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Forcing 2-Metric Dimension in the Join and Corona of Graphs

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Abstract. This study deals with the forcing subsets of 2-metric basis in graphs. Some main results generated in this study include the characterization of a 2-metric basis in graphs and the characterization of the forcing subsets of these 2-metric bases. These characterizations are used to determine values for the forcing 2-metric dimension of graphs resulting from some binary operations such as the join and corona of graphs.

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1. Introduction

Metric dimension and resolving sets, concepts initially drafted for the metric spaces introduced by Blumenthal [3] in 1953. Since then, the notion of metric dimension has been broadened to encompass both metric and geometric spaces [2, 7]. Nearly 20 years after in 1976, Harary, Melter [5] and Slater [11, 12] each separatedly discovered the idea of resolving set.

In 2019, Bailey and Yero [1] demonstrated the construction of error-correcting codes out of graphs using k-resolving sets and provided a decoding algorithm that used covering designs. A study on the idea of the k-resolving set, also known as "On 2-resolving sets in the join and corona of graphs" was published by J. Cabaro and H. Rara [4] in 2021.

The concept of forcing numbers, which was established in 1987 as a result of Klein and Randic's introduction of the study of molecular resonance structure, is another intriguing topic that has drawn the interest of several researchers [8]. Consequently, in 1991,

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Harary et. al [6] coined the term "forcing number" and presented the idea of forcing as a perfect match. In 1999, Chartrand et. al [13] initiated the investigation on the relation between forcing and dimension of a graph. The notions of a 2-resolving set and the forcing dimension of a graph serve as the inspiration for this work. We believe that this study will be tremendously beneficial to someone who is familiar with the theory of the metric dimension. The findings of this work amplified previously-revealed notions to obtain new applications in graph-to-code theory, much like the idea of a 2-resolving set, by developing a novel method for producing error-correcting codes out of graphs.

2. Terminology and Notation

In this study, we only consider graphs that are finite, simple, undirected and connected. Readers are referred to [1, 4, 9, 10] for elementary Graph Theoretic concepts.

Let G be a connected graph of order n. For an ordered set of vertices $W = \{w_1, w_2, ..., w_k\} \subseteq V(G)$ and a vertex v in G, we refer to the k-vector (ordered k-tuple) $r_G(v/W) = (d_G(v, w_1), d_G(v, w_2), ..., d_G(v, w_k))$ as the *(metric) representation of v with respect to W.*

An ordered set of vertices $W = \{w_1, w_2, ..., w_k\}$ is a *k-resolving set* for G if, for any distinct vertices $u, v \in V(G)$, the (metric) representations $r_G(u/W)$ and $r_G(v/W)$ of u and v, respectively differ in at least k positions. If $k = 1$, then the k-resolving set is called a *resolving set* for G. If $k = 2$, then the k-resolving set is called a *2-resolving set* for G.

The least size of a 2-resolving set is called a 2-*metric dimension* of G and we denote it by $dim_2(G)$. A resolving set of size $dim_2(G)$ is called a 2-*metric basis* for G.

Let G be any nontrivial connected graph and $S \subseteq V(G)$. A set $S \subseteq V(G)$ is *2-locating set* of G if it satisfies the following conditions: (i) $|[(N_G(x) \setminus N_G(y)) \cap S] \cup [(N_G(y) \setminus N_G(x)) \cap S]| \geq 2$, for all $x, y \in V(G) \setminus S$ with $x \neq y$ and (ii) $(N_G(v) \setminus N_G(w)) \cap S \neq \varnothing$ or $(N_G(w) \setminus N_G[v]) \cap S \neq \varnothing$ for all $v \in S$ and for all $w \in V(G) \setminus S$. The *2-locating number* of G, denoted by $ln_2(G)$, is the smallest cardinality of a 2-locating set of G. A 2-locating set of G of cardinality $ln_2(G)$ is referred to as an ln_2-set of G.

Let G be any nontrivial connected graph and $S \subseteq V(G)$. S is a (2,2)-*locating* (respectively (2,1)-*locating*) *set* in G if S is a 2-*locating* and $|N_G(y) \cap S| \leq |S| - 2$ ($|N_G(y) \cap S| \leq |S| - 1$, respectively), for all $y \in V(G)$. The (2,2)-*locating* (respectively (2,1)-*locating*) *number* of G, denoted by $ln_{(2,2)}(G)$ (respectively $ln_{(2,1)}(G)$), is the smallest cardinality of a (2,2)-*locating* (respectively (2,1)-*locating*) set in G. A (2,2)-*locating* (respectively (2,1)-*locating*) set in G of cardinality $ln_{(2,2)}(G)$ (respectively $ln_{(2,1)}(G)$) is referred to as an $ln_{(2,2)}$-set (respectively $ln_{(2,1)}$-set) in G.

Let W be a 2-metric basis of a graph G. A subset S of W is said to be a *forcing subset* for W if W is the unique 2-metric basis containing S. The *forcing* 2-*metric dimension* of W is given by $fdim_2(W) = \min\{|S| : S \text{ is a forcing subset for } W\}$. The *forcing* 2-*metric dimension* of G is given by

$$fdim_2(G) = \min\{fdim_2(W) : W \text{ is a 2-metric basis for } G\}.$$

Let W be an ln_2-set of a graph G. A subset S of W is said to be a *forcing subset* for

W if W is the unique ln_2-set containing S. The *forcing* 2*-locating number* of W is given by $fln_2(W) = \min\{|S| : S$ is a forcing subset for $W\}$. The *forcing* 2*-locating number* of G is given by

$$fln_2(G) = \min\{fln_2(W) : W \text{ is a } ln_2\text{-set of } G\}.$$

Let W be an $ln_{(2,2)}$-set of a graph G. A subset S of W is said to be a *forcing subset* for W if W is the unique $ln_{(2,2)}$-set containing S. The *forcing* $(2,2)$*-locating number* of W is given by $fln_{(2,2)}(W) = \min\{|S| : S$ is a forcing subset for $W\}$. The *forcing* $(2,2)$*-locating number* of G is given by

$$fln_{(2,2)}(G) = \min\{fln_{(2,2)}(W) : W \text{ is a } ln_{(2,2)}\text{-set of } G\}.$$

3. Known Results

The following known results are taken from [4].

