# Forces on Cylinders and Plates in an Oscillating Fluid ${ }^{1}$ 

Garbis H. Keulegan and Lloyd H. Carpenter


#### Abstract

The incrian and drag cocfficients of cylinders and plates in simple sinnsoidal currents are investigated. The midsedion of a reetangular basin with standing waves surging in it is selected as the locale of currents. The eylinders and plates are fixed horizont:llly and below the water surface. The average vahes of the incrtia and drag coefficients over a wave eyele show varitions when the intensity of the current and the size of the cylinders or blates are changed. Theso variatione, however, can be correlated with the period paramelor $U_{\mathrm{m}} T / D$, where $U_{\mathrm{m}}$ is the maximum intensity of the sinusoidal current, $T$ is the perind of the wave and 1 ) is the diameter of the cylinder or the width of the plate. For the colinders $U_{\mathrm{m}} T / D$ cqualing 15 is : eritical condition yielding the lowest value of the inertia cocflieient and the largest value of the drag coefficient. For the plates the higher values of the drag coefficient are associated with the smatlur vatues of $U_{m} T / D$ and the higher vahes of the mass coefficient with the larger valucs of $U_{\mathrm{m}} T / 1$. The variation of the coefficients with the phase of the wave is cx:mined and the bearing of this on the formula for the forces is discussed. The flow patterns aromd the cylinders and plates are examined photographically, and a suggestion is advanced as to the physical meaning of the parameter $U_{\mathrm{m}} T / D$.


## 1. Introduction

In a remarkable paper on the motion of pendulums Stokes showed that the expression for the force on a sphere oscillating in an unlimited viscous fluid consists of two terms, one involving the acceleration of the sphere and the other the velocity [1]. ${ }^{2}$ Furthernore, the inertia cocflicient involved in the acceleraon term is modified because of viscosity and, indeed, is augmented over the theoretical value valid for irrotational flow. The drag coefficient associated with the velocity term is modified berause of the accelcration, and its value is greater tham it would be if the siphere werc moving with a constant velocity. Subsequente to Stokes' studies, the forces on a sphere moving in a viscous fluid in an arbitrary mamer were investigated by Boussinesq and also by Basset $[2,3]$. They found that the force experienced br a sphere at a given time depends, in general, on the entire history of its uncecleration as well as the instantaneous relocity and accelcration. As an example, if a sphere is accelerated, say wilh a constant acceleration, from a position of rest to a finite velocity and is then kept at this velocity, the force during the initial instants of uniform velocity differs from the force occurring at a later time. Rayleigh has given the formula for the force for this case [4]. The forec expression of Boussinesq-Basset contains three terms, one of which is in the form of an integral involving the history of acceleration. If the integral evaluated when the acceleration is represented by sinusoidal function it then yields the modifications of the inertia and drag coefficients in Stokes' formula.

One expects quantitatively different results when the oscillating velocities are large and the flow turbulent. As yet a theoretical analysis of the problen is difficult and much of the desired information must be obtained experimentally. In this respect the experimental studies have been dealt with variously. Onc method is due to McNown and Wolf [5], who considered the force on a two-

[^0]dimensional object immersed in a flow as made up of three parts:
\[

$$
\begin{equation*}
F=A_{0}^{\prime} \rho \frac{d(k U)}{d t}+\oint_{p_{x}} l S+\frac{1}{2} C_{d} D_{\rho} U|U| \tag{1}
\end{equation*}
$$

\]

where $F$ is the force per unit length in the direction of flow: $x ; U$ the velocity at points far removed from the object; $p_{x}$, the $x$-component of the ambient pressure in the absence of the body; $d S$, an element of the surface area; $C_{d}$, the cocfficient of drag; and $k$, the virtual mass coefficient. The dimension of the body normal to the flow is $D$, and $A_{0}$ is a circular area, $A_{0}=\pi D^{2} / 4$, to which the alded mass is referred. If $A$ is the cross-scetional area of the body, $A=r A_{0}$, $r$ being a ratio, then

$$
\oint_{p, d S=\rho r A_{0}} \frac{d l}{d t}
$$

and finally

$$
F=A_{0} \rho\left[\frac{d(k U)}{d t}+r^{\prime} \frac{d U}{d t}\right]+\frac{1}{2} C_{d} D_{\rho} U|U| .
$$

In this approach the variability of the mass coefficient, $k$, is implied. Thus, introducing a new coefficient $k^{\prime}$ such that

$$
k^{\prime} \frac{d U}{d t}=\frac{d}{d t}(k U)
$$

and putting

$$
\begin{equation*}
C_{m} \stackrel{\circ}{=}\left(k^{\prime}+r\right), \tag{3}
\end{equation*}
$$

there is obtained from eq (1), the expression

$$
\begin{equation*}
F=C_{m \rho} \rho A_{0} \frac{d U}{d t}+\frac{1}{2} C_{d} D_{\rho} U|U|, \tag{4}
\end{equation*}
$$

which in fact constitutes a second approach utilized first by Morison and coinvestigators [ 6,7 ]. The form of the expression is in agreement with the Stokes formula for force on a splere oscillating in a viscous medium. In a general sensc one may still regard $C_{m}$ as a kind of mass or inertia coefficient.


A third approach was proposed by Iversen and Bulent, who considered the force on an accelerated disk moving in one direction [ 8 ]. Briefly,

$$
\begin{equation*}
F=C \rho D U^{2} \tag{5}
\end{equation*}
$$

where

$$
C=C\left(\frac{D U}{\nu}, \frac{D}{U^{2}} \frac{d U}{d t}\right)
$$

Kem has considered the case of accelerated eytinders [9] and Bugliarello that of accelcrated spheres [10], all motions heing in one direction. Here the resort is to a single coclficient $C$ and attenpts to separate the effects of acceleration and viscosity have not been shown to be successful. Accordingly, the adoption of this method can have a meaning only for monotonic motions subject to definite limitations as to initial and final conditions.

For oscillatory motions, although the forces are more accurately deseribed either using eq (2) or ey (4), the latter might be preferred provided the coefficients $C_{m}$ and $C_{d}$ could be predicted with some precision. The application of the expression to vertical piling and large submerged objects by Reid and Bretschneider stresses the necessity of having these cocfficients better determined [11].

On the basis of irrotational flow around the eylinder, $C_{m}$ should equal 2 , and one may suppose that the value of $C_{d}$ should be identical with that applicable to a constant velocity. Morison and coinvestigators have obtained the ralues of $C_{d}$ and $C_{m}$ in particular cases by considering the observed forces in the phases of the wave cycle where $I /$ or $d L^{J} / d t$ vanishcs. Such determinations show consideralle variations of $C_{m}$ from the theorctical value and of $C_{d}$ from the steady state value at the corresponding Reynolds number. Dealing with field studies at Caplen, Texas, R.O. Reid found similat rariations in $C_{m}$ and $C_{d}$ [12]. The variations in the coefficients, however, have not yet been correlated with any appropriate parameter.

The prescut investigation was undertaken with the following two objectives in mind. The first was in regard to a supplementary function $\Delta R$ that could be introduced in eq (4) for a trucr representation of force when considering the coefficients $C_{m}$ and $C_{d}$ as being constant throughout a given wave cycle. The necessity for the term $\Delta R$ is associated with the eventuality that the point values of $C_{m}$ and $C_{d}$ deriate from their average values. The second objective was to examine the possibilitr of correlating the arerage values of $C_{m}$ and $C_{d}$ with a parameter $U_{m} T / D$, where $U_{m}$ is the amplitude of the harmonically varying velocity, $T$ is the period of the oscillations, and $D$ is the diameter of a cylinder or the breadth of a rectangular plate. The mid-cross section of a large rectangular vessel with standing waves surging in it was chosen as the field of harmonically varying current. The cylinders and plates were heldi fixed horizontally, totally submerged in water and extending from one side of the vessel to the other to approximate as closely as possible, the condition of infinite length.

## 2. Fluid Forces on an Immersed Body at Rest in a Moving Liquid

It would le instructive to consider the momentum equations discussed by Mumachan for the evaluation of foree on objects immersed in a perfect liquid [1:3]. The method, however, is now gencralized to apply to imperfect liquids.

Consider the case of two-dimensional flow with $x$ horizontal and $z$ vertical. The equation of motion in the $x$-direction is

$$
\begin{equation*}
\rho\left(\frac{\partial u}{\partial t}+u \frac{\partial u}{\partial x}+w \frac{\partial u}{\partial z}\right)=\frac{\partial p_{x x}}{\partial x}+\frac{\partial p_{i x}}{\partial z} \tag{6}
\end{equation*}
$$

where $u$ and $w$ are the velocity components along the axes $x$ and $z, \rho$ the density of the liquid, $p_{x x}$ the normal stress on an elementary surface perpendicular to $x$, and $p_{z x}$ the tangential stress on an elcmentary surface normal to $z$, the stress being in the dircction of $x$. Because of the incompressibility of the liquid,

$$
\begin{equation*}
\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0 \tag{7}
\end{equation*}
$$

and eq (2) becomes

$$
\begin{equation*}
\rho \frac{\partial u}{\partial t}+\rho\left(\frac{\partial}{\partial x} u^{2}+\frac{\partial}{\partial z} u w\right)=\frac{\partial p_{x x}}{\partial x}+\frac{\partial p_{x x}}{\partial x} . \tag{8}
\end{equation*}
$$

Take the immersed cylindrical body of surface $S$, as in figure 1 , and draw a surface $S^{\prime}$ of arbitrary shape which encloses the cylinder. Let $\omega$ be the region bounded br $S$ and $S^{\prime}$ and $l$ and $n$ the direction cosines of the nomal drawn inward into the region. Integrating ey (8) throughout $\omega$, and in this making use of Green's Thcorem, one finds

$$
\begin{array}{r}
\rho \int \frac{\partial u}{\partial t} d \omega-\rho \int u(l u+n w) d S-\rho \int u(l u+n w) d S^{\prime}= \\
\quad-\int\left(l p_{x x}+n p_{z x}\right) d S-\int\left(l p_{x x}+n p_{2 x}\right) d S^{\prime} \tag{9}
\end{array}
$$



Figure 1. Notation diagram for force analysis.

