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Forces on a Sphere Accelerating in a Viscous Fluid

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PREFACE

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An expression is proposed for the resisting forces acting on a sphere moving rectilinearly in a viscous fluid:

$$\vec{F} = C_A \frac{4}{3} \pi R^3 \rho \vec{a} + \frac{1}{2} C_D \rho \pi R^2 |V| \vec{V} + C_H R^2 \sqrt{\pi\mu\rho} \int_0^t \frac{\overrightarrow{\acute{a}(t-t')}}{\sqrt{t'}} dt'$$

where R is radius of sphere, ρ is density of fluid, μ is viscosity, V is velocity of the object in a fluid, a is acceleration of the object and C_A, C_D and C_H are coefficients. The forces comprise three basic terms. The first expresses the viscous and form drags; the second indicates the effect of added mass and the last is due to the history of motion. The coefficients were determined experimentally by measuring the forces on a sphere moving sinusoidally within the range of $0 < Re < 62$. The calculated forces agreed with the measured forces very well.

FORCES ON A SPHERE ACCELERATING IN A VISCOUS FLUID

by

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INTRODUCTION

Determination of the forces exerted by a fluid flow unsteady with respect to some of its boundaries has always been a big challenge in fluid mechanics. There are many ways that the motion of the fluid can be unsteady with respect to some of its boundaries. Either the boundaries or the fluid or both can accelerate. Examples of the first case are the motion of bombs or missiles through still air or water, acceleration of airships or submarines, and the deceleration of objects upon entering water or the atmosphere. The forces exerted by long waves on piles or on the bottom of a canal, and the friction due to unsteady flow in pipes fit in the second category. The last category may involve very complicated motions such as snow and sediment transport, wind-induced oscillations of tall chimneys and suspension bridges, and motion of piles, off-shore platforms, and anchored ships in waves.

The investigation described herein was undertaken to solve problems concerned with drifting snow. Arctic buildings need to be shaped and arranged so that snowdrifts will not interfere seriously with their use. Like sediment transport in rivers, snow transport is too complicated by local geometry to permit mathematical analysis; so model studies are required. But, as will be illustrated subsequently, the usual similarity criteria cannot be met, and an attempt must be made to exert the proper total force on a model particle at each instant. Thus, the problem becomes one of determining the length, velocity, density, and viscosity scale ratios that will, on an average, cause the model particles to follow paths similar to those of particles in nature.

If the usual method of dimensional analysis is employed to find scale factors between model and nature, geometric similarity among all parts of the system plus equality between certain dimensionless parameters is required. In the case of drifting snow, the gravity force exerted on the snow particles, the buoyant force of the air, and the drag of the air determine the particle trajectory. It then follows that the Reynolds and Froude numbers, respectively, and the fluid to particle density ratio should be alike in model and prototype. Thus, for example, if the length ratio, model to prototype, is 1/100 the kinematic viscosity ratio should be 1/1000 to satisfy both Froude's and Reynold's criteria and clearly this requirement cannot be met.

For a small-scale model there is the additional difficulty of providing geometric similarity for the particles. Drifting snow in cold climates is composed of rounded hard grains, 1 mm or less in diameter. A 1/100 scale model of the snow would be dust having a grain size of a few ten-thousandths of an inch — much finer than Portland cement. Such a material may not be commercially available or it may have undesirable properties.

Illustrations of this sort are strong reminders of the fact that distorted scale models are unavoidable if drifting snow is to be modeled at all. The usual method is to adjust density, velocity, and amount of falling material until the model surface reproduces a known configuration under natural conditions. The model is then supposed to be able to serve as a forecast of changes that would be caused by limited geometric changes, such as removing or introducing obstructions to the flow. An alternative procedure (Odar, 1962), subject to experimental verification, is to reproduce the paths of particles, to duplicate threshold characteristics, and to control the amount of falling material in accordance with a certain criterion in the model.

As stated previously, in order to reproduce the paths of particles, it is necessary to exert the proper force on a model particle at each instant. In this case the scale factors can be calculated from an equation of motion. Thus, snow transport may be reproduced in a model even though the Reynolds numbers in model and under natural conditions are not alike. Moreover, it will not be necessary for the sizes of the particles to correspond to the geometric scale of the model.

The motion of a snow particle is curvilinear and three-dimensional. The velocity field around the particle depends on the turbulence characteristics and the boundary layer in the atmosphere. Both the particle and air move unsteadily and irregularly. If all of these conditions are considered, the fundamental laws governing the motion of the particle cannot be readily determined by using present knowledge. In order to find these laws simplified cases must be studied first. As the knowledge is furthered, the research can be extended to more complicated situations.

Let us consider the following simplified case:

a) The shape of a snow particle is spherical. This is an assumption which holds for a majority of the particles, since they are abraded into round grains by continuous turbulence and movement over the surface.

b) The motion of a snow particle is rectinlinear. In an open field, snow particles follow an almost rectilinear path in the absence of obstructions.

c) The sizes of turbulent eddies are very large compared to the sizes of snow particles so that the velocity field at a sufficient distance from a particle is considered uniform.

Thus, the problem reduces to the study of the arbitrary rectilinear motion of a sphere in a fluid which is otherwise moving uniformly. In this case the total force acting on the sphere can be considered as the sum of the following forces:

1. Forces exerted by gravity.
2. Forces exerted by the fluid on the object due to the relative velocity and acceleration.
3. Forces exerted by the fluid on the object when both the relative velocity and acceleration are zero.

Relative velocity and acceleration refer to the difference between the velocities and accelerations, respectively, of the sphere and of the undisturbed fluid flow at sufficient distance from the sphere. So, the equation of motion of a spherical particle in a fluid can be written as

$$\frac{\pi}{6} \rho_p D^3 \frac{d\vec{V}_p}{dt} = \frac{\pi}{6} \rho_p D^3 \vec{g} + \vec{F} - \iint_S p\, d\vec{S} \qquad (1)$$

in which ρ_p, D and $\vec{V}_p$ are the density, diameter and velocity of the particle respectively, t is the time, g is the acceleration of gravity, F is the force due to the relative motion, p is the pressure in the fluid and dS is a vector representing an area element at the surface of the particle. The first term on the left is the product of the mass and acceleration of the particle. The right side of the equation represents the summation of the forces acting on the particle. The last term on the right side represents the forces exerted by the fluid on the sphere when the relative velocity and acceleration are zero. This term can be evaluated as follows:

$$-\iint_S p\, d\vec{S} = \frac{\pi}{6} D^3 \rho \frac{d\vec{V}_f}{dt} - \frac{\pi}{6} \rho D^3 \vec{g} \qquad (2)$$

in which V_f is the velocity of the uniform fluid flow. This treatment is valid only if the flow is uniform. V_f is a function of t only. When the relative acceleration and velocity are zero, the fluid moves like a rigid body together with the sphere as shown in Figure 1.

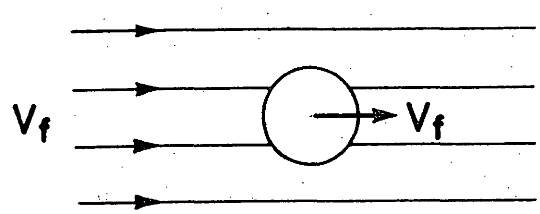

Figure 1. Motion of sphere with uniform fluid flow.

As stated previously, only the simple case of the motion of a sphere in an otherwise uniform fluid flow is investigated at present. Obviously, more complicated situations may occur during the motion of a snow particle. For instance, if the fluid flow is not uniform, there will be some local forces on the sphere due to the nonuniformity of the velocity field.

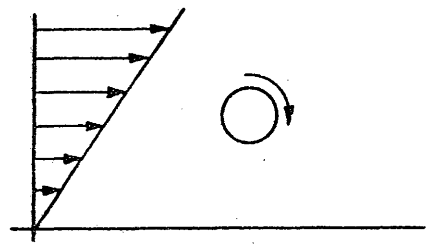

Figure 2. Motion of sphere with nonuniform fluid flow.

For the conditions illustrated in Figure 2, the sphere will spin and the forces acting on the sphere and, consequently, the motion will be different.

The following study concerns the determination of the forces exerted by the fluid due to the relative velocity and acceleration of the sphere. Subsequently previous work on rectilinear motion of bodies in a still fluid and the rectilinear motion of a fluid past fixed objects is briefly reviewed.

SURVEY OF LITERATURE

Mathematical investigations

a. Nonviscous irrotational flow. - In some fluid dynamics problems such as the lift effect of fluid flow on a body, the assumptions of inviscidity and irrotationality lead to results which agree with the experiments. However, in problems of resistance exerted on a body moving in a fluid, calculations made by using these assumptions show that the resistance is zero if the body moves with a constant speed. This result obviously does not agree with reality. If the body accelerates in the fluid, then part of the resistance can be calculated.

In order to do this, the concept of added mass is used. The force, $m \frac{dV}{dt}$, required to accelerate a body of mass m in a vacuum is increased to $(m + m_a) \frac{dV}{dt}$ when the body is accelerating in a fluid. Hence, the resistance due to nonviscous irrotational flow is

$$F = m_a \frac{dV}{dt} \qquad (3)$$

in which m_a is the added mass.

One method used to calculate the added mass utilizes the kinetic energy of an otherwise quiet fluid outside the body. This kinetic energy is equated to the kinetic energy of a hypothetical particle moving at the same speed as the body. The mass of this particle is then equal to the added mass. This suggests that the added mass is a hypothetical quantity. Poisson (1832) was the first to evaluate the added mass for a sphere. He found that it is equal to one half the mass of the fluid displaced by the sphere. This was later confirmed by Green (1833) and Stokes (1843).

Kirchoff (1869) showed that the added mass is proportional to the volume and the density of the displaced fluid and to the shape of the body. Thus, the added mass can be expressed as

$$m_a = k\rho \forall \qquad (4)$$

in which k is the shape factor and V is the volume of the body. The values of k for a sphere and cylinder are $\frac{1}{2}$ and 1 respectively.

Generally, the added mass is not a scalar quantity as shown in eq 4. If an ellipsoid is moving along one of its principal axes, for example, the kinetic energy of the fluid outside the ellipsoid depends on which axis is parallel to the motion. It can be shown that the added mass of an irregularly shaped body making a translatory motion is a second order tensor which is expressed by nine quantities. A detailed analysis of the added mass tensor is outside the scope of the present work; the reader interested in more information is referred to Birkhoff (1955).

b. Laminar motion where convective acceleration terms are omitted. — In this group of studies the convective acceleration terms in the Navier-Stokes equations are omitted. Therefore, the motions are confined to low velocities. The equations are solved and the expressions for the force on the body are determined.

Stokes (1851) investigated the oscillations of a sphere, cylinder, and an infinitely long flat plate in its plane in a fluid. The oscillations were simple harmonic. The force expressions consist of two terms, one involving the acceleration and the other involving the velocity. The acceleration term is different from the term on the right side of eq 3 in that it includes viscosity. Stokes also found a force expression which is valid for the steady uniform motion of a sphere in an infinite fluid.

