by obtaining general solutions for the corresponding free vibration problem. Many researchers have developed general solutions of the transverse vibrations of a Timoshenko beam including Traill-Nash and Collar [25], Huang [26], and Han et al. [27]. In [25–27], the general solutions are obtained for two frequency ranges, $\omega < \omega_c$ and $\omega > \omega_c$, excluding the cutoff frequency ω_c. van Rensburg and van der Merwe [28] seemed to be the first to present general solutions for three frequency ranges, $\omega < \omega_c$, $\omega = \omega_c$, and $\omega > \omega_c$. Leissa and Qatu [29] also presented the same general solutions for three frequency ranges. However, in this study, we develop a new expression of general solutions for two frequency ranges, $\omega \le \omega_c$ and $\omega \ge \omega_c$, including the cutoff frequency ω_c.

By imposing the boundary conditions for a specific problem on the general solutions, we obtain eigensolutions for the specific problem. In this study, we limited our consideration to simply supported (hinged-hinged or pinned-pinned) boundary conditions. For simply supported Timoshenko beams, Traill-Nash and Collar [25] first reported the appearance of a "second frequency spectrum" when the vibration frequency ω is larger than a specific frequency known as the cutoff frequency ω_c. They suggested that the pure shearing oscillation may occur at $\omega = \omega_c$. However, they did not present the natural frequencies in explicit analytical form. Dolph [30] presented the natural frequencies and mode shapes for both bending and shear vibrations in explicit analytical form. However, he did not present the mode shapes at $\omega = \omega_c$. Though Huang [26] presented the mode shapes for a simply supported Timoshenko beam, the mode shapes fail to satisfy the boundary conditions for bending moment as criticized by van Rensburg and van der Merwe [28]. Han et al. [27] derived the natural frequencies and mode shapes for bending and transverse shear vibrations in explicit analytical form and discussed the MAM used to obtain forced vibration responses. However, they did not investigate whether there are mode shapes at $\omega = \omega_c$ or not. van Rensburg and van der Merwe [28] derived the mode shapes for the bending and transverse shear vibrations by determining the coefficients of assumed mode shapes needed to satisfy governing equations. They reported that ω_c itself is a natural frequency and presented the mode shape at $\omega = \omega_c$, which has been recognized as the "pure shear mode" [31]. However, they did not present natural frequencies in explicit forms. Thus, in this study, we presented a complete set of natural frequencies and mode shapes for all frequency ranges in explicit forms.

To apply the MAM to a forced vibration analysis of a Timoshenko beam, the orthogonality properties of mode shapes are essential. For simply supported Timoshenko beams, Dolph [30] derived the orthogonality properties of mode shapes and other researchers [1, 32, 33] used the orthogonality properties derived by Dolph [30] for the modal analysis of forced vibration problems. However, Dolph [30] and other researchers [1, 32, 33] did not consider the orthogonality of the mode shapes at $\omega = \omega_c$. Although van Rensburg et al. [34] mentioned the existence of a mode shape at $\omega = \omega_c$, they did not include it in their free vibration analysis of a simply supported Timoshenko beam. Roux et al. [31] included the pure shear mode shape at $\omega = \omega_c$ in a series solution of the free vibration of a simply supported Timoshenko beam, but they did not apply the orthogonality properties of pure shear mode shape at $\omega = \omega_c$ to determine the coefficients of the series solution. Based on our literature survey of the modal analysis of forced vibrations of simply supported Timoshenko beams, we find that there have been no reports in which the pure shear mode shape at $\omega = \omega_c$ is considered in themodal analysis of the forced vibrations of simply supported Timoshenko beams. We also find that there have been no reports in which the vibrations of a simply supported Timoshenko beam induced by a stationary or moving bending moment are considered by using the MAM. Thus, in this study, we present the closed-form solutions of a simply supported Timoshenko beam subjected to stationary or moving bending moment, including the pure shear mode shape at $\omega = \omega_c$.

In this study, we discuss the mathematical formulation of the general solutions of the free vibration of a Timoshenko beam subjected to arbitrary boundary conditions in Section 2. The general solutions are presented for frequency ranges $\omega \le \omega_c$ and $\omega \ge \omega_c$. We then derive natural frequencies and mode shapes in explicit forms for the case of simply supported boundary conditions. Finally, we present the orthogonality properties of the mode shapes. In Section 3, we describe the MAM for the forced vibration of a simply supported Timoshenko beam subjected to arbitrary initial conditions and to stationary or moving loads (a point transverse force and a point bending moment). In Section 4, we describe our numerical results. Lastly, in Section 5, we present concluding remarks.

2. Mathematical Theory

2.1. Mathematical Model of a Timoshenko Beam. The governing equations for a Timoshenko beam of length L can be written in a matrix form as [35]

$$\mathbf{M}\frac{\partial^2 \mathbf{u}(x,t)}{\partial t^2} + \mathbf{K}\mathbf{u}(x,t) = \mathbf{f}(x,t), \tag{1}$$

where

$$\mathbf{u}(x,t) = \begin{Bmatrix} w(x,t) \\ \theta(x,t) \end{Bmatrix}, \quad \mathbf{f}(x,t) = \begin{Bmatrix} f(x,t) \\ \tau(x,t) \end{Bmatrix}, \tag{2}$$

$$\mathbf{M} = \begin{bmatrix} \rho A & 0 \\ 0 & \rho I \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} -\kappa GA\dfrac{\partial^2}{\partial x^2} & \kappa GA\dfrac{\partial}{\partial x} \\ -\kappa GA\dfrac{\partial}{\partial x} & \kappa GA - EI\dfrac{\partial^2}{\partial x^2} \end{bmatrix}, \tag{3}$$

$w(x,t)$ is the transverse displacement, $\theta(x,t)$ is the rotation of the cross section due to bending, $f(x,t)$ is the external transverse force, $\tau(x,t)$ is the external bending moment, E is Young's modulus, G is the shear modulus, ρ is the mass density, κ is the shear coefficient factor, A is the cross-sectional area, and I is the area moment of inertia. The natural and geometric boundary conditions relevant to (1) are given by

$$\begin{aligned} V(0,t) &= -V_1(t) \quad \text{or} \quad w(0,t) = w_1(t), \\ V(L,t) &= V_2(t) \quad \text{or} \quad w(L,t) = w_2(t), \\ M(0,t) &= M_1(t) \quad \text{or} \quad \theta(0,t) = \theta_1(t), \\ M(L,t) &= -M_2(t) \quad \text{or} \quad \theta(L,t) = \theta_2(t), \end{aligned} \tag{4}$$

where $V(x,t)$ and $M(x,t)$ are the resultant transverse shear force and bending moment, respectively, defined by

$$\begin{aligned} V(x,t) &= \kappa GA\left(\frac{\partial w}{\partial x} - \theta\right), \\ M(x,t) &= EI\frac{\partial \theta}{\partial x}. \end{aligned} \tag{5}$$

Finally, the initial conditions are given by

$$\begin{aligned} \mathbf{u}(x,0) &= \mathbf{g}(x), \\ \frac{\partial \mathbf{u}(x,0)}{\partial t} &= \mathbf{h}(x). \end{aligned} \tag{6}$$

2.2. General Solutions. To derive the eigenfunctions (natural modes) for a Timoshenko beam, we must first obtain the general solutions for the free vibration problem. Thus, we consider the homogeneous governing equation reduced from (1) as follows:

$$\mathbf{M}\frac{\partial^2 \mathbf{u}(x,t)}{\partial t^2} + \mathbf{K}\mathbf{u}(x,t) = \mathbf{0}. \tag{7}$$

The solution of (7) is assumed to be in the following form:

$$\mathbf{u}(x,t) = \begin{Bmatrix} W(x) \\ \Theta(x) \end{Bmatrix} e^{i\omega t} = \mathbf{U}(x)e^{i\omega t}, \tag{8}$$

where $i = \sqrt{-1}$ is the imaginary unit and ω is the angular frequency. By substituting (8) into (7), an eigenvalue problem is obtained as follows:

$$\left(-\omega^2\mathbf{M} + \mathbf{K}\right)\mathbf{U}(x) = \mathbf{0}. \tag{9}$$

We assume that the solutions of (7) are in the following form:

$$\mathbf{U}(x) = \begin{Bmatrix} a \\ b \end{Bmatrix} e^{sx}, \tag{10}$$

where s denotes the wavenumber. Substituting (10) into (9) gives algebraic equations as follows:

$$\begin{bmatrix} -\left(\kappa GAs^2 + \rho A\omega^2\right) & s\kappa GA \\ -s\kappa GA & \kappa GA - EIs^2 - \omega^2\rho I \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{11}$$

For the existence of nontrivial solutions, the determinant of the two-by-two matrix in (11) must vanish at certain values of s, that is, at eigenvalues. From this condition, a dispersion equation is obtained as follows:

$$s^4 + \omega^2\left(\frac{\rho I}{EI} + \frac{\rho A}{\kappa GA}\right)s^2 + \omega^2\left(\frac{\rho A}{EI}\frac{\rho I}{\kappa GA}\omega^2 - \frac{\rho A}{EI}\right) = 0. \tag{12}$$

In order to obtain the four eigenvalues, the above quartic equation can be reduced to a quadratic equation by replacing s^2 with ζ (where $s = \pm\sqrt{\zeta}$). By solving this quadratic equation, we can obtain four eigenvalues as follows:

$$\begin{aligned} s_1 &= -s_2 = i\beta \\ s_3 &= -s_4 = \begin{cases} \alpha & (\text{if } \omega \le \omega_c) \\ i\alpha' & (\text{if } \omega \ge \omega_c), \end{cases} \end{aligned} \tag{13}$$

where β, α, and α' are always real numbers and ω_c is the cutoff frequency defined by

$$\omega_c = \sqrt{\frac{\kappa GA}{\rho I}}, \tag{14}$$

$$\beta = \frac{1}{\sqrt{2}} \cdot \sqrt{\left(\frac{\rho I}{EI} + \frac{\rho A}{\kappa GA}\right)\omega^2 + \sqrt{\left(\frac{\rho I}{EI} - \frac{\rho A}{\kappa GA}\right)^2\omega^4 + 4\frac{\rho A}{EI}\omega^2}} \tag{15a}$$

$$\alpha = \frac{1}{\sqrt{2}} \cdot \sqrt{\sqrt{\left(\frac{\rho I}{EI} - \frac{\rho A}{\kappa GA}\right)^2\omega^4 + 4\frac{\rho A}{EI}\omega^2} - \left(\frac{\rho I}{EI} + \frac{\rho A}{\kappa GA}\right)\omega^2} \quad (\omega \le \omega_c) \tag{15b}$$

$$\alpha' = \frac{1}{\sqrt{2}} \cdot \sqrt{\left(\frac{\rho I}{EI} + \frac{\rho A}{\kappa GA}\right)\omega^2 - \sqrt{\left(\frac{\rho I}{EI} - \frac{\rho A}{\kappa GA}\right)^2\omega^4 + 4\frac{\rho A}{EI}\omega^2}} \quad (\omega \ge \omega_c). \tag{15c}$$

By using the four eigenvalues given by (13), the general solutions of (9) can be written as follows:

(i) When $0 < \omega \le \omega_c$

$$\begin{Bmatrix} W(x) \\ \Theta(x) \end{Bmatrix} = \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} e^{i\beta x} + \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} e^{-i\beta x} + \begin{Bmatrix} a_3 \\ b_3 \end{Bmatrix} e^{\alpha x} + \begin{Bmatrix} a_4 \\ b_4 \end{Bmatrix} e^{-\alpha x} \tag{16}$$

(ii) When $\omega \geq \omega_c$

$$\begin{Bmatrix} W(x) \\ \Theta(x) \end{Bmatrix} = \begin{Bmatrix} \bar{a}_1 \\ \bar{b}_1 \end{Bmatrix} e^{i\beta x} + \begin{Bmatrix} \bar{a}_2 \\ \bar{b}_2 \end{Bmatrix} e^{-i\beta x} + \begin{Bmatrix} \bar{a}_3 \\ \bar{b}_3 \end{Bmatrix} e^{i\alpha' x} + \begin{Bmatrix} \bar{a}_4 \\ \bar{b}_4 \end{Bmatrix} e^{-i\alpha' x}. \tag{17}$$

By substituting each eigenvalue into (11), we obtain the ratios a_k/b_k and $\bar{a}_k/\bar{b}_k$ ($k = 1, 2, 3, 4$). By using the results, (16) and (17) can be rewritten in terms of sinusoidal and hyperbolic functions as follows:

(i) When $0 < \omega \leq \omega_c$

$$\begin{Bmatrix} W(x) \\ \Theta(x) \end{Bmatrix} = A_1 \begin{Bmatrix} \cosh \alpha x \\ g_\alpha \sinh \alpha x \end{Bmatrix} + A_2 \begin{Bmatrix} \alpha r_G^2 \sinh \alpha x \\ \alpha r_G^2 g_\alpha \cosh \alpha x \end{Bmatrix} + A_3 \begin{Bmatrix} \cos \beta x \\ -g_\beta \sin \beta x \end{Bmatrix} + A_4 \begin{Bmatrix} \sin \beta x \\ g_\beta \cos \beta x \end{Bmatrix} \tag{18}$$

(ii) When $\omega \geq \omega_c$

$$\begin{Bmatrix} W(x) \\ \Theta(x) \end{Bmatrix} = A_1 \begin{Bmatrix} \cos \alpha' x \\ -g_{\alpha'} \sin \alpha' x \end{Bmatrix} + A_2 \begin{Bmatrix} -\alpha' r_G^2 \sin \alpha' x \\ -\alpha' r_G^2 g_{\alpha'} \cos \alpha' x \end{Bmatrix} + A_3 \begin{Bmatrix} \cos \beta x \\ -g_\beta \sin \beta x \end{Bmatrix} + A_4 \begin{Bmatrix} \sin \beta x \\ g_\beta \cos \beta x \end{Bmatrix}, \tag{19}$$

where $r_G^2 = \rho I/\rho A$ is the radius of gyration and

$$g_\beta = \frac{1}{\beta}\left(\beta^2 - \frac{\omega^2 \rho A}{\kappa G A}\right), \quad g_\alpha = \frac{1}{\alpha}\left(\alpha^2 + \frac{\omega^2 \rho A}{\kappa G A}\right), \quad g_{\alpha'} = \frac{1}{\alpha'}\left(\alpha'^2 - \frac{\omega^2 \rho A}{\kappa G A}\right). \tag{20}$$

The present expression of general solutions given by (18) and (19) is equivalent to the expression for three frequency ranges, $0 < \omega < \omega_c$, $\omega = \omega_c$, and $\omega > \omega_c$, by van Rensburg and van der Merwe [28].

