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Forcing axioms and the complexity of non-stationary ideals

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Abstract

We study the influence of strong forcing axioms on the complexity of the non-stationary ideal on ω_2 and its restrictions to certain cofinalities. Our main result shows that the strengthening MM^{++} of Martin's Maximum does not decide whether the restriction of the non-stationary ideal on ω_2 to sets of ordinals of countable cofinality is Δ_1-definable by formulas with parameters in $\mathrm{H}(\omega_3)$. The techniques developed in the proof of this result also allow us to prove analogous results for the full non-stationary ideal on ω_2 and strong forcing axioms that are compatible with CH. Finally, we answer a question of S. Friedman, Wu and Zdomskyy by showing that the Δ_1-definability of the non-stationary ideal on ω_2 is compatible with arbitrary large values of the continuum function at ω_2.

Keywords Non-stationary ideals · Δ_1-definability · Martin's Maximum · Subcomplete Forcing Axiom · stationary reflection

Mathematics Subject Classification 03E57 · 03E35 · 03E47

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1 Introduction

The fact that closed unbounded subsets generate a proper normal filter, the *club filter* on κ

$$Club_\kappa = \{A \subseteq \kappa \mid \exists C \subseteq A \, closed \, and \, unbounded \, in \, \kappa\},$$

is one of the most important combinatorial properties of uncountable regular cardinals κ. The study of the structural properties of these filters and their dual ideals, the *non-stationary ideal* on κ

$$NS_\kappa = \{A \subseteq \kappa \mid \exists C closed \, and \, unbounded \, in \, \kappa \, with \, A \cap C = \emptyset\}$$

plays a central role in modern set theory.

In [23] and [24], Mekler, Shelah and Väänänen initiated the study of the *complexity* of club filters and non-stationary ideals, leading to various results establishing interesting connections between the complexity of these objects and their structural properties. Given an uncountable regular cardinal κ, it is easy to see that both $Club_\kappa$ and NS_κ are definable by a Σ_1-formula with parameter κ, i.e. there exist Σ_1-formulas $\varphi_0(v_0, v_1)$ and $\varphi_1(v_0, v_1)$ such that $Club_\kappa = \{A \mid \varphi_0(A, \kappa)\}$ and $NS_\kappa = \{A \mid \varphi_1(A, \kappa)\}$. The results of [24] show that under CH, the $\mathbf{\Delta}_1(\mathrm{H}(\omega_2))$-definability of NS_{ω_1} (i.e. the assumption that $NS_{\omega_1} = \{A \mid \psi_1(A, z)\}$ holds for some Π_1-formula $\psi(v_0, v_1)$ and some $z \in \mathrm{H}(\omega_2)$) is equivalent to several interesting combinatorial and model-theoretic assumptions about objects of size ω_1. In particular, it is shown that this definability assumption is equivalent to the existence of a so-called *canary tree*, a tree of height and cardinality ω_1 without cofinal branches that has specific properties with respect to the ordering of such trees under order-preserving embeddings. Since the results of [23] show that the existence of a canary tree is independent of ZFC + CH, it follows that this theory is not able to determine the exact complexity of NS_{ω_1}.

The above results were later generalized to higher cardinals. If S is a stationary subset of an uncountable regular cardinal κ, then we let $NS \restriction S = NS_\delta \cap \wp(S)$ denote the *restriction* of the non-stationary ideal on δ to S. Given infinite regular cardinals $\lambda < \kappa$, we set $S^\kappa_\lambda = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \lambda\}$. In addition, if $m < n < \omega$, then we write S^n_m instead of $S^{\omega_n}_{\omega_m}$. Results of Hyttinen and Rautila in [13] showed that if κ is an infinite regular cardinal in a model of the GCH, then, in a cofinality-preserving forcing extension, the set $NS \restriction S^{\kappa^+}_\kappa$ is $\mathbf{\Delta}_1(\mathrm{H}(\kappa^{++}))$-definable. Furthermore, in [10], S. Friedman, Wu and Zdomskyy showed that for every successor cardinal in Gödel's constructible universe L, there is a cardinality-preserving forcing extension of L in which NS_κ is $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable. These results can be easily used to show that the complexity of the non-stationary ideal and its restriction is not determined by ZFC (see Lemma 1.1 and the subsequent discussion below). Finally, recent work also unveiled several interesting consequences of the $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definability of restriction of NS_κ at higher cardinals κ. In particular, this set-theoretic assumption was shown to be closely connected to model-theoretic questions dealing with Shelah's *Classification Theory* and the complexity of certain mathematical theories (see, for example, [8, Theorem 64]).

The above results strongly motivate the question whether canonical extensions of ZFC decide more about the complexity of non-stationary ideals, and this question turns out to be closely connected to important recent developments in set theory. In [8], S. Friedman, Hyttinnen and Kulikov showed that, in the constructible universe L, the sets of the form $NS \upharpoonright S$ for some stationary subset S of an uncountable regular cardinal κ are not $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable. Using the notion of *local club condensation* (see [7]), it is possible to extend this conclusion to larger canonical inner models. In another direction, S. Friedman and Wu observed in [9] that strong saturation properties of the non-stationary ideal on ω_1, i.e. the assumption that the poset $\wp(\omega_1)/NS_{\omega_1}$ has a dense subset of cardinality ω_1, imply the $\mathbf{\Delta}_1(\mathrm{H}(\omega_2))$-definability of NS_{ω_1}. Results of Woodin in [26, Chapter 6] show that NS_{ω_1} possesses these properties in certain forcing extensions of determinacy models. Finally, Schindler and his collaborators recently studied the question whether forcing axioms determine the complexity of NS_{ω_1}. In [19], Larson, Schindler and Wu showed that *Woodin's Axiom* $(*)$ (see [26, Definition 5.1]) implies that NS_{ω_1} is not $\mathbf{\Delta}_1(\mathrm{H}(\omega_2))$-definable. In combination with recent results of Asperó and Schindler in [1], this shows that MM^{++}, a natural strengthening of *Martin's Maximum*, implies that NS_{ω_1} is not $\mathbf{\Delta}_1(\mathrm{H}(\omega_2))$-definable.

The work presented in this paper is motivated by the question whether strong forcing axioms determine the complexity of the non-stationary ideal on ω_2 and its restrictions. The following result from [21] shows that all extensions of ZFC that are preserved by forcing with $<\omega_2$-directed posets are compatible with the assumption that for every stationary subset S of ω_2, the set $NS \upharpoonright S$ is not $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable. In particular, the results of [4, 17, 18] show that this statement is compatible with all standard forcing axioms, like MM^{++}. The lemma follows directly from a combination of [21, Theorem 2.1], showing that no $\mathbf{\Delta}^1_1$-definable set (see [20, Definition 1.2]) separates $Club_\kappa$ from NS_κ in the given model of set theory, and [20, Lemma 2.4], showing that $\mathbf{\Delta}^1_1$-definability coincides with $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definability at all uncountable regular cardinals κ.