Later Boussinesq (1885), Basset (1888), and Oseen (1927) studied the rectilinear motion of a sphere which is rapidly accelerating and arbitrarily moving in a viscous fluid. They found that the forces on the sphere depend not only on its instantaneous velocity and acceleration but also on an integral term which represents the effect of its entire history of acceleration. Each effect is represented by a separate term. It is necessary to note that the acceleration term does not include viscosity, although the whole expression is valid for a viscous flow. In fact the acceleration term is the same as the force expression in a nonviscous irrotational flow, namely the product of added mass and acceleration. If Basset's equation is applied to an oscillating sphere, and if the integral term is evaluated, the force expression found by Stokes (1851) can be obtained. This shows that, in a force expression valid for a specified motion, a quantity multiplied by acceleration does not necessarily represent the added mass in a viscous fluid. The added mass, a hypothetical value, has been defined for a nonviscous irrotational flow but it has not been defined for a viscous flow.

c. Laminar boundary layer cases. — In this case the velocity of the object is high and a laminar boundary layer is formed around it. Here, the shear stresses on the body are calculated by solving boundary layer equations and the pressure is calculated from nonviscous irrotational flow outside the boundary layer. The magnitude of the force exerted on the body is found by summing these stresses.

Boundary layer equations are non-linear and exact solutions are difficult to find. One method of obtaining these solutions involves successive approximations. The reader interested in this method is referred to Schlichting (1960). In this method the velocity is approximated by the sum of the terms:

$$u = u_0 + u_1 + u_2 + \ldots \tag{5}$$

Each term on the right side of eq 5 satisfies a certain partial differential equation. The boundary conditions change according to the nature of the problem. For the case of a cylinder or wedge which is suddenly accelerated from rest and kept at a constant velocity thereafter, the solutions of u_0, u_1, and u_2 are given by Schlichting (1960), Blasius (1908) and Goldstein (1936) respectively. For a constantly accelerated cylinder or wedge the solutions of u_0 and u_1 were found by Blasius (1908). For a sphere which is suddenly accelerated from rest and kept at a constant velocity thereafter, the solutions of u_0 and u_1 are given by Boltze (1908). Odar (1962) found general integral solutions for u_0 and higher order approximation terms for an arbitrary motion of a body. The body can be a cylinder or wedge with two dimensional flow around it, or a sphere with axially symmetric flow around it. Obviously the forces exerted on the body can be calculated by this method as far as the laminar boundary layer exists.

Experimental investigations

Forces acting on accelerating objects submerged in a fluid can be determined by experiments. Because of the difficulty of theoretical investigations, experiments are more often used for practical problems.

One experimental method utilizing dimensional reasoning can be credited to McNown (1957). He expressed the force acting on a two dimensional object immersed in an accelerating fluid by three terms involving the acceleration and the velocity of the undisturbed fluid and the surface integral of the fluid pressure in the absence of the object as explained in eq 1.

$$F = \frac{1}{2} C_D h l \rho V_f^2 + \iint_S p \, dS + \mathcal{V}\rho \frac{d(XV_f)}{dt} \qquad (6)$$

in which C_D is the drag coefficient and X is the added mass coefficient as defined by McNown. V_f is the velocity of the fluid undisturbed by the object; h and l are the height and the length of the object respectively.

McNown conducted experiments with two types of objects in an oscillating fluid, a flat plate placed normal to the flow, and a thin lenticular cylinder with its longitudinal axis placed parallel to the flow. He did not include the effect of viscosity in his analysis The results were exploratory rather than definitive.

O'Brien and Morison (1952) investigated the forces exerted by an oscillating fluid on a sphere located near the bottom of a tank. They assumed that the forces could be expressed by a formula similar to eq 6 except that their added mass coefficient was outside the derivative sign. The values of the drag and the added mass coefficients were determined at separate locations where the acceleration and velocity, respectively, became zero. They made eight runs. No correlation for the added mass coefficient was found. The drag coefficient was correlated with the Reynolds number. However, the correlation seems to be poor. The values of the drag coefficient are scattered. In relation to similar Reynolds numbers these values are two to four times higher than those in steady flow.

Keulegan and Carpenter (1958) investigated the forces acting on cylinders and plates in an oscillating fluid. They assumed that the force can be expressed by a formula similar to eq 6 except that they introduced a new added mass coefficient which was outside the derivative sign. Using Fourier analysis they found expressions for the drag and added mass coefficients in terms of time. The curves for drag coefficients, which are drawn from experimental data, are discontinuous and tend to go to infinity as the velocity approaches zero. No dependence on the Reynolds number was found. The Reynolds numbers based on the maximum velocities during the wave cycle and the diameters of the cylinders were quite high. They varied roughly from 6,000 to 30,000. It may be also recalled that, for a steady motion in this region, the drag coefficients do not change with the Reynolds number either, but if a wider range of Reynolds numbers is considered, it can be seen that the values of the drag coefficient will then depend on the Reynolds number.

Iversen and Balent (1951) studied the forces acting on accelerating disks moving normal to their planes in an otherwise quiet fluid. They considered the following type of equation:

$$F = \frac{1}{2} C \rho \frac{\pi}{4} d^2 V^2 \qquad (7)$$

in which a, V, and d are the acceleration, velocity, and diameter of the disk respectively, and C is a coefficient dependent on the Reynolds number, $\frac{Vd}{\nu}$, and the so-called acceleration modulus, $\frac{ad}{V^2}$. Similar studies were made by Keim (1956) on accelerating cylinders and by Bugliarello (1956) on accelerating spheres. Eq 7 is very limited in its application. For example, if a simple harmonic motion is considered, C has to be infinitely large at the place where V is zero so that F can have a finite value. The value of F also changes with the magnitude of the acceleration. It is very unlikely that all types of rectilinear motions can be expressed by eq 7.

It seems that an adequate expression for the forces acting on accelerating bodies in a fluid has not been found. In the following, rectilinear motion of a sphere will be studied and an effort will be made to determine the resistance of the fluid.

ARBITRARY RECTILINEAR MOTION OF A SPHERE

Basset (1888) solved this problem mathematically by using the Navier-Stokes equations for the case where the convective acceleration terms were disregarded and found an expression for the resisting force on the sphere:

$$F = \frac{1}{2}\left(\frac{4}{3}\pi R^3\right)\rho a + 6\pi R\mu V + 6R^2\sqrt{\pi\mu\rho}\int_0^t \frac{a(t-t')}{\sqrt{t'}}\,dt' \qquad (8)$$

in which R is the radius of the sphere, μ is the viscosity of the fluid, ρ is density of the fluid, and V and a are the velocity and acceleration of the sphere respectively. The first term on the right is the same as the steady-state viscous drag on the sphere. The second term is the same as the resistance to an accelerating sphere in a nonviscous irrotational flow and the third term is the effect of the history of acceleration. More information about evaluation of the history of acceleration and its underlying principles can be found in Odar (1962). Since the convective acceleration terms are disregarded, the motion of the sphere is confined to rapid accelerations and low velocities.

The author will introduce a new equation for the force exerted on the sphere moving in such a manner that the effect of convective acceleration is important. As seen in eq 8 three kinematic quantities, namely the velocity, acceleration and history of acceleration are responsible for the force exerted on the sphere. This can also be concluded by physical reasoning. As a matter of convenience, a simple harmonic motion of a sphere will be visualized and an effort will be made to induce an expression for the resisting force on the sphere.

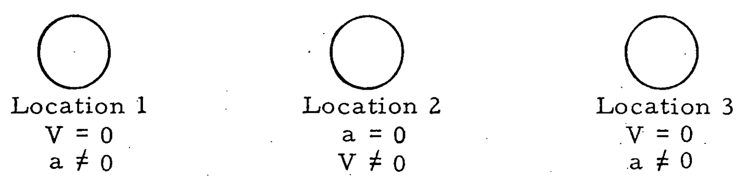

Figure 3. Simple harmonic motion.

The sphere has started from rest at location 1 (Fig. 3) at time t = 0. An instant later the velocity is almost zero and the acceleration effect is predominant. Actually, the resistance on the sphere at this time can be expressed by the added mass formula, eq 3. Generally, based on dimensional reasoning, this force can be expressed as

$$\vec{F} = C_A \frac{4}{3}\pi R^3 \rho \vec{a} \qquad (9)$$

in which C_A is the coefficient of the added mass. The values of this coefficient have to be determined by experiments. It can be a function of various dimensionless combinations which can be derived by using dimensional analysis. It is necessary to state that as t approaches zero, C_A has to approach $\frac{1}{2}$.

Now consider location 2. Since a is zero, the resistance as expressed by eq 9 will be zero. But experiments show that the resistance is not zero. Again based on dimensional reasoning, this force can be expressed as

$$\vec{F} = \tfrac{1}{2} C_D \rho \pi R^2 |V| \vec{V} \tag{10}$$

in which C_D is conveniently called the coefficient of drag. The values of this coefficient also have to be determined by experiments. It could be a function of various dimensionless combinations which can be derived by using dimensional analysis.

Now the next step is the expression of the force as the summation of these two effects, namely the acceleration and the velocity.

$$\vec{F} = C_A \frac{4}{3} \pi R^3 \rho \vec{a} + \tfrac{1}{2} C_D \rho \pi R^2 |V| \vec{V}. \tag{11}$$

This expression fits the physical picture because at the locations where a = 0 the second term gives a certain drag force, and at the locations where V = 0, the first term gives the added mass effect.

Now one more step will be taken. Suppose that the sphere started at location 1, completed its cycle, came back to location 1, and started its new cycle. It will be noticed that at this time the force acting on the sphere is different from the force recorded initially although the velocities and accelerations are the same in both instances. This shows that a third term showing the effect of the history of the motion must be included in the expression of the force.

The existence of the third term can also be visualized in the case of a sudden stop. Suppose that the sphere has stopped suddenly at some location. In this case, a = 0 and V = 0, but there will be some force acting on the sphere. This force will diminish after some time elapses, and it can be regarded as the effect of the history.

An expression for the history of motion is constructed by using the history expression in eq 8. When this is incorporated with eq 11 the expression of the resisting force becomes

$$\vec{F} = C_A \frac{4}{3} \pi R^3 \rho \vec{\bar{a}} + \tfrac{1}{2} C_D \rho \pi R^2 |V| \vec{V} + C_H R^2 \sqrt{\pi \mu \rho} \int_0^t \frac{\vec{\bar{a}}(t-t')}{\sqrt{t'}} dt' \tag{12}$$

in which C_H is the history coefficient. This coefficient also has to be evaluated by experiments and it could be a function of various dimensionless combinations which can be derived by using dimensional analysis.

It should be noted that eq 12 should converge to eq 8 found by Basset for rapid accelerations and low velocities. This also gives a clue as to the type of history expression that can be used in eq 12. In this case C_H should converge to 6 for low velocities and rapid accelerations.

The first terms of eq 8 and 12 are the added mass effects. As stated previously, $C_A = \frac{1}{2}$ at t = 0. Now another condition is imposed. For low velocities and rapid accelerations, $C_A = \frac{1}{2}$.