2.3. Natural Frequencies and Mode Shapes. To obtain the natural frequencies and mode shapes in analytical closed forms for specific boundary conditions, we considered three frequency ranges separately: (a) $0 < \omega < \omega_c$, (b) $\omega > \omega_c$, and (c) $\omega = \omega_c$. Our study was limited to the simply supported boundary conditions represented by

$$w(0,t) = w(L,t) = 0, \quad EI\frac{\partial \theta(0,t)}{\partial x} = EI\frac{\partial \theta(L,t)}{\partial x} = 0. \tag{21}$$

2.3.1. When $0 < \omega < \omega_c$. Substituting (18) into (21) gives a matrix equation as follows:

$$\mathbf{w} = \mathbf{D}\mathbf{a} = \mathbf{0}, \tag{22}$$

where

$$\mathbf{w} = \{W(0) \;\; \Theta'(0) \;\; W(L) \;\; \Theta'(L)\}^T$$

$$\mathbf{a} = \{A_1 \;\; A_2 \;\; A_3 \;\; A_4\}^T$$

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ \alpha g_\alpha & 0 & -\beta g_\beta & 0 \\ \cosh \alpha L & \dfrac{\alpha \sinh \alpha L}{r_G^{-2}} & \cos \beta L & \sin \beta L \\ \alpha g_\alpha \cosh \alpha L & \dfrac{\alpha^2 g_\alpha \sinh \alpha L}{r_G^{-2}} & -\beta g_\beta \cos \beta L & -\beta g_\beta \sin \beta L \end{bmatrix}. \tag{23}$$

From the first and second relations in (22), we obtain $A_1 = A_3 = 0$. Then, from the third and fourth relations of (22), we obtain

$$\begin{Bmatrix} W(L) \\ \Theta'(L) \end{Bmatrix} = \begin{bmatrix} \dfrac{\alpha \sinh \alpha L}{r_G^{-2}} & \sin \beta L \\ \dfrac{\alpha^2 g_\alpha \sinh \alpha L}{r_G^{-2}} & -\beta g_\beta \sin \beta L \end{bmatrix} \begin{Bmatrix} A_2 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{24}$$

For the existence of nontrivial solutions of $\{A_2, A_4\}^T$, we obtain a characteristic equation from (24) as follows:

$$\alpha r_G^2 \left(\beta g_\beta + \alpha g_\alpha\right) \sinh \alpha L \sin \beta L = 0. \tag{25}$$

Since $\alpha > 0$, $\beta g_\beta + \alpha g_\alpha \neq 0$, and $\sinh \alpha L \neq 0$, if $0 < \omega < \omega_c$, then the following condition can be obtained from (25):

$$\sin \beta_n L = 0. \tag{26}$$

From (26), we obtain

$$\beta_n = \frac{n\pi}{L} \quad \left(n = 1, 2, 3, \ldots, n_B\right). \tag{27}$$

Applying (27) to (15a), (15b), and (15c) yields natural frequencies $\omega_{B(n)}$ as follows:

$$\omega_{B(n)} = \sqrt{Z(n) - \sqrt{Z^2(n) - 4R(n)}} \quad \left(n = 1, 2, 3, \ldots, n_B\right), \tag{28}$$

where

$$Z(n) = \left(\frac{EI}{2\rho I} + \frac{\kappa GA}{2\rho A}\right)\left(\frac{n\pi}{L}\right)^2 + \frac{\kappa GA}{2\rho I},$$
$$R(n) = \frac{EI}{2\rho I}\frac{\kappa GA}{2\rho A}\left(\frac{n\pi}{L}\right)^4. \tag{29}$$

Note that n_B is the maximum value of integer n satisfying $\omega_{B(n)} < \omega_c$, which can be determined from (28) in closed form as follows:

$$n_B = \text{Integer part of } \left(n_c \equiv \frac{L}{\pi}\sqrt{\frac{\rho A}{\rho I} + \frac{\kappa GA}{EI}}\right). \tag{30}$$

To obtain the mode shapes corresponding to the natural frequencies $\omega_{B(n)}$ ($n = 1, 2, 3, \ldots, n_B$), we can determine A_2 and A_4 by substituting (27) into (24) as follows:

$$A_{2(n)} = 0,$$
$$A_{4(n)} \neq 0. \tag{31}$$

The nth mode shape corresponding to $\omega_{B(n)}$ can then be obtained from (18) in the following form:

$$\mathbf{U}_{B(n)}(x) \equiv \begin{Bmatrix} W_{B(n)}(x) \\ \Theta_{B(n)}(x) \end{Bmatrix} = A_{B(n)} \begin{Bmatrix} \sin \dfrac{n\pi x}{L} \\ g_{B(n)} \cos \dfrac{n\pi x}{L} \end{Bmatrix} \quad (n = 1, 2, 3, \ldots, n_B), \tag{32}$$

where

$$g_{B(n)} = \frac{n\pi}{L} - \frac{\omega_{B(n)}^2 \rho A}{\kappa GA}\frac{L}{n\pi}. \tag{33}$$

2.3.2. When $\omega > \omega_c$. Substituting (19) into (21) gives the following matrix equation:

$$\mathbf{w} = \mathbf{E}\mathbf{a} = \mathbf{0}, \tag{34}$$

where

$$\mathbf{E} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ -\alpha' g_{\alpha'} & 0 & -\beta g_\beta & 0 \\ \cos\alpha' L & \dfrac{-\alpha' \sin\alpha' L}{r_G^{-2}} & \cos\beta L & \sin\beta L \\ -\alpha' g_{\alpha'} \cos\alpha' L & \dfrac{\alpha'^2 g_{\alpha'} \sin\alpha' L}{r_G^{-2}} & -\beta g_\beta \cos\beta L & -\beta g_\beta \sin\beta L \end{bmatrix}. \tag{35}$$

From the first and second relations in (34), we find $A_1 = A_3 = 0$. Then, the third and fourth relations in (34) can be written as

$$\begin{Bmatrix} W(L) \\ \Theta'(L) \end{Bmatrix} = \begin{bmatrix} \dfrac{-\alpha' \sin\alpha' L}{r_G^{-2}} & \sin\beta L \\ \dfrac{\alpha'^2 g_{\alpha'} \sin\alpha' L}{r_G^{-2}} & -\beta g_\beta \sin\beta L \end{bmatrix} \begin{Bmatrix} A_2 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}. \tag{36}$$

For the existence of nontrivial solutions of $\{A_2, A_4\}^T$, we can obtain a characteristic equation from (36) as

$$\alpha' r_G^2 \left(\alpha' g_{\alpha'} - \beta g_\beta\right) \sin\alpha' L \sin\beta L = 0. \tag{37}$$

Since $\alpha' > 0$ and $(\alpha' g_{\alpha'} - \beta g_\beta) \neq 0$, if $\omega > \omega_c$, the following two conditions can be obtained from (37):

$$\sin\beta_n L = 0 \tag{38}$$

or

$$\sin\alpha'_m L = 0. \tag{39}$$

In Section 2.3.1, we derived the natural frequencies $\omega_{B(n)}$ and mode shapes $\mathbf{U}_{B(n)}$ for $0 < \omega < \omega_c$ from the same condition given by (38). Thus, the natural frequencies $\omega_{B(n)}$ and mode shapes $\mathbf{U}_{B(n)}$ for $\omega > \omega_c$ can be obtained as follows:

$$\omega_{B(n)} = \sqrt{Z(n) - \sqrt{Z^2(n) - 4R(n)}} \quad (n = n_B + 1, n_B + 2, n_B + 3, \ldots, \infty)$$
$$\mathbf{U}_{B(n)}(x) = \begin{Bmatrix} W_{B(n)}(x) \\ \Theta_{B(n)}(x) \end{Bmatrix} = A_{B(n)} \begin{Bmatrix} \sin \dfrac{n\pi x}{L} \\ g_{B(n)} \cos \dfrac{n\pi x}{L} \end{Bmatrix} \quad (n = n_B + 1, n_B + 2, \ldots, \infty). \tag{40}$$

From the second condition (39), we obtain

$$\alpha'_m = \frac{m\pi}{L} \quad (m = 1, 2, 3, \ldots, \infty). \tag{41}$$

By substituting (41) into (15), we can obtain the natural frequencies as follows:

$$\omega_{S(m)} = \sqrt{Z(m) + \sqrt{Z^2(m) - 4R(m)}} \quad (\omega_{S(m)} > \omega_c), \tag{42}$$

where $Z(m)$ and $R(m)$ are defined in (29). To derive the mode shapes corresponding to $\omega_{S(m)}$, we can determine A_2 and A_4 by substituting (41) into (36) as follows:

$$A_{2(m)} \neq 0,$$
$$A_{4(m)} = 0. \tag{43}$$

The mode shapes corresponding to $\omega_{S(m)}$ are then obtained from (19) as follows:

$$\mathbf{U}_{S(m)} \equiv \begin{Bmatrix} W_{S(m)}(x) \\ \Theta_{S(m)}(x) \end{Bmatrix} = A_{S(m)} \begin{Bmatrix} \sin \dfrac{m\pi x}{L} \\ g_{S(m)} \cos \dfrac{m\pi x}{L} \end{Bmatrix} \quad (m = 1, 2, 3, \ldots, \infty), \tag{44}$$

where

$$g_{S(m)} = \frac{m\pi}{L} - \frac{\omega_{S(m)}^2 \rho A}{\kappa GA}\frac{L}{m\pi} \quad (m = 1, 2, 3, \ldots, \infty). \tag{45}$$

2.3.3. When $\omega = \omega_c$. The general solution at $\omega = \omega_c$ can be readily obtained from either (18) or (19), by allowing ω to approach ω_c, as follows:

$$\begin{Bmatrix} W(x) \\ \Theta(x) \end{Bmatrix} = A_1 \begin{Bmatrix} 1 \\ r_G^{-2} x \end{Bmatrix} + A_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} + A_3 \begin{Bmatrix} \cos \beta_c x \\ -g_{\beta_c} \sin \beta_c x \end{Bmatrix} + A_4 \begin{Bmatrix} \sin \beta_c x \\ g_{\beta_c} \cos \beta_c x \end{Bmatrix}, \tag{46}$$

where

$$\beta_c = \beta(\omega = \omega_c) = \sqrt{\frac{\rho A}{\rho I} + \frac{\kappa G A}{E I}} = \frac{n_c \pi}{L}$$
$$g_{\beta_c} = g_\beta(\omega = \omega_c) = \frac{\kappa G A}{E I} \frac{L}{n_c \pi}. \tag{47}$$

To obtain the first two terms in (46) from either (18) or (19), L'Hospital's rule is applied.

Applying the simply supported boundary conditions given by (21) to (46) yields the following eigenvalue problem:

$$\begin{Bmatrix} W(0) \\ \Theta'(0) \\ W(L) \\ \Theta'(L) \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 1 & 0 \\ r_G^{-2} & 0 & -\beta_c g_{\beta_c} & 0 \\ 1 & 0 & \cos \beta_c L & \sin \beta_c L \\ r_G^{-2} & 0 & -\beta_c g_{\beta_c} \cos \beta_c L & -\beta_c g_{\beta_c} \sin \beta_c L \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}. \tag{48}$$

The necessary condition for the existence of a nontrivial solution of (48) (i.e., the determinant of the matrix of eigenvalue problem must vanish at the eigenvalue) is self-satisfied. Thus, we conclude that the cutoff frequency ω_c is also a natural frequency of a simply supported Timoshenko beam, which was described by van Rensburg and van der Merwe [28]. Now we must determine the mode shape corresponding to the natural frequency ω_c.

From (47), note that $\beta_c > 0$ and $g_{\beta_c} > 0$. Thus, from (48), it can be shown that the following should be satisfied: $A_1 = A_3 = 0$ and

$$A_4 \sin \beta_c L = 0$$
$$\text{or } A_4 \sin n_c \pi = 0. \tag{49}$$

To satisfy (49), we consider the following two cases.

Case 1. If n_c is not an integer, then $A_4 = 0$. In this case, the corresponding mode shape can be derived directly from (46) as follows:

$$\mathbf{U}_{S(0)}(x) \equiv \begin{Bmatrix} W_{S(0)}(x) \\ \Theta_{S(0)}(x) \end{Bmatrix} = A_{S(0)} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}. \tag{50}$$

This mode shape is identical to the pure shear mode shape presented by van Rensburg and van der Merwe [28]. Accordingly, the subscript $S(0)$ is adopted in (50) to emphasize the pure shear mode shape.

Case 2. If n_c is an integer (i.e., n_B), then $A_4 \neq 0$. In this case, the natural frequency ω_c happens to be equal to the natural frequency $\omega_{B(n_B)}$ of a bending mode shape and they become a double natural frequency. The mode shapes at this double natural frequency are given by

$$\mathbf{U}_{B(n_B)}(x) = A_{B(n_B)} \begin{Bmatrix} \sin \dfrac{n_B \pi x}{L} \\ g_{B(n_B)} \cos \dfrac{n_B \pi x}{L} \end{Bmatrix} \quad \left(\text{mode shape for } \omega_{B(n_B)}\right)$$
$$\mathbf{U}_{S(0)}(x) = A_{S(0)} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \left(\text{mode shape for } \omega_c\right). \tag{51}$$

Thus, for the modal analysis of the transverse vibrations of a simply supported Timoshenko beam subjected to a stationary or moving load, we need to consider the following three types of mode shapes:

$$\mathbf{U}_{B(n)} = A_{B(n)} \begin{Bmatrix} \sin \dfrac{n \pi x}{L} \\ g_{B(n)} \cos \dfrac{n \pi x}{L} \end{Bmatrix} \quad \left(\text{mode shapes for } \omega_{B(n)}\right)$$
$$\mathbf{U}_{S(n)} = A_{S(n)} \begin{Bmatrix} \sin \dfrac{n \pi x}{L} \\ g_{S(n)} \cos \dfrac{n \pi x}{L} \end{Bmatrix} \quad \left(\text{mode shapes for } \omega_{S(n)}\right) \tag{52}$$
$$\mathbf{U}_{S(0)} = A_{S(0)} \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} \quad \left(\text{pure shear mode shape for } \omega_c\right),$$

where $n = 1, 2, 3, \ldots, \infty$.

2.4. Orthogonality of Mode Shapes. For the modal analysis, we must derive the orthogonality properties of the mode shapes given by (52). Because any set of natural frequencies and mode shapes are the eigensolutions of the eigenvalue problem represented by (9), the mth and nth sets of eigensolutions must satisfy the following two equations separately as follows:

$$\mathbf{K}\mathbf{U}_m = \omega_m^2 \mathbf{M}\mathbf{U}_m, \quad \mathbf{K}\mathbf{U}_n = \omega_n^2 \mathbf{M}\mathbf{U}_n. \tag{53}$$

From (54), we obtain

$$\int_0^L \mathbf{U}_n^T \mathbf{K}\mathbf{U}_m dx - \int_0^L \mathbf{U}_m^T \mathbf{K}\mathbf{U}_n dx = \left(\omega_m^2 - \omega_n^2\right) \int_0^L \mathbf{U}_m^T \mathbf{M}\mathbf{U}_n dx. \tag{54}$$

By applying the simply supported boundary conditions, the left-hand side of (54) vanishes. Then, by using the definition of $\mathbf{M}$ in (3), the right-hand side of (54) can be rewritten as

$$\begin{aligned}&\left(\omega_m^2 - \omega_n^2\right) \int_0^L \mathbf{U}_m^T \mathbf{M}\mathbf{U}_n dx \\ &= \left(\omega_m^2 - \omega_n^2\right)\left(\rho A \int_0^L W_m W_n dx + \rho I \int_0^L \Theta_m \Theta_n dx\right) \\ &= 0.\end{aligned} \tag{55}$$

From (55), the orthogonality property of mode shapes with respect to $\mathbf{M}$ can be derived as follows:

$$\int_0^L \mathbf{U}_m^T \mathbf{M}\mathbf{U}_n dx = m_n \delta_{mn}, \tag{56}$$

where δ_{mn} represent the Kronecker delta symbol [36] and m_n is the modal mass. By using (56), we can derive the orthogonality property of mode shapes with respect to $\mathbf{K}$, from (53), as follows:

$$\int_0^L \mathbf{U}_m^T \mathbf{K}\mathbf{U}_n dx = m_n \omega_n^2 \delta_{mn}. \tag{57}$$

By substituting each mode shape given by (52) into (56), it can be shown that the following orthogonality properties are satisfied:

$$\begin{aligned}&\int_0^L \mathbf{U}_{B(m)}^T \mathbf{M}\mathbf{U}_{B(n)} dx = m_{B(n)} \delta_{mn} \\ &\int_0^L \mathbf{U}_{S(m)}^T \mathbf{M}\mathbf{U}_{S(n)} dx = m_{S(n)} \delta_{mn} \\ &\int_0^L \mathbf{U}_{S(0)}^T \mathbf{M}\mathbf{U}_{S(0)} dx = m_{S(0)} \\ &\int_0^L \mathbf{U}_{B(m)}^T \mathbf{M}\mathbf{U}_{S(n)} dx = 0 \\ &\int_0^L \mathbf{U}_{B(m)}^T \mathbf{M}\mathbf{U}_{S(0)} dx = 0 \\ &\int_0^L \mathbf{U}_{S(m)}^T \mathbf{M}\mathbf{U}_{S(0)} dx = 0 \\ &\qquad (m, n = 1, 2, 3, \ldots),\end{aligned}$$

$$\begin{aligned}&\int_0^L \mathbf{U}_{B(m)}^T \mathbf{K}\mathbf{U}_{B(n)} dx = m_{B(n)} \omega_{B(n)}^2 \delta_{mn} \\ &\int_0^L \mathbf{U}_{S(m)}^T \mathbf{K}\mathbf{U}_{S(n)} dx = m_{S(n)} \omega_{S(n)}^2 \delta_{mn} \\ &\int_0^L \mathbf{U}_{S(0)}^T \mathbf{K}\mathbf{U}_{S(0)} dx = m_{S(0)} \omega_c^2 \\ &\int_0^L \mathbf{U}_{B(m)}^T \mathbf{K}\mathbf{U}_{S(n)} dx = 0 \\ &\int_0^L \mathbf{U}_{B(m)}^T \mathbf{K}\mathbf{U}_{S(0)} dx = 0 \\ &\int_0^L \mathbf{U}_{S(m)}^T \mathbf{K}\mathbf{U}_{S(0)} dx = 0 \\ &\qquad (m, n = 1, 2, 3, \ldots),\end{aligned} \tag{58}$$

where the modal masses are defined by

$$\begin{aligned}m_{B(n)} &= \frac{1}{2} L \left(\rho A + \rho I g_{B(n)}^2\right) A_{B(n)}^2 \\ m_{S(n)} &= \frac{1}{2} L \left(\rho A + \rho I g_{S(n)}^2\right) A_{S(n)}^2 \\ m_{S(0)} &= L \rho I A_{S(0)}^2 \\ &\qquad (n = 1, 2, 3, \ldots).\end{aligned} \tag{59}$$

To derive normalized mode shapes (i.e., the normal modes) from (52), all modal masses given by (64) are set to unit value as follows: $m_{B(n)} = m_{S(n)} = m_{S(0)} = 1$. Then, from (64), the coefficients of each normal mode shape are determined as follows:

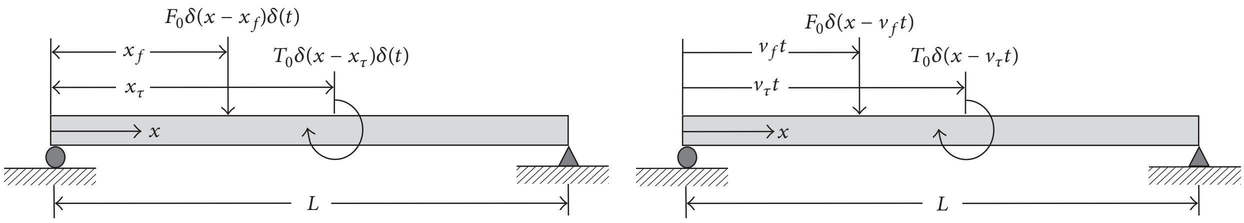
$$\begin{aligned}A_{B(n)} &= \sqrt{\frac{2}{L\left(\rho A + \rho I g_{B(n)}^2\right)}}, \\ A_{S(n)} &= \sqrt{\frac{2}{L\left(\rho A + \rho I g_{S(n)}^2\right)}} \\ A_{S(0)} &= \sqrt{\frac{1}{L \rho I}} \\ &\qquad (n = 1, 2, 3, \ldots).\end{aligned} \tag{60}$$

3. Modal Analysis of Forced Vibration

The forced vibration responses of (1) can be represented by using the normal mode summation method [37] as follows:

$$\begin{aligned}\mathbf{u}(x, t) = &\sum_{n=1}^{\infty} \mathbf{U}_{B(n)}(x)\, q_{B(n)}(t) + \sum_{n=1}^{\infty} \mathbf{U}_{S(n)}(x)\, q_{S(n)}(t) \\ &+ \mathbf{U}_{S(0)}(x)\, q_{S(0)}(t),\end{aligned} \tag{61}$$

where $q_{B(n)}(t)$, $q_{S(n)}(t)$, and $q_{S(0)}(t)$ are generalized coordinates to be determined in order to satisfy initial conditions.