Lemma 1.1 *Let κ be an uncountable cardinal with $\kappa^{<\kappa} = \kappa$ and let G be* $\mathrm{Add}(\kappa, \kappa^+)$*-generic over* V. *In* V[*G*], *no $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable subset of $\wp(\kappa)$ separates $Club_\kappa$ from NS_κ, i.e. no set X definable in this way satisfies $Club_\kappa \subseteq X \subseteq \wp(\kappa) \setminus NS_\kappa$.*

Note that, if S is a stationary subset of an uncountable regular cardinal κ, then $NS \upharpoonright S$ separates $Club_\kappa$ from NS_κ. This shows that, in $\mathrm{Add}(\kappa, \kappa^+)$-generic extensions, sets of the form $NS \upharpoonright S$ for stationary subsets S are not $\mathbf{\Delta}_1(\mathrm{H}(\kappa^+))$-definable.

In contrast, we will prove the following theorem that shows that strong forcing axioms like MM^{++} are also compatible with the existence of a $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable set that separates the club filter on ω_2 from the corresponding non-stationary ideal. The proof of this result is based on a detailed analysis of the preservation properties of a variation of a forcing iteration constructed by Hyttinen and Rautila in the consistency proofs of [13]. Our construction will also allow us to produce such models with arbitrary large 2^{ω_2}. See Sect. 2 for the meaning of the "$+\mu$" versions of forcing axioms.[1]

[1] In keeping with the prevailing convention in the literature: MM^{+} refers to MM^{+1}, but MM^{++} refers to $\mathrm{MM}^{+\omega_1}$, *not* to MM^{+2} (and similarly for PFA and other forcing axioms).

Theorem 1.2 *Let* FA *denote any one of the following forcing axioms:*

- $\mathrm{MM}^{+\mu}$*, where* μ *is a cardinal and* $0 \leq \mu \leq \omega_1$*; or*
- $\mathrm{PFA}^{+\mu}$*, where* μ *is a cardinal and* $1 \leq \mu \leq \omega_1$*.*

Assume that FA *holds, and let* θ *be a cardinal with* $\theta^{\omega_2} = \theta$*. Then there exists a* $<\omega_2$*-directed closed, cardinal-preserving poset* $\mathbb{P}$ *with the property that whenever* G *is* $\mathbb{P}$*-generic over* V*, then, in* V$[G]$*, the axiom* FA *still holds,* $2^{\omega_2} = \theta$ *and the set* $NS \restriction S^2_0$ *is* $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$*-definable.*

We will also apply the techniques developed in the proof of the above result to forcing axioms that are compatible with the continuum hypothesis, focusing on the axiom $\mathrm{FA}^+(\sigma\text{-closed})$ and the *subcomplete forcing axiom* SCFA introduced by Jensen in [16]. Note that both axioms are preserved by $<\omega_2$-directed closed forcings (see [4] and [18]) and hence Lemma 1.1 above already shows that they are compatible with the assumption that no $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable set separates $Club_{\omega_2}$ from NS_{ω_2}.

Theorem 1.3 *Let* FA *denote either the axiom* SCFA *or the axiom* $\mathrm{FA}^+(\sigma\text{-}closed)$*. Assume that* $2^\omega = \omega_1$*,* $2^{\omega_1} = \omega_2$ *and* FA *holds. Let* θ *be a cardinal satisfying* $\theta^{\omega_2} = \theta$*. Then there exists a* $<\omega_2$*-directed closed, cardinal-preserving poset* $\mathbb{P}$ *with the property that whenever* G *is* $\mathbb{P}$*-generic over* V*, then, in* V$[G]$*, the axiom* FA *still holds,* $2^{\omega_2} = \theta$ *and the set* NS_{ω_2} *is* $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$*-definable.*

This theorem also provides an affirmative answer to [10, Problem 3.3] posed by S. Friedman, Wu and Zdomskyy, by showing that the $\mathbf{\Delta}_1$-definability of NS_{ω_2} is compatible with $2^{\omega_2} \geq \omega_5$.

2 Preliminaries

This section covers well-known results, mostly related to the notions of internal approachability and Shelah's Approachability Ideal.

First we recall the "plus" versions of forcing axioms, which were first introduced by Baumgartner [2] (though the prevailing notation has changed somewhat since then). If Γ is a class of posets and μ is a cardinal, $\mathrm{FA}^{+\mu}(\Gamma)$ states that for every poset $\mathbb{P} \in \Gamma$, for every collection $\mathcal{D}$ of size ω_1 of dense subsets of $\mathbb{P}$ and for every sequence $\langle \sigma_\xi \mid \xi < \mu \rangle$ of $\mathbb{P}$-names for stationary subsets of ω_1, there is a $\mathcal{D}$-generic filter g on $\mathbb{P}$ with the property that the set $\{\alpha < \omega_1 \mid \exists p \in g \ p \Vdash_{\mathbb{P}} \text{“}\check{\alpha} \in \sigma_\xi\text{”}\}$ is stationary in ω_1 for every $\xi < \mu$. If Γ is the class of posets that preserve stationary subsets of ω_1, we write $\mathrm{MM}^{+\mu}$ instead of $\mathrm{FA}^{+\mu}(\Gamma)$; and, as mentioned earlier, MM^{++} refers to $\mathrm{MM}^{+\omega_1}$, *not* to MM^{+2}. Similar comments apply to the class of proper posets and PFA.

Definition 2.1 Let $\mathbb{P}$ be a poset and let $W \prec (\mathrm{H}(\theta), \in, \mathbb{P})$ for some sufficiently large regular cardinal θ.

(1) A condition $p \in \mathbb{P}$ is a **$(W, \mathbb{P})$-master condition** if

$$W[G] \cap \mathrm{V} = W$$

holds whenever G is $\mathbb{P}$-generic over V with $p \in G$.

(2) A set g is **$(W, \mathbb{P})$-generic** if $g \subseteq \mathbb{P} \cap W$, g is a filter on $\mathbb{P} \cap W$, and $D \cap g \neq \emptyset$ for every $D \in W$ that is a dense subset of $\mathbb{P}$.[2]
(3) A condition $p \in \mathbb{P}$ is a **$(W, \mathbb{P})$-total master condition** if the set

$$\{r \in \mathbb{P} \cap W \mid p \leq_{\mathbb{P}} r\}$$

is a $(W, \mathbb{P})$-generic filter.