The second terms of eq 12 and 8 represent the velocity effects. The second term in eq 12 should converge to the second term in Basset's equation, eq 8, for low velocities and rapid accelerations. But the second term in Basset's equation has the same value as the drag force on a sphere moving steadily and slowly. If the second term in eq 12 is assumed to have the same value as the drag force on a sphere moving steadily, when the motion becomes slow, C_D could be replaced by $\frac{24}{Re}$ and this term would be the same as the second term in Basset's equation. Also, when the acceleration becomes zero and remains so for some period of time so that the integral term in eq 12 also becomes zero, this term would then give the value of the drag force on a steadily moving sphere regardless of the magnitude of velocity. So, it will be assumed that the second term in eq 12 has the same value as the steady-state drag force and the value of C_D will be taken from the curve established by low turbulence wind tunnel tests. These values are given in Lapple, 1951, Table I.

As stated previously, the effect of the convective acceleration was not included in eq 8. In eq 12, however, the effect of convective acceleration is included. It is necessary to investigate the changes in eq 8 as the forces due to the convective acceleration become larger and larger. In order to do this, the following ratios of forces acting on a unit cube of fluid will be considered:

$$\frac{F_{\text{con. acc.}}}{F_{\text{fric.}}} = \frac{\rho u \frac{\partial u}{\partial x}}{\mu \frac{\partial^2 u}{\partial y^2}} \sim \frac{\rho \frac{V^2}{D}}{\mu \frac{V}{D^2}} = \frac{\rho V D}{\mu} \tag{13a}$$

$$\frac{F_{\text{con. acc.}}}{F_{\text{time acc.}}} = \frac{\rho u \frac{\partial u}{\partial x}}{\rho \frac{\partial u}{\partial t}} \sim \frac{\rho \frac{V^2}{D}}{\rho a} = \frac{V^2}{aD} . \tag{13b}$$

The first ratio is the well-known Reynolds number. If the forces due to convective acceleration are small compared to the shear stress, the Reynolds number will be small. For a given viscosity a small Reynolds number can be obtained with either a small velocity or a small diameter of the sphere. Experiments made for steady-state conditions indicate that, if $Re < 1$, the contribution of the convective acceleration is not significant.

The second ratio is a number which will be called the Acceleration number and denoted as Ac. This number was used by Iversen and Balent (1951) and Keim (1956), who derived it by dimensional reasoning only.

If the forces due to convective acceleration are small compared to the forces due to time acceleration, the Acceleration number will be small. A small Acceleration number can be obtained if the velocity of the sphere is small, its acceleration is high and its diameter is large. There should be a limit of the Acceleration number below which the contribution of the convective acceleration can be disregarded.

Thus, if the density and the viscosity of the fluid and the motion of the sphere are given, there may be a range for which Basset's equation can be applied. In this range $Re < 1$ and the Acceleration number is below a certain limit where the convective acceleration effects are not important.

As was the case with C_D, the variations of the coefficients C_A and C_H with the Reynolds number and the Acceleration number have to be determined by experiments. This has been done for 69 different simple harmonic motions. The tests are described in the next section.

EQUIPMENT AND TESTS

A sketch of the equipment which gives a simple harmonic motion to a sphere in a tank full of oil is shown in Figure 4.

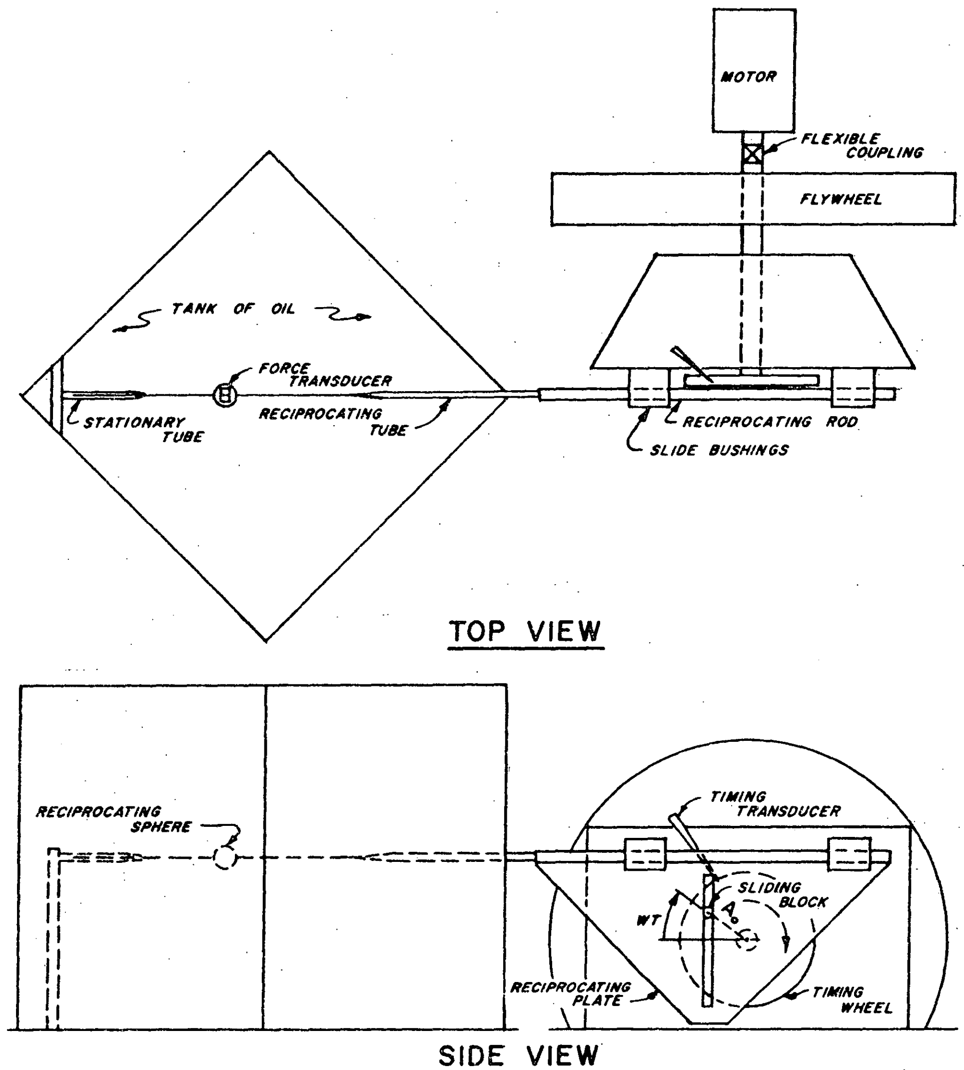

Figure 4. Equipment for producing simple harmonic motion.

As the motor turns the flywheel and timing wheel, the plate, rod, and sphere inside the tank reciprocate. The speed of the motor can be set anywhere between 17 rpm and 159 rpm. Timing accuracy is achieved by means of the transducer, as will be explained subsequently. The flywheel serves to dampen the vibrations produced by the motor.

The sliding block is attached to the timing wheel. Its location can be changed to vary the amplitude of the harmonic motion, $\underline{A}_0$. Thus, it is possible to produce a large number of different simple harmonic motions with this apparatus.

The amplifier and the recording instrument* used in this experiment are very sensitive. For example, heavy footsteps near the equipment can disturb the measurements although the floor is concrete. Disturbances caused by machinery in other parts of the building or by trucks passing by were noticed by the author, but the amplitude of these disturbances was very small. The data were taken when all other equipment in the laboratory was quiet.

The recording technique is based on the rotation of a loop of wire placed in a magnetic field. A change in the current passing through this wire rotates the loop. A light beam is reflected on the mirror attached to the loop. As the mirror rotates, the reflection of the light beam travels on a photographic paper which is moving at a certain speed. This paper is developed simultaneously. Thus, a record of the variation of current passing through the loop is obtained. The loop can rotate at frequencies of up to 3000 cps.

The travel speed of the photographic paper can be set to a selected constant value by a gear system. A wide, flat light beam projected on the photographic paper through a synchronized shutter at 0.1 sec intervals produces lines on the photographic paper. The distance measured between ten of these lines gives the travel speed of the photographic paper accurately. Thus, any malfunctioning of the gear system can be noticed immediately.

The speed of the timing wheel as well as of the motor is measured as follows:

Four strain gages constituting the four active resistances of a Wheatstone bridge are glued on a cantilever beam which is pressed against the periphery of the timing wheel. The bridge is balanced under this pressure. On the periphery of the timing wheel recesses are made at intervals of $\frac{\pi}{8}$. The relief of pressure as these recesses pass by changes the deflection of the cantilever beam causing an unbalanced current in the Wheatstone bridge. This unbalanced current is amplified and sent to a loop of wire placed in the magnetic field. As the timing wheel turns, the changes of current representing the changes of pressure at intervals of $\frac{\pi}{8}$ are registered on the moving photographic paper. Thus, since the rate of travel of the photographic paper is known accurately, the speed of the timing wheel can also be accurately determined. An extra recess showing the position of the timing wheel gives the position of the reciprocating sphere during its motion.

The transducer measuring the forces exerted on the sphere is placed inside the sphere. It consists of a small brass frame with strain gages glued on it.

As shown in Figure 5 the sphere is glued to the base of the frame and is not in contact with the reciprocating tube. The strain gages are glued on the brass beams, which are 0.015 in. thick. The beams can be considered hinged at the tube, since the hole for the tube removes most of the width of the brass beams.

The Wheatstone bridge is balanced when the sphere is not in motion. When the sphere moves in the oil, forces acting on it deflect the brass beams. This changes the resistance of the strain gages and consequently the balance of the bridge is disturbed. The unbalanced current which represents the force acting on the sphere is amplified, sent to a loop of wire in the magnetic field, and recorded as previously explained.

The distance traveled by the reflected light beam on the photographic paper is calibrated at intervals of 5 g of force. Within these intervals the variation of the distance with the force is assumed to be linear. When the recording instrument is set at attenuation 1, its highest sensitivity, 5 g of force causes the reflected light beam to travel 0.09 in. on the photographic paper.

The force transducer has a low natural frequency due to its thin beams and the large inertia of the oil inside the sphere. The beams have to be thin so that as small a force as 5 g can be measured with reasonable accuracy. It would be desirable to build a force transducer that has a high natural frequency but remains sensitive enough to measure

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* Manufactured by Consolidated Electrodynamics Corp.