(a) Stationary loads (b) Moving loads

Figure 1: A simply supported Timoshenko beam subjected to (a) a stationary impulsive point transverse force and a stationary impulsive point bending moment and (b) a moving point transverse force and a moving point bending moment.

Substituting (61) into (1) and applying the orthogonality conditions of the normal mode shapes yield the modal equations as follows:

$$\begin{aligned} \frac{d^2 q_{B(n)}}{dt^2} + \omega_{B(n)}^2 q_{B(n)} &= f_{B(n)} \\ \frac{d^2 q_{S(n)}}{dt^2} + \omega_{S(n)}^2 q_{S(n)} &= f_{S(n)} \\ \frac{d^2 q_{S(0)}}{dt^2} + \omega_c^2 q_{S(0)} &= f_{S(0)} \\ &(n = 1, 2, 3, \ldots), \end{aligned} \tag{62}$$

where the generalized forces are defined by

$$\begin{aligned} f_{B(n)} &= \int_0^L \left[W_{B(n)}(x) f(x,t) + \Theta_{B(n)}(x)\tau(x,t)\right] dx \\ f_{S(n)} &= \int_0^L \left[W_{S(n)}(x) f(x,t) + \Theta_{S(n)}(x)\tau(x,t)\right] dx \\ f_{S(0)} &= \int_0^L \Theta_{S(0)}(x)\tau(x,t)\, dx. \end{aligned} \tag{63}$$

The initial conditions for (62) can be derived from (6) by using the orthogonality properties of normal mode shapes as follows:

$$\begin{aligned} q_{B(n)}(0) &= \int_0^L \mathbf{U}_{B(n)}^T \mathbf{M}\mathbf{g}(x)\, dx, \\ \frac{dq_{B(n)}(0)}{dt} &= \int_0^L \mathbf{U}_{B(n)}^T \mathbf{M}\mathbf{h}(x)\, dx \\ q_{S(n)}(0) &= \int_0^L \mathbf{U}_{S(n)}^T \mathbf{M}\mathbf{g}(x)\, dx, \\ \frac{dq_{S(n)}(0)}{dt} &= \int_0^L \mathbf{U}_{S(n)}^T \mathbf{M}\mathbf{h}(x)\, dx \\ q_{S(0)}(0) &= \int_0^L \mathbf{U}_{S(0)}^T \mathbf{M}\mathbf{g}(x)\, dx, \\ \frac{dq_{S(0)}(0)}{dt} &= \int_0^L \mathbf{U}_{S(0)}^T \mathbf{M}\mathbf{h}(x)\, dx, \end{aligned} \tag{64}$$

where $n = 1, 2, 3, \ldots$.

By using (3), (6), and (52), we can write the initial conditions for $q_{S(0)}(t)$ as

$$\begin{aligned} q_{S(0)}(0) &= \sqrt{\frac{\rho I}{L}} \int_0^L \theta(x,0)\, dx, \\ \frac{dq_{S(0)}(0)}{dt} &= \sqrt{\frac{\rho I}{L}} \int_0^L \frac{\partial\theta(x,0)}{\partial t} dx. \end{aligned} \tag{65}$$

From (63) and (65), we note that the pure shear mode shape $\mathbf{U}_{S(0)}$ must be included in the modal analysis when a Timoshenko beam is subjected to external bending moment $\tau(x,t)$, initial rotation $\theta(x,0)$, and initial angular velocity $\partial\theta(x,0)/\partial t$. However, there have been no reports in the literature in which the external transverse force $f(x,t)$, bending moment $\tau(x,t)$, and arbitrary initial conditions were fully considered in the modal analysis of forced vibrations by taking into account the pure shear mode shape $\mathbf{U}_{S(0)}$. In this study, we derived the vibration responses of a Timoshenko beam for two cases:

(1) Case 1: when the beam is subjected to a stationary impulsive point transverse force and a stationary impulsive point bending moment.

(2) Case 2: when the beam is subjected to a moving point transverse force and a moving point bending moment.

3.1. Case 1: Stationary Impulsive Point Transverse Force and Bending Moment. As shown in Figure 1(a), a stationary impulsive point transverse force and a stationary bending moment acting on the arbitrary positions x_f and x_τ of the beam can be expressed by employing Dirac delta functions $\delta(x)$ and $\delta(t)$ [36] as follows:

$$\begin{aligned} f(x,t) &= F_0 \delta\left(x - x_f\right)\delta(t), \\ \tau(x,t) &= T_0 \delta\left(x - x_\tau\right)\delta(t), \end{aligned} \tag{66}$$

where F_0 is the magnitude of the transverse impulsive point force and T_0 is the magnitude of the impulsive point bending moment.

Substituting (66) into (63) yields the generalized forces, and substituting the results into (62) gives

$$\begin{gathered}\frac{d^2 q_{B(n)}}{dt^2}+\omega_{B(n)}^2 q_{B(n)}=Q_{B(n)}\delta(t)\\ \frac{d^2 q_{S(n)}}{dt^2}+\omega_{S(n)}^2 q_{S(n)}=Q_{S(n)}\delta(t)\\ \frac{d^2 q_{S(0)}}{dt^2}+\omega_c^2 q_{S(0)}=Q_{S(0)}\delta(t)\\ (n=1,2,3,\ldots),\end{gathered} \tag{67}$$

where

$$\begin{aligned}Q_{B(n)}&=A_{B(n)}\left(F_0\sin\frac{n\pi x_f}{L}+T_0 g_{B(n)}\cos\frac{n\pi x_\tau}{L}\right)\\ Q_{S(n)}&=A_{S(n)}\left(F_0\sin\frac{n\pi x_f}{L}+T_0 g_{S(n)}\cos\frac{n\pi x_\tau}{L}\right)\\ Q_{S(0)}&=A_{S(0)}T_0,\end{aligned} \tag{68}$$

where $A_{B(n)}$, $A_{S(n)}$, and $A_{S(0)}$ are defined by (60). We solved (67) for unknown generalized coordinates $q_{B(n)}(t)$, $q_{S(n)}(t)$, and $q_{S(0)}(t)$, and then we substituted the results into (61) to obtain the vibration responses as follows:

$$\begin{aligned}\mathbf{u}(x,t)&=\sum_{n=1}^{\infty}\mathbf{U}_{B(n)}(x)\left(I_{B(n)}+\frac{Q_{B(n)}}{\omega_{B(n)}}\sin\omega_{B(n)}t\right)\\ &\quad+\sum_{n=1}^{\infty}\mathbf{U}_{S(n)}(x)\left(I_{S(n)}+\frac{Q_{S(n)}}{\omega_{S(n)}}\sin\omega_{S(n)}t\right)\\ &\quad+\mathbf{U}_{S(0)}\left(I_{S(0)}+\frac{Q_{S(0)}}{\omega_c}\sin\omega_c t\right),\end{aligned} \tag{69}$$

where

$$\begin{aligned}I_{B(n)}&=\frac{dq_{B(n)}(0)}{dt}\frac{\sin\omega_{B(n)}t}{\omega_{B(n)}}+q_{B(n)}(0)\cos\omega_{B(n)}t\\ I_{S(n)}&=\frac{dq_{S(n)}(0)}{dt}\frac{\sin\omega_{S(n)}t}{\omega_{S(n)}}+q_{S(n)}(0)\cos\omega_{S(n)}t\\ I_{S(0)}&=\frac{dq_{S(0)}(0)}{dt}\frac{\sin\omega_c t}{\omega_c}+q_{S(0)}(0)\cos\omega_c t.\end{aligned} \tag{70}$$

Note that the vibration responses contributed by initial conditions are related to $I_{B(n)}$, $I_{S(n)}$, and $I_{S(0)}$, while the vibration responses contributed by external forces are related to $Q_{B(n)}$, $Q_{S(n)}$, and $Q_{S(0)}$. Equation (69) clearly shows that the shear mode shape $\mathbf{U}_{S(0)}$ must be taken into account when a Timoshenko beam is subjected to a stationary bending moment T_0 as well as to initial rotation $\theta(x,0)$ and angular velocity $\partial\theta(x,0)/\partial t$.

3.2. Case 2: Moving Point Transverse Force and Bending Moment. As shown in Figure 1(b), a point transverse force moving at a speed v_f and a point bending moment moving at a speed v_τ can be expressed by employing the Dirac delta function $\delta(x)$ as follows:

$$\begin{aligned}f(x,t)&=F_0\delta\left(x-v_f t\right),\\ \tau(x,t)&=T_0\delta\left(x-v_\tau t\right).\end{aligned} \tag{71}$$

By substituting (71) into (63), we obtain the generalized forces. Then, by substituting the results into (62), we obtain

$$\begin{aligned}\frac{d^2 q_{B(n)}}{dt^2}+\omega_{B(n)}^2 q_{B(n)}&=A_{B(n)}\left(F_0 H_f(t)\sin\frac{n\pi v_f t}{L}\right.\\ &\quad\left.+T_0 g_{B(n)}H_\tau(t)\cos\frac{n\pi v_\tau t}{L}\right)\\ \frac{d^2 q_{S(n)}}{dt^2}+\omega_{S(n)}^2 q_{S(n)}&=A_{S(n)}\left(F_0 H_f(t)\sin\frac{n\pi v_f t}{L}\right.\\ &\quad\left.+T_0 g_{B(n)}H_\tau(t)\cos\frac{n\pi v_\tau t}{L}\right)\\ \frac{d^2 q_{S(0)}}{dt^2}+\omega_c^2 q_{S(0)}&=A_{S(n)}T_0 H_\tau(t),\end{aligned} \tag{72}$$

where $n=1,2,3,\ldots$, and

$$\begin{aligned}H_f(t)&=h(t)-h\left(t-\frac{L}{v_f}\right),\\ H_\tau(t)&=h(t)-h\left(t-\frac{L}{v_\tau}\right),\end{aligned} \tag{73}$$

where $h(t)$ denotes the Heaviside step function as defined by [36]

$$h(t)=\begin{cases}0 & (t<0)\\ 1 & (t\geq 0).\end{cases} \tag{74}$$

We solved (72) to obtain the following results:

$$\begin{aligned}q_{B(n)}(t)&=I_{B(n)}+\gamma_{B(n)}\left[J_1\left(\omega_{B(n)},v_f\right)h(t)\right.\\ &\quad\left.+J_2\left(\omega_{B(n)},v_f\right)h\left(t-\frac{L}{v_f}\right)\right]\\ &\quad+\eta_{B(n)}\left[J_3\left(\omega_{B(n)},v_\tau\right)h(t)\right.\\ &\quad\left.+J_4\left(\omega_{B(n)},v_\tau\right)h\left(t-\frac{L}{v_\tau}\right)\right]\\ q_{S(n)}(t)&=I_{S(n)}+\gamma_{S(n)}\left[J_1\left(\omega_{S(n)},v_f\right)h(t)\right.\\ &\quad\left.+J_2\left(\omega_{S(n)},v_f\right)h\left(t-\frac{L}{v_f}\right)\right]\\ &\quad+\eta_{S(n)}\left[J_3\left(\omega_{S(n)},v_\tau\right)h(t)\right.\\ &\quad\left.+J_4\left(\omega_{S(n)},v_\tau\right)h\left(t-\frac{L}{v_\tau}\right)\right]\\ q_{S(0)}(t)&=I_{S(0)}+\frac{A_{S(0)}}{\omega_c^2}T_0\left\{\left(1-\cos\omega_c t\right)h(t)\right.\\ &\quad\left.-\left[1-\cos\omega_c\left(t-\frac{L}{v_\tau}\right)\right]h\left(t-\frac{L}{v_\tau}\right)\right\},\end{aligned} \tag{75}$$

where $I_{B(n)}$, $I_{S(n)}$, and $I_{S(0)}$ are defined by (70), and

$$\gamma_{B(n)} = \frac{A_{B(n)}F_0}{J_0\left(\omega_{B(n)}, v_f\right)},$$
$$\gamma_{S(n)} = \frac{A_{S(n)}F_0}{J_0\left(\omega_{S(n)}, v_f\right)}$$
$$\eta_{B(n)} = \frac{A_{B(n)}T_0 g_{B(n)}}{J_0\left(\omega_{B(n)}, v_\tau\right)},$$
$$\eta_{S(n)} = \frac{A_{S(n)}T_0 g_{S(n)}}{J_0\left(\omega_{S(n)}, v_\tau\right)},$$
$$J_0(\omega, v) = \omega^3 - \omega\left(\frac{n\pi v}{L}\right)^2 \tag{76}$$
$$J_1(\omega, v) = \omega \sin\frac{n\pi v t}{L} - \frac{n\pi v}{L}\sin\omega t$$
$$J_2(\omega, v) = \frac{n\pi v}{L}\sin\omega\left(t - \frac{L}{v}\right)\cos n\pi - \omega\sin\frac{n\pi v t}{L}$$
$$J_3(\omega, v) = \omega\left(\cos\frac{n\pi v t}{L} - \cos\omega t\right)$$
$$J_4(\omega, v) = \omega\left(\cos\frac{n\pi v t}{L} - \cos\omega\left(t - \frac{L}{v}\right)\cos n\pi\right).$$

By substituting (75) into (61), we obtain the forced vibration responses of a simply supported Timoshenko beam subjected to a moving point transverse force and a moving point bending moment, with arbitrary initial conditions.