Remark 1. *For any connected nontrivial graph G of order $n \geq 2$, $2 \leq ln_2(G) \leq n$. Moreover, $ln_2(K_n) = n$, for $n \geq 2$.*

Theorem 1. *Let G be a connected nontrivial graph. Then $ln_2(G) = 2$ if and only if $G \cong P_2$ or $G \cong P_3$.*

Proposition 1. *$dim_2(G) = 2$ if and only if $G \cong P_n$, $n \geq 2$.*

Example 1. Let n be a positive integer. Then P_n and C_n

$$ln_2(P_n) = \begin{cases} \dfrac{n}{2} + 1, & \textit{if } n \geq 2 \text{ and } n \text{ is even,} \\ \dfrac{n+1}{2}, & \textit{if } n \geq 3 \text{ and } n \text{ is odd, and} \end{cases}$$

$$ln_2(C_n) = \begin{cases} \dfrac{n}{2}, & \textit{if } n \geq 6 \text{ and } n \text{ is even,} \\ \dfrac{n+1}{2}, & \textit{if } n \geq 5 \text{ and } n \text{ is odd.} \end{cases}$$

Example 2. The formulas below give the (2,2)-locating number of the path P_n and cycle C_n.

$$ln_{(2,2)}(P_n) = \begin{cases} 4, & \text{if } n = 5, \\ \dfrac{n}{2} + 1, & \text{if } n \geq 6 \text{ and } n \text{ is even,} \\ \dfrac{n+1}{2}, & \text{if } n \geq 7 \text{ and } n \text{ is odd, and} \end{cases}$$

$$ln_{(2,2)}(C_n) = \begin{cases} \dfrac{n}{2}, & \text{if } n \geq 8 \text{ and } n \text{ is even,} \\ \dfrac{n+1}{2}, & \text{if } n \geq 7 \text{ and } n \text{ is odd.} \end{cases}$$

Theorem 2. *Let G be a connected graph of order greater than 3 and let $K_1 = \langle v \rangle$. Then $S \subseteq V(K_1 + G)$ is a 2-resolving set of $K_1 + G$ if and only if either $v \notin S$ and S is a (2,2)-locating set in G or $S = \{v\} \cup T$ is (2,1)-locating set in G.*

Theorem 3. *Let G and H be nontrivial connected graphs. A proper subset S of $V(G+H)$ is a 2-resolving set in $G + H$ if and only if $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are 2-locating sets in G and H, respectively, where S_G or S_H is (2,2)-locating set or S_G and S_H are (2,1)-locating sets.*

Theorem 4. *Let G and H be nontrivial connected graphs. A set $S \subseteq V(G \circ H)$ is a 2-resolving set of $G \circ H$ if and only if $S = A \cup B$, where $A \subseteq V(G)$ and $B = \bigcup\{S_v : S_v \text{ is a 2-resolving set of } H^v, \text{ for all } v \in V(G)\}$.*

Remark 2. *Let G and H be non-trivial connected graphs, $C \subseteq V(G \circ H)$ and $S_v = V(H^v) \cap C$ where $v \in V(G)$. For each $x \in V(H^v) \setminus S_v$ and $z \in S_v$,*

$$d_{G \circ H}(x, z) = \begin{cases} 1, & \textit{if } z \in N_{H^v}(x), \\ 2, & \textit{otherwise.} \end{cases}$$

4. Forcing 2-Metric Dimension of Some Special Graphs

Remark 3. Let G be a nontrivial connected graph. Then

$$0 \leq fdim_2(G) \leq dim_2(G).$$

Remark 4. Let G be a connected graph. Then
(i) $fdim_2(G) = 0$ if and only if G has a unique 2-metric basis, and
(ii) $fdim_2(G) = 1$ if and only if G has at least two 2-metric bases, one of which, say B, that contains an element not in any 2-metric basis for G.

Theorem 5. Let G be a connected graph. Then $fdim_2(G) = dim_2(G)$ if and only if for all 2-metric basis D for G and for each $u \in D$, there exists $v_u \in V(G) \setminus D$ such that $\big[D \setminus \{u\}\big] \cup \{v_u\}$ is a 2-metric basis for G.

Proof: Suppose that $fdim_2(G) = dim_2(G)$. Let D be a 2-metric basis for G such that $fdim_2(G) = |D| = dim_2(G)$, that is, D is the only forcing subset for itself. Let $u \in D$. Since $D \setminus \{u\}$ is not a forcing subset for D, there exists a $v_u \in V(G) \setminus D$ such that $\big[D \setminus \{u\}\big] \cup \{v_u\}$ is a 2-metric basis for G.

Conversely, suppose that every 2-metric basis for G satisfies the given condition. Let D be a 2-metric basis for G such that $fdim_2(G) = fdim_2(D)$. Suppose further that D has a forcing subset J with $|J| < |D|$, that is, $D = J \cup K$ where $K = \{w \in B : w \notin J\}$. Pick $w \in K$. By assumption, there exists $v_w \in V(G) \setminus D$ such that $\big[D \setminus \{w\}\big] \cup \{v_w\} = T$ is a 2-metric basis for G. Hence, $T = J \cup M$, where $M = \big[K \setminus \{w\}\big] \cup \{v_w\}$, is a 2-metric basis containing J, a contradiction. Hence, D is the only forcing subset for D. Therefore, $fdim_2(G) = dim_2(G)$. □

Proposition 2. For any complete graph K_n with $n \geq 1$ vertices,

$$fdim_2(K_n) = 0.$$

Proof: By definition of K_n, $V(K_n)$ is the only minimum 2-resolving set for K_n. By Remark 4 (i), $fdim_2(K_n) = 0$. $\square$

Proposition 3. For any path P_n with $n \geq 2$ vertices,

$$fdim_2(P_n) = 0.$$

Proof: Suppose that $P_n = [v_1, v_2, \ldots, v_n]$. By Proposition 1, $dim_2(P_n) = 2$ for all $n \geq 2$. We claim that $S = \{v_1, v_n\}$ is the unique 2-metric basis of P_n. Suppose $S' = \{v_i, v_j\}$ where $1 \leq i < j \leq n$ and $S' \neq S$. Consider the following cases:

Case 1. Suppose $i = 1$.

If $j = 2$, then $r_{P_n}(v_1/S') = (0, 1)$ and $r_{P_n}(v_3/S') = (2, 1)$. If $2 < j < n$, then $r_{P_n}(v_{j-1}/S') = (j-2, 1)$ and $r_{P_n}(v_{j+1}/S') = (j, 1)$.

Case 2. Suppose $1 < i < j < n$.

If $j = i+1$, then $r_{P_n}(v_i/S') = (0, 1)$, $r_{P_n}(v_{j+1}/S') = (2, 1)$, $r_{P_n}(v_{i-1}/S') = (1, 2)$, and $r_{P_n}(v_j/S') = (1, 0)$. If $j > i+1$, then $r_{P_n}(v_{i-1}/S') = (1, j-i+1)$ and $r_{P_n}(v_{i+1}/S') = (1, j-i-1)$.