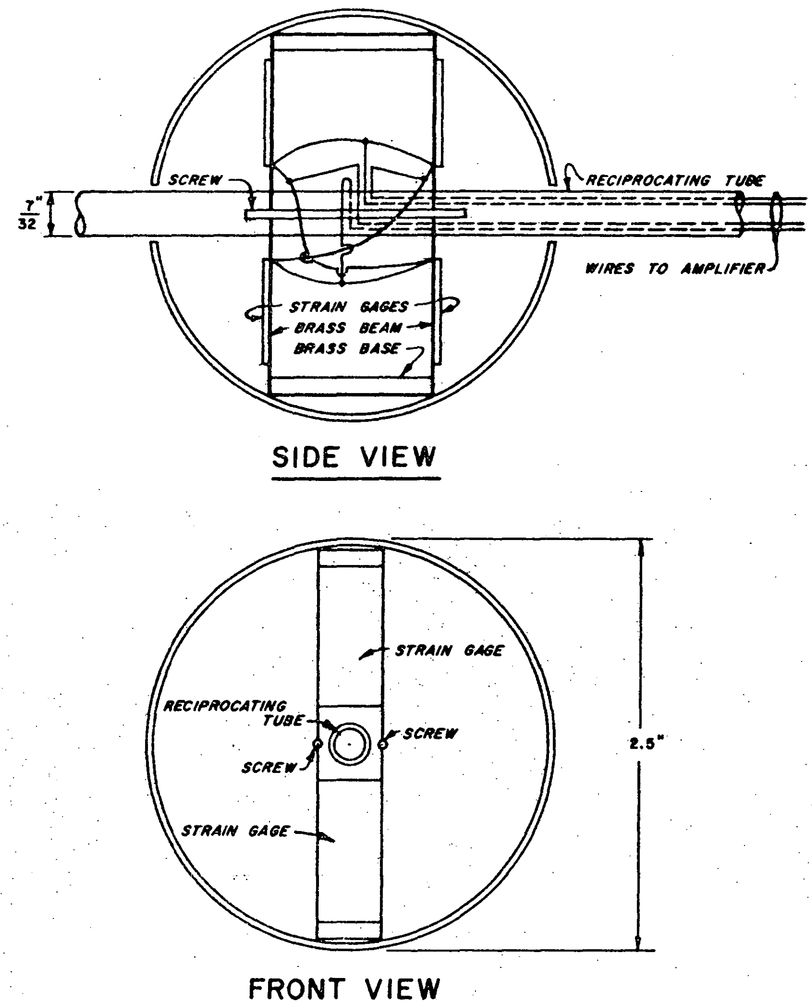

Figure 5. Transducer for measuring forces exerted on the sphere.

small forces accurately. The natural frequency of the transducer, estimated experimentally, is about 30-40 cps.

During the experiments a slight knock occurred in the apparatus. This knock was due to small but unavoidable clearances such as that between the surfaces of the sliding block and the slot in the reciprocating plate. Without such a clearance the block cannot move inside the slot. Another such clearance is inside the ball bearing used in the sliding block. After the knock the sphere oscillated with its natural frequency. The inertia forces due to the knock and the subsequent low frequency oscillation were always small and in some experiments they were not even recorded by the sensitive instrument that was used. When the midpoints of the recorded low frequency vibrations in the force vs time diagram were connected, the anticipated force vs time diagram for simple harmonic motion was obtained.

Different attenuations, expressed in degrees of sensitivity, were used during the recording. For forces lower than 70 g, attenuation 1 was used. At 70 g the light beam travels a distance of 1.5 in., the maximum distance the reflected light beam can travel on the recording. For the forces lower than 130 g, 200 g and 300 g, attenuations 2, 3 and 5 were used respectively. The transducer was not loaded to more than 300 g.

A human error of 0.01 in. in measuring the force ordinate would cause errors of 0.5 g, 0.9 g, 1.3 g and 2 g in attenuations 1, 2, 3 and 5 respectively. A human error of 0.01 in. in measuring the time abscissa in high speeds at attenuation 3 could cause an error of up to 4 g. Very high attention was required in measuring the forces to minimize these errors.

Considering the low frequency of the force transducer, the smoothing of the low frequency vibrations after the knock, and the possible human errors, it was assumed that the forces can be measured to within a 2% or 2 g accuracy, whichever is greater. The author believes that it is possible to draw conclusions from experimental data taken with this accuracy.

The viscosity and density of the oil are 1.65×10^{-2} slug/ft-sec and 1.725 slug/ft^3 respectively. The tests were conducted for amplitudes $A_0 = 1$, 2, 3, and 4 in. and for a wide range of speeds. A run was made by selecting a certain A_0 and setting the speed of the motor to a certain value. The forces acting on the sphere were measured for 69 different runs. During the tests the Reynolds numbers varied from 0 to 62. The Acceleration number varies from 0 to ∞ during a cycle of a simple harmonic motion.

The force transducer measures the forces exerted on the sphere by the fluid outside as well as the inertia forces inside the sphere. The inertia forces are due to the acceleration of the mass of oil inside the sphere, the mass of the sphere itself, and the mass of some parts of the transducing system. In order to find this total accelerating mass, the sphere was filled with oil and reciprocated in air. Since the clearance between the shell of the sphere and the reciprocating tube is very small, there was no noticeable oil leakage. The shell of the sphere is translucent and any free surface inside the sphere can be observed. A recording showing the magnitude of this inertia force was made as the sphere reciprocated. The total accelerating mass, the effective mass of the sphere, was calculated to be exactly 148 g. This figure was checked roughly by holding the tube in the vertical position. In this position, however, the oil started to leak immediately and the weight was down to 145 g by the time a measurement could be made.

EVALUATION OF THE COEFFICIENTS

The motion of the sphere can be expressed as

$$x = A_0 \cos wt \tag{14a}$$

$$V = -A_0 w \sin wt \tag{14b}$$

$$a = -A_0 w^2 \cos wt \tag{14c}$$

in which $\underline{x}$, $\underline{V}$, and $\underline{a}$ are the displacement, velocity and acceleration of the sphere respectively.

If these quantities are introduced in eq 12,

$$F = -C_A \frac{4}{3} \pi R^3 \rho A_0 w^2 \cos wt - \frac{1}{2} C_D \rho \pi R^2 A_0 w^2 |\sin wt| \sin wt -$$

$$-C_H R^2 \sqrt{\pi \rho \mu} \int_0^t \frac{A_0 w^2 \cos w(t-t')}{\sqrt{t'}} dt' \tag{15}$$

will be obtained.

For convenience the oscillations after a long period of time from the start of the motion are considered. Thus, the upper limit of the integral in eq 15 is changed to ∞. This change is made according to the principles of superposition (Odar, 1962). The value of the integral is

$$\int_0^\infty \frac{A_0 w^2 \cos w(t-t')}{\sqrt{t'}} dt' = \frac{A_0 w \sqrt{\pi w}}{\sqrt{2}} (\cos wt + \sin wt).$$

The forces due to the added mass, drag, history and inertia are denoted by F_A, F_D, F_H and F_I respectively. Thus

$$F_A = -C_A \frac{4}{3} \pi R^3 \rho A_0 w^2 \cos wt \qquad (16a)$$

$$F_D = -\frac{1}{2} C_D \rho \pi R^2 w^2 A_0^2 |\sin wt| \sin wt \qquad (16b)$$

$$F_H = -C_H R^2 \pi \sqrt{\frac{\mu\rho}{2}} A_0 w^{\frac{3}{2}} (\cos wt + \sin wt) \qquad (16c)$$

$$F_I = -m_{eff} A_0 w^2 \cos wt. \qquad (16d)$$

As stated previously, C_D is assumed to be the steady-state drag coefficient dependent on the Reynolds number, calculated by using instantaneous velocity. In order to find the variation of C_A with the Reynolds number or the Acceleration number, or both, the location where $F_H = 0$ is chosen. According to eq 16, $wt = \frac{3\pi}{4}$ where $F_H = 0$.

At this location the force measured by the transducer is

$$F = F_A + F_D + F_I.$$

Hence, by measuring the total force at location $wt = \frac{3\pi}{4}$, and calculating F_D and F_I, F_A and consequently C_A can be evaluated. Figure 6 shows the results. The tests showed that C_A changes with the Acceleration number only.

Using these plotted values of C_A, the total force at location $wt = \frac{3\pi}{4}$ was then calculated for all runs, as shown in Table I in Appendix A, to check the accuracy of the C_A values and to illustrate that they do not depend on Re. The consistency of the data and the independence of Re may be judged by comparing the total force so calculated with the measured force (last two columns). A measured force represents the average of two measurements at π intervals.

The values of C_D were taken from Particle Dynamics (Lapple, 1951, Table I). It is remarkable that these values of C_D, the steady-state drag coefficient based on the instantaneous velocity and the diameter of the sphere, lead to values of C_A which are independent of Re.

It is assumed that the values of C_A shown in Figure 6 apply throughout the entire cycle of the motion. Thus, the value of C_A at any desired location can be found from Figure 6 by calculating the Acceleration number from the absolute value of the instantaneous acceleration, together with the velocity and the diameter of the sphere.

The next step is to determine the nature of C_H. It is clear from eq 16c that the absolute value of F_H is a maximum at location $wt = \frac{\pi}{4}$. According to the assumptions made, the values of C_D and C_A at location $wt = \frac{\pi}{4}$ will be the same as those at location $wt = \frac{3\pi}{4}$. Therefore, F_D will have the same value as at location $wt = \frac{3\pi}{4}$ and the magnitude of F_A will be the same as at location $wt = \frac{3\pi}{4}$ but its sign will be reversed (see eq 16a and 16b).

F_A, F_D and F_I were calculated from eq 16a, 16b and 16d. The difference between the total measured force at location $wt = \frac{\pi}{4}$ and the calculated $F_D + F_A + F_I$ gives the value of F_H. Then, using eq 16c, C_H was evaluated. The tests showed that C_H changed with the Acceleration number only (Fig. 7).

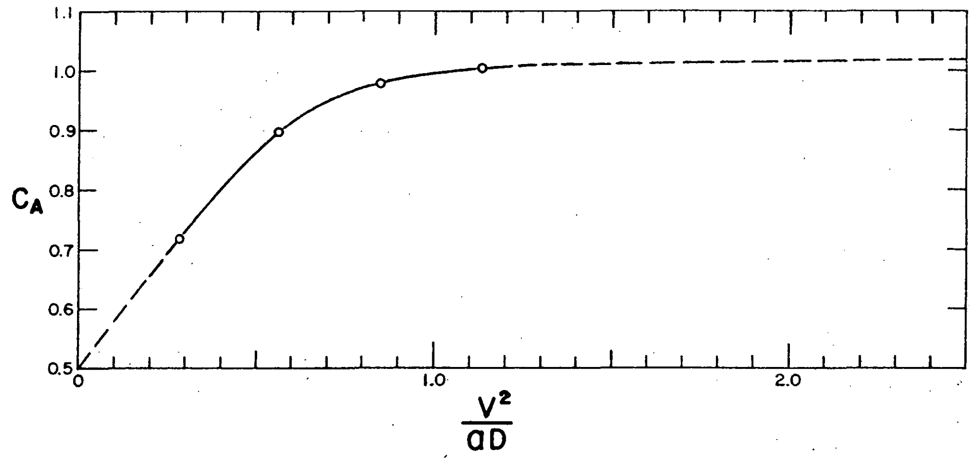

Figure 6. Variation of $\underline{C}_A$ with Acceleration number. Each point is the average $\underline{C}_A$ calculated from at least ten runs at different values of Re.