The following result is well-known:

Lemma 2.2 *Let $\mathbb{P}$ be a poset, let $W \prec (\mathrm{H}(\theta), \in, \mathbb{P})$, let μ be an ordinal with $\mu \subseteq W$, and let $\dot{f} \in W$ be a $\mathbb{P}$-name for a function from μ to the ground model* V. *If p is a $(W, \mathbb{P})$-total master condition and G is $\mathbb{P}$-generic over* V *with $p \in G$, then $\dot{f}^G \in \mathrm{V}$.*

Proof Fix a $(W, \mathbb{P})$-total master condition p and a filter G on $\mathbb{P}$ that is generic over V and contains the condition p. Let $g = \{r \in \mathbb{P} \cap W \mid p \leq_{\mathbb{P}} r\}$ denote the $(W, \mathbb{P})$-generic filter induced by p. Then $G \cap W$ is a $(W, \mathbb{P})$-generic filter extending g and therefore standard arguments show that $G \cap W = g$. By elementarity, there is a sequence $\langle A_\xi \mid \xi < \mu \rangle \in W$ of maximal antichains in $\mathbb{P}$ with the property that for every $\xi < \mu$, each condition in A_ξ decides the value of $\dot{f}$ at ξ. Since $\mu \subseteq W$, this shows that for all $\xi < \mu$, the unique condition in $A_\xi \cap g$ decides the value of $\dot{f}$ at ξ. But the sequence $\langle A_\xi \mid \xi < \mu \rangle$ and the filter g are both elements of V and hence the function $\dot{f}^G$ is also in the ground model. □

We state a definition that will be used extensively in the following arguments:

Definition 2.3 Given an infinite regular cardinal κ, we let IA_κ denote the class of all sets W with the property that there exists a sequence $\vec{N} = \langle N_\alpha \mid \alpha < \kappa \rangle$ that satisfies the following statements:

(1) The sequence $\vec{N}$ is $\subseteq$-increasing and $\subseteq$-continuous.
(2) $W = \bigcup \{N_\alpha \mid \alpha < \kappa\}$.
(3) $|N_\alpha| < \kappa$ for all $\alpha < \kappa$.
(4) Every proper initial segment of $\vec{N}$ is an element of W.

Remark 2.4 If $\vec{N}$ witnesses that W is an element of IA_κ and $W \prec \mathrm{H}(\theta)$ for some $\theta > \kappa$, then $\kappa \subseteq W$. This is because we have $\vec{N} \restriction \alpha \in W$ for every $\alpha < \kappa$, and the domain of $\vec{N} \restriction \alpha$, namely α, is definable from the parameter $\vec{N} \restriction \alpha$.

In what follows, if τ is a regular uncountable cardinal, $\wp_\tau(H)$ refers to the set of all $W \subseteq H$ with $|W| < \tau$, and $\wp^*_\tau(H)$ denotes the set

$$\{W \in \wp_\tau(H) \mid W \cap \tau \in \tau\}.$$

The set $\wp^*_\tau(\mathrm{H}(\theta))$ contains a club in the sense of Jech (see [14]), but not necessarily in the sense of Shelah (see [6]).

[2] Sometimes the requirement that $g \subseteq W$ is dropped, but then one has the demand that $D \cap g \cap W \neq \emptyset$ holds for each dense $D \in W$.

Remark 2.5 In the above situation, if $W \in \wp^*_\tau(\mathrm{H}(\theta))$, $W \prec \mathrm{H}(\theta)$, and $x \in W$ with $|x| < \tau$, then $x \subseteq W$.[3]

Lemma 2.6 *If κ is a regular and uncountable cardinal, then IA_κ is stationary in $\wp_{\kappa^+}(\mathrm{H}(\theta))$ for all sufficiently large regular θ.*

Proof Given a first-order structure $\mathfrak{A} = (\mathrm{H}(\theta), \in, \kappa, \ldots)$ in a countable language, recursively construct a $\subseteq$-continuous and $\subseteq$-increasing sequence $\vec{N} = \langle N_\alpha \mid \alpha < \kappa\rangle$ of elementary substructures of $\mathfrak{A}$ of cardinality less than κ such that $\vec{N} \restriction \alpha \in N_{\alpha+1}$ for all $\alpha < \kappa$. Then $\vec{N}$ witnesses that its union is contained in IA_κ. □

Lemma 2.7 *Let $\mathbb{P}$ be a poset, let $\kappa < \theta$ be infinite regular cardinals with $\mathbb{P} \in \mathrm{H}(\theta)$, let $\lhd$ be a well-ordering of $\mathrm{H}(\theta)$, let $W \prec (\mathrm{H}(\theta), \in, \mathbb{P}, \lhd)$ with $W \in IA_\kappa$, and let $p \in \mathbb{P} \cap W$.*

(1) If $\mathbb{P}$ is $<\kappa$-closed, then there exists a $(W, \mathbb{P})$-generic filter that contains p.
(2) If $\mathbb{P}$ is $<\kappa^+$-closed, then there exists a $(W, \mathbb{P})$-total master condition below p.

Proof Let $\vec{N} = \langle N_\alpha \mid \alpha < \kappa\rangle$ witness that W is an element of IA_κ.

(1) Assuming that $\mathbb{P}$ is $<\kappa$-closed. Using the closure of $\mathbb{P}$ and the fact that each N_α has cardinality less than κ, we can recursively construct a descending sequence $\vec{p} = \langle p_\alpha \mid \alpha < \kappa\rangle$ of conditions below p in $\mathbb{P}$ such that the following statements hold for all $\alpha < \kappa$:

(a) The condition $p_{\alpha+1}$ is the $\lhd$-least element of $\mathbb{P}$ below p_α that is an element of every open dense set that belongs to N_α.[4]
(b) If α is a limit ordinal, then p_α is the $\lhd$-least lower bound of the sequence $\langle p_\ell \mid \ell < \alpha\rangle$.

Then every proper initial segment of $\vec{p}$ is definable from a proper initial segment of $\vec{N}$, and hence every proper initial segment of $\vec{p}$ is in W. In particular, we know that $p_{\alpha+1} \in W$ for all $\alpha < \kappa$. It follows that the filter in $\mathbb{P}$ generated by the subset $\{p_\alpha \mid \alpha < \kappa\}$ is $(W, \mathbb{P})$-generic.

(2) Now, assume that $\mathbb{P}$ is $<\kappa^+$-closed and repeat the above construction of the sequence $\vec{p}$. Then $\vec{p}$ has a lower bound in $\mathbb{P}$, and this lower bound is clearly a $(W, \mathbb{P})$-total master condition. □

Next we discuss one variant of proper forcing.

Definition 2.8 Let κ be an infinite regular cardinal.

(1) A poset $\mathbb{P}$ is ***IA_κ*-proper** if for all sufficiently large regular cardinals θ, all $W \prec (\mathrm{H}(\theta), \in, \mathbb{P})$ with $W \in \mathrm{IA}_\kappa$ and all $p \in \mathbb{P} \cap W$, there is a $(W, \mathbb{P})$-master condition below p.
(2) A poset $\mathbb{P}$ is ***IA_κ*-totally proper** if for all sufficiently large regular cardinals θ, all $W \prec (\mathrm{H}(\theta), \in, \mathbb{P})$ with $W \in \mathrm{IA}_\kappa$ and all $p \in \mathbb{P} \cap W$, there is a $(W, \mathbb{P})$-total master condition below p.

[3] Note that this could fail if W were allowed to have non-transitive intersection with τ.

[4] We do not require here that $p_{\alpha+1}$ is a total master condition for N_α. That is, if $D \in N_\alpha$ is dense, the upward closure of $p_{\alpha+1}$ is only required to meet D, not necessarily $D \cap N_\alpha$.

It is well-known that IA_κ-proper posets preserve all stationary subsets of $S^{\kappa^+}_\kappa$ that lie in the approachability ideal $I[\kappa^+]$ defined below. Since we could not find a reference for exactly what is needed in our arguments, we sketch the proof below. Note that it is possible for IA_κ-proper (even IA_κ-totally proper) posets to destroy the stationarity of some subsets of $S^{\kappa^+}_\kappa$ (see [5]). So IA_κ-total properness is, in general, strictly weaker than κ^+-Jensen completeness (defined in the next section), because $<\kappa^+$-closed forcings preserve all stationary subsets of κ^+.