By Cases 1 and 2, S' is not a 2-resolving set for P_n. Thus, S is unique. Hence, by Remark 4 (i), $fdim_2(P_n) = 0$ for all $n \geq 4$. Therefore, $fdim_2(P_n) = 0$ for all $n \geq 2$ vertices. $\square$

Proposition 4. For any cycle C_n with $n \geq 3$ vertices,

$$fdim_2(C_n) = \begin{cases} 0, & \text{if } n = 3, 4 \\ 3, & \text{if } n \geq 5. \end{cases}$$

Proof: Suppose that $C_n = [u_1, u_2, \ldots, u_n]$. If $n = 3$ or 4, then $V(C_n)$ is the only 2-resolving set for C_n. By Remark 4 (i), $fdim_2(C_n) = 0$. Let $n \geq 5$. By Proposition 1, $dim_2(C_n) > 2$. Since $\{u_1, u_2, u_3\}$ is a 2-resolving set for C_n, $dim_2(C_n) = 3$. Let $S = \{u_a, u_b, u_c\}$ be a 2-metric basis for C_n. Hence, for all 2-metric basis S for C_n and for each $u_k \in S$, there exists $u_l \in V(C_n) \setminus S$ such that $(S \setminus \{u_k\}) \cup \{u_l\}$ is a 2-metric basis for C_n. By Theorem 5, $fdim_2(C_n) = dim_2(C_n) = 3$. $\square$

5. Forcing 2-Locating and (2,2)-Locating Numbers of Some Special Graphs

Remark 5. Let G be a connected graph. Then

(i) $fln_2(G) = 0$ if and only if G has a unique ln_2-set, and

(ii) $fln_2(G) = 1$ if and only if G has at least two ln_2-sets, one of which, say B, that contains an element not in any ln_2-sets of G.

Theorem 6. Let G be a connected graph. Then $fln_2(G) = ln_2(G)$ if and only if for all ln_2-set S of G and for each $u \in S$, there exists $v_u \in V(G) \setminus S$ such that $\big[S \setminus \{u\}\big] \cup \{v_u\}$ is a 2-locating set of G.

Proof: Suppose that $fln_2(G) = ln_2(G)$. Let S be an ln_2-set of G such that $fln_2(G) = |S| = ln_2(G)$, that is, S is the only forcing subset for itself. Let $u \in S$. Since $S \setminus \{u\}$ is not a forcing subset for S, there exists a $v_u \in V(G) \setminus S$ such that $\big[S \setminus \{u\}\big] \cup \{v_u\}$ is an ln_2-set of G.

Conversely, suppose that every ln_2-set of G satisfies the given condition. Let S be a 2-locating set of G such that $fln_2(G) = fln_2(S)$. Suppose further that S has a forcing subset H with $|H| < |S|$, that is, $S = H \cup I$ where $I = \{w \in S : w \notin H\}$. Pick $w \in I$. By assumption, there exists $v_w \in V(G) \setminus S$ such that $\big[S \setminus \{w\}\big] \cup \{v_w\} = J$ is an ln_2-set of G. Hence, $J = H \cup T$, where $T = \big[I \setminus \{w\}\big] \cup \{v_w\}$, is an ln_2-set containing H, a contradiction. Hence, S is the only forcing subset for S. Therefore, $fln_2(G) = ln_2(G)$. $\square$

Proposition 5. For any complete graph K_n with $n > 1$ vertices,

$$fln_2(K_n) = 0.$$

Proof: By Remark 1, $V(K_n)$ is the only ln_2-set of K_n. By Remark 5 (i), $fln_2(K_n) = 0$. $\square$

Remark 6. Let S be a 2-locating set of $P_n = [u_1, u_2, \ldots, u_{n-1}, u_n]$ where $n \geq 2$. Then
(i) $\{u_1, u_2\} \cap S \neq \varnothing$.
(ii) $\{u_{n-1}, u_n\} \cap S \neq \varnothing$.
(iii) $\{u_i, u_{i+1}, u_{i+2}\} \cap S \neq \varnothing$ where $1 \leq i < n$.

Proposition 6. For any path P_n with $n \geq 2$ vertices,

$$fln_2(P_n) = \begin{cases} 0, & \text{if } n = 2, 3 \text{ and } n \geq 7 \text{ is odd,} \\ 1, & \text{if } n = 5, \\ 2, & \text{if } n \geq 6 \text{ is even,} \\ 3, & \text{if } n = 4. \end{cases}$$

Proof: Suppose that $P_n = [u_1, u_2, \ldots, u_n]$. By Theorem 1, if $n = 2$, then $V(P_2)$ is the only ln_2-set of P_2 and if $n = 3$, then $M = \{u_1, u_3\}$ is the only ln_2-set of P_3. Thus, by Remark 5 (i), $fln_2(P_2) = fln_2(P_3) = 0$. Suppose that $n = 5$. By Example 1, $ln_2(P_5) = 3$. Then by Remark 6, the ln_2-sets of P_5 are $N_1 = \{u_1, u_3, u_5\}$ and $N_2 = \{u_2, u_3, u_4\}$ with $u_1 \in N_1$ and $u_1 \notin N_2$. Thus, by Remark 5 (ii), $fln_2(N_1) = 1 = fln_2(P_5)$. If $n = 4$, then by Remark 6, $R_1 = \{u_1, u_2, u_4\}, R_2 = \{u_1, u_3, u_4\}, R_3 = \{u_1, u_2, u_3\}$ and $R_4 = \{u_2, u_3, u_4\}$ are the ln_2-sets of P_4. Clearly, none of the singletons and doubletons is a forcing subset for an ln_2-set. Thus, $fln_2(P_4) = 3$. Suppose that $n = 6$. By Remark 6, the ln_2-sets of P_6 are

$$S_1 = \{u_1, u_2, u_4, u_6\},$$

$$
\begin{aligned}
S_2 &= \{u_1, u_3, u_4, u_5\},\\
S_3 &= \{u_1, u_3, u_4, u_6\},\\
S_4 &= \{u_1, u_3, u_5, u_6\},\\
S_5 &= \{u_2, u_3, u_4, u_5\} \text{ and}\\
S_6 &= \{u_2, u_3, u_4, u_6\}.
\end{aligned}
$$

It can be verified that $\{u_1, u_2\}$ is the forcing subset of S_1 and the minimum forcing subset of P_6. Thus, $fln_2(P_6) = 2$. Next, suppose that $n \geq 7$ and n is odd. By Remark 6, $T = \{u_1, u_3, u_5, \ldots, u_{n-2}, u_n\}$ is the only ln_2-set of P_n. Thus, $fln_2(T) = fln_2(P_n) = 0$, by Remark 5 (i).

Now, suppose that $n \geq 8$ and n is even. Then the ln_2-sets of P_n are

$$
\begin{aligned}
M_1 &= \{u_1, u_2, u_4, \ldots, u_{n-2}, u_n\},\\
M_2 &= \{u_1, u_3, u_4, \ldots, u_{n-2}, u_n\},\\
M_3 &= \{u_1, u_3, u_5, \ldots, u_{n-2}, u_{n-1}\},\\
M_4 &= \{u_1, u_3, u_5, \ldots, u_{n-2}, u_n\},\\
M_5 &= \{u_1, u_3, u_5, \ldots, u_{n-1}, u_n\} \text{ and}\\
M_6 &= \{u_2, u_3, u_4, \ldots, u_{n-2}, u_n\}
\end{aligned}
$$

by Remark 6. Since $\{u_2, u_3\} \subseteq M_6$ and not contained in any other ln_2-sets of P_n, $\{u_2, u_3\}$ is the forcing subset of M_6 and the minimum forcing subset of P_n. Hence, $fln_2(P_n) = 2$. $\square$

Remark 7. Let W be a 2-locating set of $C_n = [v_1, v_2, \ldots, v_n, v_1]$ where $n \geq 3$. Then

(i) $S \cap W \neq \varnothing$ for all $S \subseteq V(C_n)$ with $\langle S \rangle = P_3$.