To consider the effects of damping on the vibrations of a simply supported Timoshenko beam subjected to a moving point transvers force and a moving point bending moment, the left-hand side of (1) was modified to include a proportional viscous damping term given in the form of $\mathbf{C}\partial\mathbf{u}(x,t)/\partial t$, where $\mathbf{C} = \chi_1\mathbf{M} + \chi_2\mathbf{K}$. Note that $\mathbf{M}$ and $\mathbf{K}$ are the linear operators defined by (3), and χ_1 and χ_2 are damping parameters. By considering the proportional viscous damping, we can obtain damped solutions, instead of (75), as follows:

$$\begin{aligned}
q_{B(n)} &= I_{Bd(n)} + \gamma_{Bd(n)}\left\{J_{1d}\left(\omega_{B(n)}, v_f, \xi_{B(n)}\right)h(t)\right.\\
&\quad\left.+ J_{2d}\left(\omega_{B(n)}, v_f, \xi_{B(n)}\right)h\left(t - \frac{L}{v_f}\right)\right\}\\
&\quad+ \eta_{Bd(n)}\left\{J_{3d}\left(\omega_{B(n)}, v_\tau, \xi_{B(n)}\right)h(t)\right.\\
&\quad\left.+ J_{4d}\left(\omega_{B(n)}, v_\tau, \xi_{B(n)}\right)h\left(t - \frac{L}{v_\tau}\right)\right\}\\
q_{S(n)} &= I_{Sd(n)} + \gamma_{Sd(n)}\left\{J_{1d}\left(\omega_{S(n)}, v_f, \xi_{S(n)}\right)h(t)\right.\\
&\quad\left.+ J_{2d}\left(\omega_{S(n)}, v_f, \xi_{S(n)}\right)h\left(t - \frac{L}{v_f}\right)\right\}\\
&\quad+ \eta_{Sd(n)}\left\{J_{3d}\left(\omega_{S(n)}, v_\tau, \xi_{S(n)}\right)h(t)\right.\\
&\quad\left.+ J_{4d}\left(\omega_{S(n)}, v_\tau, \xi_{S(n)}\right)h\left(t - \frac{L}{v_\tau}\right)\right\}\\
q_{S(0)} &= I_{Sd(0)} + \frac{A_{S(0)}T_0}{\omega_{cd}^2}\left\{\Gamma_1 h(t) - \Gamma_2 h\left(t - \frac{L}{v_\tau}\right)\right\},
\end{aligned} \tag{77}$$

where

$$\begin{aligned}
\xi_{B(n)} &= \frac{\chi_1}{2\omega_{B(n)}} + \frac{\chi_2}{2}\omega_{B(n)},\\
\xi_{S(n)} &= \frac{\chi_1}{2\omega_{S(n)}} + \frac{\chi_2}{2}\omega_{S(n)},\\
\xi_c &= \frac{\chi_1}{2\omega_c} + \frac{\chi_2}{2}\omega_c\\
\omega_{Bd(n)} &= \omega_{B(n)}\sqrt{1 - \xi_{B(n)}^2},\\
\omega_{Sd(n)} &= \omega_{S(n)}\sqrt{1 - \xi_{S(n)}^2},\\
\omega_{cd} &= \omega_c\sqrt{1 - \xi_c^2}
\end{aligned} \tag{78}$$

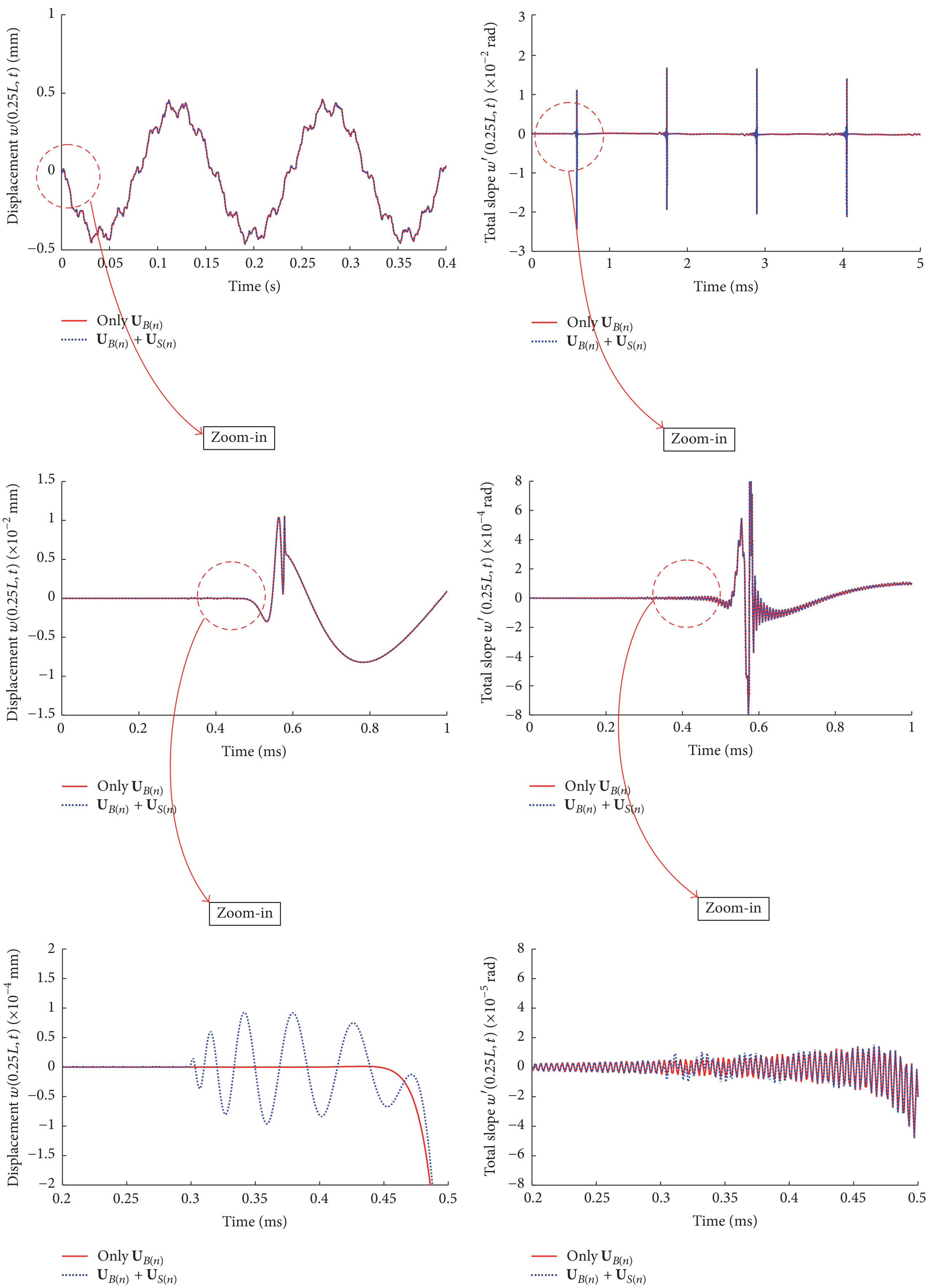
and other symbols used in (77) are defined in the Appendix.

4. Numerical Results and Discussion

For all numerical results presented in this study, we reconsidered the uniform simply supported Timoshenko beam that was considered by Esmailzadeh and Ghorashi [22]. The geometric and material properties of the example beam are as follows: length $L = 4.352$ m, cross-sectional area $A = 1.31 \times 10^{-3}$ m^2, area moment of inertia $I = 5.71 \times 10^{-7}$ m^4, Young's modulus $E = 2.02 \times 10^{11}$ N/m^2, shear modulus $G = 7.7 \times 10^{10}$ N/m^2, mass density $\rho = 15267$ kg/m^3, and shear correction factor $\kappa = 0.7$. For the analyses of forced vibrations and waves, we assumed that a point transverse force and a point moment applied on the example beam have the following magnitudes: $F_0 = 1$ N and $T_0 = 1$ N·m. We also assumed that the example beam has null initial conditions.

Table 1 shows the natural frequencies $\omega_{B(n)}$ and $\omega_{S(n)}$ in Hz and the corresponding mode shape parameters $g_{B(n)}$ and $g_{S(n)}$. The cutoff frequency of this example beam was found to be 14323.70 Hz. Accordingly, the number of natural frequencies $\omega_{B(n)}$ below the cutoff frequency ω_c is $n_c = 74$.

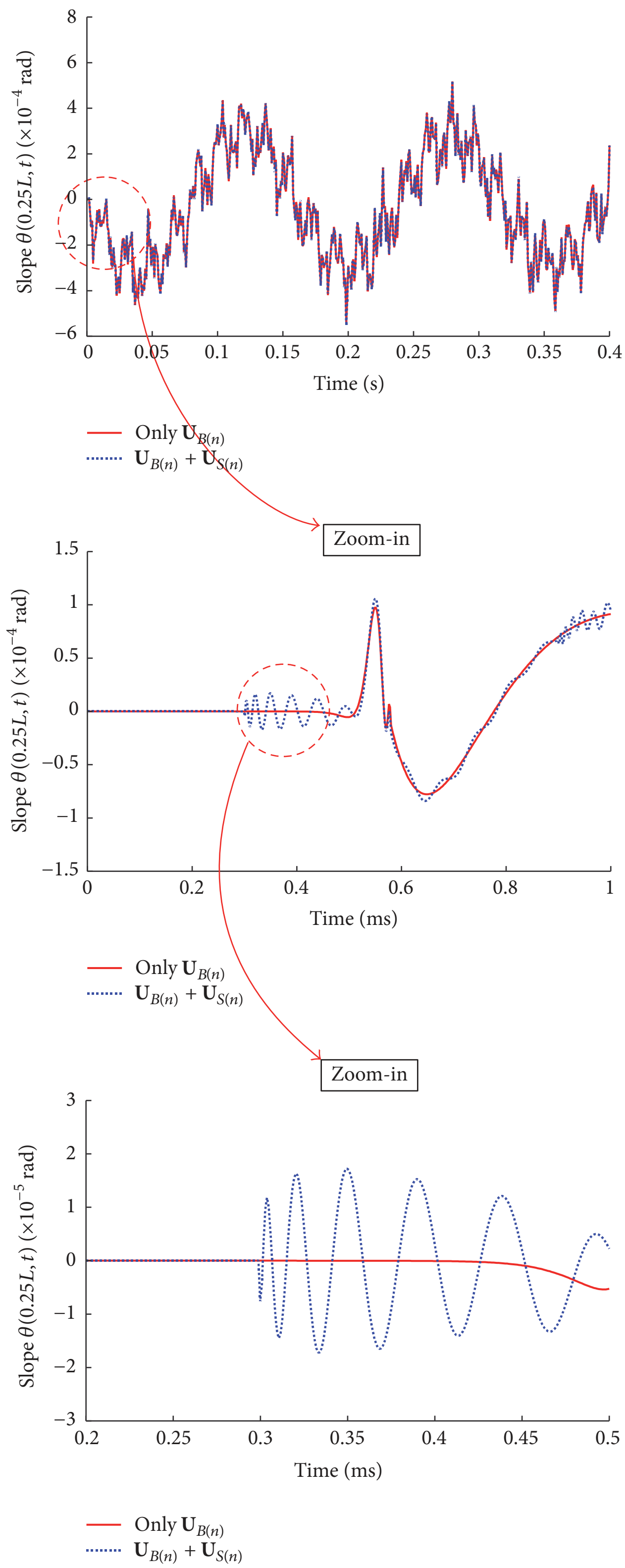
Figure 2 shows the dynamic responses predicted at $x/L = 0.25$ when the example Timoshenko beam is subjected to a stationary impulsive point transverse force F_0 applied at its middle point ($x/L = 0.5$). The responses are the transverse displacement $w(0.25L, t)$, the total slope $w'(0.25L, t)$, the slope due to bending $\theta(0.25L, t)$, and the shear angle due to transverse shear force $\psi(0.25L, t)$. The shear angle is defined by $\psi(x,t) = w'(x,t) - \theta(x,t)$ [37]. The long-term dynamic responses (vibration responses) are displayed at the top of Figure 2, while the short-term dynamic responses (wave propagations) are displayed at the middle and bottom of Figure 2. To investigate the contribution of each type of mode shape to the total dynamic responses, the dynamic responses obtained by taking into account only the mode shapes $\mathbf{U}_{B(n)}$ are compared with those obtained by taking into account both mode shapes, $\mathbf{U}_{B(n)}$ and $\mathbf{U}_{S(n)}$. Note that the mode shape $\mathbf{U}_{S(0)}$ has no effect on the dynamic responses when a simply supported Timoshenko beam is subjected to stationary or moving transverse forces, as can be readily checked with (67) and (72). From Figure 2, we investigated the following:



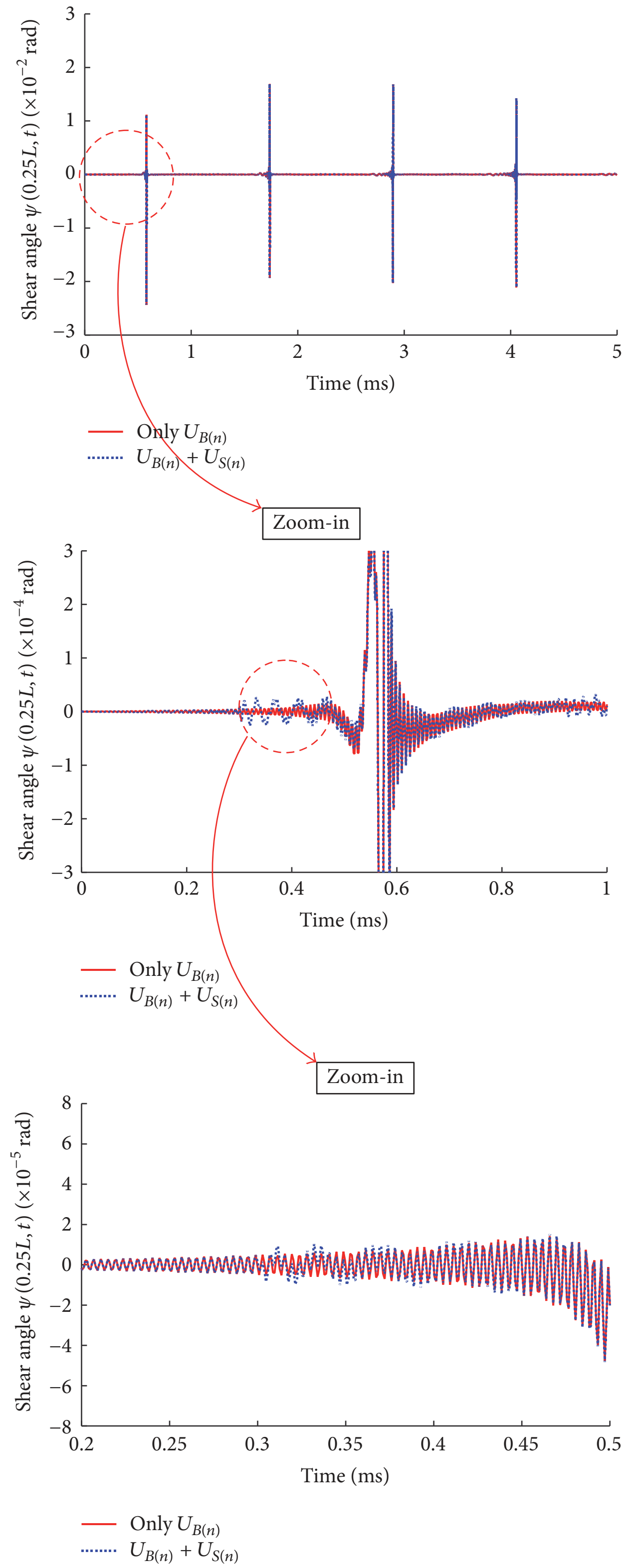
(a) Transverse displacement (w)

(b) Total slope (w')

Figure 2: Continued.



(c) Slope due to bending (θ)



(d) Shear angle (ψ)

Figure 2: Dynamic responses at $x/L = 0.25$ of a simply supported beam subjected to a stationary impulsive point transverse force applied at $x/L = 0.5$: (a) transverse displacement (w); (b) total slope (w'); (c) slope due to bending (θ); (d) shear angle (ψ).