Over the surface $S$ of the inmersed body $l u+m w$ vanishes because the body is at rest. Also $\int\left(l p_{x x}+\right.$ $\left.2 p_{t x}\right) \cdot \mid S=r^{\prime}$, that is, the $x$-component of the forece cxerted on the solid by the moving liquid. It may be assumed that if $S^{\prime}$ is removed sufficiently from the body the tangential stress $p_{2 x}$ on $S^{\prime}$ vanislics and the normal stress $p_{F}$ reduces to the hydrostatic pressure $-p$. Solving for $F$,

$$
\begin{equation*}
F=-\rho \int \frac{\partial u}{\partial t^{\prime}} d \omega+\rho \int u(l u+n w) d S^{\prime}+\int l p d S^{\prime} . \tag{10}
\end{equation*}
$$

The later relation may be given in another form, suitable for the present purpose. Select the bounding surfuce $S^{\prime}$ as the rectangular strip slown in figure 1 . The planc $S_{1}^{\prime}$ to the left of the cylinder passes through t1 e point $x=-x_{1}$ and the plane $S_{2}^{\prime}$ to the right passes through $x=x_{1}$. Denoting the horizontal velocil: connponcrits at the points $P_{1}^{\prime}$ and $P_{2}$ with the cominon elevation $z_{1}$ by $u_{1}$ and $u_{2}$, and the pressurcs by $p_{1}$ and $p_{2}$, eq (10) now reduces to

$$
\begin{equation*}
F=-\rho \int_{\omega} \frac{\partial u t}{\partial t} d \omega+\rho \int_{-\infty}^{+\infty}\left(u_{1}^{2}-u_{2}^{2}\right) d z_{1}+\int_{-\infty}^{+\infty}\left(p_{1}-p_{2}\right) d z_{1} \tag{11}
\end{equation*}
$$

which is the momentum equation of familiar form.
This may be specialized to evaluate the foree on a circular cylinder when the motion is irrotational. Letting $U$ be the undisturbed velocity and referring
to Lainb [14],

$$
\left.\begin{array}{l}
u=U\left[1+\frac{a^{2}}{r^{2}} \cos 2 \theta\right]  \tag{12}\\
w=-U \frac{a^{2}}{r^{2}} \sin 2 \theta \\
\underset{\rho}{p}=\frac{d U}{d t}\left(r+\frac{a^{2}}{r}\right) \sin \theta-\frac{1}{2}\left(u^{2}+w^{2}\right)
\end{array}\right\},
$$

where $a$ is the radius of the cylinder, $r$ is radial distance, and $\theta$ is the angle between a radius rector and the vertical line $x=0$ passing through the center of
cylinder. Clearly, $u_{1}=u_{2}$ and the momentum aation, eq (11), reduces to

$$
\begin{equation*}
F_{1}=-\rho \int_{\omega} \frac{\partial u}{\partial t} d \omega+\int_{-\infty}^{\infty}\left(p_{1}-p_{2}\right) d z_{1} \tag{13}
\end{equation*}
$$

Introducing the values of $u$ and $p$ from eq (12), and omitting the straightforward but somewhat lengthy evaluations, the result is

$$
F_{1}=2 \pi \frac{d U}{d t} a^{2} \rho,
$$

$r$ in terms of the diameter $D$ of the cylinder

$$
\begin{equation*}
F_{1}=C_{m} \frac{\rho \pi D^{2}}{4} \frac{d U}{d t}, \tag{14}
\end{equation*}
$$

where $C_{m}=2$.
Next, suppose that the undisturbed velocity is constant tad that the body experiences a drag. With the liquid extending to infinity and ignoring the variation of pressures from the shcdlding eddies, or, more properly, assuming that the surfaces $S_{1}^{\prime \prime}$ and $S_{2}^{\prime}$ are far removed from the cylinder, $p_{1}=p_{2}$, and cq (11) rcduces to

$$
\begin{equation*}
F_{2}=\rho \int_{-\infty}^{+\infty}\left(u_{1}^{2}-u_{2}^{2}\right) d z_{1} \tag{15}
\end{equation*}
$$

The velocity $u_{1}=U$, and $u_{2}=m U$, where $m$ is dependent on $z_{1} / D$ and on Reynolds number $U D / \nu$. Thus,

$$
\begin{equation*}
F_{2}=C_{a} \rho D \frac{U^{2}}{2} \tag{16}
\end{equation*}
$$

where

$$
C_{d}=2 \int_{-\infty}^{+\infty}\left(1-m^{2}\right) d \frac{z_{1}}{D}
$$

It :ppears that in ordinary eases where the flow departs from irrotationality and becomes unstcady and edtlying, oq (11) is still the basis for craluating the force, since the first and third integrals nay be associatel with accelcration and the second with dray. That is, the coefficients $C_{m}$ and $C_{d}$ are derived from eq (13) and (15) provided the velocitios and pressures can be given. The force of the statenent is only academic, sinee in the flows involving separation sud intermittent eddy lormation the pressures and velocities are not known and the integrations in eq (11), at present, cannot be carried out. Nevertheless, experience suggests that eq (4) remains uscful at least for sinusoidal motions, if allowance can be made for the variations in $\mathbf{C}_{m}$ and $\mathrm{C}_{a}$.
Had one carried out the integrations in eq (11) for an extended plate using the known velocity cxpressions derived from the Kirchoff solution for the impact on a lamina, definite values for $\mathbf{C}_{m}$ and $\mathbf{C}_{d}$ would have resulted. This would have shown in principle the existence of a relation between $C_{m}$ and $\mathrm{C}_{d}$ in the absence of eddy formation. In the Kirchofl solution the wake is of infinite length and this is causc for concern. McNown overcomes this difficulty by considering the case of a closed wake as between two plates and finds a rclation between $k$ and $C_{d}$ or between $C_{m}$ and $C_{d}[15]$. This result is very significant as it points to the path to be followed in analytical approaches taking into account also the effect of the eddy processes. With cylinders the chatuging separation seats are a cause of added difficulty.
Meanwhile, the tasks of the experimental investigations become more necessary. Not only are the needs of the applied arts to be fulfilled, but also therce must be clarification as regards thic flow processes during unsteady flows.

## 3. Cylinder in a Field of Sinusoidal Motion

Forces on a cylinder admit an casier representation when the undisturbed portion of the flow, infinite in extent is varying harmonically. Tet the velocity be given by

$$
\begin{equation*}
U=-U_{m} \cos \sigma t \tag{17}
\end{equation*}
$$

where $\zeta_{m}$ is the scmimmplitude of the emrent, $T^{\prime}$ the period of the altcrmations, and $\sigma=2 \pi / T$. The force on the eylinder per unit length $F$ is in gencral

$$
\begin{equation*}
F=f\left(t, T, U_{m}, D, \rho, \nu\right) \tag{18}
\end{equation*}
$$

Gronping the variables on the basis of dimensional reasoning

$$
\frac{F}{\rho U_{m}^{2} D}=f\left(\frac{t}{T}, \frac{U_{m} T}{D}, \frac{U_{m} D}{\nu}\right)
$$

or introducing

$$
\begin{align*}
& \quad \theta=2 \pi t / T,=\omega t  \tag{19}\\
& \frac{F}{\rho U_{m}^{2} \bar{D}}=f\left(\theta, \frac{U_{m} T}{D}, \frac{U_{m} D}{\nu}\right) \tag{20}
\end{align*}
$$

where $\mathrm{U}_{m} D / \nu$ is a Revnolds number and $U_{m} T / D$ will be termed the "period parameter." Bearing in mind that $F$ is periodic, and that because of flow symmetry

$$
F^{\prime}(\theta)=-F^{\prime}(\theta+\pi)
$$

we have

$$
\begin{align*}
\frac{F}{\rho U_{m}^{2} D}= & A_{1} \sin \theta+A_{3} \sin 3 \theta+A_{5} \sin 5 \theta+\ldots \\
& +B_{1} \cos \theta+B_{3} \cos 3 \theta+B_{5} \cos 5 \theta+\ldots \tag{21}
\end{align*}
$$

Here the coeflicients $A_{1}, A_{3} \ldots$, and $B_{1}, \beta_{3} \ldots$ are independent of $\theta$, and are at most functions of $\mathrm{U}_{m} Z^{\prime} / D$ and $U_{m} D / \nu$. A smo method of approach in the analysis of the observed force curve is to resort to a Fourier analysis to determine the coefficients $A_{1}$. . $B_{1}$. . .

$$
\begin{equation*}
A_{n}=\frac{1}{\pi} \int_{0}^{2 \pi} \frac{F \sin n \theta}{\rho U_{m}^{2} D} d \theta \tag{22}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{n}=\frac{1}{\pi} \int_{0}^{2 \pi} \frac{r^{\prime} \cos n \theta}{\rho U_{m}^{2} D} d \theta \tag{23}
\end{equation*}
$$

Once the coefficients are obtained, their dependence on $U_{m} T / D$ and $U_{m} D / \nu$ may be establishcd, provided the observational data are of sufficient number and of large extent.

The above general and fundamental relation, eq (21), may be reconciled with eq (4), which is the form which Morison and coinvestigators Reid, Bretschneider and others, have adopted in their numerous studies. Introducing $U$ from eq (17) into eq (4)

$$
\left.\frac{F}{\rho l_{m}^{2} D}=\frac{\pi}{4} C_{m} \cdot \frac{D_{\sigma}}{U_{m}} \sin \theta-\frac{C_{d}}{2} \right\rvert\, \cos \theta ; \cos \theta
$$

By the rule of Fourier .
$|\cos \theta| \cos \theta=\sum_{n=0}^{\infty} \frac{\int_{0}^{2 \pi}|\cos \theta| \cos \theta \cos n \theta d \theta}{\int_{0}^{2 \pi} \cos ^{2} n \theta d \theta}$
where

$$
\begin{align*}
& a_{n}=0 \text { for } n \text { even, } \\
& a_{n}=(-1)^{\frac{n+1}{2}} \frac{8}{n\left(n^{2}-4\right) \pi} \text { for } n \text { odd, } \\
& a_{1}=\frac{8}{3 \pi}, a_{3}=\frac{8}{15 \pi}, a_{5}=-\frac{8}{105 \pi}, \ldots \tag{25}
\end{align*}
$$

Introducing this in eq (21), and writing

$$
\left.\begin{array}{l}
B_{1}^{\prime}=\frac{B_{1}}{a_{1}}  \tag{26}\\
B_{3}^{\prime}=B_{3}-\frac{a_{3}}{a_{1}} B_{1} \\
B_{5}^{\prime}=B_{5}-\frac{a_{5}}{a_{1}} B_{1}
\end{array}\right\}
$$

one has
$\frac{F}{\rho U_{m}^{2} D}=A_{1} \sin \theta+A_{3} \sin 3 \theta+A_{\mathbf{5}} \sin 5 \theta+\ldots$
$+B_{1}^{\prime}|\cos \theta| \cos \theta+B_{3}^{\prime} \cos 3 \theta+B_{5}^{\prime} \cos 5 \theta+$
Now eq (24) and (27) may be compared. One can write

$$
\frac{\pi}{4} C_{m} \cdot \frac{D_{\sigma}}{U_{m}}=A_{1}+A_{3} \frac{\sin 3 \theta}{\sin \theta}+A_{5} \frac{\sin 5 \theta}{\sin \theta} \ldots
$$

and

$$
\frac{C_{d}}{2}=-B_{1}^{\prime}-B_{3}^{\prime} \frac{\cos 3 \theta}{|\cos \theta| \cos \theta}-\frac{B_{5}^{\prime} \cos 5 \theta}{|\cos \theta| \cos \theta}+\ldots
$$

or

$$
\begin{align*}
& C_{m}(\theta)=\frac{2}{\pi^{2}} \frac{U_{m} T}{D}\left[A_{1}+A_{3}+A_{5}+2\left(A_{3}\right.\right. \\
&\left.\left.\quad+A_{5}\right) \cos 2 \theta+2 A_{5} \cos 4 \theta+\ldots\right] \tag{28}
\end{align*}
$$

and
$C_{d}(\theta)=-2 B_{1}^{\prime}+\frac{2}{|\cos \theta|}\left[2\left(B_{3}^{\prime}-B_{5}^{\prime}\right)+4\left(B_{5}^{\prime}-B_{3}^{\prime}\right) \cos 2 \theta\right.$

$$
\begin{equation*}
\left.-4 B_{5}^{\prime} \cos 4 \theta+\ldots\right] \tag{29}
\end{equation*}
$$