Definition 2.9 (Shelah) Let κ be an infinite regular cardinal.

(1) Given a sequence $\vec{z} = \langle z_\alpha \mid \alpha < \kappa^+\rangle$ a sequence of elements of $[\kappa^+]^{<\kappa}$, an ordinal $\gamma < \kappa^+$ is called **approachable with respect to $\vec{z}$** if there exists a sequence

$$\vec{\alpha} = \langle \alpha_\xi \mid \xi < \mathrm{cof}(\gamma)\rangle$$

cofinal in γ such that every proper initial segment of $\vec{\alpha}$ is equal to z_α for some $\alpha < \gamma$.

(2) The **Approachability ideal $I[\kappa^+]$** on κ^+ is the (possibly non-proper) *normal* ideal generated by sets of the form

$$A_{\vec{z}} = \{\gamma < \kappa^+ \mid \gamma \textit{ is approachable with respect to } \vec{z}\}$$

for some sequence $\vec{z} \in {}^{\kappa^+}([\kappa^+]^{<\kappa})$.

Note that a subset X of κ^+ is an element of $I[\kappa^+]$ if and only if there exists some club $D \subseteq \kappa^+$ and some sequence $\vec{z} \in {}^{\kappa^+}([\kappa^+]^{<\kappa})$ such that every $\gamma \in D \cap X$ is approachable with respect to $\vec{z}$. In the following, we will make use of several facts about $I[\kappa^+]$. Throughout this section, κ denotes a regular cardinal.

Lemma 2.10 ([5]) *Suppose $\kappa^{<\kappa} \leq \kappa^+$, and let $\langle z_\alpha \mid \alpha < \kappa^+\rangle$ be an enumeration of $[\kappa^+]^{<\kappa}$.*[5] *Define*

$$M_{\vec{z}} = \{\gamma \in S^{\kappa^+}_\kappa \mid \gamma \textit{ is approachable with respect to } \vec{z}\}.$$

Then the following statements hold:

(1) $M_{\vec{z}}$ is a stationary subset of $S^{\kappa^+}_\kappa$.
(2) $M_{\vec{z}} \in I[\kappa^+]$.
(3) $M_{\vec{z}}$ is a maximum element of $I[\kappa^+] \cap \wp(S^{\kappa^+}_\kappa)$ mod NS, *i.e. whenever S is a stationary subset of $S^{\kappa^+}_\kappa$ such that $S \in I[\kappa^+]$, then $S \setminus M_{\vec{z}}$ is non-stationary.*
(4) If $\kappa^{<\kappa} = \kappa$, then $S^{\kappa^+}_\kappa \setminus M_{\vec{z}}$ is non-stationary. In particular, $\kappa^{<\kappa} = \kappa$ implies that $S^{\kappa^+}_\kappa \in I[\kappa^+]$.

Proof (1) Fix a sufficiently large regular cardinal θ and a well-ordering $\lhd$ of $\mathrm{H}(\theta)$. Fix $W \in \mathrm{IA}_\kappa$ with $W \prec (\mathrm{H}(\theta), \in, \lhd, \vec{z})$ and let $\langle N_\alpha \mid \alpha < \kappa\rangle$ be a sequence witnessing

[5] Note that such an enumeration exists by our cardinal arithmetic assumption.

that $W \in \mathrm{IA}_\kappa$. Given $\alpha < \kappa$, set $\gamma_\alpha = \sup(N_\alpha \cap \kappa^+) < \kappa^+$. Then $\vec{\gamma} = \langle \gamma_\alpha \mid \alpha < \kappa \rangle$ enumerates a cofinal subset of $W \cap \kappa^+$ of order-type κ and every proper initial segment of this sequence is an element of W.

Moreover, each proper initial segment of $\vec{\gamma}$ is an element of $[\kappa^+]^{<\kappa}$, and hence an element of

$$W \cap \{z_\alpha \mid \alpha < \kappa^+\} \;=\; \{z_\alpha \mid \alpha < W \cap \kappa^+\}.$$

This shows that $W \cap \kappa^+$ is approachable with respect to $\vec{z}$. Since Lemma 2.6 shows that there are stationarily-many $W \in \mathrm{IA}_\kappa$ with $W \prec (\mathrm{H}(\theta), \in, \lhd, \vec{z})$, these computations allow us to conclude that $M_{\vec{z}}$ is a stationary subset of $S^{\kappa^+}_\kappa$.

(2) Since $M_{\vec{z}} \subseteq A_{\vec{z}} \in I[\kappa^+]$, the statement $M_{\vec{z}} \in I[\kappa^+]$ holds trivially.

(3) Now, suppose that $S \in I[\kappa^+]$ is a stationary subset of $S^{\kappa^+}_\kappa$. By earlier remarks, there is a sequence $\vec{u} = \langle u_\alpha \mid \alpha < \kappa^+ \rangle$ of elements of $[\kappa^+]^{<\kappa}$ and club subset D of κ^+ with the property that every $\gamma \in D \cap S$ is approachable with respect to $\vec{u}$. Define

$$E \;=\; \{\gamma \in S \mid \mathrm{Hull}^{(\mathrm{H}(\theta), \in, \vec{u}, \vec{z}, D)}(\gamma) \cap \kappa^+ = \gamma\}.$$

Then $S \setminus E$ is non-stationary. Fix $\gamma \in E$ and set $M(\gamma) = \mathrm{Hull}^{(\mathrm{H}(\theta), \in, \vec{u}, \vec{z}, D)}(\gamma)$. Since $\gamma \in S \in I[\kappa^+]$, there is a cofinal sequence $\vec{\beta} = \langle \beta_\ell \mid \ell < \kappa \rangle$ in γ such that every proper initial segment of $\vec{\beta}$ appears in $\vec{u} \restriction \gamma$. But since $\vec{z}$ enumerates all of $[\kappa^+]^{<\kappa}$, the fact that $\vec{u}, \vec{z} \in M(\gamma) \prec (\mathrm{H}(\theta), \in)$ implies that for every $\alpha < \gamma$ there is a $k(\alpha) < \gamma$ with $u_\alpha = z_{k(\alpha)}$, i.e.

$$\{u_\alpha \mid \alpha < \gamma\} \;\subseteq\; \{z_\alpha \mid \alpha < \gamma\}.$$

In particular, every proper initial segment of $\vec{\beta}$ appears in $\vec{z}$ before γ and therefore γ is approachable with respect to $\vec{z}$. These computations show that $S \setminus M_{\vec{z}} \subseteq S \setminus E$ is non-stationary in κ^+.