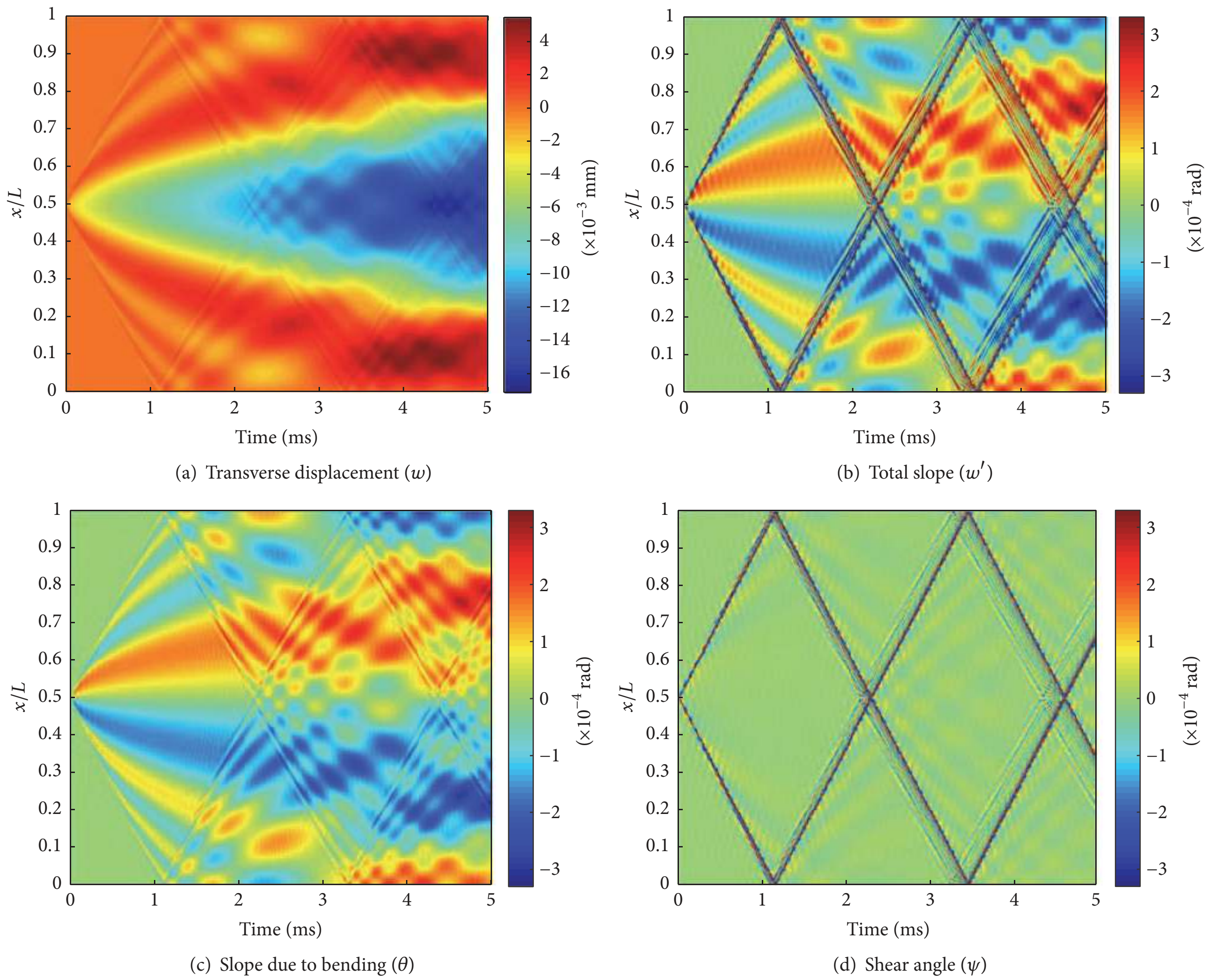
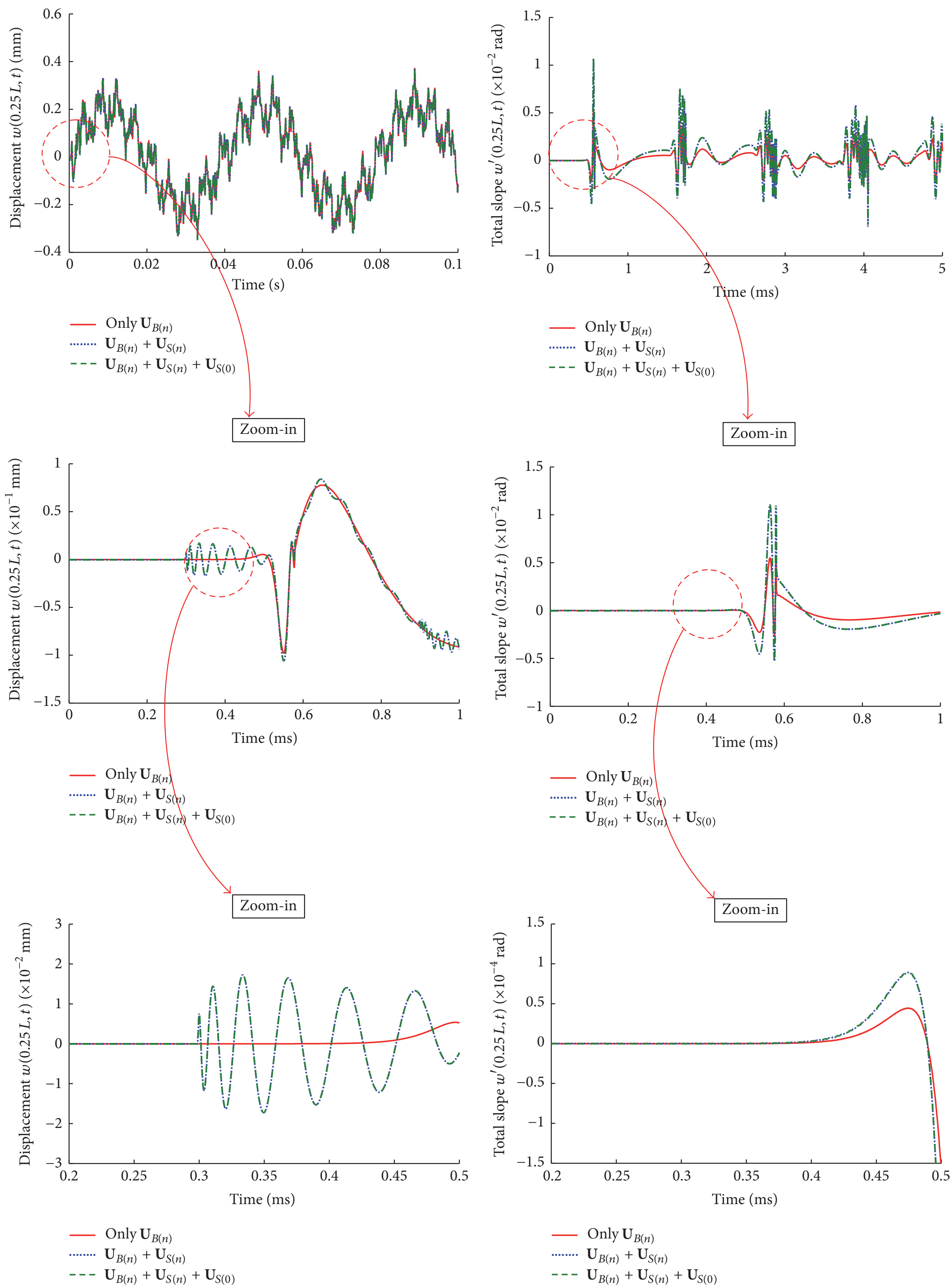


FIGURE 3: Wave propagations in a simply supported beam subjected to a stationary impulsive point transverse force applied at $x/L = 0.5$: (a) transverse displacement (w); (b) total slope (w'); (c) slope due to bending (θ); (d) shear angle (ψ).

(1) although the effects of the mode shapes $\mathbf{U}_{S(n)}$ on the long-term dynamic responses are not significant, the mode shapes $\mathbf{U}_{S(n)}$ must be taken into account to capture accurate wave characteristics in the short-term dynamic responses. (2) There are multiple sharp peaks in the long-term time histories of the total slope w' and shear angle ψ. As shown in Figures 2(b) and 2(d), the sharp peaks appear repeatedly at about 0.6 ms, 1.5 ms, 2.7 ms, 3.9 ms, and so on. Comparing Figures 2(b) and 2(d) with the corresponding short-term time histories (wave propagations) shown in Figures 3(b) and 3(d) shows that the sharp peaks are mainly due to the propagation of shear waves.

Figure 4 shows the dynamic responses predicted at $x/L = 0.25$ for a case in which the example Timoshenko beam is subjected to a stationary impulsive point moment T_0 applied at its middle point ($x/L = 0.5$). From Figure 4, we investigated the following: (1) although the effects of mode shapes $\mathbf{U}_{S(n)}$ on the long-term transverse displacement are not significant, the mode shapes $\mathbf{U}_{S(n)}$ must be taken into account to capture accurate wave characteristics in the short-term dynamic transverse displacement. The mode shape $\mathbf{U}_{S(0)}$ has no influence on the transverse displacement, as suggested by (52), (67), and (68). (2) To accurately predict the slope due to the bending moment and shear angle due to transverse shear force, mode shapes $\mathbf{U}_{S(n)}$ and $\mathbf{U}_{S(0)}$ must both be taken into account in the computation. The mode shape $\mathbf{U}_{S(0)}$ was found to be especially important for the accurate prediction of short-time wave propagations.

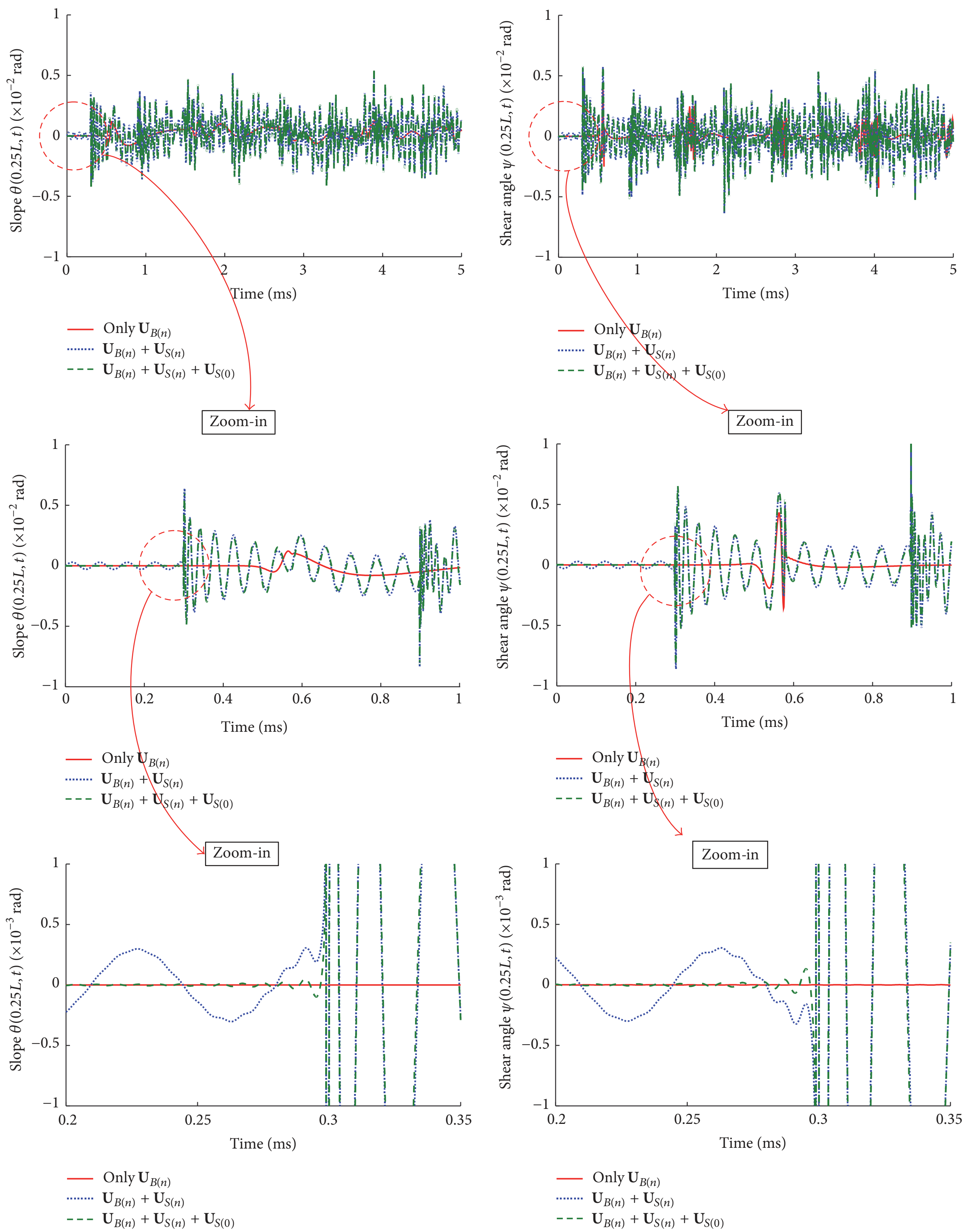
Figure 5 shows the deformed shapes of the example Timoshenko beam at five different times ($t/T_A = 0, 0.25, 0.5, 0.75, 1$) when a point transverse force F_0 is moving on the beam, where T_A denotes the time required for a moving load to cross the beam from the left end ($x/L = 0$) to the right end ($x/L = 1$). Similarly, Figure 6 shows the deformed shapes of the same beam when a point bending moment T_0 is moving on the beam. In both Figures 5 and 6, the deformed shapes are shown for four constant moving speeds: $0.25v_{cr}$, $0.5v_{cr}$, v_{cr}, and $1.5v_{cr}$, where v_{cr} denotes



(a) Transverse displacement (w)

(b) Total slope (w')

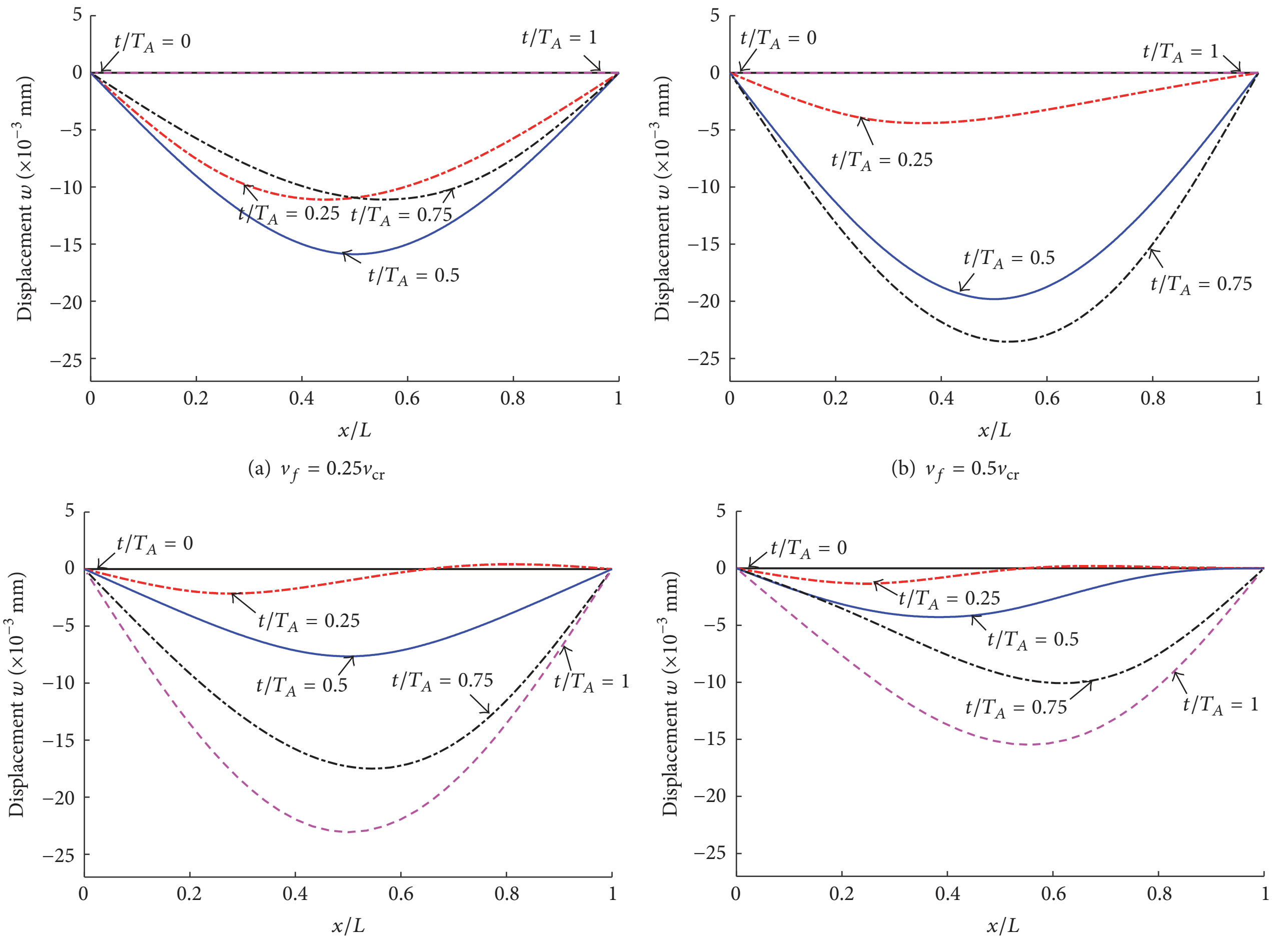
Figure 4: Continued.



(c) Slope due to bending (θ)

(d) Shear angle (ψ)

FIGURE 4: Dynamic responses at $x/L = 0.25$ of a simply supported beam subjected to a stationary impulsive point bending moment applied at $x/L = 0.5$: (a) transverse displacement (w); (b) total slope (w'); (c) slope due to bending (θ); (d) shear angle (ψ).



(a) $v_f = 0.25v_{cr}$ (b) $v_f = 0.5v_{cr}$

(c) $v_f = v_{cr}$ (d) $v_f = 1.5v_{cr}$

FIGURE 5: Deformed shapes of a simply supported Timoshenko beam at five different times ($t/T_A = 0$, 0.25, 0.5, 0.75, and 1) when the beam is subjected to a point transverse force moving at four different constant speeds: (a) $v_f = 0.25v_{cr}$; (b) $v_f = 0.5v_{cr}$; (c) $v_f = v_{cr}$; (d) $v_f = 1.5v_{cr}$, where v_{cr} is the lowest critical speed.

TABLE 1: Natural frequencies and mode shape parameters of a simply supported Timoshenko beam.