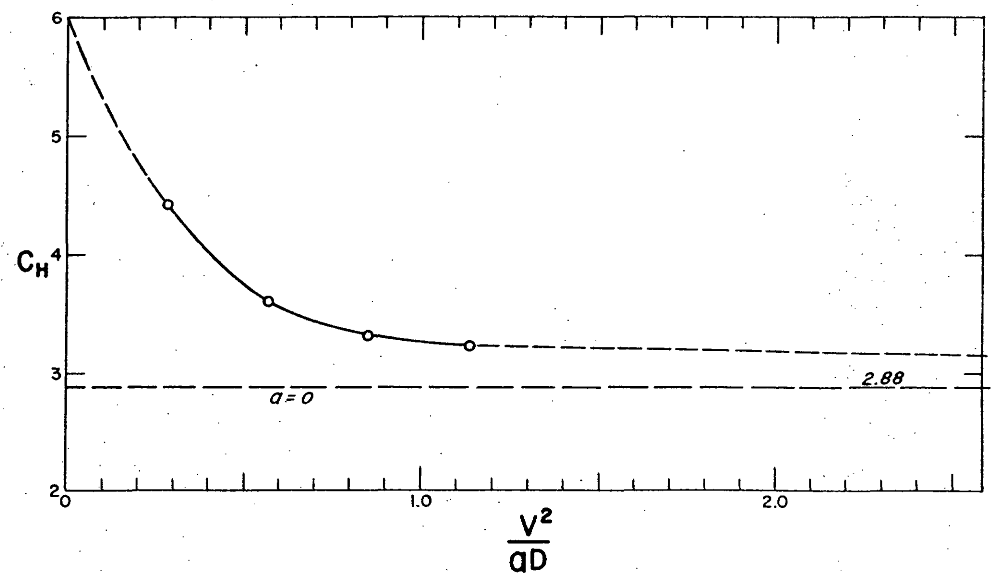

Figure 7. Variation of $\underline{C}_H$ with Acceleration number. Each point is an average calculated from at least ten runs at different values of Re.

Using the plotted values of $\underline{C}_H$ and the values of $\underline{C}_A$ in Figure 6, the total force at location $wt = \frac{\pi}{4}$ was then calculated for all the runs, as shown in Table II in Appendix A, to check the accuracy of the $\underline{C}_H$ values and to illustrate that they do not depend on Re. The consistency of the data and the independence of Re may be judged by comparing the total force so calculated with the measured force (last two columns).

Now, as in the case of $\underline{C}_A$, it will be assumed that the values of $\underline{C}_H$ shown in Figure 7 apply throughout the entire cycle of the motion. Thus, the value of $\underline{C}_H$ at any desired location can be found from Figure 7.

The asymptotic line in Figure 7 was found from the investigation of forces at the location where a = 0. Since a = 0, $F_A = 0$ also. Thus, the only resisting forces acting on the sphere at this location are $\underline{F}_D$ and $\underline{F}_H$. The value of $\underline{F}_D$, calculated from eq 16b, subtracted from measured total force at this location gives $\underline{F}_H$, and $\underline{C}_H$ is obtained from eq 16c. For several runs the value of $\underline{C}_H$ at this location was found to be 2.88. A constant value was expected by the author, because at the location where a = 0, Ac is constant, namely ∞.

Using this value of $\underline{C}_H$, the total force at location $wt = \frac{\pi}{2}$ was then calculated for all runs (Table III) to check the accuracy of the value $C_H = 2.88$. The consistency of the data may be judged by comparing the total force so calculated with the measured force (last two columns).

The next step is the investigation of the forces at the location where V = 0. At this location $\underline{F}_D = 0$ and Ac = 0. Since the velocity is zero and the acceleration is high, Basset's equation can be applied. Using Basset's values, $C_A = \frac{1}{2}$ and $C_H = 6$, the total force at location $wt = 2\pi$ was calculated for all runs (Table IV) to check these theoretical values of $\underline{C}_A$ and $\underline{C}_H$. The consistency of the data may be judged by comparing the total force so calculated with the measured force (last two columns).

Finally, the forces were calculated at intervals of $\frac{\pi}{8}$ by using the plotted values of $\underline{C}_A$ and $\underline{C}_D$ in Figures 6 and 7 respectively, and the known values of $\underline{C}_D$ for the steady motion. These values are compared with the measured forces in Table V. The agreement was found to be excellent, although some of the values of $\underline{C}_A$ and $\underline{C}_H$ were taken from the dashed parts of the curves. However, the author believes that further experiments are necessary to accurately establish the dashed portions of the curves.

Figure 8 shows an example of the agreement between the calculated and the measured forces. The curves coincide.

CONCLUSION

The experiments show that the assumed form of eq 12 for the resisting force on a sphere moving rectilinearly is entirely satisfactory for the full range of the tests. Eq 12 is

$$\vec{F} = C_A \frac{4}{3}\pi R^3 \rho \vec{a} + \tfrac{1}{2} C_D \rho \pi R^2 |V| \vec{V} + C_H R^2 \sqrt{\pi\rho\mu} \int_0^t \frac{\overrightarrow{a(t-t')}}{\sqrt{t'}}\, dt'.$$

The resisting force is due to the added mass, drag and the history of the acceleration.

At the beginning of the analysis it was assumed that the drag term in eq 12 was the same as the steady-state drag term. This assumption was made because of the analogy between eq 12 and Basset's equation for the resisting force on a rapidly accelerating and arbitrarily moving sphere

$$F = \frac{1}{2}\frac{4}{3}\pi R^3 \rho \vec{a} + 6\pi R \mu V + 6 R^2 \sqrt{\pi\rho\mu} \int_0^t \frac{a(t-t')}{\sqrt{t'}}\, dt'.$$

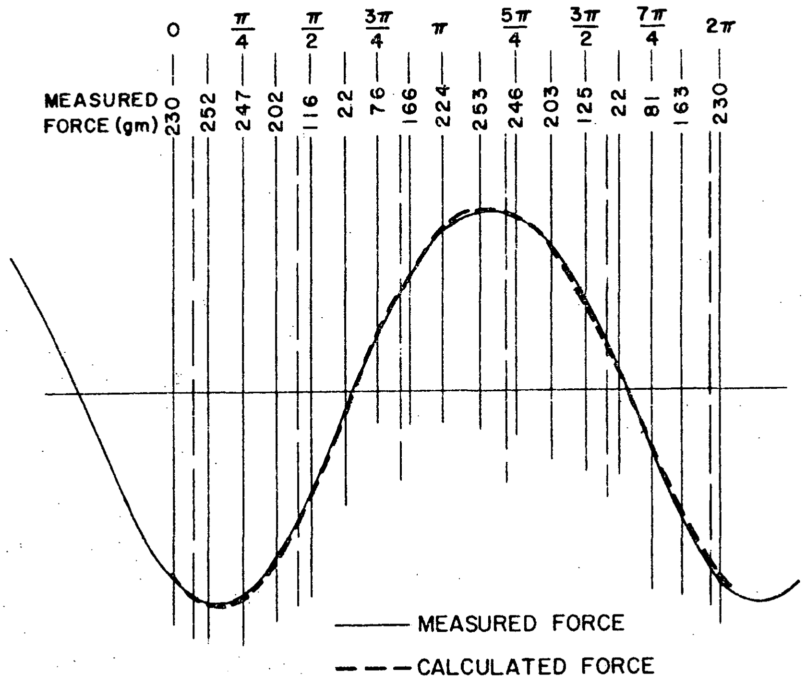

Figure 8. Example of record of force, showing calculated curve for comparison.

The drag term in Basset's equation is the same as the steady-state drag term, although the acceleration of the sphere is very large. Note that Basset's equation was derived for a sphere moving rectilinearly in such a manner that the contribution of convective acceleration was omitted. In eq 12 the contribution of the convective acceleration is included by introducing $\underline{C}_D$, $\underline{C}_A$, and $\underline{C}_H$. Eq 12 should converge to Basset's equation if the velocity and not the acceleration of the sphere is reduced. Also, when the motion becomes steady, eq 12 should reduce to the expression for steady-state drag.

The coefficient of the added mass $\underline{C}_A$ is dependent on the Acceleration number (Fig. 6). $\underline{C}_A$ does not depend on the Reynolds number. This is in agreement with the fact that $\underline{C}_A$ is independent of viscosity just after a sphere has started rapidly from rest. Note that $\underline{C}_A$ changes with the Acceleration number and is different from the theoretical value of $\frac{1}{2}$ valid for nonviscous irrotational flow. In the present study no assumption about the irrotationality of the flow is made.

The history term is the product of the coefficient of history $\underline{C}_H$ and the history expression derived by Basset. $\underline{C}_H$ is also dependent on the Acceleration number (Fig. 7), but not on the Reynolds number.

If Basset's equation is to be applied to a given motion, the Reynolds number should be less than 1 and the Acceleration number should be below a certain limit. It was impossible to determine this limit with the experiments made by the author, but it is apparently between Ac = 0 and Ac = 0.283. The curves of $\underline{C}_A$ and $\underline{C}_H$ which were shown hypothetically by dashed lines in Figures 6 and 7 could be different when this limit is established.

If the effect of the convective acceleration becomes negligibly small, eq 12 converges to Basset's equation which is valid for a rapidly accelerating and arbitrarily but slowly moving sphere along a straight line. It is necessary to make further experiments with different motions to find out whether the values of $\underline{C}_A$ and $\underline{C}_H$ as plotted can be used for an arbitrary rectilinear motion of a sphere.

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APPENDIX A.

Table I. Comparison of the measured and the calculated forces at location $wt = \frac{3\pi}{4}$.