(4) Now, assume that $\kappa^{<\kappa} = \kappa$. Then $|\eta^{<\kappa}| = \kappa$ for every $\eta < \kappa^+$ and hence there is a function $f : \kappa^+ \longrightarrow \kappa^+$ with the property that for all $\eta < \kappa^+$, every element of $[\eta]^{<\kappa}$ is enumerated by $\vec{z} \restriction f(\eta)$. Let D denote the club of all $\kappa < \gamma < \kappa^+$ such that

$$\mathrm{Hull}^{(\mathrm{H}(\theta), \in, \vec{z}, f)}(\gamma) \;\cap\; \kappa^+ \;=\; \gamma.$$

Pick $\gamma \in D \cap S^{\kappa^+}_\kappa$, and set $W(\gamma) = \mathrm{Hull}^{(\mathrm{H}(\theta), \in, \vec{z}, f)}(\gamma)$. Fix a cofinal sequence $\vec{\alpha}$ in γ of order-type κ in γ, and some $\xi < \kappa$. Since $\mathrm{cof}(\gamma) = \kappa$, there is $\eta < \gamma$ with $\alpha_\ell < \eta$ for all $\ell < \xi$ and $\vec{\alpha} \restriction \xi = z_\zeta$ for some $\zeta < f(\eta)$. Moreover, since $\eta \in W(\gamma)$ and $|f(\eta)| \leq \kappa \subseteq W(\gamma)$, elementarity implies that $f(\eta) \in W(\gamma)$ and $f(\eta) \subseteq W(\gamma)$. Since $\vec{z}$ and ζ are both elements of $W(\gamma)$, we can conclude that $z_\zeta \in W(\gamma)$. Hence γ is approachable with respect to $\vec{z}$. □

The next few lemmas address stationary set preservation when GCH may fail to hold.

Lemma 2.11 *The class IA_κ is projective stationary over*

$$\mathcal{S} = \{T \subseteq S^{\kappa^+}_\kappa \mid T\, is\, stationary\, and\, T \in I[\kappa^+]\},$$

i.e. if $T \in \mathcal{S}$, then for every sufficiently large regular cardinal θ and every function $F : [\mathrm{H}(\theta)]^{<\omega} \longrightarrow \mathrm{H}(\theta)$, there exists $W \in IA_\kappa$ such that $W \cap \kappa^+ \in T$ and W is closed under F.

Proof Fix $T \in \mathcal{S}$. Then there is a club D in κ^+ and a sequence $\vec{z} \in {}^{\kappa^+}[\kappa^+]^{<\kappa}$ such that every element of $D \cap T$ is approachable with respect to $\vec{z}$.

Fix a regular ϑ with $F \in \mathrm{H}(\vartheta)$, let $\lhd$ be a well-ordering of $\mathrm{H}(\vartheta)$ and set

$$\mathfrak{A} = (\mathrm{H}(\vartheta), \in, \lhd, \vec{z}, D, F, T).$$

Pick $\gamma \in D \cap T$ and $W \prec \mathfrak{A}$ with $\gamma = W \cap \kappa^+$, which is possible because T is stationary. Since $\gamma \in T \cap D$, there is an increasing sequence $\vec{\beta} = \langle \beta_\alpha \mid \alpha < \kappa\rangle$ that is cofinal in γ and has the property that every proper initial segment of $\vec{\beta}$ is equal to z_α for some $\alpha < \gamma$. Since $\vec{z} \in W$ and $W \cap \kappa^+ = \gamma$, it follows that every proper initial segment of $\vec{\beta}$ is an element of W. Recursively define a sequence $\vec{N} = \langle N_\alpha \mid \alpha < \kappa\rangle$ as follows:

- Given $\alpha < \kappa$, let $N_{\alpha+1}$ be the $\lhd$-least element of $[\mathrm{H}(\theta)]^{<\kappa}$ such that $N_{\alpha+1}$ is closed under F, $\langle N_\ell \mid \ell \leq \alpha\rangle \in N_{\alpha+1}$, $\alpha \subseteq N_{\alpha+1}$, and $\sup(N_{\alpha+1} \cap \kappa^+) \geq \beta_\alpha$.
- If $\alpha < \kappa$ is a limit ordinal, then $N_\alpha = \bigcup\{N_\ell \mid \ell < \alpha\}$.

Set $N = \bigcup\{N_\alpha \mid \alpha < \kappa\}$. Then $N \in \mathrm{IA}_\kappa$, N is closed under F, and

$$\sup(N \cap \kappa^+) \geq \sup_{\alpha<\kappa} \beta_\alpha = \gamma. \tag{1}$$

On the other hand, for each $\alpha < \kappa$, the sequence $\langle N_\ell \mid \ell \leq \alpha\rangle$ is definable in $\mathfrak{A}$ from the parameter $\langle \beta_\ell \mid \ell \leq \alpha\rangle$, which is an element of W by the above remarks. Hence every proper initial segment of $\vec{N}$ is an element of W and, in particular, we know that

$$\sup(N_\alpha \cap \kappa^+) < \gamma = W \cap \kappa^+$$

for all $\alpha < \kappa$.

It follows that $\sup(N \cap \kappa^+) \leq \gamma$. Combined with (1), this shows that $\sup(N \cap \kappa^+) = \gamma$. Finally, since $\alpha \subseteq N_{\alpha+1}$ for all $\alpha < \kappa$, it follows that $\kappa \subseteq N$ and hence we know that $N \cap \kappa^+$ is transitive. This allows us to conclude that

$$N \cap \kappa^+ = \sup(N \cap \kappa^+) = \gamma,$$

completing the proof of the lemma. □

The following lemma is one way to salvage stationary set preservation in the non-GCH context.

Lemma 2.12 *Let $\mathbb{P}$ be a IA_κ-proper poset and let $T \subseteq S^{\kappa^+}_\kappa$ be stationary with $T \in I[\kappa^+]$. Then forcing with $\mathbb{P}$ preserves the stationarity of T.*

Proof Set $\tau = \kappa^+$. Let $\dot{C}$ be a $\mathbb{P}$-name for a club in τ, let $p \in \mathbb{P}$ and let θ be a sufficiently large regular cardinal. Using Lemma 2.11, we find $\gamma \in T$ and $W \prec (\mathrm{H}(\theta), \in, p, \dot{C}, \mathbb{P})$ with $W \in \mathrm{IA}_\kappa$ and $W \cap \tau = \gamma$. By our assumptions, there is a $(W, \mathbb{P})$-master condition q below p in $\mathbb{P}$. Let G be $\mathbb{P}$-generic over V with $q \in G$. Then $W[G] \cap \tau = W \cap \tau = \gamma$. Moreover, since $\dot{C} \in W$, we now know that $\dot{C}^G \cap W[G]$ is unbounded in γ and hence $\gamma \in \dot{C}^G \cap T$.

These computations show that, in the ground model V, we have

$$q \Vdash_{\mathbb{P}} \text{“}\dot{C} \cap \check{T} \neq \emptyset\text{”}$$

for densely-many conditions q in $\mathbb{P}$. □

3 Generalizing a lemma of Jensen

The notion of IA_κ-properness, defined in Sect. 2, is a non-GCH analogue of the notion of κ-*properness* introduced in [13, Definition 3.4] . This notion will be important to proving that tails of the iteration described in Sect. 4 do not add cofinal branches to a certain tree, and that argument will closely follow the corresponding arguments of [13].