Mode number (n)	Natural frequencies (Hz)		Mode shape parameters (rad/m)	
	$\omega_{B(n)}$	$\omega_{S(n)}$	$g_{B(n)}$	$g_{S(n)}$
1	6.29	14331.42	0.72	−3180.85
2	25.14	14354.54	1.44	−1594.48
3	56.41	14392.93	2.15	−1067.48
4	99.92	14446.40	2.86	−805.32
5	155.39	14514.66	3.53	−649.08
10	598.86	15064.35	6.66	−344.31
20	2122.10	17004.83	10.95	−209.53
30	4125.86	19679.14	12.87	−178.31
40	6335.86	22782.05	13.33	−172.12
50	8624.20	26151.69	13.05	−175.79
60	10935.20	29699.86	12.44	−184.42
70	13245.32	33374.30	11.71	−195.95
74	14166.78	34871.53	11.41	−201.13
75	14396.84	35247.95	11.33	−202.47
80	15545.26	37141.61	10.96	−209.36
90	17832.00	40979.20	10.24	−224.06

Note: the cutoff frequency is $\omega_c = 14323.7$ Hz.

the lowest critical speed that can be obtained for a simply supported beam by equating the time period of the first mode to the time needed to pass through a double length of the beam as follows [12]: $v_{cr} = 2f_1 L = 54.79$ m/s, where f_1 is the first natural frequency in Hz. Note that a sufficient number of mode shapes $\mathbf{U}_{B(n)}$ and $\mathbf{U}_{S(n)}$, including the pure shear mode shape $\mathbf{U}_{S(0)}$, were considered in order to obtain the deformed shapes shown in Figures 5 and 6. From Figures 5 and 6, we investigated the following: (1) the deformed shapes strongly depend on the speed of a moving load, and (2) the upward deformation does not seem to be significant when a beam is subjected to a moving point downward transverse force, whereas it can be significant when the beam is subjected to a moving point bending moment. The vibration responses or the deformed shapes of a Timoshenko beam were obtained from (61) by using the generalized coordinates computed from (75). As shown in (75), the generalized coordinates are the functions of $J_0(\omega, v)$, $J_1(\omega, v)$, $J_2(\omega, v)$, $J_3(\omega, v)$, and $J_4(\omega, v)$, which are dependent on the speed of a moving load. This is why the deformed shapes shown in Figures 5 and 6 strongly depend on the speed of a moving load.

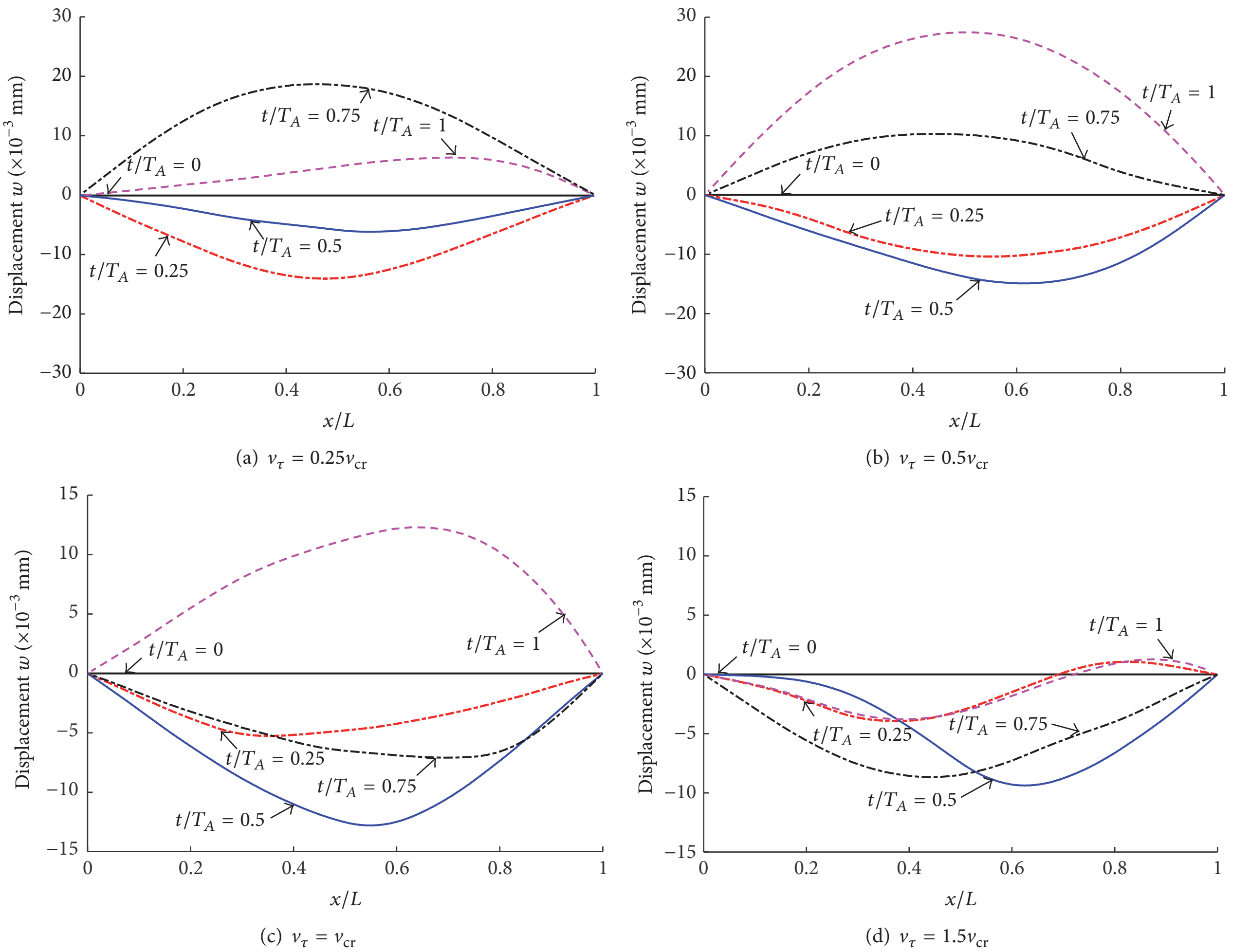


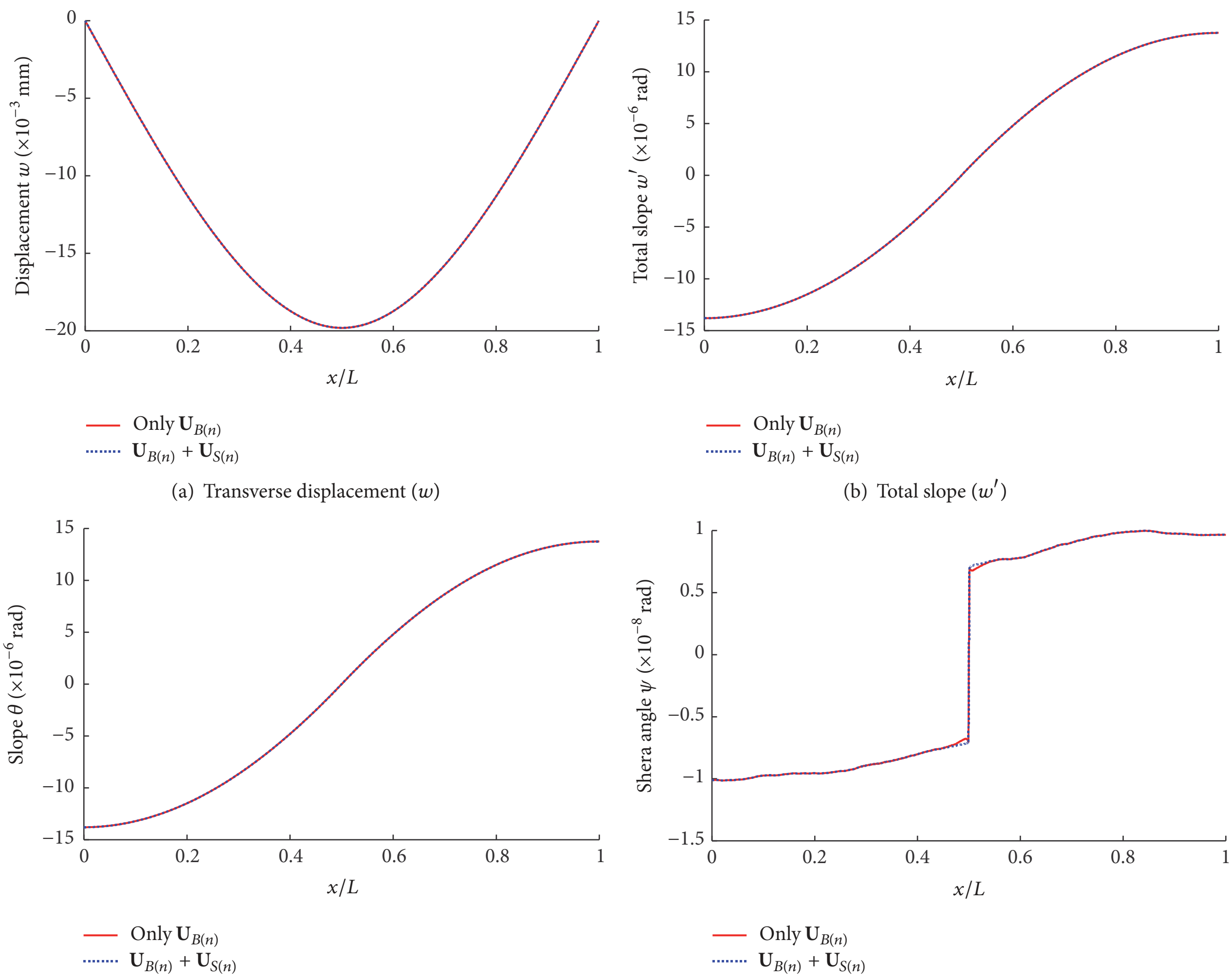
FIGURE 6: Deformed shapes of a simply supported Timoshenko beam at five different times ($t/T_A = 0$, 0.25, 0.5, 0.75, and 1) when the beam is subjected to a point bending moment moving at four different constant speeds: (a) $v_\tau = 0.25v_{cr}$; (b) $v_\tau = 0.5v_{cr}$; (c) $v_\tau = v_{cr}$; (d) $v_\tau = 1.5v_{cr}$, where v_{cr} is the lowest critical speed.

Figures 7 and 8 show the distributions of the transverse displacement w, the total slope w', the slope due to bending θ, and the shear angle due to transverse shear force ψ at $t = 0.5T_A$ when the example Timoshenko beam is subjected to a moving point transverse force and a moving point bending moment, respectively. We assumed that the moving point transverse force and bending moment have the same moving speed of $0.5v_{cr}$. From Figures 7 and 8, we investigated the following: (1) the deformed shapes generated by a moving point transverse force are quite different from those generated by a moving point bending moment, and (2) the shear angle ψ generated by a moving point transverse force can be well predicted by using the mode shapes $\mathbf{U}_{B(n)}$ only. However, for accurate prediction of the shear angle generated by a moving point bending moment, the mode shapes $\mathbf{U}_{S(n)}$ and $\mathbf{U}_{S(0)}$ must be considered. Note that there is a step in the curves in Figure 7(d) to satisfy the force equilibrium at the middle of the beam at which a moving point transverse force arrives at an instant of $t = 0.5T_A$.

Figures 9 and 10 show the time histories of the transverse displacement w, the total slope w', the slope due to bending θ, and the shear angle due to transverse shear force ψ at $x/L = 0.5$ when the example Timoshenko beam is subjected to the same moving point transverse force and bending moment, respectively. From Figures 9 and 10, we investigated the following: (1) the dynamic responses due to a moving point transverse force are quite different from those due to a moving point bending moment and (2) the time history of the shear angle ψ due to a moving point transverse force can be accurately predicted by using the mode shapes $\mathbf{U}_{B(n)}$ only. However, the mode shapes $\mathbf{U}_{S(n)}$ and $\mathbf{U}_{S(0)}$ must be considered for the accurate prediction of the shear angle due to a moving point bending moment.

Based on Figures 7–10, we investigated the following: (1) the long-term dynamic responses can be well predicted by using the mode shapes $\mathbf{U}_{B(n)}$ only, and (2) the mode shapes $\mathbf{U}_{S(n)}$ and $\mathbf{U}_{S(0)}$ must be considered in a prediction of accurate shear angles due to the transverse shear forces.

To verify the accuracy of the present MAM, the dynamic responses of a simply supported Timoshenko beam obtained by the present MAM and the frequency-domain spectral element method (SEM) are compared in Figure 11. We assumed that the beam is subjected to a point transverse force moving at three different constant speeds v_f: (a) $v_f = 0.5v_{cr}$,



(a) Transverse displacement (w)

(b) Total slope (w')

(c) Slope due to bending (θ)

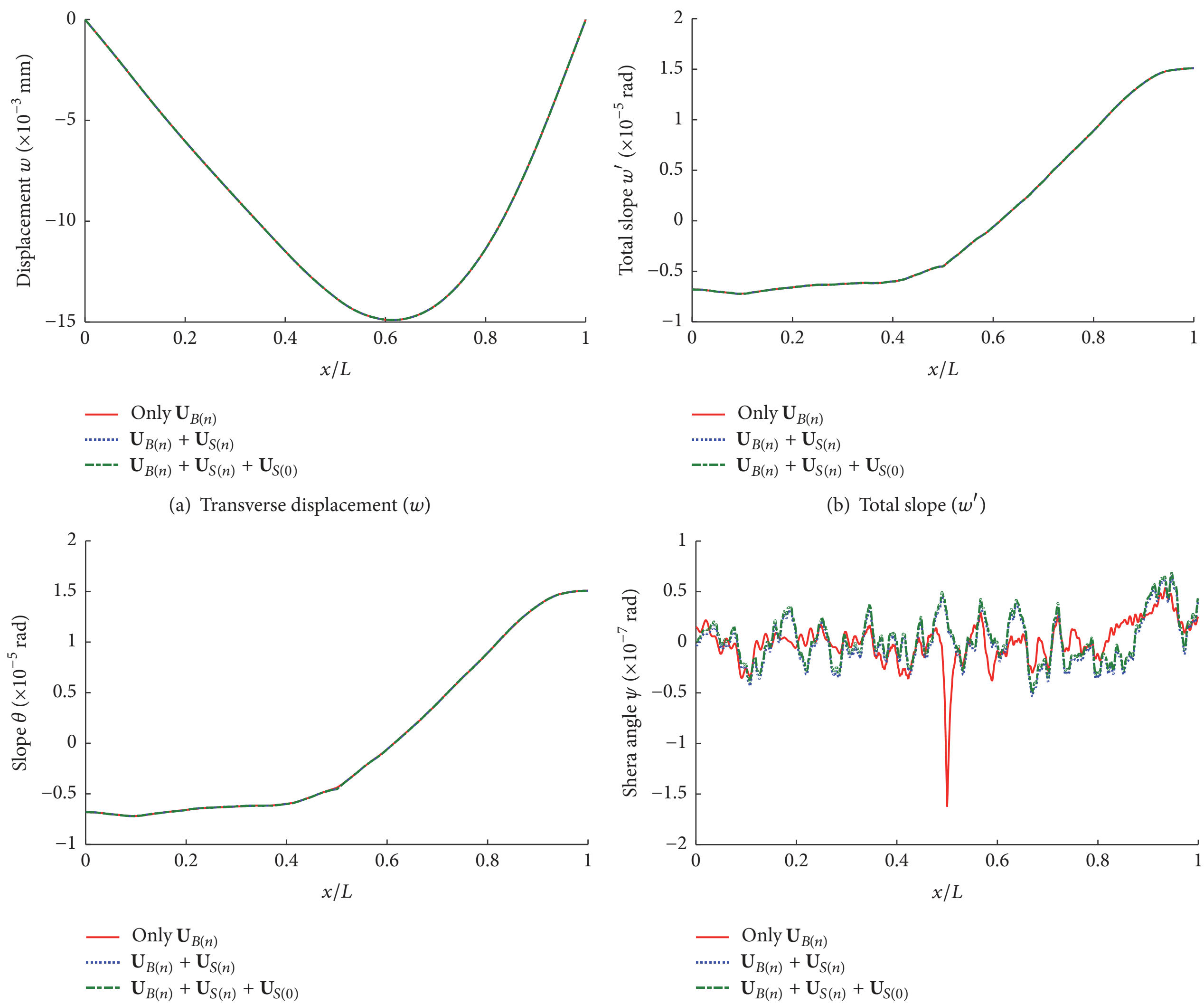
(d) Shear angle (ψ)

Figure 7: Deformed shapes of a simply supported Timoshenko beam at $t = 0.5T_A$ when the beam is subjected to a point transverse force moving at a constant speed of $v_f = 0.5v_{\mathrm{cr}}$: (a) transverse displacement (w); (b) total slope (w'); (c) slope due to bending (θ); (d) shear angle (ψ).