(ii) If $v_j, v_{j+1} \in W$, then $v_{j+3}, v_{j+5}, \ldots, v_{j-2} \in W$ where $1 \leq j \leq n$ and $n + k \equiv k \pmod{n}$ for any positive integer k.

Proposition 7. For any cycle C_n with $n \geq 3$ vertices,

$$
fln_2(C_n) = \begin{cases} 0, & \text{if } n = 3, 4,\\ 1, & \text{if } n \geq 6 \text{ is even},\\ 2, & \text{if } n \geq 7 \text{ is odd},\\ 3, & \text{if } n = 5. \end{cases}
$$

Proof: Suppose that $C_n = [u_1, u_2, \ldots, u_n, u_1]$. Note that $C_3 = K_3$ and by Proposition 5, $fln_2(C_3) = 0$. If $n = 4$, then $V(C_4)$ is the only 2-locating set of C_4. Thus, by Remark 5 (i), $fln_2(C_4) = 0$. Suppose that $n = 5$. By Example 1, $ln_2(C_5) = 3$. Then $B_{i,j} = V(C_5) \setminus \{v_i, v_j\}$ for all $i, j \in \{1, 2, \ldots, 5\}$ and $i \neq j$ are the ln_2-sets of C_5. Thus, for every $v_k \in B_{i,j}$ where $k \neq i, j$ there exists $v_i \in B_{j,k}$ such that $[B_{i,j} \setminus \{v_k\}] \cup \{v_i\} = B_{j,k}$ is an ln_2-set of C_5. Hence, by Theorem 6, $fln_2(C_5) = 3$.

Next, suppose $n \geq 6$ and n is even. Then by Example 1, $ln_2(C_n) = \dfrac{n}{2}$. Thus, C_n has ln_2-sets $D_1 = \{u_1, u_3, u_5, \ldots, u_{n-1}\}$ and $D_2 = \{u_2, u_4, u_6, \ldots, u_n\}$. It can be seen that D_1

is the only ln_2-set containing the vertex u_1. Thus, by Remark 5 (ii), $fln_2(C_n) = 1$. Now, suppose that $n \geq 7$ and n is odd. By Example 1, $ln_2(C_n) = \dfrac{n+1}{2}$. Hence, by Remark 7, the ln_2-sets of C_n is of the form

$$S_i = \{u_i, u_{i+1}, u_{i+3}, u_{i+5}, \ldots, u_{i-4}, u_{i-2}\}$$

where $1 \leq i \leq n$ and $n+k \equiv k \pmod{n}$ for any positive integer k. Observe that no single vertex is contained in a unique ln_2-set of C_n. Thus, $fln_2(C_n) > 1$. It can be verified that $\{u_i, u_{i+1}\}$ is uniquely contained in S_i. Hence, $fln_2(S_i) = 2 = fln_2(C_n)$. □

Remark 8. Let G be a connected graph. Then
(i) $fln_{(2,2)}(G) = 0$ if and only if G has a unique $ln_{(2,2)}$-set, and
(ii) $fln_{(2,2)}(G) = 1$ if and only if G has at least two $ln_{(2,2)}$-sets, one of which, say B, that contains an element not in any $ln_{(2,2)}$-sets of G.

Theorem 7. Let G be a connected graph. Then $fln_{(2,2)}(G) = ln_{(2,2)}(G)$ if and only if for all $(2,2)$-locating set S of G and for each $u \in S$, there exists $v_u \in V(G) \setminus S$ such that $\big[S \setminus \{u\}\big] \cup \{v_u\}$ is an $ln_{(2,2)}$-set of G.

Proof: Suppose that $fln_{(2,2)}(G) = ln_{(2,2)}(G)$. Let S be an $ln_{(2,2)}$-set of G such that $fln_{(2,2)}(G) = |S| = ln_{(2,2)}(G)$ that is, S is the only forcing subset for itself. Let $u \in S$. Since $S \setminus \{u\}$ is not a forcing subset for S, there exists a $v_u \in V(G) \setminus S$ such that $\big[S \setminus \{u\}\big] \cup \{v_u\}$ is an $ln_{(2,2)}$-set of G.

Conversely, suppose that every $ln_{(2,2)}$-set of G satisfies the given condition. Let S be an $ln_{(2,2)}$-set of G such that $fln_{(2,2)}(G) = fln_{(2,2)}(S)$. Suppose further that S has a forcing subset D with $|D| < |S|$, that is, $S = D \cup I$ where $I = \{w \in S : w \notin D\}$. Pick $w \in I$. By assumption, there exists $v_w \in V(G) \setminus S$ such that $\big[S \setminus \{w\}\big] \cup \{v_w\} = J$ is an $ln_{(2,2)}$-set of G. Hence, $J = D \cup T$, where $T = \big[I \setminus \{w\}\big] \cup \{v_w\}$, is an $ln_{(2,2)}$-set containing D, a contradiction. Hence, S is the only forcing subset for S. Therefore, $fln_{(2,2)}(G) = ln_{(2,2)}(G)$. □

Remark 9. A (2,2)-locating set in G does not exist for some graph G. In particular, if $\gamma(G) = 1$, then G has no (2,2)-locating set.

Proposition 8. For any path P_n with $n \geq 4$ vertices,

$$fln_{(2,2)}(P_n) = \begin{cases} 0, & \text{if } n = 4 \text{ and } n \geq 7 \text{ is odd,} \\ 2, & \text{if } n \geq 8 \text{ is even,} \\ 3, & \text{if } n = 6, \\ 4, & \text{if } n = 5. \end{cases}$$

Proof: Suppose that $P_n = [v_1, v_2, \ldots, v_n]$. If $n = 4$, then $V(P_4)$ is the only $ln_{(2,2)}$-set of P_4. Thus, by Remark 8 (i), $fln_{(2,2)}(P_4) = 0$. Suppose that $n = 5$. By Example 2, $ln_{(2,2)}(P_5) = 4$. Then $Q_1 = \{v_1, v_2, v_3, v_4\}, Q_2 = \{v_1, v_2, v_3, v_5\}, Q_3 = \{v_1, v_2, v_4, v_5\}$,