Thus if $A_{3}, A_{5}$, and $B_{3}^{\prime}, B_{5}^{\prime}$ vanish, the coefficients of mass and drag remain constant for all the phases
in the wave cycle and

$$
\begin{equation*}
C_{m}=\frac{2}{\pi^{2}} \frac{U_{m} T}{D} A_{1}=\frac{2}{\pi^{3}} \cdot \frac{U_{m} T}{D} \int_{0}^{2 \pi} \frac{F \sin \theta_{1} l \theta}{\rho U_{m}^{2} D} \tag{30}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{d}=-2 B_{1}^{\prime}=-\frac{3}{4} \int_{0}^{2 \pi} \frac{F \cos \theta d \theta}{\rho U_{m}^{2} D} \tag{31}
\end{equation*}
$$

Th the event that these corfficients vary with the phase $\theta$ of the wayc gycle, the values given by eq (30) and (31) are in a sense the weighted averages

$$
\begin{equation*}
C_{n}=\frac{1}{\pi} \int_{0}^{2 \pi} C_{m}(\theta) \sin ^{2} \theta d \theta \tag{32}
\end{equation*}
$$

and

$$
\begin{equation*}
C_{d}=+\frac{3}{4} \int_{0}^{2 \pi} C_{d}(\theta)|\cos \theta| \cos ^{2} \theta d \theta \tag{33}
\end{equation*}
$$

With the above possibilitics in mind, it is preferable to adopt the expressions

$$
\begin{equation*}
\frac{F}{\rho U_{n}^{2} D}=A_{1} \sin \theta+B_{1}^{\prime} \cos \theta|\cos \theta|+\Delta R \tag{34}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{F}{\rho U_{m}^{2} D}=\frac{\pi}{4} C_{m} \cdot \frac{D_{\sigma}}{U_{m}} \sin \theta-\frac{C_{d}}{2}|\cos \theta| \cos \theta+\Delta R \tag{34a}
\end{equation*}
$$

where $A_{1}, B_{1}^{\prime}, C_{m}$, and $C_{d}$ are constant, and $\Delta R$ lias the value

$$
\begin{equation*}
\Delta R=A_{3} \sin 3 \theta+A_{3} \sin 5 \theta+1 B_{3}^{\prime} \cos 3 \theta+B_{5}^{\prime} \cos 5 \theta . \tag{35}
\end{equation*}
$$

The function $\Delta R$ will be referred to as the remainder function, and then this remainder function is obtained by subtracting the computed values of $A_{1}$ $\sin \theta$ and " $B_{1}^{\prime}|\cos \theta| \cos \theta$ from the observed $F / \rho U_{m}^{2} D$. The remainder thus obtained may be examinced in regard to its Fourier structure and also as to its magnitude.

## 4. Characteristics of the Experimental Waves

The region under the nodal area of a standing wave that may be realized in a rectingular vessel furnishes a velocity field of simple lammonic motion in the velocity component $U$. This circumstance is not seriously modified even when the surges are moderately high.

Taking the $x$-axis in the plane surface of the undisturbed water, the $z$-axis vertical and upwards and the origin at one end of the basin, (see fig. 2), the surface elevation as reckoned from the undisturbed level, according to the second-approximation theory, from Miche [16], is

$$
\begin{gather*}
h=a \cos k x \sin \sigma t+a \frac{a k}{4} N_{1} \cos 2 k x- \\
a \frac{a k}{4} N_{2} \cos 2 k x \cos 2 \sigma t, \tag{36}
\end{gather*}
$$

wherc

$$
N_{1}=\frac{\cosh 2 k \cdot H}{\sinh 2 k \bar{I} I}
$$

and

$$
N_{2}=\frac{\cosh ^{2} k H(\cosh 2 k H+2)}{\sinh ^{2} k H} \frac{\sinh k H}{}
$$

Here $k=\pi / L$, $L$ being the length of the basin; $\sigma=$ $2 \pi / T, T$ being the period of oscillation: $I I$ the depth of water; and a the scmiwave lieight, that is, the mean value of the extreme cad deflections in a cycle. The expression for the period is the same as in the first-approximation theory, that is,

$$
\begin{equation*}
\sigma^{2}=g k \tanh k H . \tag{37}
\end{equation*}
$$

Focusing attention on the basin end $x=0$, the surface displacement is
$h=a \sin \sigma t+a \frac{a k}{4} N_{1}-a \frac{a k}{4} N_{2} \cos 2 \sigma t ; \quad x=0$.
Thus, the nıaximum elevation, occurring at $t=\pi / 2 \sigma$, is

$$
\begin{equation*}
h_{1}=a+a \frac{a k}{4}\left[N_{1}+N_{2}\right] \tag{39}
\end{equation*}
$$

and the maximum depression, at $t=3 \pi / 2 \sigma$, is

$$
h_{2}=-a+a \frac{a k}{4}\left[N_{1}+N_{2}\right]
$$

The ratio of the elevation to the depression is
$\frac{h_{1}}{h_{2}}=-\left(1+\frac{H k}{4}\left[N_{1}+N_{2}\right] \frac{a}{h}\right) /\left(1-\frac{H k}{4}\left[N_{1}+N_{2}\right] \frac{a}{\bar{H}}\right)$,
and accordingly its valuc increases with wave height. The surface configuration for $t=0$ is

$$
\begin{equation*}
h=a \frac{a k}{4}\left[N_{1}-N_{2} \cos 2 k x\right], \quad t=0 \tag{41}
\end{equation*}
$$



Figure 2. Notation diagram for wave profile.

This represents a positive hump at the center of the basin and depressions at the emds. As a result, the duration of time that the surface of the water at one ond of the basin is found to be above the mulisturbed level is shorter than the duration that it is below. This matter has a bearing on the manner of fixing the reference tinn of the forec cycles studied, and requires further discussion.

At a small positive time $t=\tau_{0}$, the elevation $h$ is nil, and this is the time when the wave in its upward surge reaches the undisturbed level.
Since $\sigma \tau_{0}$ is a small angle, $\sin \sigma \tau_{0}=\sigma \tau_{0}$, and from cq (38)

$$
\begin{equation*}
\sigma \tau_{0}=\left(N_{2}-N_{1}\right) \frac{a k}{4}-\frac{1}{2}\left(N_{2}-N_{1}\right) \frac{a^{3} k^{3}}{16} N_{2} \tag{42}
\end{equation*}
$$

At a later time, $t=T / 2+\tau_{1}$, onec more $h=0$. This is the time when the wave in its downward surge reaches the undisturbed level. Since $\sigma \tau_{1}$ is also a small angle, $\sin \sigma \tau_{1}=\sigma \tau_{1}$, and from eq (38)

$$
\sigma \tau_{1}=\left(N_{1}-N_{2}\right) \frac{a k}{4}+\frac{1}{2}\left(N_{1}-N_{2}\right)^{2} \frac{a^{3} k^{3}}{16} N_{2},
$$

and, thus,

$$
\begin{equation*}
\tau_{1}=-\tau_{0} \tag{43}
\end{equation*}
$$

Let $T_{i}$ denote the duration of time that the surface of the water at the end of the basin, $x=0$, is above the undisturbed level, and $T_{0}$ the duration below the same level. Accordingly,
and

$$
T_{i}+T_{0}=T,
$$

$$
\frac{2\left(T_{0}-T_{i}\right)}{T}=2\left(1-\frac{2 T_{i}}{T}\right)
$$

By definition

$$
T_{i}=\frac{T}{2}+\tau_{1}-\tau_{0}
$$

and in view of eq (43)

$$
T_{i}=\frac{T}{2}-2 \tau_{0}
$$

or

$$
\frac{2 T_{t}}{T}=1-\frac{2 \sigma \tau_{0}}{\pi}
$$

and, thus

$$
\frac{2\left(T_{0}-T_{i}\right)}{T}=\frac{4 \sigma \tau_{0}}{\pi} .
$$

Introducing the value of $\sigma \tau_{0}$ from eq (42)
$\frac{2\left(T_{0}-T_{4}\right)}{T}=\left(N_{2}-N_{1}\right) \frac{k H}{\pi} \frac{a}{H}-\left(N_{2}-N_{1}\right)^{2} N_{2} \frac{H^{3} k^{3}}{8 \pi}\left(\frac{a}{H}\right)^{3}$.

If the instant, when the upsurging wave at the end, $x=0$, reaches the level of whe undisturbed water is observed, this then determines the instant $t=\tau_{n}$ : Since $4 \tau_{0}=T_{0}-T_{i}$, the valuc of $\tau_{0}$ may be obtained from the time durations that the water surface is below or above the still level. If on the other hand these observations bave not been made, theu $\tau_{0}$ must be ohtained from eq (42), introducing in it the wave height $a$ of the obseryed surge deflections.
The expressions for the particle velocities within the order of the approximations considered are from Miche [16],
$u=\frac{\text { gak }}{\sigma} \frac{\cosh k(z+H)}{\cosh k H} \sin k x \cos \sigma t$

$$
\begin{equation*}
-\frac{3}{4} \frac{g a^{2} k^{2}}{\sigma} \frac{\cosh 2 k(z+H)}{\sinh ^{2} k H \sinh 2 k H} \sin 2 k x \sin 2 \sigma t \tag{45}
\end{equation*}
$$

and
$w=\frac{g a k}{\sigma} \frac{\sinh k(z+H)}{\cosh k H} \cos k x \cos \sigma t$

$$
\begin{equation*}
+\frac{3}{4} \frac{g}{\sigma} a^{2} k^{2} \frac{\sinh 2 k(z+H)}{\sinh ^{2} k H \sinh 2 k H} \cos 2 k x \sin 2 \sigma t . \tag{46}
\end{equation*}
$$

At the vertical plane through the midsection of the basin, that is, at thic plane $x=L / 2$ or $k x=\pi / 2$, the relocities are

$$
\begin{equation*}
u=\frac{g a k}{\sigma} \frac{\cosh k(z+H)}{\cosh k H} \cos \sigma t \tag{47}
\end{equation*}
$$

and

$$
\begin{equation*}
w=-\frac{3}{4} \frac{g}{\sigma} a^{2} k^{2} \frac{\sinh 2 k(z+H)}{\sinh ^{2} k H \sinh 2 k H} \sin 2 \sigma t . \tag{48}
\end{equation*}
$$

Thus at the chamel midsection, the horizontal component of the particle velocitics is simple harmonic. The vertical component is also simple harmonic except that the frequency is twice as large. The cffect of vertieal velocity decreases with wive lecight. It is further reduced by lowering the object in the basin. Denoting the position of the object in the basin by $z_{1}$ and putting

$$
\begin{equation*}
U_{m}=\frac{g a k}{\sigma} \frac{\cosh k\left(z_{1}+H\right)}{\cosh k H} \tag{49}
\end{equation*}
$$

the velocity components are

$$
\begin{equation*}
u=-U_{m} \cos \sigma t \tag{50}
\end{equation*}
$$

and

$$
\begin{equation*}
w=-\frac{3}{4} k H \frac{\sinh k\left(z_{1}+H\right)}{\sinh ^{3} k H} \cdot \frac{a}{H} U_{m} \sin 2 \sigma t . \tag{51}
\end{equation*}
$$

It is inferred that $w$ becomes less significant when $k: 1 H$ is larger than 0.9 . This limits the leng $t_{1}$ of the basin for a given depth of water. For studies of wave forces in basins of greater length or with water of less depth the present theory proves inadequate.

All the experiments discussed subsequently were made in a basin of length $L=242 \mathrm{~cm}$ and water
depth $H=70 \mathrm{~cm}$. The oljeets, cylinders or plates, were phaced 25 cm below the water surface, that is $z_{1}=-25 \mathrm{~cm}$ in the midsection plane $x_{1}=121 \mathrm{~cm}$.