However, IA_κ-properness (in the case $\kappa = \omega_1$) is **not** sufficient for ensuring the preservation of forcing axioms that we need for the proofs of our main results. There are examples of IA_{ω_1}-proper forcings that destroy, for example, the Proper Forcing Axiom.[6] On the other hand, $<\omega_2$-directed closed posets preserve all standard forcing axioms (see [17] and [18]). In this section, we generalize a result of Jensen, yielding a property that is forcing equivalent to $<\omega_2$-directed closure, but often easier to verify than $<\omega_2$-directed closure.

In [16], Jensen defines a poset $\mathbb{P}$ to be **complete** if for every sufficiently large θ, there are club-many $W \in \wp_{\omega_1}(\mathrm{H}(\theta))$ such that every $(W, \mathbb{P})$-generic filter has a lower bound in $\mathbb{P}$.[7] He then proves:

Lemma 3.1 *(Jensen) The following statements are equivalent for every poset $\mathbb{P}$:*

(1) *The poset $\mathbb{P}$ is complete.*
(2) *The poset $\mathbb{P}$ is forcing equivalent to a σ-closed poset.*

We will generalize a version of this lemma to larger cardinals, and, in fact, characterize directed closure (see Lemma 3.6 below).[8] However there are a few technicalities

[6] E.g. if $2^{\omega_1} = \omega_2$ then there is a natural IA_{ω_1}-proper poset that forces the *Approachability Property* to hold at ω_2, hence destroys the Proper Forcing Axiom. This poset is just the natural poset to shoot an ω_1-club through the set M described in Lemma 2.10.

[7] Jensen's notes say this is equivalent to a definition of Shelah in [25, Chapter 10].

[8] Note that σ-closure is equivalent to σ-directed closure, so the distinction is only important at larger cardinals.

to address. Note that for any $W \in \wp_{\omega_1}(\mathrm{H}(\theta))$, the fact that W is countable ensures that there always exist $(W, \mathbb{P})$-generic filters, regardless of what $\mathbb{P}$ is. In particular, the phrase "… *every* $(W, \mathbb{P})$*-generic filter* …" is never vacuous, if W is countable. Of course, for uncountable W, it may happen that (depending on the poset $\mathbb{P}$) there do not exist any $(W, \mathbb{P})$-generic filters at all; e.g. if $W \prec \mathrm{H}(\theta)$ and $\omega_1 \subseteq W$, then there does not exist a $(W, \mathrm{Col}(\omega, \omega_1))$-generic filter.

Definition 3.2 Given a regular uncountable cardinal τ, a poset $\mathbb{P}$ is **τ-Jensen-complete** if the following statements hold for all sufficiently large regular cardinals θ:

(1) For every $p \in \mathbb{P}$, there are stationarily-many $W \in \wp^*_\tau(\mathrm{H}(\theta))$ with the property that there exists a $(W, \mathbb{P})$-generic filter including p.
(2) For all but non-stationarily many $W \in \wp^*_\tau(\mathrm{H}(\theta))$, every $(W, \mathbb{P})$-generic filter has a lower bound in $\mathbb{P}$.[9]

Remark 3.3 Note that clause (1) of Definition 3.2 always holds true for $\tau = \omega_1$, and is hence redundant in that case. In particular, for $\tau = \omega_1$, Definition 3.2 is equivalent to Jensen's definition of completeness.

Remark 3.4 In combination, the clauses (1) and (2) of Definition 3.2 imply that the poset $\mathbb{P}$ is *totally proper* on a stationary subset of $\wp^*_\tau(\mathrm{H}(\theta))$; i.e. that there are stationarily-many $W \in \wp^*_\tau(\mathrm{H}(\theta))$ such that every condition in $\mathbb{P} \cap W$ can be extended to a $(W, \mathbb{P})$-total master condition in the sense of Definition 2.1. This conclusion, however, is strictly weaker than τ-Jensen-completeness, since (for example with $\tau = \omega_1$) shooting a club through a bistationary subset of ω_1 has the latter property but is not $<\omega_1$-closed. In the case $\tau = \omega_2$, if $2^{\omega_1} = \omega_2$, then shooting an ω_1-club through the set M described in Lemma 2.10 is IA_{ω_1}-totally proper, but forces the approachability property to hold at ω_2. In particular, this forcing destroys the Proper Forcing Axiom, and PFA is preserved by ω_2-Jensen-complete forcings (by [17] and Lemma 3.6 below).

Lemma 3.5 *If κ is an infinite cardinal and $\mathbb{P}$ is a $<\kappa$-closed poset, then clause (1) of Definition 3.2 holds for $\tau = \kappa^+$ and $\mathbb{P}$.*

Proof This follows immediately from Lemmas 2.6 and 2.7. □

Next, we state our generalization of Jensen's lemma. Its Corollary 2 will be used in the proof of Theorem 4.2 below.

Lemma 3.6 *Given a poset $\mathbb{P}$ and a successor cardinal τ, the following statements are equivalent:*

(1) *The poset $\mathbb{P}$ is forcing equivalent to a $<\tau$-directed closed poset.*
(2) *The poset $\mathbb{P}$ is forcing equivalent to a τ-Jensen-complete poset.*

Proof First, assume that $\mathbb{P}$ is $<\tau$-directed closed. Then, in particular, $\mathbb{P}$ is $<\tau$-closed, and hence Lemma 3.5 ensures that clause (1) of Definition 3.2 holds for $\mathbb{P}$. But then the directed closure of $\mathbb{P}$ ensures that *any* $(W, \mathbb{P})$-generic filter for *any* $W \in \wp^*_\tau(\mathrm{H}(\theta))$

[9] Note that this clause is allowed to be **vacuously** true for some elements W of $\wp^*_\tau(\mathrm{H}(\theta))$, even for stationarily-many such sets W.

has a lower bound in $\mathbb{P}$, and hence clause (2) of Definition 3.2 holds for $\mathbb{P}$ as well. This shows that $\mathbb{P}$ is τ-Jensen-complete.

Now, suppose that $\mathbb{P}$ is τ-Jensen-complete. Let $F : [\mathrm{H}(\theta)]^{<\omega} \longrightarrow \mathrm{H}(\theta)$ generate a club witnessing clause (2) of Definition 3.2, i.e. whenever $W \in \wp^*_\tau(\mathrm{H}(\theta))$ and W is closed under F, then any $(W, \mathbb{P})$-generic filter has a lower bound. We may assume that F also codes a well-ordering $\lhd$ of $\mathrm{H}(\theta)$, i.e. if W is closed under F, then $W \prec (\mathrm{H}(\theta), \in, \lhd))$ holds.

Define a poset $\mathbb{Q}$, whose conditions are pairs (M, g) satisfying the following statements:

- $M \in \wp^*_\tau(\mathrm{H}(\theta))$.
- M is closed under F.
- $g \subseteq M \cap \mathbb{P}$ is an $(M, \mathbb{P})$-generic filter.

and whose ordering is given by:

$$(N, h) \leq_{\mathbb{Q}} (M, g) \iff N \supseteq M \ \wedge \ h \cap M = g.$$

Note that $\leq_{\mathbb{Q}}$ is transitive and clause (1) of Definition 3.2 ensures that $\mathbb{Q}$ is nonempty.