$Q_4 = \{v_1, v_3, v_4, v_5\}$ and $Q_5 = \{v_2, v_3, v_4, v_5\}$ are the $ln_{(2,2)}$-sets of P_5. Clearly, for every $v_i \in Q_j$ there exists $v_k \in V(P_5) \setminus Q_j$ such that $[Q_j \setminus \{v_i\}] \cup \{v_k\}$ where $i, j, k \in \{1, 2, 3, 4, 5\}$ is an $ln(2, 2)$-set of P_5. Thus, by Theorem 7, $fln_{(2,2)}(P_5) = 4$. Suppose that $n = 6$. By Example 2, $ln_{(2,2)}(P_6) = 4$. Then the $ln_{(2,2)}$-sets of P_6 are $R_1 = \{v_1, v_2, v_4, v_5\}, R_2 = \{v_1, v_2, v_4, v_6\}, R_3 = \{v_1, v_3, v_4, v_5\}, R_4 = \{v_1, v_3, v_4, v_6\}$, $R_5 = \{v_1, v_3, v_5, v_6\}, R_6 = \{v_2, v_3, v_4, v_5\}$ and $R_7 = \{v_2, v_3, v_4, v_6\}$. Note that $\{v_3, v_5, v_6\}$ is the forcing subset of R_5 and the minimum forcing subset of P_6. Thus, $fln_{(2,2)}(P_6) = 3$. Now, suppose that $n \geq 7$ and n is odd. By Example 2, $ln_{(2,2)}(P_n) = \dfrac{n+1}{2}$. Then $S = \{v_1, v_3, v_5, \ldots, v_{n-2}, v_n\}$ is the only $ln(2, 2)$-set of P_n. Thus, by Remark 8 (i), $fln_{(2,2)}(S) = 0 = fln_{(2,2)}(P_n)$.

Next, suppose that $n \geq 8$ and n is even. Then

$$\begin{aligned}
T_1 &= \{v_1, v_2, v_4, \ldots, v_{n-2}, v_n\},\\
T_2 &= \{v_1, v_3, v_4, \ldots, v_{n-2}, v_n\},\\
T_3 &= \{v_1, v_3, v_5, \ldots, v_{n-3}, v_{n-2}, v_{n-1}\},\\
T_4 &= \{v_1, v_3, v_5, \ldots, v_{n-3}, v_{n-2}, v_n\},\\
T_5 &= \{v_1, v_3, v_5, \ldots, v_{n-1}, v_n\} \text{ and}\\
T_6 &= \{v_2, v_3, v_4, \ldots, v_{n-2}, v_n\}
\end{aligned}$$

are the $ln(2, 2)$-sets of P_n. Hence, no vertex of P_n is contained in a unique $ln(2, 2)$-set. Thus, $fln_{(2,2)}(P_n) \geq 2$. It can be seen that $\{v_1, v_2\}, \{v_{n-2}, v_{n-1}\}, \{v_{n-1}, v_n\}$ and $\{v_2, v_3\}$ are uniquely contained in T_1, T_3, T_4 and T_5, respectively. Therefore,

$$fln_{(2,2)}(T_1) = fln_{(2,2)}(T_3) = fln_{(2,2)}(T_4) = fln_{(2,2)}(T_5) = 2 = fln_{(2,2)}(P_n).$$

□

Proposition 9. For any cycle C_n with $n \geq 4$ vertices,

$$fln_{(2,2)}(C_n) = \begin{cases} 0, & \text{if } n = 4,\\ 1, & \text{if } n \geq 8 \text{ is even},\\ 2, & \text{if } n \geq 7 \text{ is odd},\\ 4, & \text{if } n = 5 \text{ and } 6. \end{cases}$$

Proof: Suppose that $C_n = [v_1, v_2, \ldots, v_n, v_1]$. If $n = 4$, then $V(C_4)$ is the only $ln(2, 2)$-set of C_4. Thus, by Remark 8 (i), $fln_{(2,2)}(C_4) = 0$. If $n = 5$, then the $ln(2, 2)$-sets are $B_1 = \{v_1, v_2, v_3, v_4\}, B_2 = \{v_1, v_2, v_3, v_5\}, B_3 = \{v_1, v_2, v_4, v_5\}, B_4 = \{v_1, v_3, v_4, v_5\}$ and $B_5 = \{v_2, v_3, v_4, v_5\}$. Clearly, for each $v_i \in B_j$ there exists $v_k \in V(C_5) \setminus B_j$ such that $[B_j \setminus \{v_i\}] \cup \{v_k\}$ where $i, j, k \in \{1, 2, 3, 4, 5\}$ is an $ln(2, 2)$-set of C_5. Thus, by Theorem 7, $fln_{(2,2)}(C_5) = 4$. Suppose that $n = 6$. Then $E_{i,j} = V(C_6) \setminus \{v_i, v_j\}$ for all $i, j \in \{1, 2, \ldots, 6\}$ are the $ln_{(2,2)}$-sets of C_6. Thus, for every $v_k \in E_{i,j}$ where $k \neq i, j$ there exists $v_i \in V(C_6) \setminus E_{i,j}$ such that $[E_{i,j} \setminus \{v_k\}] \cup \{v_i\} = E_{j,k}$ is an $ln_{(2,2)}$-set of C_6. Hence, by Theorem 7, $fln_{(2,2)}(C_6) = 4$.

Next, suppose that $n \geq 7$ and n is odd. By Examples 1 and 2, $ln_2(C_n) = \dfrac{n+1}{2}$ $= ln_{(2,2)}(C_n)$. By similar argument as in the proof of Proposition 7, $fln_{(2,2)}(C_n) = 2$. Now, suppose that $n \geq 8$ and n is even. By Example 2, $ln_{(2,2)}(C_n) = \dfrac{n}{2}$. Then $F_1 = \{v_1, v_3, v_5, \ldots, v_{n-3}, v_{n-1}\}$ and $F_2 = \{v_2, v_4, v_6, \ldots, v_{n-2}, v_n\}$ are the only $ln(2,2)$-sets of C_n with $v_1 \in F_1$ and $v_1 \notin F_2$. Thus, by Remark 8(ii), $fln_{(2,2)}(F_1)$ $= 1 = fln_{(2,2)}(C_n)$. □

6. Forcing 2-Metric Dimension in the Join of Graphs

The *join* of two graphs G and H, denoted by $G + H$, is the graph with vertex-set $V(G+H) = V(G)\dot{\cup}V(H)$ and edge-set $E(G+H) = E(G)\dot{\cup}E(H) \cup \{uv : u \in V(G), v \in V(H)\}$.

In view of Theorem 2, we have the following theorem.

Theorem 8. Let G be a connected graph with $|V(G)| \geq 3$ and let $K_1 = \langle v \rangle$. Then a proper subset S of $V(K_1 + G)$ is a 2-metric basis of $K_1 + G$ if and only if one of the following holds:
(i) S is an $ln_{(2,2)}$-set of G,
(ii) $S = \{v\} \cup T$, where T is an $ln_{(2,1)}$-set of G.