For this condition. $71 h:=0.908, N_{1}=1.054$, and $N_{2}=3.322$. From eq (40), the ratio of end deflections reduces to

$$
\begin{equation*}
\frac{h_{1}}{h_{2}}=-\frac{1+0.993 a / H}{1-0.993 a / H} \tag{52}
\end{equation*}
$$

The graph of this equation is shown in figure 3, and ralues from observation are given by circles. The ngreement between theory and obscrvation is satisfactory for $a / H$ less than 0.3 . With this restriction in mind, the valuc of the semiwave height $a$ may be inferred from (39), that is,

$$
\begin{equation*}
\frac{h_{1}}{a}=1+0.993 a / H \tag{53}
\end{equation*}
$$

During the tests the elevation $h_{1}$ was most casily observed.

From eq (49) the relation between the current semiamplitude and the wave height, in egs units, is

$$
\begin{equation*}
U_{m}=3.43 a \tag{54}
\end{equation*}
$$

At $z_{1}$ the horizontal velocity is not unifom in the vertical direction. In the absence of a cylinder, with $z$ measured in centimeters,

$$
\frac{1}{U_{m}} \frac{d U_{m}}{d z}=0.00685
$$

Thus, if $\Delta U_{m}$ be the difference in the maximnm velocities at two points differing in elevation by $D$, then

$$
\frac{\Delta U_{m}}{U_{m}}=0.48 \mathrm{D} / \mathrm{H}
$$

For the largest cylinder used in the experiments, $D=7.62 \mathrm{~cm}$, the value of the ratio $\Delta U_{m} / U_{m}$ is 0.052 .
The maximum value, during the cycle, of the vertical velocity component is given by

$$
w_{m} / U_{m}=0.38 a / H
$$

The majority of the experiments were made with $a$ less than 10 cm . For these cases, $w_{m} / U_{m}$ is less than 0.055 .
From eq (44) the proportion of time that the surface of the water at onc end of the channel is above or bclow the undisturbed level is given by

$$
\begin{equation*}
\frac{T_{0}-T_{i}}{T}=0.328 \frac{a}{\bar{H}}-0.254\left(\frac{a}{\bar{H}}\right)^{3} . \tag{55}
\end{equation*}
$$

The graph of this expression is shown in figure 4 and values from observation are given by circles. For the observations, there was introduced into the basin at each end a parnllel-wire resistance clectrode, the bare parts of the wires being about 5 cm long and placed in a horizontal position just touching the
surface of water at rest. The time that the electric current was traversing the electrodes gave the time that the water surface was above the undisturbed level, as in figure 5.


Figule 3. Variation of end deflections with wave height.


Figure 4: Difference in the durations of the end elevations and depressions.


Figure 5. Time record of the durations of the end deflections.

## 5. Dynamometer and Calibration

The sketch of the dymanometer assembly is shown in figure (i. The rigid and massive base it for supporting purposes is firmly attached to the sted frame of the rocking basin dircetly above the water. The dymamometer itself consists of a pendular frame to which is attached the objeet to be immersed in water. a crlinder or a rectangmar phate. The frane is constructed of brass angles and is strong enongh to resist torsional and fleximal deformation. 'The pivot depressions: located at the upper corners of the frame, consist of small hores of 1 mm in diancter in a bronze bedding. The bores are centered abont polished sted conical points emerging from the supporting basc. At a lower level two dirahminum annular rings of rectangular cross section are clanped to the frame and to the base.

These rings constitnte clastic elements for measuring the forces. To indicate the ring deformations two pairs of strain gages, SR-4, 120 ohms, are glued to cath of the rings, inside and ontside, and at diametrically opposite points. The four strain gage elements form the bridge which is led to a mimiersal analyser. The latter is relayed to one of the channels of a two-channel magnetic oscillorraph. 'The second channel is reserved for timing obscrations. A similar connection is adopted for the other ring. By having fom strain gages on each ring the sensitivity is increased and no corrections are neded for temperature changes. Two different sets of rings are used for momsuring fores of diflorent magnitudes. The method of calibration may be inferred from the sketch in figure 7 . The sum of the forces on the two rings equals 0.625 times the load appliced to the frame. The ring deformations are examined for loads producing tension and compression. The indications of the ring deformations as read from the oscillograph record are lincir as shown in figure 7. The calibrations were repeated before cach rim to
gramd against accidental changes in the strain-gage behavior.


Fiaure 6. Dynamometer assembly (dimensions in centimeters).


Fiotre: 7. Calibration of the strain gages.


Figure 8. An example of oscillograph record of forces.
Run 82, $U_{m} T / D=15.6$.

## 6. Record of Forces and the Reduction

An cxample of two oscillograph recorts of the forecs, one from each ring, and of the timing is shown in figures. The nearly sinusoidal traces relate to the forers acting on the rings; the others, in stcps, give the time sectuence. The: incidence of the larger dicflections indicates the time that the parallel wire elcetrode at the basin end $x=0$ was imnersed; and the ineidence of 110 deflection indicates the time that the clectrode was out of the water. The point $P$ where the greater deflections appear to conimence gives the instant that the upsurging water reached The undisturbed level. Thus the point $l^{\prime}$ gives the time $t=T_{0}$, the value of which was computed from cq ( 55 ), $4 \tau_{0}=T_{0}-T_{t}$, after introducing the semiwave height of the wave. This value was transferred to the record to mark the origin of time, $t=0$, shown by the line $A A^{\prime}$. The line $B B^{\prime}$ indicates the cnd of the wave crecle and corresponds to $t=T$. To establish the correspondence of the records from the two rings, the timing marks appearing at the lower edges of the records were used.
At the time the record of the forces was being tilken the wave elcration $h_{1}$ was read visually against a paper scale altached to the end wall of the basin. The water surface was readily discermable through the lucite walls of the basin. The magnitude of the semiwave height $a$ was dcduced from eq (53), using the observed value $h_{1}$. Maximum current $C_{m}$ was deduced from eq (54).

The sum of the corresponding readings of the sinusoidal tracings in figure 8 gives the magnitude of the forces acting on the two rings when the calibration is applied. Tiking moments about the dummometer pivot point, the total force $X$ on the crlinder is obtained. This is clivided by the length of the cyiinder to give $F$. The time history of the reduced forec, $F / \rho U_{n}{ }^{2} D$, is shown in figure 9 .


Ficidre 9. An example of a curve of reduced forces on a cylinder.
Run 82. $U_{m} T / D=15.6$. Run 82. $U_{m} T / D=15.6$.

## 7. Inertia and the Drag Coefficients of Cylinders and Plates

Considering the force data in timensionless form, such as shown in figure: 9 , the cocfficients $A_{1}$ and $B_{1}^{\prime}$ of ca, (34) were determined by the method implied in eq (30) and (31). The desired integrations were carricd out in the form of summations by giving to the differential multiplier $d \theta$ the incrementin value $\Delta \theta=0.05 \pi$. The valucs of $A_{1}$ and $B_{1}{ }^{\prime}$ thus fomm are entered in table 5 for the cylinders and in talle: 6 for the plates. Tables 1 and 2 contain the diancters of the cylinders, or the width of the plates, the values of the maximum currents and the water temperatures. Next the values of the inertia cocflicient, $C_{m}$, were determined on the basis of eq (30), and the ralucs of the drag coefficient, $C_{d}$, on the basis of eq (31). These results are entered in table 3 for the cylinders and in table 4 for the plates. These tables also contain the Reynolds number $U_{m} D / \nu$ and the pariod parameter $U_{m} T / D$.

The incrtia coefficient $C_{m}$ varies from the theoretical value 2 as the diameter of the cylimer is changed, or with a given cylinder as the maximum current is varied. Similar variations occur in the drag coefficient $C_{d}$, the changes being in the form of additions to the value experienced in ste:rdy flow: A correlation between the coefficients and Reynolds number $U_{m} D / \nu$ does not appear to exist. On the other hand, when these coefficients are related to the period parameter $U_{m} T / D$, definite and regular dependencics are found. This is illustrated in figures 10 and 11 for the cylinders, and in figures 12 and 13 for the plates.

Table 1. Cyliniders
$[T=2.075 \mathrm{sacl}$

| Run | I) | $0_{0}$ | $\theta$ | IRun | I) | $U_{m}$ | $\theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | in. | $\mathrm{cm} / \mathrm{sec}$ | ${ }^{\circ} \mathrm{C}$ |  | in. | cm.sec | ${ }^{\circ} \mathrm{C}$ |
| 1. | 3 | 34.2 | 23.1 | 31 | 1 | 71. 7 | 30.0 |
| 2. | 3 | 30.2 | 22.11 | 31 | 1 | 58.7 | 310 |
| 3 | 3 | 27.7 | ${ }^{23} 30$ | 32 | 1 | 4 4 .3 | 30.0 |
| 4. | 3 | 24.5 | 22.5 | 33. | 1 | 30.0 | 30. 0 |
|  | 3 | 21, 1 | 22.5 | 34. | 0. 75 | 70.7 | 30.0 |
|  | 3 | 19.2 | 22.11 | 35 | 75 | 933. 8 | 30.0 |
|  | 3 | 15.8 | 22.11 | 341 | .75 | 53.6 | 310 |
| 8 | 3 | 13.1 | 23.13 | 137. | . 75 | 45.3 | 310 |
| 9. | 3 | 10.0 | 23.0 | 38 | . 75 | 35.1 | 310.0 |
| 10. | 2.5 | 33.1 | 24.0 | 35. | .5 | 33.4 | 30.0 |
|  | 2. 5 | 27.4 | 24.0 | 40 | 5 | \%8. 7 | 30.0 |
| 12. | 2.5 | 20.7 | 24.0 | 41 | 5 | 48.0 | 310 |
|  | 2. 5 | 13. 0 | 24. 0 | 78 | 1, 75 | 24.6 | 2 c |
|  | 2. 5 | 10.3 | 24.0 | 79 | 1. 75 | 24,0 | 2\% 11 |
|  | 2 | 41.5 | 24.0 | 80. | 1. 75 | 17.7 | 22.0 |
| 15. | 2 | 3.5. 4 | 24.0 | 81. | 1. 75 | 14.4 | 22.0 |
| 17 | 2 | 27.5 | 24.0 | 82 | 1.5 | 28.7 | 20.5 |
| 18 | 2 | 19.1 | 24.8 | 83 | 1,5 | $\underline{2.5}$ | 21.5 |
| 19 | 2 | 23.5 | 24.8 | 84. | 1.5 | 20.2 | 21.5 |
|  | 1.5 | 53.2 | 26.0 | 85. | 1.5 | 14.6 | 20.5 |
| 21. | 1.5 | 43.4 | 26.0 |  | 0.5 | $66^{4} 4$ | 21.0 |
| 22. | 1.5 | 33.3 4 | 26.19 | 87. | . 5 | 54.3 | 21.0 |
| 23 | 1. 5 | $2 \mathrm{S}$. | 26.1 | 88. | . 5 | 44. 6 | 21.0 |
|  | 1. 5 | 19.4 | 26.0 | 84 | . 5 | 32.6 | 21.0 |
| 25 | 1. 25 | 62.9 | 24.0 | 90 | . 75 | 54.0 | 12.0 |
| 28. | 1. 25 | 54.5 | 28.0 | 91. | . 75 | 49.6 | 12.0 |
|  | 1, 2.5 | 43.8 | 24.0 | 92. | .75 | 46.0 | 12.0 |
|  | 1.25 | 35. ${ }^{\mathbf{1}}$ | ${ }^{29} 9$ |  | .75 | 41.0 | 12.0 |