Claim 3.7 *$\mathbb{Q}$ is $<\tau$-directed closed.*

Proof of Claim 3.7 Let $\{(M_i, g_i) \mid i \in I\}$ be a directed set of conditions in $\mathbb{Q}$ with $|I| < \tau$. Set $M = \bigcup_{i \in I} M_i$ and $g = \bigcup_{i \in I} g_i$. We will show that (M, g) is a condition in $\mathbb{Q}$ below all (M_i, g_i).

The regularity of τ ensures that $M \in \wp^*_\tau(\mathrm{H}(\theta))$, and M is closed under F, because each M_i is closed under F and the collection $\langle M_i \mid i \in I\rangle$ is $\subseteq$-directed. In addition, we have $g \subseteq M \cap \mathbb{P}$ and g clearly has the property that $D \cap g \neq \emptyset$ for every dense $D \in M$, because each such D lies in some M_i and $g_i \subseteq g$ is an $(M_i, \mathbb{P})$-generic filter. Finally, the fact that g is a filter follows easily from the fact that the given collection is directed and each g_i is a filter. This shows that (M, g) is a condition in $\mathbb{Q}$.

Now, fix $i \in I$. Then $M \supseteq M_i$, $g \cap M_i$ is a filter on $M_i \cap \mathbb{P}$, and $g \cap M_i \supseteq g_i$. But since g_i is $(M_i, \mathbb{P})$-generic, we know that g_i is a $\subseteq$-maximal filter on $M_i \cap \mathbb{P}$. In particular, we can conclude that $g \cap M_i = g_i$. This computation shows that $(M, g) \leq_{\mathbb{Q}} (M_i, g_i)$. □

Claim 3.8 *The poset $\mathbb{Q}$ is forcing equivalent to $\mathbb{P}$.*

Proof of Claim 3.8 It is easy to see that the boolean completions of τ-Jensen-complete posets are themselves τ-Jensen-complete. Therefore, we may assume that $\mathbb{P}$ is a complete boolean algebra. For each condition (M, g) in $\mathbb{Q}$, let $p_{M,g}$ be the $\mathbb{P}$-greatest lower bound of g. This conditions exists and is non-zero, because M is closed under F, g is $(M, \mathbb{P})$-generic, $\mathbb{P}$ is a complete boolean algebra, and because of clause (2) of Definition 3.2. In the following, we will show that the map

$$e : \mathbb{Q} \longrightarrow \mathbb{P}; \ (M, g) \longmapsto p_{M,g}$$

is a dense embedding, which will finish the proof of the claim.

First, we show that e is order-preserving. Suppose that $(N, h) \leq_{\mathbb{Q}} (M, g)$. Then $N \supseteq M$ and $g = h \cap M$. Since $g \subseteq h$ and $p_{N,h}$ is a lower bound of h, it follows that $p_{N,h}$ is also a lower bound of g. But $p_{M,g}$ is the greatest lower bound of g, and hence

$$e(N, h) = p_{N,h} \leq_{\mathbb{P}} p_{M,g} = e(M, g).$$

Next, we show that e preserves incompatibility. Suppose (M_0, g_0) and (M_1, g_1) are conditions in $\mathbb{Q}$ with the property that there is a condition p in $\mathbb{P}$ that extends both $e(M_0, g_0)$ and $e(M_1, g_1)$. By clause (1) of Definition 3.2, there we can find $W \in \wp^*_\tau(\mathrm{H}(\theta))$ such that $p, g_0, g_1, M_0, M_1 \in W$, W is closed under F, and there exists a $(W, \mathbb{P})$-generic filter G with $p \in G$. Since $W \cap \tau$ is transitive and $|M_i| < \tau$ for all $i < 2$, it follows that $M_0 \cup M_1 \subseteq W$. Furthermore, since p is below both p_{M_0,g_0} and p_{M_1,g_1} and $M_i \cap \mathbb{P} \subseteq W \cap \mathbb{P}$ for all $i < 2$, the fact that g_0 and g_1 are maximal filters in $M_0 \cap \mathbb{P}$ and $M_1 \cap \mathbb{P}$, respectively, implies that $G \cap M_0 = g_0$ and $G \cap M_1 = g_1$. Hence (W, G) is a condition in $\mathbb{Q}$ that lies below both (M_0, g_0) and (M_1, g_1).

Finally, we show that the range of e is dense in $\mathbb{P}$. Fix a condition p in $\mathbb{P}$. By clause (1) of Definition 3.2, there is a $W \in \wp^*_\tau(\mathrm{H}(\theta))$ such that W is closed under F, and there exists a $(W, \mathbb{P})$-generic filter G with $p \in G$. Then (W, G) is a condition in $\mathbb{Q}$, and $e(W, g) = p_{W,G}$ is stronger than p. □

This completes the proof of the lemma. □

Corollary 1 *Given a successor cardinal τ, all τ-Jensen-complete posets are $<\tau$-distributive.* □

Remark 3.9 Another common way to verify the $<\tau$-distributivity of a given poset $\mathbb{P}$ is the following weaker version of τ-Jensen completeness: if for every $p \in \mathbb{P}$, there are stationarily-many $W \in \wp^*_\tau(\mathrm{H}(\theta))$ such that there is a $(W, \mathbb{P})$-*total* master condition below p (see Definition 2.1), then $\mathbb{P}$ is $<\tau$-distributive. Note that this weaker version would not suffice for our purposes, however, because we seem to need $<\tau$-directed closure (or a close approximation of it) to prove Theorem 4.2.

Corollary 2 *Let κ be an infinite regular cardinal and set $\tau = \kappa^+$. If $\mathbb{P}$ is $<\kappa$-closed poset with the property that for all but non-stationarily many $W \in \wp^*_\tau(\mathrm{H}(\theta))$, every $(W, \mathbb{P})$-generic filter has a lower bound in $\mathbb{P}$, then the poset $\mathbb{P}$ is forcing equivalent to a $<\tau$-directed closed poset.*

Proof By Lemma 3.5, the $<\kappa$-closure of $\mathbb{P}$ ensures that clause (1) of Definition 3.2 holds. Since clause (2) of Definition 3.2 holds by assumption, this implies $\mathbb{P}$ is τ-Jensen-complete and Lemma 3.6 yields the desired conclusion. □

In particular, if a poset $\mathbb{P}$ satisfies the assumptions of the above corollary for $\kappa = \omega_1$, then forcing with $\mathbb{P}$ preserves all standard forcing axioms.

4 The main technical result

In this section, we will prove the main technical result of our paper. It directly extends the main results of [13] and [23]. In the next section, we will use it to prove the two theorems stated in the introduction.

Definition 4.1 If κ be an infinite regular cardinal and let $S \subseteq S^{\kappa^+}_\kappa$. Then we let $T(S)$ denote the tree that consists of all $t \in {}^{<\kappa^+}\kappa^+$ such that $\mathrm{dom}(t)$ is a successor ordinal, $\mathrm{ran}(t) \subseteq S$, t is strictly increasing, and t is continuous at all points of cofinality κ in its domain and is ordered by end-extension.