(b) $v_f = v_{\mathrm{cr}}$, and (c) $v_f = 1.5v_{\mathrm{cr}}$. The SEM is known as an exact element method that provides extremely accurate solutions to one-dimensional structural dynamics problems [38]. Song et al. [24] applied the SEM to a moving load problem to verify its high accuracy. Figure 11 shows that the dynamic responses obtained by using the present MAM are almost identical to those obtained by using the SEM.

To investigate the effects of the shear deformation and rotary inertia on the dynamic responses of a beam, the transverse displacements at $x/L = 0.25$ of a simply supported beam obtained by using Timoshenko beam theory and Bernoulli-Euler beam theory are compared in Figure 12. We assumed that the simply supported beam is subjected to a stationary impulsive point transverse force at $x/L = 0.5$. It is well known that the phase velocity of flexural waves in a Bernoulli-Euler beam increases indefinitely with increasing wave number (or frequency). On the other hand, for a Timoshenko beam, the phase velocity of flexural waves has a finite maximum value, while the phase velocity of transverse shear waves, which is infinite at a wave number of zero, gradually decreases to a limit value with increasing wave number [39]. Figure 12(a) demonstrates that the time history of transverse displacement of a Bernoulli-Euler beam starts from $t = 0$ because the wave modes of infinite or nearly infinite phase speeds generated by the impulsive point transverse force applied at $x/L = 0.5$ can reach immediately the measurement point $x/L = 0.25$. On the other hand, the time history of transverse displacement of a Timoshenko beam delays starting because the flexural waves in a Timoshenko beam have finite values. As the shear deformation is completely neglected in Bernoulli-Euler beam model, Figure 12(b) demonstrates that, as expected, the time history of shear angle exists only in case of the Timoshenko beam.

Finally, the effects of damping on the dynamic responses of a simply supported Timoshenko beam subjected to a point transverse force moving at a constant speed $v_f = 0.5v_{\mathrm{cr}}$ are investigated by comparing the dynamic responses obtained by considering and without considering damping. Figure 13 shows the comparison of the dynamic responses at three locations ($x/L = 0.25$, 0.5, and 0.75) obtained by considering



(a) Transverse displacement (w)

(b) Total slope (w')

(c) Slope due to bending (θ)

(d) Shear angle (ψ)

FIGURE 8: Deformed shapes of a simply supported Timoshenko beam at $t = 0.5T_A$ when the beam is subjected to a point bending moment moving at a constant speed of $v_\tau = 0.5v_{cr}$: (a) transverse displacement (w); (b) total slope (w'); (c) slope due to bending (θ); (d) shear angle (ψ).

and without considering the proportional viscous damping, where we used $\chi_1 = 0$, 9.89 and 39.55 when $\chi_2 = 0$. We can observe from Figure 13 that the amplitudes of transverse displacements in general decrease due to damping.

5. Conclusions

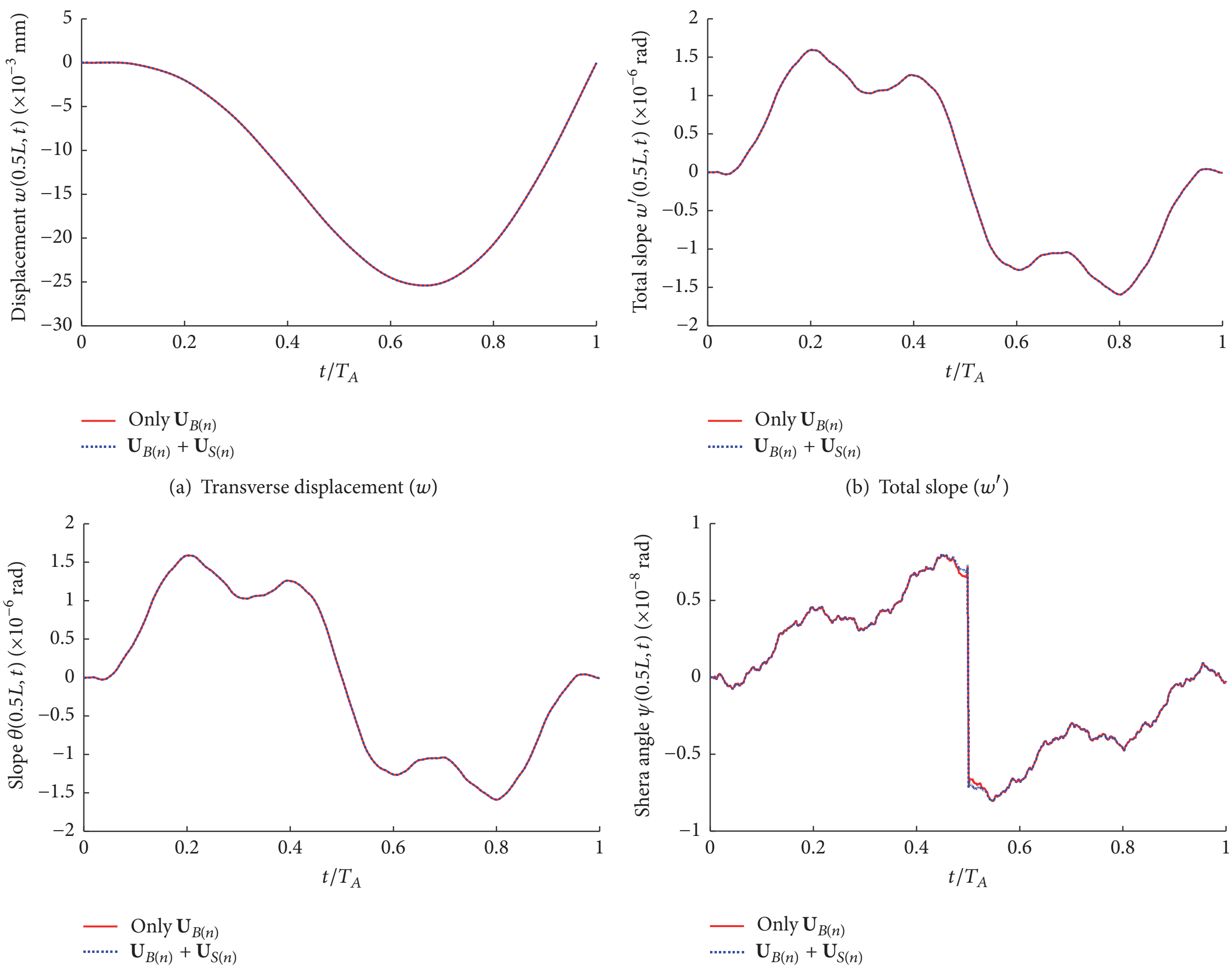
We examined general solutions, natural frequencies, mode shapes, and the orthogonality properties of mode shapes for simply supported Timoshenko beams. We also presented the forced vibration responses of a simply supported Timoshenko beam in analytical closed form when the beam is subjected to arbitrary initial conditions and to stationary or moving transverse forces and bending moments. The new findings made in this study are summarized as follows:

(1) A complete set of natural frequencies and mode shapes are presented in closed forms for all frequency ranges: $0 < \omega \le \omega_c$ and $\omega \ge \omega_c$, where ω_c is the cutoff frequency.

(2) It is shown that three types of mode shapes (denoted by $\mathbf{U}_{B(n)}$, $\mathbf{U}_{S(n)}$, and $\mathbf{U}_{S(0)}$) are required for a modal analysis of the forced vibrations of a Timoshenko beam subjected to arbitrary initial conditions and to arbitrary stationary or moving loads.

(3) It is found that, in addition to the mode shapes $\mathbf{U}_{B(n)}$ and $\mathbf{U}_{S(n)}$, the pure shear mode shape $\mathbf{U}_{S(0)}$ must be included in the modal analysis when a Timoshenko beam is subjected to external bending moments or to the initial rotation and angular velocity.

(4) In general, the long-term dynamic responses (vibrations) due to stationary and moving transverse forces can be well predicted by using only the bending mode shapes $\mathbf{U}_{B(n)}$, but this is not true for stationary and



(a) Transverse displacement (w)

(b) Total slope (w')

(c) Slope due to bending (θ)

(d) Shear angle (ψ)

FIGURE 9: Dynamic responses at $x/L = 0.5$ of a simply supported Timoshenko beam subjected to a point transverse force moving at a constant speed of $v_f = 0.5v_{cr}$: (a) transverse displacement (w); (b) total slope (w'); (c) slope due to bending (θ); (d) shear angle (ψ).

moving moments. It is necessary to take into account the shear mode shapes $\mathbf{U}_{S(n)}$ to accurately predict the short-term dynamic responses (wave propagations) due to stationary and moving transverse forces or bending moments.

(5) The deformed shapes of a Timoshenko beam strongly depend on the speeds of moving loads. The upward deformation does not seem to be significant when the beam is subjected to a moving point downward transverse force, whereas it can be significant when the beam is subjected to a moving point bending moment.

(6) The effects of shear deformation and rotatory inertia are investigated by comparing short-term dynamic responses (waves) obtained by using Bernoulli-Euler beam theory and Timoshenko beam theory.

(7) Numerical results show that the amplitudes of transverse displacements in general decrease due to damping.

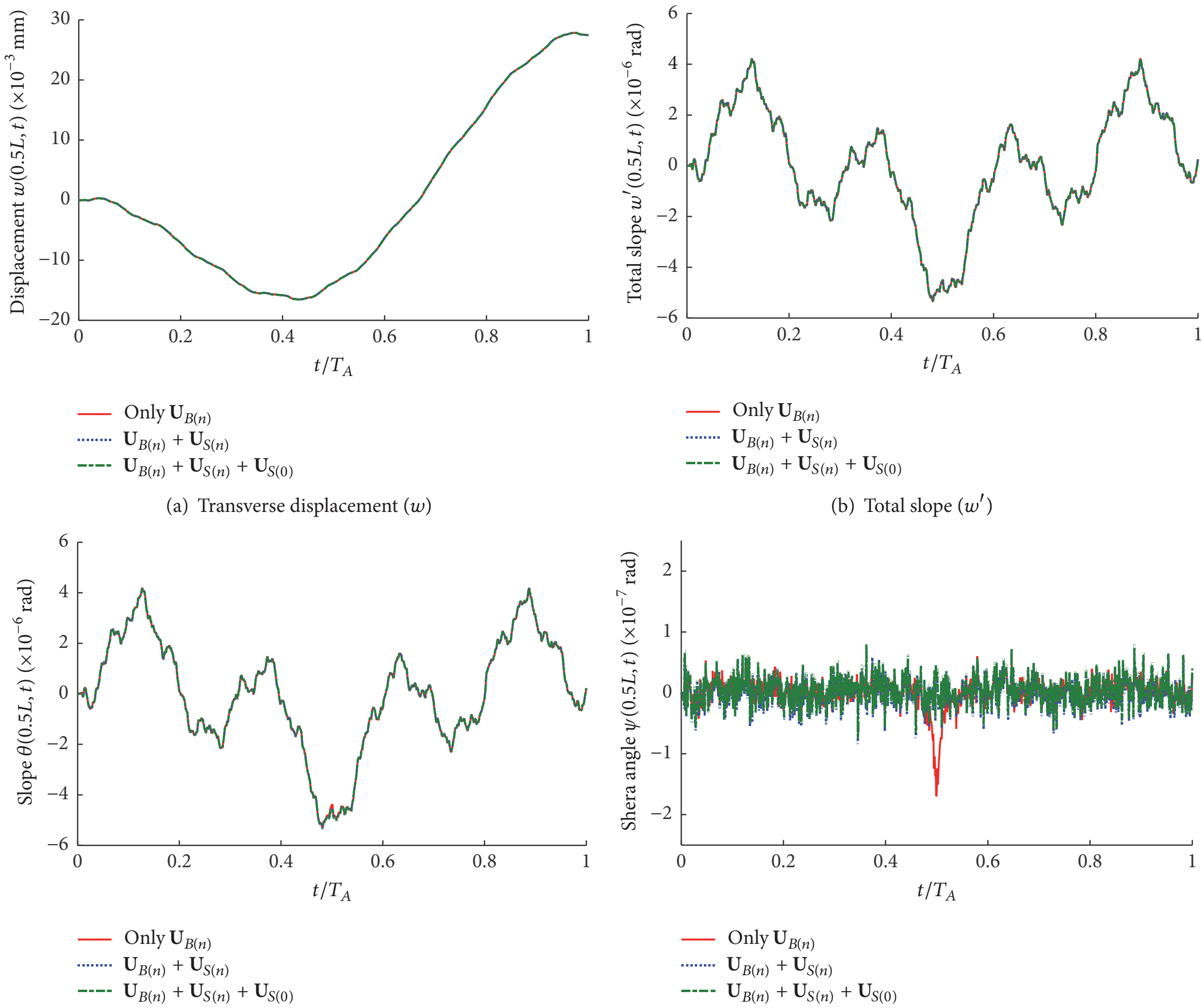
Appendix

Symbols Used in (77)

The symbols used in (77) are defined by

$$I_{Bd(n)} = e^{-\xi_{B(n)}\omega_{B(n)}t}\left[\left\{\xi_{B(n)}\omega_{B(n)}q_{B(n)}(0) + \frac{dq_{B(n)}(0)}{dt}\right\}\frac{\sin\omega_{Bd(n)}t}{\omega_{Bd(n)}} + q_{B(n)}(0)\cos\omega_{Bd(n)}t\right]$$

$$I_{Sd(n)} = e^{-\xi_{S(n)}\omega_{S(n)}t}\left[\left\{\xi_{S(n)}\omega_{S(n)}q_{S(n)}(0) + \frac{dq_{S(n)}(0)}{dt}\right\}\frac{\sin\omega_{Sd(n)}t}{\omega_{Sd(n)}} + q_{S(n)}(0)\cos\omega_{Sd(n)}t\right]$$



(a) Transverse displacement (w)
(b) Total slope (w')
(c) Slope due to bending (θ)
(d) Shear angle (ψ)

Figure 10: Dynamic responses at $x/L = 0.5$ of a simply supported Timoshenko beam subjected to a point bending moment moving at a constant speed of $v_\tau = 0.5v_{\mathrm{cr}}$: (a) transverse displacement (w); (b) total slope (w'); (c) slope due to bending (θ); (d) shear angle (ψ).

$$I_{Sd(0)} = e^{-\xi_c \omega_c t}\left[\left\{\xi_c \omega_c q_{S(0)}(0) + \frac{dq_{S(0)}(0)}{dt}\right\}\frac{\sin \omega_{cd} t}{\omega_{cd}} + q_{S(0)}(0)\cos \omega_{cd} t\right]$$

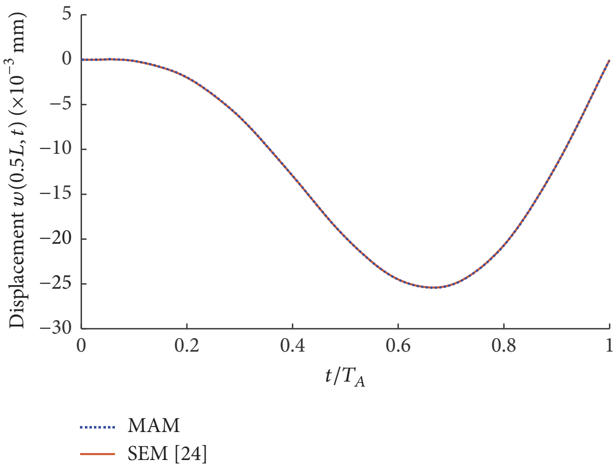
$$\gamma_{Bd(n)} = \frac{A_{B(n)} F_0}{J_{0d}\left(\omega_{B(n)}, v_f, \xi_{B(n)}\right)},$$

$$\gamma_{Sd(n)} = \frac{A_{S(n)} F_0}{J_{0d}\left(\omega_{B(n)}, v_f, \xi_{B(n)}\right)}$$

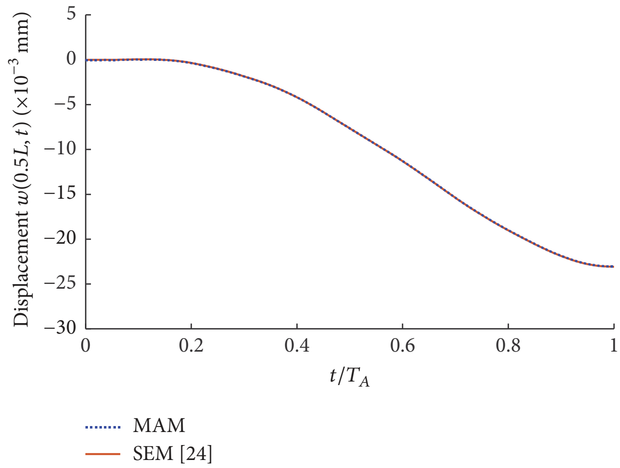
$$\eta_{Bd(n)} = \frac{A_{B(n)} T_0 g_{B(n)}}{J_{0d}\left(\omega_{B(n)}, v_f, \xi_{B(n)}\right)},$$

$$\eta_{Sd(n)} = \frac{A_{S(n)} T_0 g_{S(n)}}{J_{0d}\left(\omega_{B(n)}, v_f, \xi_{B(n)}\right)}$$