Run	w (rad/sec)	Re	C_D	F_D (g)	Ac	C_A (avg)	$F_A + F_I$ Cal. (g)	Force Cal. (g)	Force Meas. (g)
1	1.67	4.28	8.85	-4.58	0.566	0.907	2.64	-1.94	-0.00
2	1.94	4.96	6.98	-4.85	0.566	0.907	3.52	-1.33	-2.00
3	2.26	5.80	6.20	-5.90	0.566	0.907	4.80	-1.10	-1.00
4	2.62	6.70	5.52	-7.03	0.566	0.907	6.45	-0.58	-0.00
5	2.96	7.60	4.97	-8.11	0.566	0.907	8.21	0.10	1.00
6	3.29	8.40	4.62	-9.26	0.566	0.907	10.15	0.89	2.20
7	3.75	9.58	4.24	-11.00	0.566	0.907	13.20	2.20	3.00
8	4.13	10.60	3.87	-12.10	0.566	0.907	16.10	4.00	5.00
9	4.61	11.28	3.70	-13.35	0.566	0.907	19.30	5.95	5.80
10	5.10	13.10	3.35	-16.20	0.566	0.907	24.50	8.30	8.20
11	5.68	14.55	3.13	-18.75	0.566	0.907	30.30	11.55	10.00
11a	5.65	14.40	3.15	-18.60	0.566	0.907	30.00	11.40	12.00
12	6.26	16.00	2.94	-21.40	0.566	0.907	36.90	15.50	16.20
13	6.90	17.65	2.76	-24.10	0.566	0.907	44.70	20.60	18.00
14	7.65	19.61	2.58	-28.00	0.566	0.907	55.20	27.20	27.50
15	8.50	21.70	2.42	-32.20	0.566	0.907	67.90	35.70	36.20
15a	8.43	21.60	2.42	-32.00	0.566	0.907	66.90	34.90	35.00
16	9.43	24.15	2.27	-37.40	0.566	0.907	83.50	45.90	48.00
17	10.55	27.00	2.12	-43.80	0.566	0.907	104.50	60.70	58.00
18	11.90	30.40	1.99	-52.00	0.566	0.907	133.00	81.00	79.00
19	13.30	33.80	1.90	-62.50	0.566	0.907	166.00	103.50	100.00
20	1.62	2.07	14.15	-1.72	0.283	0.720	1.13	-0.59	-0.00
21	1.94	2.48	12.10	-2.12	0.283	0.720	1.61	-0.51	-0.00
22	2.26	2.88	10.75	-2.53	0.283	0.720	2.18	-0.35	-0.00
23	2.61	3.34	9.60	-3.04	0.283	0.720	2.92	-0.12	-0.00
24	2.99	3.82	8.65	-3.57	0.283	0.720	3.82	0.25	0.70
25	3.34	4.27	7.95	-4.10	0.283	0.720	4.97	0.87	1.20
26	3.74	4.78	7.20	-4.65	0.283	0.720	6.00	1.35	2.10
27	4.15	5.30	6.61	-5.26	0.283	0.720	7.35	2.09	3.00
28	4.88	6.22	5.85	-6.42	0.283	0.720	10.15	3.73	4.00
29	5.10	6.50	5.65	-6.78	0.283	0.720	11.10	4.32	5.50
30	5.64	7.20	5.20	-7.60	0.283	0.720	13.60	6.00	7.00
31	6.25	8.00	4.80	-8.66	0.283	0.720	16.70	8.04	9.00
32	6.89	8.72	4.50	-9.90	0.283	0.720	20.40	10.50	11.00
33	7.65	9.80	4.18	-11.30	0.283	0.720	25.00	13.70	15.30
34	8.55	10.82	3.82	-12.92	0.283	0.720	31.20	18.30	20.50
35	9.43	12.00	3.55	-14.58	0.283	0.720	38.00	24.42	24.70
36	10.50	13.40	3.32	-16.90	0.283	0.720	47.20	30.30	33.00
37	11.75	15.00	3.08	-19.60	0.283	0.720	59.10	39.50	38.50
38	13.30	16.95	2.83	-23.00	0.283	0.720	75.90	52.90	55.00
39	15.21	19.45	2.60	-27.90	0.283	0.720	99.20	71.30	67.00
40	16.40	20.95	2.48	-30.80	0.283	0.720	114.80	84.00	84.00
41	1.68	6.45	5.70	-6.70	0.848	0.980	4.10	-2.60	-2.30
42	1.95	7.50	5.02	-8.03	0.848	0.980	5.55	-2.48	-1.70
43	2.27	8.70	4.51	-9.68	0.848	0.980	7.48	-2.20	-1.50
44	2.60	9.96	4.15	-11.70	0.848	0.980	9.80	-1.90	-1.50
45	2.96	11.40	3.70	-13.50	0.848	0.980	12.70	-0.80	0.00

Table I. (Cont'd) Comparison of the measured and the calculated forces at location $wt = \frac{3\pi}{4}$.

Run	w (rad/sec)	Re	C_D	F_D (g)	Ac	C_A (avg)	$F_A + F_I$ Cal. (g)	Force Cal. (g)	Force Meas. (g)
46	3.35	12.90	3.40	-16.00	0.848	0.980	16.30	0.30	0.35
47	3.74	14.35	3.17	-18.50	0.848	0.980	20.30	1.80	3.00
48	4.15	15.90	2.93	-21.00	0.848	0.980	25.00	4.00	4.10
49	4.63	17.80	2.74	-24.50	0.848	0.980	31.20	6.70	8.00
50	5.10	19.60	2.58	-28.10	0.848	0.980	37.90	9.80	11.00
51	5.64	21.60	2.43	-32.20	0.848	0.980	46.00	13.80	13.50
52	6.41	24.10	2.28	-37.40	0.848	0.980	55.80	18.40	19.70
53	6.87	26.40	2.16	-42.50	0.848	0.980	68.50	26.00	25.00
54	7.65	29.40	2.02	-49.40	0.848	0.980	85.00	35.60	32.00
55	8.43	32.30	1.95	-57.70	0.848	0.980	103.00	45.30	40.70
56	9.37	36.00	1.81	-66.50	0.848	0.980	127.00	60.50	61.00
57	10.48	40.20	1.70	-77.50	0.848	0.980	152.50	75.00	74.50
58	2.56	13.10	3.35	-16.30	1.130	1.003	12.60	-3.70	-3.60
59	2.94	15.05	3.07	-19.80	1.130	1.003	16.80	-3.00	-1.50
60	3.26	16.07	2.86	-22.70	1.130	1.003	20.70	-2.00	-1.30
61	3.74	19.00	2.67	-27.20	1.130	1.003	27.40	0.20	1.00
62	4.08	20.90	2.48	-30.80	1.130	1.003	32.80	2.00	3.00
63	4.60	23.60	2.30	-36.20	1.130	1.003	41.50	5.30	5.00
64	5.09	26.00	2.17	-41.60	1.130	1.003	50.70	9.10	8.00
65	5.63	28.80	2.04	-48.00	1.130	1.003	62.10	14.10	12.20
66	6.22	31.90	1.94	-55.80	1.130	1.003	76.00	20.20	22.00
67	6.36	35.30	1.83	-64.40	1.130	1.003	93.00	28.60	28.00
68	7.63	39.00	1.73	-74.80	1.130	1.003	114.00	39.20	38.00
69	8.49	43.40	1.62	-86.40	1.130	1.003	141.00	54.60	51.00

Table II. Comparison of the measured and the calculated forces at location wt = $\frac{\pi}{4}$.

Run	w (rad/sec)	Re	F_D (g)	$F_A + F_I$ (g)	Ac	C_H (avg)	F_H (g)	Force Cal. (g)	Force Meas. (g)
1	1.67	4.28	-4.58	-2.64	0.566	3.61	-3.39	-10.61	-9.80
2	1.94	4.96	-4.85	-3.52	0.566	3.61	-3.96	-12.33	-12.10
3	2.26	5.80	-5.90	-4.80	0.566	3.61	-5.33	-16.03	-16.80
4	2.62	6.70	-7.03	-6.45	0.566	3.61	-6.65	-20.13	-20.50
5	2.96	7.60	-8.11	-8.21	0.566	3.61	-8.00	-24.32	-25.00
6	3.29	8.40	-9.26	-10.15	0.566	3.61	-9.38	-30.69	-30.70
7	3.75	9.58	-11.00	-13.20	0.566	3.61	-11.40	-35.60	-36.20
8	4.13	10.60	-12.10	-16.10	0.566	3.61	-13.20	-41.40	-41.70
9	4.61	11.28	-13.35	-19.30	0.566	3.61	-15.50	-48.15	-50.30
10	5.10	13.10	-16.20	-24.50	0.566	3.61	-18.10	-58.80	-60.10
11	5.68	14.55	-18.75	-30.30	0.566	3.61	-21.10	-70.15	-69.20
11a	5.65	14.40	-18.60	-30.00	0.566	3.61	-21.00	-69.60	-71.10
12	6.26	16.00	-21.40	-36.90	0.566	3.61	-24.50	-82.80	-82.70
13	6.90	17.65	-24.10	-44.70	0.566	3.61	-28.50	-97.30	-97.00
14	7.65	19.61	-28.00	-55.20	0.566	3.61	-33.20	-116.40	-117.20
15	8.50	21.70	-32.20	-67.90	0.566	3.61	-38.30	-138.40	-136.80
15a	8.43	21.60	-32.00	-66.90	0.566	3.61	-38.40	-137.30	-137.50
16	9.43	24.15	-37.40	-83.50	0.566	3.61	-45.40	-166.30	-168.50
17	10.55	27.00	-43.80	-104.50	0.566	3.61	-53.50	-201.80	-202.70
18	11.90	30.40	-52.00	-133.00	0.566	3.61	-64.30	-249.30	-246.50
19	13.30	33.80	-62.50	-166.00	0.566	3.61	-76.90	-305.40	-301.50
20	1.62	2.07	-1.72	-1.13	0.283	4.43	-1.98	-4.83	-5.10
21	1.94	2.48	-2.12	-1.61	0.283	4.43	-2.58	-6.31	-6.65
22	2.26	2.88	-2.53	-2.18	0.283	4.43	-3.28	-7.99	-7.50
23	2.61	3.34	-3.04	-2.92	0.283	4.43	-4.07	-10.03	-9.70
24	2.99	3.82	-3.57	-3.82	0.283	4.43	-5.00	-12.39	-12.00
25	3.34	4.27	-4.10	-4.97	0.283	4.43	-5.87	-14.94	-14.90
26	3.74	4.78	-4.65	-6.00	0.283	4.43	-6.95	-17.60	-17.90
27	4.15	5.30	-5.26	-7.35	0.283	4.43	-8.16	-20.77	-20.85
28	4.88	6.22	-6.42	-10.15	0.283	4.43	-10.32	-26.89	-25.40
29	5.10	6.50	-6.78	-11.10	0.283	4.43	-11.05	-28.93	-29.00
30	5.64	7.20	-7.60	-13.60	0.283	4.43	-12.90	-34.13	-34.35
31	6.25	8.00	-8.66	-16.70	0.283	4.43	-15.00	-40.36	-40.30
32	6.89	8.72	-9.90	-20.40	0.283	4.43	-17.35	-47.65	-47.50
33	7.65	9.80	-11.30	-25.00	0.283	4.43	-20.30	-56.60	-56.30
34	8.55	10.82	-12.92	-31.20	0.283	4.43	-24.00	-68.10	-67.80
35	9.43	12.00	-14.58	-38.00	0.283	4.43	-27.70	-80.15	-79.40
36	10.50	13.40	-16.90	-47.20	0.283	4.43	-32.70	-96.80	-97.00
37	11.75	15.00	-19.60	-59.10	0.283	4.43	-38.00	-116.70	-116.30
38	13.30	16.95	-23.00	-75.90	0.283	4.43	-46.80	-145.70	-146.50
39	15.21	19.45	-27.90	-99.20	0.283	4.43	-57.20	-184.30	-187.50
40	16.40	20.95	-30.80	-114.80	0.283	4.43	-64.00	-209.60	-209.00
41	1.68	6.45	-6.70	-4.10	0.848	3.33	-4.72	-15.52	-16.00
42	1.95	7.50	-8.03	-5.55	0.848	3.33	-5.90	-19.43	-19.00
43	2.27	8.70	-9.68	-7.48	0.848	3.33	-7.20	-24.36	-24.40
44	2.60	9.96	-11.70	-9.80	0.848	3.33	-9.06	-30.56	-31.20
45	2.96	11.40	-13.50	-12.70	0.848	3.33	-11.10	-37.30	-37.60

Table II. (Cont'd) Comparison of the measured and calculated forces at location $wt = \frac{\pi}{4}$.