Table 2. Plates


| Run | D | $U_{m}$ | $\theta$ | Ruı | D | $U_{m}$ | $\theta$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
|  | in. | cm/sec | ${ }^{\circ} \mathrm{C}$ |  | in. | $\mathrm{cm} / \mathrm{sec}$ | ${ }^{\circ} \mathrm{C}$ |
| 42. | 3 | 14. 1 | 23.0 | 60. | 1.5 | 13.7 | 24.8 |
| 43. | 3 | 12, 8 | 23.0 | 61. | 1.25 | 3.0 | 28.5 |
| 4. | 3 | U. 8 | 23.0 | 6.2 | 1.25 | 29. 5 | 25. 5 |
| 45. | 3 | 8.0 | 23.0 | 63 | 1.25 | 22.0 | 28.5 |
| 46. | 3 | G. 4 | 23.0 |  | 1. 25 | 16.1 | 28.5 |
|  | 2.5 | 18.5 | 24.0 | 65 | 1 | 41.2 | 30.4 |
|  | 2.5 | 15.8 | 24.0 | nit. | 1 | 34.5 | 311. 4 |
|  | 2. 5 | 13.0 | 24.0 | 67. | 1 | 27.4 | 311.4 |
| 50 | 2. 5 | 11.3 | 24.0 | 68. | 1 | 18.9 | 30: 4 |
|  | 2.5 | 6. 5 | 24.0 |  | 0.75 | 57.1 | 30.8 |
| 52. | $\frac{2}{2}$ | 21.6 | 27.0 | 70 | 75 | 47.0 | 30.8 |
| 53. | 2 | 18.9 | 27.0 | 71 | . 75 | 37. 7 | 30.8 |
| 5. | 2 | 16.1 | 27.0 | 72 | . 75 | 27.4 | 30.8 |
| 5.5 | 2 | 13.0 | 27.0 | 73 | . 5 | 72.4 | 30.0 |
| 56. | 2 | 9.9 | 27.0 |  | . 5 | 63.6 | 30.0 |
|  | 1.5 | 30.0 | 24.8 |  | . 5 | 54.0 | 30.0 |
|  | 1.5 | 25.0 | 24.8 |  | . 5 | 4.5 .3 | 30.0 |
| 59 | 1.5 | 18.9 | 24.8 |  | 5 | 35.8 | 30.0 |

For the cylinders, as one passes from the sinall values of the period parameter to the larger values, the inertia coefficient commences to fall from the initial value 2 to a minimum valne of 1.00 at $U_{m} T / D=15$ and theu gradually increases to a value of 2.5 at $U_{m} T / D=120$. In regard to the coefficient of drag, there is an increase from the initial value 0.9 to a maximum value 2.5 at $U_{m} T / D=1$ 5 and then there is a gradual deerease to the value obtained in steady flow. It appears that for the cylinders the narrow region around $U_{m} T / D=15$ is a critical one.

As regards the plates, the course of the variations of $C_{m}$ and $C_{d}$ with the period parameter is of a very different kind. It will be noticed that $C_{m}$ first increases, then decreases and finally rises again to a value of nearly 4.5. The most remarkable behavior, however, is in regard to $C_{d}$. The coefficient of drag, starting with an unusually large value, 10 , decreases rapidly at first and then gradually for

Table 3. Cylinders

| Run | $C_{m}$ | $C d$ | $U_{m} T / D$ | $U_{m} D / \nu$ | Run | $C_{m}$ | $C_{d}$ | $U_{m} T / D$ | $U_{m} D^{\prime / p}$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1------- | 1.11 | 1.24 | 9.9 | $293 \times 10^{3}$ | 30. | 1. 66 | 1.09 | 58.6 | $227 \times 10^{3}$ |
| 2. | 1. 44 | 1.14 | 8.2 | $2: 39$ | 31. | 1. 70 | 1. 29 | 47. 9 | 18.5 |
| 3. | 1. 49 | 1. 32 | 7. 6 | 225 | 32 | 1. 68 | 1.40 | 37.0 | 143 |
| 4. | 1. 70 | 1.13 | 6.7 | 196 | 331 | 1. 6.4 | 1.49 | 29.4 | 114 |
| 5. | 1.88 | 1.00 | 5.7 | 109 | 43. | 1. 82 | 1.10 | 77.0 | 167 |
| $\mathrm{f}_{7}$ | 1. 95 | 0.91 | 5. 2 | 1.52 | 35 | 1. 61 | 1.39 | 69.5 | 151 |
| 7. | 2.05 | 1.23 | 4.3 | 125 | 35. | 1.183 | 1.42 | 58.3 | 127 |
| 8. | 2. 10 | 1. 01 | 3.6 | 106 | 37 | 1. 64 | 1. 4.5 | 49.3 | 107 |
| 9 | 2. 14 | 0.70 | 2. 7 | 81 | 38 | 1. 8.4 | 1. 50 | 41.5 | 90 |
| 10 | 0.74 | 1. 69 | 10.8 | 229 | 39.-- | 2. 54 | 1.07 | 119.9 | 110 |
| 11.- | 1. 14 | 1.91 | 8.9 | 159 | 40. | 2. 3.5 | 1. 29 | $0 \times 8$ | 93 |
| 13 | 1. 71 | 1.36 | 6.8 | 143 | 41. | 2. 15 | 1.42 | 7 T .5 | 76 |
| 13 | 2. 02 | 1.15 | 4.3 | 90 | 78 | 0.82 | 1. 69 | 12.9 | 127 |
| 14 | 2. 06 | 1.12 | 3.4 | 71 | 70. | . 84 | 2.18 | 11.2 | 111 |
| 15 | 0. 72 | 1. 73 | 17.0 | 230 | 80 | 1.41 | 2. Mi; | 8.3 | 82 |
| 16...- | . 70 | 1. 98 | 11.5 | 196 | 81. | 1.78 | 1.75 | 6. 7 | 67 |
| 17. | . 83 | 2. 18 | 11.2 | 152 | 82 | 0.80 | 2. 18 | 15. 4 | 109 |
| 18. | 1. 50 | 1.89 | 7.8 | 108 | 8 | . 88 | 2. 28 | 13.7 | 96 |
| 19 | 1.10 | 1.97 | 9.6 | 132 |  | . 87 | $\cdots$ | 11.0 | 77 |
| 20. | 1.02 | 1.30 | 29.0 | 231 |  | 1.40 | 2. 18 | 7.9 | 56 |
| 21--- | 1. 02 | 1.49 | 23.6 | 159 | 8 sr . | 2.52 | 1.18 | 108. 1 | 85 |
| 22. | 0.85 | 1.73 | 18. 2 | 145 | 8. | 2. 00 | 1.21 | 81.5 | 71 |
| 23 | . 74 | 2.15 | 14.0 | 112 | 88 | 2. 32 | 1.13 | 72.9 | 57 |
| 24 | . 87 | 2.21 | 10.6 | 8.4 | 49 | 2. 26 | 1.54 | $5: 3$ | 42 |
| 25. | 1.24 | - 1.15 | 41.1 | 218 | 90 | 1. 82 | 1. 28 | 58.8 | 83 |
| 26. | 1.27 | 1. 23 | 35. 5 | 206 |  | 1.81 | 1.38 | 540 | 76 |
| 27 | 1.40 1.26 | 1. 44 | 29.6 2.3 | 152 138 |  | 1.81 | 1. 42 | 50. 1 | 71 |
| 29 | 0.87 | 1.75 | $\underline{17.7}$ | 138 |  | 1. 76 | 1.54 | 4.1. 7 | 63 |

Table 4. Plates

| Run | $C_{\text {m }}$ | $C_{d}$ | $U_{m} T / D$ | $U_{m} D / p$ | Run | $C_{m}$ | $C_{d}$ | $U_{m} T / D$ | $U_{m} D / \nu$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 42. | 1.94 | 8.75 | 3.8 | $114 \times 10^{2}$ | 60. | 2.51 | 5. 15 | 7.5 | $58 \times 10^{2}$ |
| 4. | 1. 74 | 8.81 | 3. 5 | 103 | 61. | 2.14 | 3.25 | 24.2 | 142 |
| 44 | 1. 56 | 9.76 | 2. 7 | 80 | 62. | 1.07 | 3.94 | 19.3 | 113 |
| 4.5 | 1. 51 | 10.21 | 2. 2 | G5 | 63. | 1. 4.3 | 4.09 | 14.3 | 84 |
| 46. | 1.35 | 11.55 | 1.7 | 52 | 64 | 2.25 | 4.43 | 10.5 | 62 |
| 47. | 2. 28 | 5. 50 | f. 1 | 128 | 65. | 2. 45 | 3. 13 | 33.6 | 131 |
| 48 | 2. 12 | 7. 096 | 5.2 | 109 | 66. | 2. 10 | 3.55 | 28. 2 | 110 |
| 49. | 2.00 | 8.01 | 4. 2 | 90 | 67. | 2.01 | 3. 68 | 22.4 | 87 |
| 5 | 1.91 | 8. 64 | 3.4 | 71 | 68. | 1. 56 | 4. 38 | 15.4 | f0 |
| 51. | 1.57 | 11.44 | 2.1 | 45 | 69. | 3. 17 | 2. 4.3 | 62.2 | 138 |
| 52-- | 2. 22 | 5. 41 | 8.8 | 128 | 70. | 2. 88 | 2.86 | 51.2 | 113 |
| 53 | 2.44 | 5. 48 | 7.7 | 112 |  | 2.89 | 3. 06 | 41.1 | 91 |
| 54. | 2.42 | 6.31 | 6. 6 | 95 | 72 | 2. 71 | 3. 36 | 29.9 | 96 |
| 55 | 2. 17 | 7.25 | 5. 3 | 77 | 73 | 4.96 | 1.81 | 118.2 | 114 |
| 56. | 2.16 | 8. 04 | 4. 1 | 59 |  | 4. 09 | 2.03 | 104.0 | 101 |
| 57. | 0.95 | 4. 11 | 16.3 | 127 |  | 4. 00 | 2. 32 | 88.3 | 85 Pe |
| 58. | 1.07 | 4. 28 | 13. 6 | 106 | 76 | 3. 58 | 2. 45 | 74.0 | 72 |
| 59. | 2.08 | 4.61 | 10. 3 | 80 | 77. | 3. 70 | 2. 59 | 58.6 | 57 |