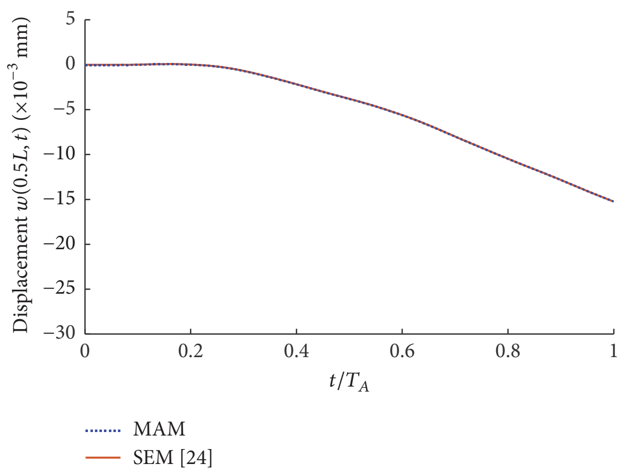
$$J_{0d}(\omega, v, \xi) = \omega\sqrt{1-\xi^2}\left\{\omega^2\left(1-\xi^2\right) - \left(\frac{n\pi v}{L}\right)^2\right\}$$



(a) at $x/L = 0.25$



(b) at $x/L = 0.5$



(c) at $x/L = 0.75$

Figure 11: Comparison of the transverse displacements $w(x,t)$ obtained by the present MAM and the SEM in [24] at $x/L = 0.5$ of a simply supported Timoshenko beam subjected to a point transverse force moving at three different constant speeds v_f: (a) $v_f = 0.5v_{\text{cr}}$; (b) $v_f = v_{\text{cr}}$; (c) $v_f = 1.5v_{\text{cr}}$.

$$J_{1d}(\omega,v,\xi) = \left\{\frac{K_{11}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\sin\frac{n\pi vt}{L} - \frac{K_{12}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\cos\frac{n\pi vt}{L}\right\}$$

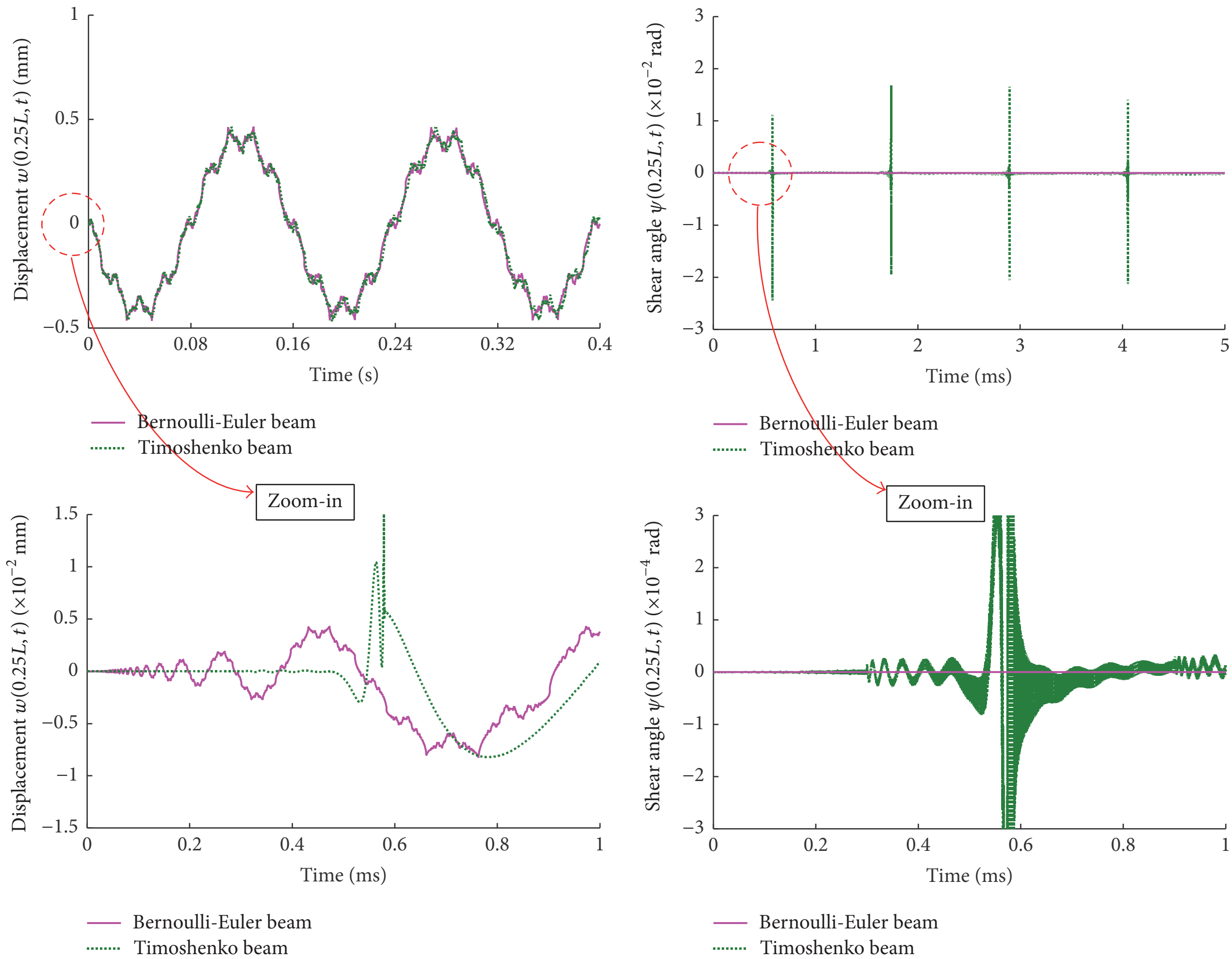
$$+ e^{-\xi\omega t}\left\{\frac{K_{13}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\sin\omega\sqrt{1-\xi^2}t + \frac{K_{11}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\cos\omega\sqrt{1-\xi^2}t\right\}$$

$$J_{2d}(\omega,v,\xi) = \left\{\frac{K_{11}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\sin\frac{n\pi vt}{L} - \frac{K_{12}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\cos\frac{n\pi vt}{L}\right\} - e^{-\xi\omega(t-L/v)}\left\{\frac{K_{21}(\omega,v,\xi)}{2K_{01}(\omega,v,\xi)} + \frac{K_{22}(\omega,v,\xi)}{2K_{02}(\omega,v,\xi)}\right\}$$

$$J_{3d}(\omega,v,\xi) = \left\{\frac{K_{11}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\cos\frac{n\pi vt}{L} + \frac{K_{12}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\sin\frac{n\pi vt}{L}\right\}$$

$$- e^{-\xi\omega t}\left\{\frac{K_{11}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\cos\omega\sqrt{1-\xi^2}t + \frac{K_{14}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\sin\omega\sqrt{1-\xi^2}t\right\}$$

$$J_{4d}(\omega,v,\xi) = -\left\{\frac{K_{11}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\cos\frac{n\pi vt}{L} + \frac{K_{12}(\omega,v,\xi)}{K_0(\omega,v,\xi)}\sin\frac{n\pi vt}{L}\right\} + e^{-\xi\omega(t-L/v)}\left\{\frac{K_{23}(\omega,v,\xi)}{2K_{01}(\omega,v,\xi)} + \frac{K_{24}(\omega,v,\xi)}{2K_{02}(\omega,v,\xi)}\right\}$$



(a) Transverse displacement (w) (b) Shear angle (ψ)

FIGURE 12: Comparison of the dynamic responses at $x/L = 0.25$ of a simply supported beam subjected to a stationary impulsive point transverse force applied at $x/L = 0.5$ obtained by Timoshenko beam theory and Bernoulli-Euler beam theory: (a) transverse displacement (w); (b) shear angle (ψ).

$$\Gamma_1 = \frac{e^{-\xi_c \omega_c t}\left[e^{\xi_c \omega_c t}\omega_{cd}^2 - \xi_c \omega_c \omega_{cd} \sin \omega_{cd} t - \omega_{cd}^2 \cos \omega_{cd} t\right]}{\omega_{cd}^2 + \xi_c^2 \omega_c^2}$$

$$\Gamma_2 = \frac{e^{\xi_c \omega_c t}\omega_{cd}^2 - e^{\xi_c \omega_c L/v_\tau}\left[\xi_c \omega_c \omega_{cd} \sin \omega_{cd}\left(t - L/v_\tau\right) + \omega_{cd}^2 \cos \omega_{cd}\left(t - L/v_\tau\right)\right]}{\omega_{cd}^2 + \xi_c^2 \omega_c^2}, \tag{A.1}$$

where

$$K_0(\omega, v, \xi) = K_{01}(\omega, v, \xi) K_{02}(\omega, v, \xi)$$

$$K_{01}(\omega, v, \xi) = \left(n\pi v - \omega\sqrt{1-\xi^2}L\right)^2 + \xi^2\omega^2 L^2$$

$$K_{02}(\omega, v, \xi) = \left(n\pi v + \omega\sqrt{1-\xi^2}L\right)^2 + \xi^2\omega^2 L^2$$

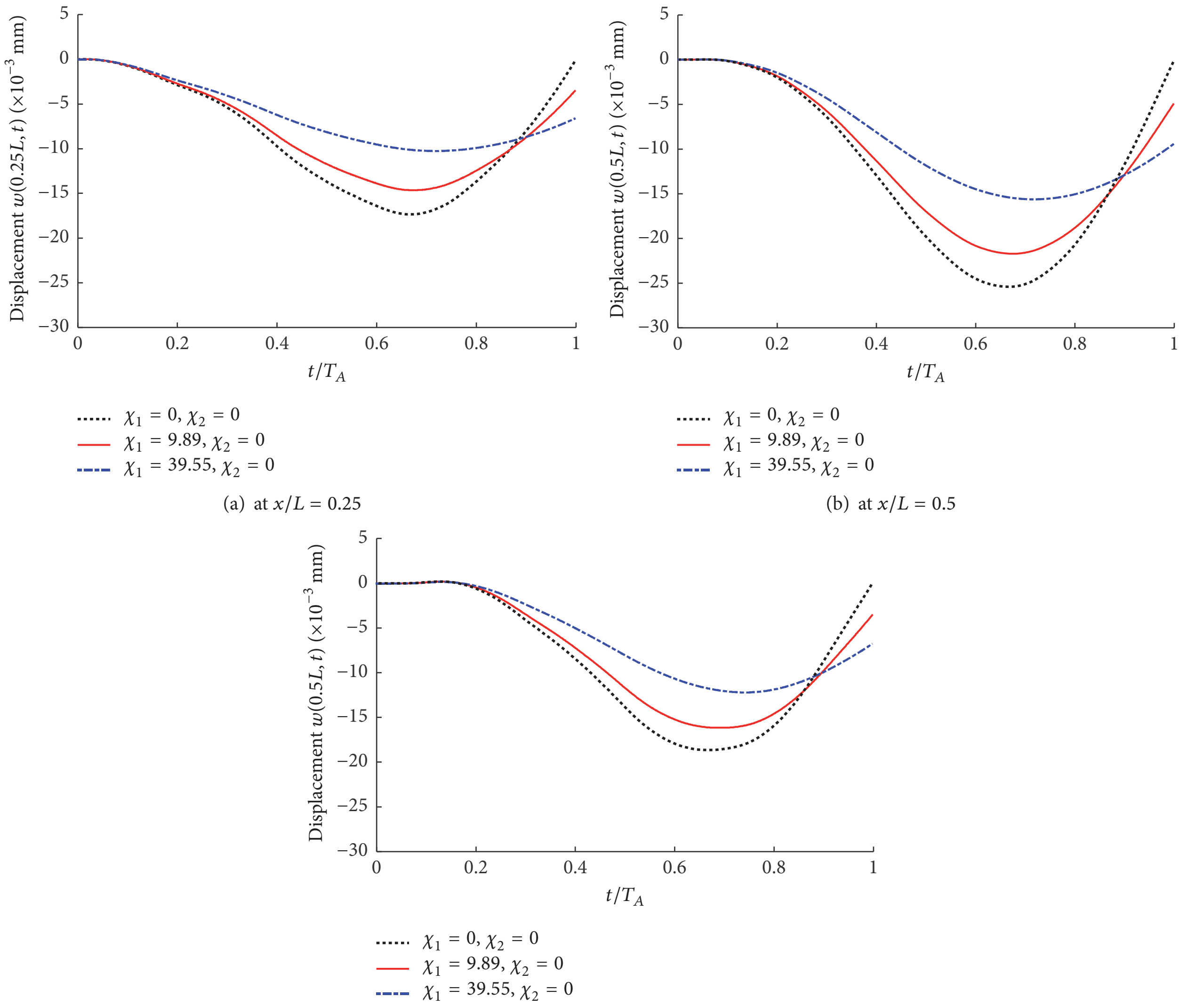
$$K_{11}(\omega, v, \xi) = \omega\sqrt{1-\xi^2}\left(\omega^2 L^2 - n^2\pi^2 v^2\right)$$

$$\cdot \left\{\omega^2 L^2\left(1-\xi^2\right) - n^2\pi^2 v^2\right\}$$

$$K_{12}(\omega, v, \xi) = 2n\pi v L \omega^2 \xi\sqrt{1-\xi^2}\left\{\omega^2 L^2\left(1-\xi^2\right) - n^2\pi^2 v^2\right\}$$

$$K_{13}(\omega, v, \xi) = -\left(\frac{n\pi v}{L}\right)\left\{\omega^2 L^2\left(1-\xi^2\right) - n^2\pi^2 v^2\right\}$$

$$\cdot \left\{\omega^2 L^2\left(1-2\xi^2\right) - n^2\pi^2 v^2\right\}$$



(a) at $x/L = 0.25$

(b) at $x/L = 0.5$

(c) at $x/L = 0.75$

Figure 13: Effects of viscous damping on the transverse displacements at three locations of a simply supported Timoshenko beam subjected to a point transverse force moving at a constant speed of $v_f = 0.5v_{\mathrm{cr}}$ when $\chi_1 = 0, 9.89$ and 39.55 with $\chi_2 = 0$: (a) at $x/L = 0.25$; (b) at $x/L = 0.5$; (c) at $x/L = 0.75$.

$$K_{14}(\omega, v, \xi) = -\omega\xi\left(\omega^2 L^2 + n^2\pi^2 v^2\right)\left(\omega^2\left(1-\xi^2\right) - n^2\pi^2 v^2\right)$$

$$K_{21}(\omega, v, \xi) = \left\{\omega^2\left(1-\xi^2\right) - \left(\frac{n\pi v}{L}\right)^2\right\} \cdot \left[-\xi\omega L^2 \cos\left\{\omega\sqrt{1-\xi^2}\left(t-\frac{L}{v}\right) + n\pi\right\} + \left(\omega L^2\sqrt{1-\xi^2} - n\pi v L\right) \cdot \sin\left\{\omega\sqrt{1-\xi^2}\left(t-\frac{L}{v}\right) + n\pi\right\}\right]$$

$$K_{22}(\omega, v, \xi) = \left\{\omega^2\left(1-\xi^2\right) - \left(\frac{n\pi v}{L}\right)^2\right\}\left[\xi\omega L^2 \cdot \cos\left\{\omega\sqrt{1-\xi^2}\left(t-\frac{L}{v}\right) - n\pi\right\} - \left(\omega L^2\sqrt{1-\xi^2} + n\pi v L\right) \cdot \sin\left\{\omega\sqrt{1-\xi^2}\left(t-\frac{L}{v}\right) - n\pi\right\}\right]$$

$$K_{23}(\omega, v, \xi) = \left\{\omega^2\left(1-\xi^2\right) - \left(\frac{n\pi v}{L}\right)^2\right\}\left[\xi\omega L^2 \cdot \sin\left\{\omega\sqrt{1-\xi^2}\left(t-\frac{L}{v}\right) + n\pi\right\} + \left(\omega L^2\sqrt{1-\xi^2} - n\pi v L\right) \cdot \cos\left\{\omega\sqrt{1-\xi^2}\left(t-\frac{L}{v}\right) + n\pi\right\}\right]$$

$$
\begin{aligned}
K_{24}(\omega, v, \xi) = {} & \left\{\omega^2\left(1-\xi^2\right)-\left(\frac{n\pi v}{L}\right)^2\right\}\left[\xi\omega L^2\right. \\
& \cdot \sin\left\{\omega\sqrt{1-\xi^2}\left(t-\frac{L}{v}\right)-n\pi\right\} \\
& +\left(\omega L^2\sqrt{1-\xi^2}+n\pi v L\right) \\
& \left.\cdot \cos\left\{\omega\sqrt{1-\xi^2}\left(t-\frac{L}{v}\right)-n\pi\right\}\right].
\end{aligned} \tag{A.2}
$$

Competing Interests

The authors declare that there is no conflict of interests regarding the publication of this article.

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