Run	w (rad/sec)	Re	F_D (g)	$F_A + F_I$ (g)	Ac	C_H (avg)	F_H (g)	Force Cal. (g)	Force Meas. (g)
46	3.35	12.90	-16.00	-16.30	0.848	3.33	-13.30	-45.60	-45.50
47	3.74	14.35	-18.50	-20.30	0.848	3.33	-15.70	-54.50	-54.80
48	4.15	15.90	-21.00	-25.00	0.848	3.33	-18.40	-64.40	-65.20
49	4.63	17.80	-24.50	-31.20	0.848	3.33	-21.60	-77.30	-77.40
50	5.10	19.60	-28.10	-37.90	0.848	3.33	-25.00	-91.00	-91.20
51	5.64	21.60	-32.20	-46.00	0.848	3.33	-29.10	-107.30	-108.20
52	6.41	24.10	-37.40	-55.80	0.848	3.33	-33.90	-127.10	-126.00
53	6.87	26.40	-42.50	-68.50	0.848	3.33	-39.10	-150.10	-150.00
54	7.65	29.40	-49.40	-85.00	0.848	3.33	-45.80	-180.20	-179.50
55	8.43	32.30	-57.70	-103.00	0.848	3.33	-53.00	-213.70	-211.50
56	9.37	36.00	-66.50	-127.00	0.848	3.33	-61.60	-255.10	-254.50
57	10.48	40.20	-77.50	-152.50	0.848	3.33	-73.50	-303.50	-301.50
58	2.56	13.10	-16.30	-12.60	1.130	3.24	-11.50	-40.40	-41.20
59	2.94	15.05	-19.80	-16.80	1.130	3.24	-14.20	-50.80	-50.30
60	3.26	16.07	-22.70	-20.70	1.130	3.24	-16.10	-59.50	-60.10
61	3.74	19.00	-27.20	-27.40	1.130	3.24	-20.20	-74.80	-75.50
62	4.08	20.90	-30.80	-32.80	1.130	3.24	-23.30	-86.90	-87.90
63	4.60	23.60	-36.20	-41.50	1.130	3.24	-27.90	-105.60	-106.70
64	5.09	26.00	-41.60	-50.70	1.130	3.24	-32.10	-124.40	-124.50
65	5.63	28.80	-48.00	-62.10	1.130	3.24	-37.60	-148.30	-149.00
66	6.22	31.90	-55.80	-76.00	1.130	3.24	-43.80	-175.60	-175.00
67	6.36	35.30	-64.40	-93.00	1.130	3.24	-50.20	-207.60	-206.00
68	7.63	39.00	-74.80	-114.00	1.130	3.24	-59.30	-246.10	-247.00
69	8.49	43.40	-86.40	-141.00	1.130	3.24	-69.10	-296.50	-294.80

Table III. Comparison of the measured and the calculated forces at location wt = $\frac{\pi}{2}$ (á = 0).

Ac → ∞ C_H = 2.88

Run	w (rad/sec)	Re	F_D (g)	F_H (g)	Force Cal. (g)	Force Meas. (g)
1	1.67	6.04	-6.20	-1.92	-8.12	-8.20
2	1.94	7.00	-7.36	-2.24	-9.60	-9.50
3	2.26	8.15	-8.90	-3.00	-11.90	-12.00
4	2.62	9.50	-10.85	-3.75	-14.60	-15.00
5	2.96	10.70	-12.45	-4.52	-16.97	-18.40
6	3.29	11.90	-14.30	-5.30	-19.60	-20.60
7	3.75	13.50	-17.20	-6.45	-23.65	-24.30
8	4.13	15.00	-19.50	-7.45	-26.95	-27.80
9	4.61	16.70	-22.50	-8.78	-31.28	-33.20
10	5.10	18.40	-25.80	-10.20	-36.00	-37.70
11	5.68	20.50	-29.90	-11.95	-41.85	-42.50
11a	5.65	20.40	-29.80	-12.20	-42.00	-42.50
12	6.26	22.70	-34.30	-13.80	-48.10	-48.60
13	6.90	25.00	-39.30	-16.10	-55.40	-57.20
14	7.65	27.70	-45.00	-18.80	-63.80	-70.50
15	8.50	30.70	-52.90	-21.70	-74.60	-77.20
15a	8.43	30.50	-52.00	-21.70	-73.70	-79.00
16	9.43	34.00	-61.00	-25.60	-86.60	-93.00
17	10.55	38.20	-72.30	-30.20	-102.50	-109.50
18	11.90	43.00	-85.60	-36.30	-121.90	-120.50
19	13.30	47.10	-104.00	-43.50	-147.50	-149.50
20	1.62	2.93	-2.57	-0.92	-3.49	-3.70
21	1.94	3.50	-3.24	-1.19	-4.43	-5.30
22	2.26	4.08	-3.88	-1.50	-5.38	-6.00
23	2.61	4.71	-4.60	-1.87	-6.47	-7.00
24	2.99	5.40	-5.40	-2.30	-7.70	-8.00
25	3.34	6.03	-6.18	-2.70	-8.88	-9.10
26	3.74	6.75	-7.10	-3.19	-10.29	-10.70
27	4.15	7.50	-8.06	-3.75	-11.81	-12.00
28	4.88	8.80	-9.95	-4.76	-14.66	-16.00
29	5.10	9.22	-10.46	-5.10	-15.56	-16.40
30	5.64	10.20	-11.90	-5.95	-17.85	-17.50
31	6.25	11.28	-13.40	-7.15	-20.55	-20.90
32	6.89	12.47	-15.30	-7.96	-23.26	-24.20
33	7.65	13.85	-17.62	-9.38	-27.00	-28.50
34	8.55	15.45	-20.40	-11.10	-31.50	-31.20
35	9.43	17.05	-23.20	-12.75	-35.95	-38.50
36	10.50	19.00	-26.80	-15.10	-41.90	-42.10
37	11.75	21.40	-31.20	-17.50	-48.70	-49.50
38	13.30	24.10	-37.40	-21.60	-59.00	-61.00
39	15.21	27.60	-46.20	-26.40	-72.60	-76.50
40	16.40	29.60	-50.30	-29.40	-79.70	-84.50
41	1.68	9.10	-10.20	-2.88	-13.08	-13.00
42	1.95	10.60	-12.31	-3.62	-15.93	-16.30

Table III. (Cont'd) Comparison of the measured and the calculated forces at location $wt = \frac{\pi}{2}$ (a = 0).

$Ac \rightarrow \infty$ $C_H = 2.88$

Run	w (rad/sec)	Re	F_D (g)	F_H (g)	Force Cal. (g)	Force Meas. (g)
43	2.27	12.30	-14.90	-4.55	-19.45	-19.80
44	2.60	14.10	-17.85	-5.55	-23.40	-23.50
45	2.96	16.10	-21.20	-6.78	-27.98	-28.60
46	3.35	18.10	-24.00	-8.12	-32.12	-33.00
47	3.74	20.30	-29.40	-9.60	-39.00	-38.90
48	4.15	22.50	-33.80	-11.30	-45.10	-44.70
49	4.63	25.10	-39.70	-13.20	-52.90	-52.00
50	5.10	27.65	-45.20	-15.00	-60.20	-60.70
51	5.64	30.60	-52.50	-18.20	-70.70	-71.00
52	6.22	33.70	-60.00	-20.70	-80.70	-81.00
53	6.87	37.30	-69.60	-23.90	-93.50	-94.00
54	7.65	41.50	-81.00	-28.10	-107.10	-109.50
55	8.43	45.80	-93.80	-32.50	-126.30	-125.50
56	9.37	51.00	-108.50	-37.60	-146.10	-147.00
57	10.48	56.80	-128.00	-44.80	-172.80	-173.00
58	2.56	18.50	-25.90	-7.24	-33.14	-33.50
59	2.94	21.20	-31.50	-8.80	-40.30	-40.20
60	3.26	23.60	-36.00	-10.40	-46.40	-44.00
61	3.74	27.00	-43.60	-12.70	-56.30	-52.50
62	4.09	29.50	-49.60	-14.60	-64.20	-62.00
63	4.60	33.20	-58.50	-17.50	-76.00	-74.50
64	5.09	36.80	-68.00	-20.20	-88.20	-84.50
65	5.63	40.70	-78.60	-23.70	-102.30	-102.00
66	6.22	45.00	-90.00	-27.40	-117.40	-116.00
67	6.88	49.60	-104.50	-31.70	-135.20	-135.50
68	7.63	55.00	-122.00	-37.30	-159.30	-160.00
69	8.49	61.30	-143.50	-43.40	-186.90	-184.00

Table IV. Comparison of the measured and the calculated forces

at location wt = 2 π (V = 0).

$C_A = 0.5^{5}$ $C_H = 6$ Ac = 0

Run	w (rad/sec)	$F_A + F_I$ (g)	F_H (g)	Force Cal. (g)	Force Meas. (g)
1	1.67	-2.99	-3.99	-6.98	-7.10
2	1.94	-4.04	-4.65	-8.69	-8.70
3	2.26	-5.46	-6.28	-11.74	-11.50
4	2.62	-7.35	-7.80	-15.15	-15.85
5	2.96	-9.40	-9.43	-18.83	-18.80
6	3.29	-11.60	-11.00	-22.60	-24.30
7	3.75	-15.10	-13.40	-28.50	-29.30
8	4.13	-18.35	-15.50	-33.85	-34.10
9	4.61	-22.90	-18.30	-41.20	-41.30
10	5.10	-27.90	-21.30	-49.20	-49.80
11	5.68	-34.60	-24.90	-59.50	-59.60
11a	5.65	-34.20	-24.70	-58.90	-58.90
12	6.26	-42.20	-28.80	-71.00	-70.70
13	6.90	-47.80	-33.50	-81.30	-82.70
14	7.65	-62.80	-39.10	-101.90	-102.70
15	8.50	-77.70	-45.10	-122.80	-123.20
15a	8.43	-76.30	-45.10	-121.40	-121.00
16	9.43	-95.30	-53.50	-148.80	-153.00
17	10.55	-119.50	-63.00	-182.50	-188.50
18	11.90	-152.50	-75.70	-228.20	-227.00
19	13.30	-190.00	-90.70	-280.70	-281.00
20	1.62	-1.41	-1.90	-3.31	-3.70
21	1.94	-2.02	-2.48	-4.50	-5.00
22	2.26	-2.73	-3.14	-5.87	-6.00
23	2.61	-3.66	-3.90	-7.56	-7.30
24	2.99	-4.79	-4.79	-9.58	-8.80
25	3.34	-5.98	-5.63	-11.61	-10.60
26	3.74	-7.48	-6.65	-14.13	-14.00
27	4.15	-9.23	-7.82	-17.05	-17.50
28	4.88	-12.80	-11.05	-21.80	-20.60
29	5.10	-13.95	-10.62	-24.57	-25.30
30	5.64	-17.00	-12.35	-29.35	-29.60
31	6.25	-20.90	-14.40	-35.30	-35.50
32	6.89	-25.50	-16.60	-42.10	-42.70
33	7.65	-31.40	-19.50	-50.90	-51.40
34	8.55	-39.30	-23.00	-62.30	-62.00
35	9.43	-47.60	-26.60	-74.20	-75.20
36	10.50	-59.20	-31.40	-90.60	-90.30
37	11.75	-74.80	-36.40	-111.20	-112.00
38	13.30	-95.00	-45.00	-140.00	-141.00
39	15.21	-124.50	-54.80	-179.30	-177.50
40	16.40	-144.00	-61.10	-205.10	-201.00
41	1.68	-4.54	-6.05	-10.59	-10.50
42	1.95	-6.16	-7.53	-13.69	-13.70

Table IV. (Cont'd) Comparison of the measured and the calculated forces at location wt = 2 π (V = 0).