Table 5. Cylinders

| Ruv | A, | $B_{1}^{\prime}$ | $A_{3}$ | $n_{3}^{\prime}$ | As | $B_{5}^{\prime}$ | $U_{m} T / D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 1. | 0. 56 | -0, 612 | -0.11 | -0. 14 | -0.113 | -0.08 | 0.0 |
| 2 | . 87 | -. $5:$ | -. 01 | -. 113 | -.192 | -. 06 | 8.1 |
| 3 | . 47 | -. 86 | - 0is | . 110 | -. 02 | -. 6.4 | 7. |
| 4. | 1.26 | -. 56 | -. 03 | +.105 | -.110 | -. 113 | $8 \cdot 7$ |
| 5 | 1. 172 | -. 51 | -. 02 | . 05 | .00 | -. 03 | 5.7 |
| 6. | 1. S 4 | $-.46$ | +. 02 | . 16 | . 010 | $-.01$ | 5. 2 |
| 7 | 2. 35 | -. 62 | . 01 | . 12 | $-.01$ | -. 0.2 | 4. 3 |
| 8 | ${ }^{2} .91$ | $-.51$ | . 03 | 1.1 | +. 01 | . 00 | 3.6 |
| 9. | 3.87 | $-.35$ | . 01 | . 18 | -. 02 | +.02 | 2.7 |
| 10 | 0.34 | -. 84 | -. 19 | -. 02 | -. 04 | -. 08 | 10.8 |
| 11 | . 63 | -. 80 | -. 14 | $-.05$ | -. 04 | -. 09 | 8.9 |
| 12 | 1. 2.5 | -. 68 | $-.07$ | $-.02$ | . 00 | -. 06 | 6.8 |
| 13. | 2. 34 | -. 58 | +. 10 | +. 13 | +.01 | -. 01 | 4.3 |
| 14. | 3.03 | $-.56$ | . 01 | . 14 | . 00 | +. 01 | 3.4 |
|  | 0.21 | $-.96$ | -. 18 | . 00 | -. 04 | -. 05 | 17.0 |
| 16 | 24 | -. 99 | -. 22 | $-.01$ | $-.02$ | $-.07$ | 14.5 |
| 17 | . 36 | -1.09 | -. 19 | -. 0.5 | -. 02 | -. 12 | 11.2 |
| 18. | 05 | -0.95 | -. 14 | -. 04 | -. 02 | -. 11 | 7.8 |
| 19 | 57 | -. 99 | -. 15 | -. 06 | -. 01 | -. 11 | 9.6 |
| 20 | 17 | -. 65 | -. 05 | . 00 | -. 01 | $-.03$ | 29.0 |
| 21 | . 21 | -. 74 | -. 0. | . 00 | . 00 | -. 04 | 23. 0 |
| 22 | . 23 | $-.87$ | -. 15 | -. 05 | -. 04 | -. 017 | 18.2 |
| 23 | . 26 | -1.08 | -. 22 | $-.04$ | . 00 | $-.08$ | 14.0 |
| 24 | . 41 | $-1.10$ | -. 20 | $-.07$ | . 00 | -. 11 | 10.6 |
| 25. | . 15 | -0.58 | . 00 | +. 01 | .00 | $-.01$ | 41.1 |
| 26 | . 18 | -. 62 | -. 01 | 01 | . 09 | -. 02 | 35.6 |
| 27 | . 24 | -. 73 | -. 01 | .12 | . 010 | -. 02 | 28.6 |
| 28. | . 27 | -. 76 | -. 04 | . 010 | . 00 | -. 05 | 23.3 |
| 29. | . 24 | -. 87 | $-.13$ | -. 104 | -. 04 | -. 07 | 17.7 |
| 30 | -14 | -. 55 | +. 02 | +. 01 | 00 | $-.01$ | 58.6 |
| 31. | . 17 | -. 64 | . 02 | . 01 | -. 01 | $-.01$ | 47.9 |
| 32. | 22 | -. 70 | . 01 | . 13 | . CH | -. 02 | 37.11 |
| 33 | . 27 | -. 74 | . 01 | . 102 | .00 | -.122 | -274 |
| 34 | 12 | -. 55 | . 03 | . 01 | -. 01 | -. 01 | $\div 7.0$ |
| 35 | . 11 | -. 60 | . 01 | . 01 | -. 01 | -. 01 | 69.5 |
| 36. | 14 | -. 71 | . 01 | . 02 | . 00 | -. 02 | 58.3 |
| 37 | 16 | -. 72 | . 01 | . 02 | -. 01 | -. 01 | 49.3 |
| 38. | 22 | -. 75 | . 02 | . 02 | . (k) | -. 01 | 41.5 |
| 39 | 10 | -. 54 | . 03 | . 110 | . 141 | -. 01 | 119.9 |
| 40. | 12 | -. 65 | . 02 | . 01 | . 00 | -. 01 | 95.8 |
| 41. | . 14 | -. 71 | . 01 | . 02 | . 00 | -. 01 | 785 |
| 78 | . 31 | -.99 | $-.18$ | $-.05$ | $-.01$ | -. 10 | 12. |
| 79 | 34 | -1.04 | -. 19 | -. 122 | -. 0.3 | -. 12 | 11:2 |
| 80 | 84 | $-1.03$ | -. 15 | -. 10 | -. 01 | -. 13 | 8.3 |
| 81 | 1. 30 | $-0.88$ | -. 07 | -. 10 | . 00 | $-10$ | 6.7 |
| 82 | 0. 2.5 | -1.02 | $-.22$ | -. 0.3 | -. 04 | -. 06 | 15.f |
| 88. | . 28 | -1.14 | -. 24 | $-.01$ | -. 01 | -. 10 | 13.7 |
| 84 | . 39 | -1.18 | -. 23 | -. 0.5 | -. 02 | -. 1.1 | 11.19 |
| 85. | . 91 | -1.09 | $-.15$ | $-.04$ | . 010 | $-.12$ | 7.9 |
| 86 | . 11 | -0. 80 | +. 03 | +. 01 | (0) | -. 01 | 1145.1 |
| 87. | . 14 | -. 05 | . 02 | . 01 | -. 01 | -. 01 | 89, |
| 88. | . 10 | -. 72 | . 01 | (12) | -. 01 | -. 01 | -2.9 |
| 89 | . 21 | -. 77 | . 01 | - . 0.4 | +. 01 | -. 011 | S3. 2 |
| 90 | . 15 | -. 64 | . 02 | . 01 | . 00 | -. 01 | 58.5 |
| 91 | . 17 | -. 69 | . 02 | . 01 | . 00 | -. 01 | 54.0 |
| 92 | . 18 | -. 71 | . 02 | 02 | . 00 | . 00 | 50.1 |
| 83 | . 19 | $-.76$ | . 02 | 02 | . 00 | -. 01 | 44.7 |



Figure 10. Variation of inertia coeffcient of cylinders.


「Tablf, 6. Plates

| Run | $A_{1}$ | $\boldsymbol{H}_{1}^{\prime}$ | A1 | $B_{8}^{\prime}$ | As | $B_{5}$ | $U_{m} T / D$ |
| :---: | :---: | :---: | :---: | :---: | :---: | :---: | :---: |
| 12 | 2. 50 | -4.38 | 0. 58 | 1). 56 | 0.11 | -0.145 | 3.8 |
| 43. | 2. 44 | -4.41 | . 58 | . 47 | . 0.4 | -. 11 | 3.5 |
| 41. | 2.85 | -4.89 | 47 | . 40 | -. 112 | -. 18 | 2.7 |
| $4 i$ | 3.42 | -5. 11 | 5:3 | . 45 | -. 01 | $-.17$ | 2.2 |
| 46. | 3.83 | -5.77 | . 43 | . 48 | $-.02$ | $-.16$ | 1.7 |
| 47. | 1.80 | -2.75 | . 37 | . 58 | -.0x | - 03 | 0. 1 |
| 4. | 2. 03 | -3.53 | . 4.5 | . 59 | $-.14$ | -. 02 | 5.2 |
| 49 | 2. 32 | -4. 11 | . 53 | 64 | -.132 | -. 02 | 4. 2 |
| 80 | 2.81 | -4.32 | . 8.8 | . 52 | +. 07 | -. 0.16 | 3.4 |
| 51. | 3.63 | $-5.72$ | . 61 | 55 | . 01 | -. 14 | 2.1 |
| 52 | 1.24 | -2.61 | . 35 | . 42 | -. 0.5 | +. 12 | 8.8 |
| 53 | 1.56 | -2.74 | . 31 | . 49 | -. 016 | -. 0.3 | 7.7 |
| 54 | 1.81 | -3.16 | . 38 | . 5 | -. 05 | -. 02 | 6. 6 |
| 55. | 2.01 | -3.63 | . 4.5 | . 84 | -. 01 | -. 03 | 5. 3 |
| 56. | 2. 63 | -4.02 | . 66 | . 60 | . 00 | -. 01 | 4.1 |
| 57. | 0.29 | -2.05 | . 29 | . 33 | +. 02 | +. 07 | 16.3 |
| 58. | . 39 | -2.14 | . 30 | . 31 | . 00 | . 013 | 13. 6 |
| 89 | 1. 00 | -2. 31 | . 29 | . 43 | -. 08 | (1) | 10.3 |
| 60 | 1. 66 | -2.58 | . 30 | 54 | -. 07 | -. 04 | 7.5 |
| ${ }_{61}$ | 0.44 | -1.62 | . 11 | . 18 | -. 05 | -. 02 | 24.2 |
| i2, | . 27 | -1.97 | . 18 | . 19 | +. 02 | +. 07 | 19.3 |
| 63. | . 49 | -2. 04 | . 31 | . 315 | -. 02 | . 04 | 14.3 |
| 1.4 | 1. 0.5 | -2.22 | . 311 | 4.5 | -.06 | -. 01 | 10.5 |
| 65. | 0. 36 | -1.56 | . 08 | . 13 | -. 0.5 | -. 01 | 33.6 |
| 06 | . 37 | $-1.77$ | . 09 | .16 | -. 04 | -. 01 | 28.2 |
| ${ }^{6} 7$. | . 44 | -1.84 |  |  | -. 02 | +.03 | 22.4 |
| 68 | . 50 | -2.19 | . 32 | . 38 | -. 01 | . 06 | 15.4 |
| 96. | . 25 | -1.21 | . 0.5 | . 05 | $-.02$ | -. 04 | [i2. 2 |
| 70. | . 28 | $-1.43$ | . 06 | . 08 | -. 03 | -. 04 | 51.2 |
| 71. | . 35 | -1.53 | . 08 | . 11 | -. 04 | $-.04$ | 41.1 |
| 72 | 45 | -1.68 | . 11 | . 13 | -. 03 | $-.01$ | 29.9 |
| 73. | 21 | -0.91 | . 06 | . 03 | -. 01 | -. 03 | 118.2 |
| 74 | 19 | $-1.01$ | . 07 | . 03 | -. 01 | -. 03 | 104.0 |
| 73 | 22 | $-1.16$ | 05 | . 05 | -. 02 | -. 0.3 | 88.3 |
| 76 | 24 | -1.22 | . 05 | . 05 | -. 02 | -. 02 | 74.0 |
| 77 | . 31 | -1.29 | . 07 | . 06 | $-.03$ | -. 03 | 58.6 |



Figure 11. Variation of drag coefficient of cylinders. Diameter (inches):
Correspondug symbol:
0 $\stackrel{3.5}{\Delta}$
increasing period parameter. The final value is almost identical with that found for steady flow. It is perhaps important to mention that O'Brien and Morison [17] noted equally large values of drag coefficients for spheres subjected to the action of progressive waves. It will be noted that the larger ralues of $C_{d}$ are associated with the smaller values of $O_{m}$, and the larger values of $C_{m}$ with the smaller values of $C_{d}$. Because the drag coefficient is large when $U_{m} T / D$ is small and the variation of $C_{m}$ is rclatively moderate, the wave forces on plates are essentially due to drag, and the inertia effects play a small role almost independent of the period parameter.


Figure 12. Variation of inertia coefficient of plates.



Figene 13. Variation of drag cocfficient of plates.