Theorem 9. Let $K_1 = \langle v \rangle$ and G a connected graph with $|V(G)| \geq 3$ and $ln_{(2,2)}(G) = ln_{(2,1)}(G)$. Then

$$fdim_2(K_1 + G) = \begin{cases} 0, & \text{if } G \text{ has a unique } ln_{(2,2)}\text{-set ,} \\ fln_{(2,2)}(G), & \text{if } G \text{ has no unique } ln_{(2,2)}\text{-set.} \end{cases}$$

Proof: Suppose that G has a unique $ln_{(2,2)}$-set, say T. Since $ln_{(2,2)}(G) = ln_{(2,1)}(G)$, by Theorem 8, T is a unique 2-metric basis for K_1+G. By Remark 4 (i), $fdim_2(K_1+G) = 0$. Now, suppose that G has at least two $ln_{(2,2)}$-sets. Let A be an $ln_{(2,2)}$-set of G and let F be a forcing subset for A such that

$$fln_{(2,2)}(G) = fln_{(2,2)}(A) = |F|.$$

By Theorem 8, A is a 2-metric basis of $K_1 + G$. Thus,

$$fdim_2(K_1 + G) \leq fln_{(2,2)}(A) = fln_{(2,2)}(G).$$

Let S_0 be a 2-metric basis for $K_1 + G$ such that $fdim_2(K_1 + G) = fdim_2(S_0)$. By Theorem 8, S_0 is an $ln_{(2,2)}$-set of G. Let F_0 be a forcing subset for S_0 with $|F_0| = fdim_2(S_0)$. Hence,

$$fdim_2(K_1 + G) = fdim_2(S_0) = |F_0| \geq fln_{(2,2)}(S_0) \geq fln_{(2,2)}(G).$$

Therefore, $fdim_2(K_1 + G) = fln_{(2,2)}(G)$. □

Example 3. (1.) For the fan $F_n = K_1 + P_n$, where $n \geq 2$,

$$fdim_2(F_n) = \begin{cases} 0, & \text{if } n = 2,3 \text{ and } n \geq 7 \text{ is odd,} \\ 1, & \text{if } n = 5, \\ 2, & \text{if } n \geq 8 \text{ is even,} \\ 3, & \text{if } n = 4,6. \end{cases}$$

(2.) For the wheel $W_n = K_1 + C_n$, where $n \geq 3$,

$$fdim_2(W_n) = \begin{cases} 0, & \text{if } n = 3,4, \\ 1, & \text{if } n \geq 8 \text{ is even,} \\ 2, & \text{if } n \geq 7 \text{ is odd,} \\ 3, & \text{if } n = 5,6. \end{cases}$$

(3.) For the star $S_n = K_1 + \overline{K_n}$ of order $n + 1$,

$$fdim_2(S_n) = 0.$$

As a consequence of Theorem 3, we have the following results.

Theorem 10. Let G and H be nontrivial connected graphs such that $ln_{(2,2)}(G) = ln_{(2,1)}(G)$ and $ln_{(2,2)}(H) = ln_{(2,1)}(H)$. A proper subset S of $V(G + H)$ is a 2-metric basis for $G + H$ if and only if $S = S_G \cup S_H$ where $S_G = V(G) \cap S$ and $S_H = V(H) \cap S$ are ln_2-sets of G and H, respectively such that S_G or S_H is an $ln_{(2,2)}$-set. In particular,

$$dim_2(G + H) = min\{ln_{(2,2)}(G) + ln_2(H), ln_2(G) + ln_{(2,2)}(H)\}.$$

Theorem 11. Let G and H be nontrivial graphs such that $(2, 2)$-locating set of G and H do not exist. Then $S \subseteq V(G + H)$ is a 2-metric basis for $G + H$ if and only if $S = S_G \cup S_H$ where S_G and S_H are $ln_{(2,1)}$-sets of G and H, respectively. In particular, $dim_2(G + H) = ln_{(2,1)}(G) + ln_{(2,1)}(H)$.

Theorem 12. Let G and H be nontrivial graphs such that $ln_{(2,2)}(G) = ln_{(2,1)}(G)$ and $ln_{(2,2)}(H) = ln_{(2,1)}(H)$. Then

$$fdim_2(G + H) = min\{fln_{(2,2)}(G) + fln_2(H), fln_2(G) + fln_{(2,2)}(H)\}.$$

Proof: Suppose that $ln_{(2,2)}(G) + ln_2(H) < ln_2(G) + ln_{(2,2)}(H)$. By Theorem 10, $dim_2(G + H) = ln_{(2,2)}(G) + ln_2(H)$. Let S_G be an $ln_{(2,2)}$-set of G with $fln_{(2,2)}(G) = fln_{(2,2)}(S_G)$ and F_G be a forcing subset of S_G with $fln_{(2,2)}(S_G) = |F_G|$. Let S_H be an ln_2-set of H such that $fln_2(H) = fln_2(S_H)$ and F_H be a forcing subset of S_H with $fln_2(S_H) = |F_H|$. By Theorem 10, $S = S_G \cup S_H$ is a 2-metric basis of $G + H$. We claim that $F_G \cup F_H$ is a forcing subset of S. Suppose there exists a 2-metric basis $S' \neq S$ of $G+H$ with $F_G \cup F_H \subseteq S'$. By Theorem 10, $S' = S'_G \cup S'_H$

where S'_G is an $ln_{(2,2)}$-set of G and S'_H is an ln_2-set of H. Since $F_G \cup F_H \subseteq S'$, $F_G \cap F_H = \varnothing$ and $S'_G \cap S'_H = \varnothing$, hence, F_G is not a forcing subset of S_G or F_H is not a forcing subset of S_H, a contradiction. Thus, $F_G \cup F_H$ is a forcing subset of S. This implies that

$$\begin{aligned} fdim_2(G+H) &\leq fdim_2(S) \\ &\leq |F_G \cup F_H| \\ &= |F_G| + |F_H| \\ &= fln_{(2,2)}(S_G) + fln_2(S_H) \\ &= fln_{(2,2)}(G) + fln_2(H). \end{aligned}$$

Now, let W be a 2-metric basis for $G+H$ with $fdim_2(G+H) = fdim_2(W)$ and let F be a forcing subset of W such that $fdim_2(W) = |F|$. Then by Theorem 10, $W = W_G \cup W_H$ where W_G is an $ln_{(2,2)}$-set of G and W_H is an ln_2-set of H or W_G is an ln_2-set of G and W_H is an $ln_{(2,2)}$-set of H. Since $F \subseteq W = W_G \cup W_H$, consider the following cases:

Case 1. $F \subseteq W_G$

Then W_H is a unique ln_2-set of H and F is a forcing subset of W_G. For if there exists $ln_{(2,2)}$-set $W'_G \neq W_G$ of G and $F \subseteq W'_G$, then $F \subseteq W' = W'_G \cup W_H$. By Theorem 10, $W' \neq W$ is a 2-metric basis for $G+H$, a contradiction.