$C_A = 0.5$ $C_H = 6$ Ac = 0

Run	w (rad/sec)	$F_A + F_I$ (g)	F_H (g)	Force Cal. (g)	Force Meas. (g)
43	2.27	-8.30	-9.48	-17.78	-18.40
44	2.60	-10.90	-11.56	-22.46	-23.10
45	2.96	-14.10	-14.10	-28.20	-28.90
46	3.35	-18.00	-16.95	-34.95	-35.30
47	3.74	-22.50	-20.00	-42.50	-42.60
48	4.15	-27.70	-23.50	-51.20	-52.20
49	4.63	-34.60	-27.50	-62.10	-61.50
50	5.10	-41.80	-31.70	-73.50	-74.70
51	5.64	-51.20	-37.10	-88.30	-87.30
52	6.22	-62.10	-43.20	-105.30	-107.70
53	6.87	-76.10	-49.90	-126.00	-124.00
54	7.65	-94.40	-58.50	-152.90	-153.50
55	8.43	-114.60	-67.80	-182.40	-182.00
56	9.37	-141.50	-78.60	-220.10	-219.00
57	10.48	-176.50	-93.50	-270.00	-269.00
58	2.56	-14.08	-15.10	-29.18	-29.80
59	2.94	-18.50	-18.60	-37.10	-37.60
60	3.26	-22.80	-21.80	-44.60	-45.40
61	3.74	-30.10	-26.70	-56.80	-57.20
62	4.09	-35.70	-30.60	-66.30	-68.00
63	4.60	-45.40	-36.50	-81.90	-81.00
64	5.09	-55.40	-42.00	-97.40	-97.70
65	5.63	-68.00	-49.50	-117.50	-115.50
66	6.22	-83.00	-57.50	-140.50	-140.00
67	6.88	-101.50	-66.50	-168.00	-167.50
68	7.63	-125.00	-77.90	-202.90	-203.50
69	8.49	-153.50	-90.70	-244.20	-242.00

Table V. Comparison of the measured and computed forces.

Run	w (rad/sec)	wt (rad)	Re	Ac	F_D (g)	$F_A + F_I$ (g)	F_H (g)	Force Cal. (g)	Force Meas. (g)
9	4.61	0	0.00	0.000	0.00	-22.90	-18.30	-41.20	-41.30
9	4.61	$\pi/8$	6.40	0.1270	-6.65	-22.30	-20.40	-49.35	-49.70
9	4.61	$\pi/4$	11.28	0.5660	-13.35	-19.30	-15.50	-48.15	-50.30
9	4.61	$3\pi/8$	15.50	1.7750	-20.25	-11.30	-12.75	-44.30	-44.75
9	4.61	$\pi/2$	16.70	∞	-22.50	00.00	-8.75	-31.28	-33.20
9	4.61	$5\pi/8$	15.50	1.775	-20.25	11.30	-5.30	-14.25	-16.20
9	4.61	$3\pi/4$	11.28	0.5660	-13.35	19.30	0.00	5.95	5.80
9	4.61	$7\pi/8$	6.40	0.1270	-6.65	22.30	8.50	24.15	25.35
18	11.90	0	0.00	0.0000	0.00	-152.50	-75.70	-228.20	-227.00
18	11.90	$\pi/8$	16.50	0.1270	-22.20	-147.50	-85.00	-254.70	-252.50
18	11.90	$\pi/4$	30.40	0.5660	-52.00	-133.00	-64.30	-249.30	-246.50
18	11.90	$3\pi/8$	39.80	1.7750	-76.50	-75.00	-52.80	-204.30	-202.50
18	11.90	$\pi/2$	43.00	∞	-85.60	00.00	-36.30	-121.90	-120.50
18	11.90	$5\pi/8$	39.80	1.7750	-76.50	75.00	-22.00	-23.50	-22.00
18	11.90	$3\pi/4$	30.40	0.5660	-52.00	133.00	00.00	81.00	79.00
18	11.90	$7\pi/8$	16.50	0.1270	-22.20	147.50	35.20	160.50	164.50
29	5.10	0	0.00	0.0000	0.00	-13.95	-10.62	-24.57	-25.30
29	5.10	$\pi/8$	3.54	0.0634	-3.29	-13.30	-12.80	-29.39	-29.70
29	5.10	$\pi/4$	6.50	0.2830	-6.78	-11.10	-11.05	-28.93	-29.00
29	5.10	$3\pi/8$	8.53	0.8900	-9.45	-6.80	-7.62	-23.87	-24.60
29	5.10	$\pi/2$	9.22	∞	-10.46	0.00	-5.10	-15.56	-16.40
29	5.10	$5\pi/8$	8.53	0.8900	-9.45	6.80	-3.18	-5.83	-6.75
29	5.10	$3\pi/4$	6.50	0.2830	-6.78	11.10	00.00	4.32	5.50
29	5.10	$7\pi/8$	3.54	0.0634	-3.29	13.30	5.30	15.31	16.35
38	13.30	0	0.00	0.0000	0.00	-95.00	-45.00	-140.00	-141.00
38	13.30	$\pi/8$	9.22	0.0634	-10.40	-90.60	-53.70	-154.70	-154.50
38	13.30	$\pi/4$	16.95	0.2830	-23.00	-75.90	-46.80	-145.70	-146.50
38	13.30	$3\pi/8$	22.30	0.8900	-33.40	-46.30	-32.20	-111.90	-113.00
38	13.30	$\pi/2$	24.10	∞	-37.40	00.00	-21.60	59.00	-61.00
38	13.30	$5\pi/8$	22.30	0.8900	-33.40	46.30	-18.45	-0.50	-6.00
38	13.30	$3\pi/4$	16.95	0.2830	-23.00	75.90	-00.00	52.90	55.00
38	13.30	$7\pi/8$	9.22	0.0634	-10.40	90.60	21.30	101.50	103.50
50	5.10	0	0.00	0.0000	00.00	-41.80	-31.70	-73.50	-74.70
50	5.10	$\pi/8$	10.60	0.1900	-12.30	-42.00	-33.20	-87.50	-89.00
50	5.10	$\pi/4$	19.60	0.8480	-28.10	-37.90	-25.00	-91.00	-91.20
50	5.10	$3\pi/8$	25.60	2.6600	-40.60	-20.80	-22.00	-83.40	-82.00
50	5.10	$\pi/2$	27.65	∞	-45.20	00.00	-15.00	-60.20	-60.70
50	5.10	$5\pi/8$	25.60	2.6600	-40.60	20.80	-9.14	-28.94	-29.00
50	5.10	$3\pi/4$	19.60	0.848	-28.10	37.90	00.00	9.80	11.00
50	5.10	$7\pi/8$	10.60	0.1900	-12.30	42.00	13.75	43.45	46.00
56	9.37	0	0.00	0.000	00.00	-141.50	-78.60	-220.10	-219.00
56	9.37	$\pi/8$	19.50	0.1900	-27.75	-142.00	-82.20	-251.95	-255.50
56	9.37	$\pi/4$	36.00	0.8480	-66.50	-127.00	-61.60	-255.10	-254.50
56	9.37	$3\pi/8$	47.00	2.6600	-97.20	-70.00	-54.40	-221.60	-220.50
56	9.37	$\pi/2$	51.00	∞	-108.50	00.00	-37.60	-146.10	-147.00
56	9.37	$5\pi/8$	47.00	2.6600	-97.20	70.00	-22.70	-49.90	-48.50
56	9.37	$3\pi/4$	36.00	0.8480	-66.50	127.00	00.00	60.50	61.00
56	9.37	$7\pi/8$	19.50	0.1900	-27.75	142.00	34.20	148.45	152.00

Table V. (Cont'd) Comparison of the measured and computed forces.

Run	w (rad/sec)	wt(rad)	Re	Ac	F_D (g)	$F_A + F_I$ (g)	F_H (g)	Force Cal. (g)	Force Meas. (g)
65	5.63	0	00.00	0.0000	00.00	-68.00	-49.50	-117.50	-115.50
65	5.63	π/8	15.60	0.2540	-20.50	-69.60	-48.90	-139.10	-141.50
65	5.63	π/4	28.80	1.1300	-48.00	-62.10	-37.60	-148.30	-149.00
65	5.63	3π/8	37.60	3.5500	-70.90	-33.70	-33.90	-138.50	-136.50
65	5.63	π/2	40.70	∞	-78.60	00.00	-23.70	-102.30	-102.00
65	5.63	5π/8	37.60	3.5500	-70.90	33.70	-14.10	-52.30	-52.30
65	5.63	3π/4	28.80	1.1300	-48.00	62.10	00.00	14.10	12.20
65	5.63	7π/8	15.60	0.2540	-20.50	69.60	20.30	69.40	68.10
68	7.63	0	00.00	0.0000	00.00	-125.00	-77.90	-202.90	-203.50
68	7.63	π/8	21.10	0.2540	-31.00	-128.20	-76.80	-235.00	-238.50
68	7.63	π/4	39.00	1.1300	-74.00	-114.00	-59.30	-246.10	-247.00
68	7.63	3π/8	51.00	3.5500	-109.00	-62.00	-53.30	-224.30	-219.50
68	7.63	π/2	55.00	∞	-122.00	00.00	-37.30	-159.30	-160.00
68	7.63	5π/8	51.00	3.5500	-109.00	62.00	-22.20	-69.20	-69.00
68	7.63	3π/4	39.00	1.1300	-74.00	114.00	00.00	39.20	38.00
68	7.63	7π/4	21.10	0.2540	-31.00	128.20	31.80	129.00	132.00

APPENDIX B.

NOTATION

a	Acceleration of the object
Ac	Acceleration number = $\frac{V^2}{aD}$
A_0	Amplitude of the harmonic motion
C	Coefficient
C_D	Drag coefficient
C_A	Added mass coefficient
C_H	History coefficient
D	Diameter of the particle or the sphere
d	Diameter of the disk
F	Total resisting force
F_D	Drag force
F_A	Added mass force
F_H	History force
g	Acceleration of gravity
h	Height of the object
k	Shape factor
l	Length of the object
m_a	Added mass
m_{eff}	Effective mass
p	Pressure in a fluid
R	Radius of the sphere
Re	Reynolds number
S	Surface of the particle
t	Time
t'	Dummy variable
u	Velocity of the fluid in the $\underline{x}$ direction (rectangular coordinates)
$\forall$	Volume of the object
V_p	Velocity of the particle
V	Velocity of the object in a fluid
V_f	Velocity of the fluid undisturbed by the object
w	Angular velocity
X	McNown's added mass coefficient
ρ	Density of the fluid
ρ_p	Density of the particle
μ	Viscosity
ν	Kinematic viscosity