## 8. Variations of the Remainder Function and $C_{m}$ and $C_{d}$ During the Wave Cycle

The values of $C_{m}$ and $C_{d}$ given in tables 3 and 4 are average values for the entire wave cyele, and in some cases local values may differ from the average. Where the inertia and drag cocfficients, $C_{m}$ and $C_{d}$, cach have the same constant value at all phases of the wave cyde, eq (24) should suffice to describe adequately the magnitude of the forces at every phase. On the other hand, should $C_{m}$ and $C_{d}$ vary with the different phases, the forces are better represented by eci (34a). The variations in $C_{m}$ and $C_{d}$ should lead to the remainder force function $\Delta R$. The examples of the remainder function $\Delta R$ are given in figures 14 and 15 where $\Delta R$ is the difference between $F / \rho U_{m}^{2} D-A_{1} \sin \theta$ and $B_{1}^{\prime \prime} \cos \theta|\cos \theta|$ in conformity with eq (34). Once a curve of $\Delta R$ as a function of $\theta$ is obtained, its structure in Fourier components may be considered and the coefficients $A_{3}, A_{5} \ldots$, and $B_{3}^{\prime}, B_{5}^{\prime}$ mar be obtained. These detcrminations are given in tables 5 and 6 and in figures 16, 17, 18, and 19.

Now for the determinations of the local values of $C_{m}$ and $C_{d}$, two methods are available. The first gives the point values of the coefficients in a ware evcle as determined from the observed values of $F^{\prime} / \rho U_{m}^{2} D$, using eq (24). Two sets of evaluations


Figune 14. Evaluation of remainder force $\Delta R$ for a cylinder. Run 82, $U_{m} T / D=15.6$


Figure 15. Evaluation of remainder force $\Delta R$ for a plate. Run 5i, $U_{m} T / D=6.6$.


Figure 16. Variation of coefficients of the remainder force of. cylinders.



Figure 17. Variation of coefficients of the remainder furce of plates.


Figure 18. Fariation of copflcients of the remainder force of cylinders.


Figure 19. Variation of coefficients of the remainder forcc of plates.

```
lol
```

were made, the basic suppositions being as follows: It was assumed in the first craluation that for $\theta_{1}=\pi / 2+\alpha$ and $\theta_{2}=\pi / 2-\alpha$, where $\alpha$ is an angle less than $\pi / 2$, the coefficients $C_{m}$ and $C_{d}$ each have equal values, since these are the phases where the accelerations, $d u / d t$, are equal, and the currents $u$, are equal in absolute value although of opposing signs. This is true also for $\theta_{1}=3 \pi / 2+\alpha$ and $\theta_{2}=3 \pi / 2-\alpha$. In the second evaluation, it was assumed that for $\theta=\pi+\beta$ and $\theta=\pi-\beta$, where $\beta$ is an angle less than $\pi$,
the coofficients $\mathrm{C}_{\mathrm{m}}$ und $C_{d}$ canch have equal values, sinee these are the phases where the euremis, $u$, are equal and the aceclerations, du/dt, are equal in absolute value, allhough of opposing signs. Also since we know, the values of the cocflicients $A_{1}, A_{3}$, $A_{5}$ and $B_{1}^{\prime}, B_{3}^{\prime}, B_{5}^{\prime}$, whe curves of $C_{j n}$ and $C_{a}^{\prime}$ as function of 0 may be obtained by using eat (28) and (29). The laterer is the second method and is mathematically equivalent to the assumptions made above.
In the cylinder data the agrecoment between the: observed and computed force is satisf:ctory when $U_{m} T / D$ is smesll. The computation was based on eq (24), introducing the values of $C_{m}$ and $C_{a}$ from the tables. Figure 20 illustrates this agrecment. The local values of $C_{m}(\theta)$ and $C_{d}(\theta)$ for this casc are shown in figure 21. The first determinations discussed above are show by circles and squares, wherens the curves are determined by the second me.lind. It is seen that $C_{m}(\theta)$ is independent of the phase $t / T$ and that the coedficient $C_{d}(\theta)$ is constant except in short ranges of the phenses $t / T=0.25$ and 0.75 . This is expected, for at these plases the eurrent $u$ vanishos. The values of $C_{m}$ and $C_{d}$ determined by eq (30) and (31) are given in the caption.

The: agrement between the observed and computed forces is also satisfactory when the period parameter is large. This is illustrated in figure 22. The local valuts of the coefficients for this case are shown in fiyure 2:3. Herc asain, allowing small deriations, $C_{m}(\theta)$ is praetically indepentent of the phase $t / T$ and differs very little from the value given in table 3. On the other hand, considerahle variations are obtained between the observed and computed values of the forces in those cases where the period parmeter is near $C_{m} T / D=15$, as shown in figure 24. The local values of the corfficients for this case are shown in figure 25 . Now (? ${ }^{2}(\theta)$ varics considerably with the phate: $1 / T$. the smaller values oceurring at $t / T=0.0,0.50$. and 1 , and the larger values at $t / T=0.25$ and 0.75 . Also, $C_{a}(\theta)$ apparars to be considerably augmented at the phaises where the velocity $u$ vanishes, Hat is, at $t / T=0.25$ and 0.75 . The example slown is typical for all the cases where $U_{m} T / D$ is in the neigliborthood of $U_{m} T / D=15$. In the example shown in figure $25, C_{m}(\theta)$ shows large neg:tive ralues at the points $t / T=0$, 0.5 , and 1.0 . Tlie significance of this is not clear. It is believed, however, that the presence of negative values is not related altogether to the observational methods that were used.

For the plates deviations were always found between the observed ralues of the forees and the values computed on the basis of eq (24). An example is given in figure 26 . The local values of $C_{m}(\theta)$ and $C_{d}(\theta)$ for this case are shown in figure 27. An additional example is given in figure 28. What is shown in these figures is typical for all the rmns made with the plates: The coefficient $C_{m}^{\prime}(\theta)$ undergoes consider:able variation in valne for varying $t / T$, the greater values oceurring at $t / T=0,0.5$, and 1.0 and the lesser values a $t / t=0.25$ and 0.75 . Furthermore, the increase in $C_{d}(\theta)$ at the points $t / T=0.25$ and 0.75 is very decided.


Figure 20. Comparison of measured and compulcd forces on a Run 9, $U_{m} T / D=3.0$.

(8)

Frgure 21. An cxample of variution of the incria and drag coefficionls of a cylinder during a wave cycle.
Run 9, $U_{m} T / D=3.0, C_{m}=2.14, C_{d}=0.70$.


Figure: 22. Comparison of mcasured and compulcd forces on a Run $93, U_{\mathrm{m}} T / \dot{D}=44.7$.



Figure 23. An example of variation of the incrlia and drag coefficients of a cylinder during a wave cycle.
Run 93, $U_{m} T / D=44.7, C_{m}=1.76, C_{d}=1.54$.


Figure 24. Comparison of measurcd and compuled forces on a Run S2, $\boldsymbol{U}_{\mathrm{m}} T / D=15.6$. cylinder.



Figure 25. An example of variation of the inertia and drag coefficients of a cylinder during a wave cycle.
Run 82, $U_{m} T / D=15.6, C_{m}=0.80, C_{d}=2.05$.
For the cylinder data, as long as the period parameter is sufficiently small, or sufficiently large, the forces may be computed on the busis of eq (24); the remainder function, $\Delta R$, is small. For period parameters in the neighborhood of the critical value, $U_{m} T / D=15$, the representation of forces is more exact using eq (34a); the remainder function is of significance. For the plate data the remainder may not be disregarded, in particular when the period parameter is small.


Figure 26. Comparison of measured and compuled forces on a Run 54, $U_{m} T / D=6.6$.



Figure 27. An example of variation of the incrlia and drag cocfficients of a plate during a wave cycle.
Run 54, $U_{m} T / D=6.6, C_{m}=2.42, C_{d}=6.31$.


Figure 28. An example of varialion of the inerlia and drag coefficients of a plate during a wave cycle.
Run $69, U_{m} T / D=62.2, C_{m}=3.17, C d=2.43$.

## 9. Flow Pattern Around Cylinders and Plates

The flow patterns around the cylinders and plates for varying values of $U_{m} T / D$ were examincal, becanse they may have had a bearing on the fact that the mature of forees during a cycle is significantly uffected by the period parameter. The flow patuern was visually examined by introducing a jet of colored liquid on one side of the immersed object. The disposition of the liquid close to the oljject during the eyclic motion was recorded by a still camera and allso ly a motion-picture camera. Some of these piciures are shown in figures 29 and 30 .

Figure 29, a and b, were taken with the 3 -inch cylinder, the first corresponding to $U_{m} T / D=4$, the sccond to a larger value $U_{m} T / D=10$. When the period parameter is small there is no separation of flow; the liquid near the cylinder clings to the cylinder, and the partitioning of flow from above and below is symmetrical. It will be remembered that at low period parameter the inertia coefficient is about equal to the theoretical value 2 , and drag is negligible. As $U_{m} T / D$ is increased there is separation of flow at the top surface of the cylinder during the relatively longer time that the flow continues in one direction. Although not risible in the picture, somewhat later, but prior to the reversal of current, liquid coming mound the cylinder from below moves upward and, although transforming into an eddy, remains close to the cylinder.

Figure 29, e. illnstrates the flow pattern for $U_{m} T / D=17$ with the 2 -inch cylinder. Note the complete separation at the upper surface of the cylinder with the following flow around the lower surfice directed upward with the subsequent eddy formation.

A completely different picture is oblained for large period parameter, as shown in figure 29, d, taken with the $1 / 2$-inch cylinder: $U_{m} T / D=110$. Here one is confronted with the regular Karman vortices. The eddies are separated alternately from above and below.

With plates the flow patterns are decidedly different, especially for small period parameter. Figure 30, a and b, show the 3-inch plate, the first corresponding to $U_{m} T / D=1$, the second to a larger value, $U_{m} T / D=4$. Eddies are formed almost simultancously at the upper and lower edges of the plates. For the smaller value of $U_{m} T / D$ the eddies are apparently concentrated nearer the edges of the plate. Perhaps the large values of the drag coefficient for small period parameter are associated with the behavior of the eddies in this case, but the question is left open for another occasion.

Figure 30, c, illustrates the flow pattern for $U_{m} T / D=15$ with the $1 \frac{1}{2}$-inch plate. Here the eddy formation is no longer symmetrical, the separation oceurring first at the upper edge of the plate followed by an eddy formed at the lower edge, remaining close to the plate.

Again the Karman vortices arc obtained for large period parameter as shown in figure 30d taken with the $1 / 2$-inch plate, $U_{m} T / D=110$.


它子
Figure 29. Flow pallerns around cylinders.
(A) $D=3$ in., $U_{m} T / D=4:(\mathrm{B}) \quad D=3$ in., $U_{m} T / D=10$;
(C) $D=2$ in.. $U_{m} T / D=1$; (D) $D=0.5$ in., $U_{m} T / D=110$.


Figure 30. Flow palterns around plates
(A) $D=3 \mathrm{in} . U_{m} T / D=1 ;(B) D=3$ in.. $U_{m} T / D=4$. ${ }^{\text {(C) }} D=1.5$ in., $U_{m} T / D=1 \xi^{(D)} D=0.5$ in.. $U_{m} T / D=110$.