Case 2. $F \subseteq W_H$

Then W_G is a unique $ln_{(2,2)}$-set of G. By similar argument as in Case 1, F is a forcing subset of W_H.

Case 3. $F = A \cup B$ where $A \subseteq W_G$ and $B \subseteq W_H$.

Then A is a forcing subset of W_G since if there exists an $ln_{(2,2)}$-set $W'_G \neq W_G$ of G and $A \subseteq W'_G$, then $W' = W'_G \cup W_H$ is a 2-metric basis of $G+H$ with $F \subseteq W' \neq W$, a contradiction. Similary, B is a forcing subset of W_H. Therefore in any case $F = F_{W_G} \cup F_{W_H}$ where F_{W_G} and F_{W_H} are forcing subsets of W_G and W_H, respectively where it may happen that F_{W_G} or F_{W_H} is empty. Hence,

$$\begin{aligned} fdim_2(G+H) &= fdim_2(W) \\ &= |F| \\ &= |F_{W_G} \cup F_{W_H}| \\ &= |F_{W_G}| + |F_{W_H}| \\ &\geq fln_{(2,2)}(W_G) + fln_2(W_H) \\ &\geq fln_{(2,2)}(G) + fln_2(H). \end{aligned}$$

Consequently, $fdim_2(G+H) = fln_{(2,2)}(G) + fln_2(H)$. Similarly, if $ln_2(G) + ln_{(2,2)}(H) < ln_{(2,2)}(G) + ln_2(H)$, then,

$$fdim_2(G+H) = fln_2(G) + fln_{(2,2)}(H).$$

□

Example 4. Consider the join of two graphs C_4 and P_7. Note that $ln_{(2,2)}(C_4) = 4 = ln_{(2,1)}(C_4)$ and $ln_{(2,2)}(P_7) = 4 = ln_{(2,1)}(P_7)$. Since C_4 and P_7 both having unique $ln_{(2,2)}$-sets, $fdim_2(C_4 + P_7) = fln_{(2,2)}(C_4) = fln_{(2,2)}(P_7) = 0$.

Example 5. Consider the join of two graphs C_7 and P_6. Then $ln_{(2,2)}(C_7) = 4 = ln_{(2,1)}(C_7)$ and $ln_{(2,2)}(P_6) = 3 = ln_{(2,1)}(P_6)$. Since C_7 and P_6 have no unique $ln_{(2,2)}$-sets, $fdim_2(C_7 + P_6) = min\{fln_{(2,2)}(C_7) + fln_2(P_6), fln_2(C_7) + fln_{(2,2)}(P_6)\} = min\{2+2, 2+3\} = min\{4,5\} = 4$.

Theorem 13. Let G and H be nontrivial connected graphs where both G and H have no $(2,2)$-locating sets. Then

$$fdim_2(G+H) = \begin{cases} 0, & \text{if both } G \text{ and } H \text{ have} \\ & \text{unique } ln_{(2,1)}\text{-sets,} \\ fln_{(2,1)}(G) + fln_{(2,1)}(H), & \text{otherwise.} \end{cases}$$

Proof: Suppose that both G and H have unique $ln_{(2,1)}$-sets. Let S_G and S_H be the unique $ln_{(2,1)}$-sets of G and H, respectively. By Theorem 11, $S = S_G \cup S_H$ is the unique 2-metric basis for $G + H$. By Remark 4 (i), $fdim_2(G + H) = 0$. Next, suppose that G has no unique $ln_{(2,1)}$-set. Let S_G and S_H be $ln_{(2,1)}$-sets of G and H, respectively. Thus, by Theorem 11, $S = S_G \cup S_H$ is a 2-metric basis for $G + H$. Let F_G and F_H be forcing subsets of S_G and S_H, respectively such that $fln_{(2,1)}(G) = fln_{(2,1)}(S_G) = |F_G|$ and $fln_{(2,1)}(H) = fln_{(2,1)}(S_H) = |F_H|$. We claim that $F_G \cup F_H$ is a forcing subset of $S = S_G \cup S_H$. Clearly, $F_G \cup F_H \subseteq S_G \cup S_H = S$. Suppose there exists 2-metric basis $S' \neq S$ with $F_G \cup F_H \subseteq S'$ for $G + H$. By Theorem 11, $S' = S'_G \cup S'_H$ where S'_G and S'_H are $ln_{(2,1)}$-sets of G and H, respectively. Since $S' \neq S, S_G \neq S'_G$ or $S_H \neq S'_H$. Also, since $F_G \cup F_H \subseteq S'$, $F_H \subseteq S'_H$ and $F_G \subseteq S'_G$. This is a contradiction since F_G and F_H are forcing subsets of S_G and S_H, respectively. Hence, $F_G \cup F_H$ is a forcing subset of S. Thus,

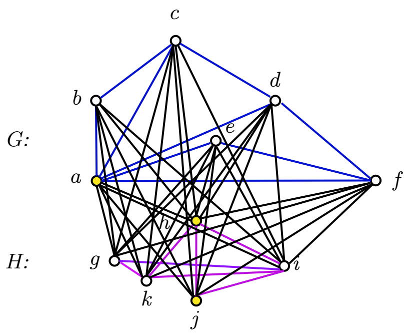
$$\begin{aligned} fdim_2(G+H) &\leq fdim_2(S) \\ &\leq |F_G \cup F_H| \\ &= |F_G| + |F_H| \\ &= fln_{(2,1)}(G) + fln_{(2,1)}(H). \end{aligned}$$

Next, suppose that S_0 is a 2-metric basis for $G + H$ such that $fdim_2(G + H) = fdim_2(S_0)$. By Theorem 11, $S_0 = S^0_G \cup S^0_H$ where S^0_G and S^0_H are $ln_{(2,1)}$-sets of G and H, respectively. Let $F_0 = F^0_G \cup F^0_H$, where $F^0_G \subseteq S^0_G$ and $F^0_H \subseteq S^0_H$, be a forcing subset of S_0 such that $fdim_2(S_0) = |F_0|$. Suppose $F^0_G \subseteq D^0_G$ for some $ln_{(2,1)}$-set D^0_G of G with $D^0_G \neq S^0_G$. Then $S'_0 = D^0_G \cup S^0_H$ is a 2-metric basis for $G + H$, $S'_0 \neq S_0$, and $F_0 \subseteq S'_0$. This contradicts the assumption that F_0 is a forcing subset of S_0. Thus, F^0_G is a forcing subset of S^0_G. Similarly, F^0_H is a forcing subset of S^0_H. Hence,

$$\begin{aligned} fdim_2(G+H) = fdim_2(S_0) = |F_0| &= |F^0_G \cup F^0_H| \\ &= |F^0_G| + |F^0_H| \\ &\geq fln_{(2,1)}(S^0_G) + fln_{(2,1)}(S^0_H) \\ &\geq fln_{(2,1)}(G) + fln_{(2,1)}(H). \end{aligned}$$

Therefore, $fdim_2(G + H) = fln_{(2,1)}(G) + fln_{(2,1)}(H)$. $\square$

Example 6. Consider the join of two graphs G and H, where (2,2)-locating sets do not exist for both graphs.