Fagure 30. Flow palter's around plates.
(A) $D=3$ in. $U_{n} T / H=1 ;(B) \quad D=3$ in.. $U_{m} T / L=4$ :
(C) $D=1.3$ iin, $U_{m} T / \nu=15{ }^{\circ}$ (D) $D=0.5$ in. $U_{m} T / D=110$

Tho eddy :pporarances discussed above surgest the following interpretation as to the physical meaning of $l_{m}{ }_{m} T / D$. If one defines a length, $l$, as the distance that a fluid particle would move in one direction in the absence of the cylinder, $l=U_{m} 7 / \pi$. Thus,

$$
\frac{U_{m} T}{D}=\frac{\pi l}{D},
$$

and accordingly the period parameter is proportionsal to the ratio of the distance traversed by a particle during a half eycle to the diameter of the cylinder. When the period parameter equals $1 \overline{5}, l / D$ is 4.8 . Prhaps when $U_{m} T / D$ is smaller than 15 , the distance travoled by a particle is not large chough to form complete eddies. When it equals 15 . the distance suffices to form a single eddy, and when much larger than 15 the greater distances allow the formation of numerous vortices of the Karman vortex street. One con hardly refrain from pointing to the similarity between the period parameter and the Stroulial number, and as suggested by McNown and Keulegan [18], the product of Stroulial and period parameter numbers fumishes an alternate parameter as serviceable as the period parameter number. If $T_{s}$ be the duration for the shedding of a single eddy, then the Strouhal number $f D / U=S$ may be written as $D /\left(2 T_{s}^{\prime} U\right)=S$, since the number of alternative eddies shed during a second is $2 f$ and $2 f T_{s}$ equals 1 second. One may suppose that the relation is satisfied approximately also for sinusoidal motions: provided $U$ is replaced by $U_{m} / 2$. Hence, the Stroulial number for sinusoidal inotion is $D /\left(J_{m} T_{s}\right)=S$.

Mintiply the two sides by the wave parameter number $U_{m} T / D$,

$$
\frac{T}{T_{s}}=S \frac{U_{m} T}{D}
$$

For eylinders, ignoring the dependence of $S$ on the Reynolds number,

$$
T / T_{s}=0.2 \frac{U_{m} T}{D}
$$

As noted previously for the cylinders: $C_{m}$ attains its least value, slightly less than unity, at about $U_{m} T / D=12.5$. This corresponds to the condition that $T / T_{s}=2$, nearly, and suggests that during a half cycle, that is, during a connplete motion of fluid particle in one direction, a single eddy is formed and is separated (see also the figure 29, b). Obviously, the process of eddy shedding lias a very significant bearing on the variations of the so-called coefficients of mass and drag, and account needs to be taken of this in the theoretical formulation of the basic proces.

## 10. Maximum Force During a Wave Cycle

In engineering applications the main interest is in' the magnitude of the maximum force experienced during a wave cycle. If the remainder function is neglected, the expression
$F / \rho U_{m}^{2} D=A_{1} \sin \theta+B_{1}^{\prime}|\cos \theta| \cos \theta$
instend of the eq (24), may be milized to evaluate the maximun force $F^{m}$ nad also its phase. If the m:ximum force $F_{m} / \rho U_{m}^{2} D$ oceurs at $\theta=\theta_{m}$, the phase 111:ly be defined as

$$
\Phi=\pi-\theta_{m}
$$

The maximum value of the computed force is given by

$$
\frac{F_{m}}{\rho U_{m}^{2} D}=A_{1} \sin \theta_{m}+B_{1}^{\prime}\left|\cos \theta_{m}\right| \cos \theta_{m}
$$

where $\theta_{m}$ satisfies the relation
$A_{1}+2 B_{1}^{\prime} \sin \theta_{m}=0$, or $\sin \theta_{m}=-\frac{A_{1}}{2 B_{1}^{n}}, \quad$ for $\frac{\pi}{2}<\theta_{m}<\frac{3 \pi}{2}$.
As the coefficients $A_{1}$ and $B_{1}^{\prime \prime}$ are functions of $U_{m} T / D$ only, then $F_{m} / \rho U_{m}^{2} D$ and $\Phi$ both are functions of $U_{m} T / D$. For greater accuracy, the remainder function $\Delta R$ must be considered, but then the evaluations become somewhat involved. If these evaluations are made. the maximum force and phase are again functions of the period parameter.

An alternative procedure is the direct establish ment of the maximuni force and phase by merely taking these quantities from the reduced force curves of this investig:tion. Such readings for the culinders are given in figure 31 and for the plates in figure 32.


Figure 31. Variations of the magnitude and phase of the maximum force on cylinders.
$\begin{array}{l:lllllllll}\text { Piameter (inches): } & 3 & 2.5 & 2 & 1.75 & 1.5 & 1.25 & 1 & 0.75 & 0.5 \\ \text { Corresponding symbol: } & \stackrel{-}{0} & \Delta & \square & 0 & 0 & \Delta & & & +\end{array}$


Figure 32. Variations of the magnitude and phase of the maximum force on plates.

Table 7. Cylinders

| $U_{m} T / D$ | $\frac{F_{m}}{\rho U_{m}^{2} D}$ | $\underset{\text { (degrees) }}{\Phi}$ | $U_{m} T / D$ | $\frac{F_{m}}{\rho U_{m}^{2} D}$ | (degrees) |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2.5 | 4.00 | 82. 0 | 12.5 | 1. 28 | -6. 8 |
| 3. ${ }^{5}$ | 3.45 | 81.2 | 15.0 | 1. 13 | -6. 0 |
| 4.0 | 2.68 | 80.8 80.0 | 20.0 | 1.03 | -5.0 |
| 4.5 | 2.34 | 79.0 | 25.0 | . 80 | 1.0 |
| 5. 0 | 2.10 | 78.0 | 30.0 | . 73 | 4.0 |
| 5. 5 | 1.83 | if. 6 | 35.0 | . 70 | 6.5 |
| 6. 0 | 1. 60 | 75.0 | 40.0 | . 68 | 8.0 |
| 6.5 | 1.42 | 72.5 | 50.0 | . 66 | 8. 0 |
| 7.0 | 1.30 | 65.0 | ${ }^{6} 0.0$ | . 65 | 8.0 |
| 7.5 | 1.20 | $3 \mathrm{5}$. | 70.0 | . 63 | 8.3 |
| 8.0 | 1. 20 | 5.11 | 81.15 | . 13 | 8.7 |
| 9.0 | 1. 25 | -3.0 | 91. 0 | -62 | S. 9 |
| 10.0 | 1.29 | -6.0 | 100.0 | . 62 | 9.0 |

Table 8. Plates

| $U_{m} T / D$. | $\frac{F_{m}}{\rho U_{m}^{z} D}$ | $\stackrel{\boldsymbol{\text { (degrees) }}}{ }$ | $U_{m} T / D$ | $\frac{F_{m}}{\rho U_{m}^{2} D}$ | (degrecs) |
| :---: | :---: | :---: | :---: | :---: | :---: |
| 2.0 | 6. 50 | 24.0 | 10.0 | 2.71 | 34.9 |
| 2.5 | 6.00 | 25.6 | 12.5 | 2.44 | 32. 0 |
| 3.0 | 5. 5.5 | ${ }^{27.0}$ | 15.0 | 2.25 | 28.5 |
| 3.5 | 5. 20 | 28.2 | 17.5 | 2. 10 | 236 |
| 4.0 | 4.85 | 29.5 | 20.0 | 1.96 | 23.5 |
| 4. 5 | 4.55 | 30.8 | 25.0 | 1.76 | 18.3 |
| 5.0 | 4. 30 | 32.0 | 30.0 | 1.63 | 13.6 |
| 5.5 | 4.05 | 33.0 | 35.0 | 1.53 | 11.1 |
| 6.0 | 3.82 | 33.6 | 40.0 | 1.45 | 9.7 |
| 7.0 | 3.43 | 34.9 | 50.0 | 1.33 | 8.9 |
| 8.0 | 3.10 | 35.5 | 60.0 | 1.25 | 8.7 |
| 9.0 | 2.86 | 35.5 | 70.0 | 1. 18 | 9.0 |
|  |  |  | 80.0 | 1.11 | 9.5 |
|  | --..-.-. |  | 90.0 | 1.06 | 10.1 |
|  |  | -------- | 100.0 | 1.02 | 10.8 |

Washington, December 2, 1957.

For reference purposes, the data of the curves is given in tables 7 and 8, and can be used directly. In ul future communication the forces on cylinders hedd in vertical positions will be computed on the basis of the data in these tables and will be compared with the laboratory observations already completed as a matter of concrete illustration.

The authors gratefully acknowledge the suggestions of G. B. Schubauer, the valuable and extensive endeavors of J. W. Lowry, a former collcaguc, in carefully examining the force records and preparing the corresponcling charts and the diligence and resourcefulness of Victor Brame in carrying out the experiments.

## 11. References

[1] G. G. Stokes, On the effect of the internal friction of fluids on the motion of pendulums, Trans. Cambridge Phil. Soc. 9, 8 (1851). (eq (51)).
[2] J. Boussinesq, Sur la résistance ${ }^{2}$ d'une sphère solide Compt. rend. 100, 935 (1885).
[3] A. $\dot{\text { B. Basset, O }}$ On the motion of a sphere in a viscous liquid Phil. Trans. Roy. Soc. London 179, 43 (1888).
[4] Lord Rayleigh, On the motion of solid bodies through viscous liquid Phil. M:g. [6] 21, 687 (1911).
[5] J. S. MeNowih and S. W. Wolf, Resistance to unsteady flow: I. Aualysis of trsts with flat plate, Engineering Research Institute, The University of Michigan, $24.46-\mathrm{I}-\mathrm{P}$, Junc 1956. Internal report to Sandia Curporation.
[6] J. M. Morison, M. I. O'Brien, J. W. Johnson, and S. A. Schaaf, The forces exerted hy surface waves on piles, J. Petrol. Technol. Am. Inst. Mining Engrs. 189, 149 (1950)
[7] J. R. Morison, J. W. Johnson, and M. P. O'Brien, Experimental studies of forers on piles. Coastal Enginerering, Proc. Fourth Conference (19.3).
[8] 1H. W. Irerson and R. Balert, A corrolating modulus for flind resist:unce in accelcrated motion, J. Appl. Phy. 22, 324 (1951).
[9] S. R. Ficim, Fluid resistanco to cylinders in accelerated motion. J. of Hyrraulics Div., Proc. Am. Soc. Civil Engrs. 82, Paper 1113 (195().
[10] G. Bugliarclio, The resistance to accelerated motion of spheres in water, Rieerca Sici. 26, 437 (1956).
[11] R. O. Reid and C. L. Bretschmeider, The design wave in deep water or shallow water, storm tide, and forces on vertical piling and liurge subrnerged objects. A. and M. College of Texas, Dept, of Oceanogriphy, Teeh. Report on Contract NOy-27474, DA-49-005-eng-18, and N7onr-48704, $36 \mathrm{pr} ., \mathrm{l}$ :ch. 1954 (unpublished).
[12] R. O. Reid, Analysis of wave forec experiments at Caplen, Texas (Sun Oil Company platform). A. and M. College of Texas, Dept. of Oceanography, Tech. Report on Contract NOy-27474, 49 pp., January 1956 (Unpublished).
[13] F. D. Mornaghan, Bul. Natl. Research Council 84, 46 (Washington, D. C., 1932).
[14] Sir Horace Lamb, Hydrodynamics, Sixth Ed., p. 77, Dover Publications.
[15] J. S. McNown, Drag in stendy flow, The university of Michigan and Sandia Corporation, Proc. Intern. Congress Appl. Mech. Brussells (1957).
[16] R. Miehe, Mouvements oudulatoires de la mer en profoundeur constante ou decroissante, Ann. Ponts et Chaussées 114, 25 (1944).
[17] M. P. O'Brien and J. R. Morison, The forces exerted by waves on objects. Trans. Am. Geophys. Union 33, 33 (.1952).
[18] J. S. MeNown and G. H. Keulegan, Vortex formation and resistance in periodic motion (publication pending).


[^0]:    Inrestigation sponsored by the Omee of Naral Rescarch
    ${ }^{2}$ Figures in brackets indicate the literature references at the end of this paper.