The join G + H

Note that G has $ln_{(2,1)}$-sets $W_1 = \{a, c, d, u_f\}$, $W_2 = \{b, c, d, e\}$, $W_3 = \{b, c, d, f\}$, $W_4 = \{b, d, e, f\}$ and $W_5 = \{c, d, e, f\}$. On the other hand, H have $ln_{(2,1)}$-sets $Z_1 = \{g, h, j, k\}, Z_2 = \{g, h, i, k\}$ and $Z_3 = \{g, i, j, k\}$. Clearly, $a \in W_1$ and $a \notin W_2$, W_3, W_4, W_5. Also, $\{h, j\} \subseteq Z_1$ and $\{h, j\} \nsubseteq Z_2, Z_3$. Thus, $fln_{(2,1)}(G) = 1$ and $fln_{(2,1)}(H) = 2$. Hence,

$$fdim_2(G + H) = fln_{(2,1)}(G) + fln_{(2,1)}(H) = 1 + 2 = 3.$$

7. Forcing 2-Metric Dimension in the Corona of Graphs

The *corona* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G of order n and n copies of H, and then joining every vertex of the ith copy of H to the ith vertex of G. For $v \in V(G)$, denote by H^v the copy of H whose vertices are attached one by one to the vertex v. Subsequently, denote by $v + H^v$ the subgraph of the corona $G \circ H$ corresponding to the join $\langle\{v\}\rangle + H^v, v \in V(G)$.

Let $S = \bigcup_{v \in V(G)} S_v \subseteq V(G \circ H)$ be a 2-resolving set of $G \circ H$ and $p, q \in V(H^v)$ where $p \neq q$ for $v \in V(G)$. Then $r_{G \circ H}(p/S)$ and $r_{G \circ H}(q/S)$ differ in at least 2-positions. By Remark 2, $r_{H^v}(p/S_v)$ and $r_{H^v}(q/S_v)$ must differ in at least two distinct positions. By definition of $G \circ H$, if $p, q \in V(H^v) \setminus S_v$, then there exist at least two distinct vertices $r, s \in V(H^v) \cap S_v$ such that either $r, s \in N_{H^v}(p) \setminus N_{H^v}(q)$ or $r, s \in N_{H^v}(q) \setminus N_{H^v}(p)$ or $r \in N_{H^v}(p) \setminus N_{H^v}(q)$ and $s \in N_{H^v}(q) \setminus N_{H^v}(p)$. Similarly, if $p \in V(H^v) \setminus S_v$ and $q \in S_v$, there exists a vertex $w \in V(H^v) \cap S_v$ such that $w \in N_{H^v}(p) \setminus N_{H^v}(q)$ or $w \in N_{H^v}(q) \setminus N_{H^v}(p)$. Hence, S_v is a 2-locating set of H^v. Thus, Theorem 4 is modified in the next result.

Theorem 14. Let G and H be nontrivial connected graphs. Then $S \subseteq V(G \circ H)$ is a 2-metric basis for $G \circ H$ if and only if $S = \bigcup_{v \in V(G)} S_v$ where S_v is an ln_2-set of H^v for all $v \in V(G)$. In particular,

$$dim_2(G \circ H) = |V(G)| ln_2(H).$$

Theorem 15. Let G and H be nontrivial connected graphs. Then

$$fdim_2(G \circ H) = \begin{cases} 0, & \text{if } H \text{ has a unique } ln_2\text{-set ,} \\ |V(G)|fln_2(H), & \text{if } H \text{ has no unique } ln_2\text{-set.} \end{cases}$$

Proof: Suppose H has a unique ln_2-set. For each $v \in V(G)$, let $M_v \subseteq V(H^v)$ be the unique ln_2-set of H^v. By Theorem 15, $S = \bigcup_{v \in V(G)} M_v$ is the unique 2-metric basis for $G \circ H$. Thus, by Remark 4 (i), $fdim_2(G \circ H) = 0$. Suppose H does not have unique ln_2-set. For each $v \in V(G)$, let $Q_v \subseteq V(H^v)$ be an ln_2-set of H^v such that $fln_2(H^v) = fln_2(Q_v)$. Let $M_{(Q_v)} \subseteq Q_v$ be a forcing subset for Q_v with $fln_2(Q_v) = |M_{(Q_v)}|$. Then by Theorem 15, $S_Q = \bigcup_{v \in V(G)} Q_v$ is a 2-metric basis of $G \circ H$. Let $C = \bigcup_{v \in V(G)} M_{(Q_v)}$. Then C is a forcing subset for S_Q. Thus,

$$\begin{aligned} fdim_2(G \circ H) &\leq fdim_2(S_Q) \\ &\leq |C| \\ &= |V(G)||M_{(Q_v)}| \\ &= |V(G)|fln_2(Q_v) \\ &= |V(G)|fln_2(H). \end{aligned}$$

Next, let S' be a 2-metric basis for $G \circ H$ such that $fdim_2(G \circ H) = fdim_2(S')$. Then by Theorem 15, $S' = \bigcup_{v \in V(G)} R_v$ where R_v is an ln_2-set of H^v for each $v \in V(G)$. Let C' be a forcing subset of S' such that $fdim_2(S') = |C'|$. We claim that $C' \cap R_v = C_v$ is a forcing subset for R_v for all $v \in V(G)$. Suppose there exists $w \in V(G)$ such that $C' \cap R_w = C_w$ is not a forcing subset for R_w. Let R'_w be an ln_2-set of H^w with $C_w \subseteq R'_w$ and $R'_w \neq R_w$. Then

$$S'' = \left(\bigcup_{v \in V(G) \setminus \{w\}} R_v \right) \cup R'_w$$

is a 2-metric basis for $G \circ H$ with $S' \neq S''$ and $C' \subseteq S''$, a contradiction. Thus, C_v is a forcing subset for R_v for each $v \in V(G)$. Let $C' = \bigcup_{v \in V(G)} C_v$. Then

$$fdim_2(G \circ H) = |C'| = \sum_{v \in V(G)} |C_v| \geq \sum_{v \in V(G)} fln_2(H^v) = |V(G)|fln_2(H).$$

Therefore, $fdim_2(G \circ H) = |V(G)|fln_2(H)$. □

Example 7. Consider the corona of two graphs P_3 and C_4. Since C_4 has a unique ln_2-set, $fdim_2(P_3 \circ C_4) = |V(P_3)|fln_2(C_4) = 3 \cdot 0 = 0$.

Example 8. Consider the corona of two graphs K_3 and C_5. Since C_5 has no unique ln_2-set, $fdim_2(K_3 \circ C_5) = |V(K_3)|fln_2(C_5) = 3 \cdot 3 = 9$.

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