Note that, in the situation of the above definition, the tree $T(S)$ has height κ^+ and contains a cofinal branch if and only if the set S contains a κ-club.

We are now ready to state the aspired result.

Theorem 4.2 *Given an infinite regular cardinal κ, there is a partial order $\mathbb{P}$ with the following properties:*

(1) $\mathbb{P}$ is κ^+-Jensen complete.[10]

(2) $\mathbb{P}$ satisfies the $(2^\kappa)^+$-chain condition.

(3) If G is $\mathbb{P}$-generic over V, then, in $V[G]$, there is a subtree T of ${}^{<\kappa^+}\kappa^+$ of height κ^+ without cofinal branches such that the following statements hold:

(a) If S is bistationary in $S^{\kappa^+}_\kappa$ and $S^{\kappa^+}_\kappa \setminus S$ contains a stationary set in $I[\kappa^+]$, then there is an order-preserving function from $T(S)$ to T.

(b) Assume that $\kappa^{<\kappa} \leq \kappa^+$ holds in V. If $M \in \mathrm{V}$ is a maximum element of $I[\kappa^+] \cap \wp(S^{\kappa^+}_\kappa)$ mod NS in V,[11] *then the following statements hold in $\mathrm{V}[G]$:*

(i) M is a maximum element of $I[\kappa^+] \cap \wp(S^{\kappa^+}_\kappa)$ mod NS.

(ii) If S is a bistationary in $S^{\kappa^+}_\kappa$ and $M \setminus S$ is stationary, then there is an order-preserving function from $T(S)$ to T.

Note that the above theorem directly generalizes the main result of [13]: if $\kappa^{<\kappa} = \kappa$ holds, then part (4) of Lemma 2.10 shows that $S^{\kappa^+}_\kappa$ is an element of $I[\kappa^+]$, and hence $S^{\kappa^+}_\kappa$ is a maximum element of $I[\kappa^+] \cap \wp(S^{\kappa^+}_\kappa)$ mod NS.

Now, if G is $\mathbb{P}$-generic over V and $T \in \mathrm{V}[G]$ is the tree given by the theorem, then there is an order-preserving function from the tree $T(S)$ to T in $\mathrm{V}[G]$ for every bistationary subset S of $S^{\kappa^+}_\kappa$ in $\mathrm{V}[G]$.

In particular, this shows that T is a *κ-canary tree* (see [13, Definition 3.1]) in $\mathrm{V}[G]$, i.e. if S is a stationary subset of $S^{\kappa^+}_\kappa$ and $\mathbb{P}$ is a $<\kappa^+$-distributive poset that forces $\kappa^+ \setminus S$ to contain a club subset, then forcing with $\mathbb{P}$ adds a cofinal branch to T.

For the remainder of this section, fix an infinite regular cardinal κ. Until further notice, we do **not** make any cardinal arithmetic assumptions.

In the following, we closely follow the arguments on pages 1684–1692 of Hyttinen-Rautila in [13], which assumed GCH (in particular, their arguments heavily rely on the assumption $\kappa^{<\kappa} = \kappa$). We also follow their notation as closely as possible.

Definition 4.3 ([13]) We let $\mathbb{Q}_0$ denote the poset that consists of functions f such that $\mathrm{dom}(f) \subseteq S^{\kappa^+}_\kappa$, $|\mathrm{dom}(f)| \leq \kappa$, $f(\delta)$ is a function from δ to δ for all $\delta \in \mathrm{dom}(f)$, and whenever $\delta < \eta$ are both in the domain of f, then $f(\delta) \not\subseteq f(\eta)$,[12] and whose ordering is given by reversed inclusion.

[10] In particular, Lemma 3.6 shows that the poset $\mathbb{P}$ is forcing equivalent to a $<\kappa^+$-directed closed poset.

[11] Such a subset M exists by Lemma 2.10.

[12] Since $f(\delta) : \delta \longrightarrow \delta$ and $f(\eta) : \eta \longrightarrow \eta$, this just means that $f(\delta)$ and $f(\eta)$ disagree at some $\xi < \delta$.

Proposition 4.4 *The poset $\mathbb{Q}_0$ is $<\kappa^+$-directed closed.*

Proof This statement follows directly from the fact that the union f of a coherent collection of conditions in $\mathbb{Q}_0$ still has the required property that $f(\eta) \not\subseteq f(\beta)$ for all $\eta < \beta$ in the domain of f, and, if the union is of size less than κ^+, then the domain of f has size less than κ^+ too. □

Definition 4.5 ([13]) If G_0 is $\mathbb{Q}_0$-generic over V, then, in V$[G_0]$, we define the following subtree of ${}^{<\kappa^+}\kappa^+$:

$$\mathcal{T}(G_0) \;=\; \{h \in {}^{<\kappa^+}\kappa^+ \mid \forall \delta \in S^{\kappa^+}_\kappa \;\; h \restriction \delta \neq (\bigcup G_0)(\delta)\}.$$

In the following, we let $\dot{\mathcal{T}}(\dot{G}_0)$ denote the canonical $\mathbb{Q}_0$-name for $\mathcal{T}(G_0)$.

Remark 4.6 In the situation of the above definition, the tree $\mathcal{T}(G_0)$ has height κ^+, since for any $\beta \in S^{\kappa^+}_\kappa$, the function f with domain β and constant value $\beta + 1$ has the property that for all $\delta \in S^{\kappa^+}_\kappa$ with $\delta \leq \mathrm{dom}(f)$, the restriction $f \restriction \delta$ is not a function from δ to δ and hence cannot be the same as the function $(\bigcup G_0)(\delta)$.

Lemma 4.7 *If G_0 is $\mathbb{Q}_0$-generic over* V*, then the tree $\mathcal{T}(G_0)$ has no cofinal branches in* V$[G_0]$.

Proof Work in V and assume, towards a contradiction, that a condition f in $\mathbb{Q}_0$ forces a $\mathbb{Q}_0$-name $\dot{b}$ to be a cofinal branch through $\dot{\mathcal{T}}(\dot{G}_0)$. Using Proposition 4.4, easy closure arguments allow us to find $\lambda \in S^{\kappa^+}_\kappa$, a function $h : \lambda \longrightarrow \lambda$ and a condition g below f in $\mathbb{Q}_0$ such that $\lambda = \sup(\mathrm{dom}(g))$ and g forces h to be the restriction of $\dot{b}$ to λ. By the definition of $\mathcal{T}(G_0)$, this implies that $h \restriction \delta \neq g(\delta)$ holds for all $\delta \in \mathrm{dom}(g)$ and we can conclude that $g \cup \{(\lambda, h)\}$ is a condition in $\mathbb{Q}_0$ below g. But this condition forces that h is not contained in $\dot{\mathcal{T}}(\dot{G}_0)$, a contradiction. □

The following poset, again taken from [13], adds an order preserving function from $T(S)$ to $\mathcal{T}(G_0)$. The role of clause 3(a)i is to add such a function with initial segments. However, the role of clauses 3(a)ii through 3(a)vi is not obvious; roughly, with the exception of clause 3(a)iv, these properties allow us to verify κ^+-Jensen completeness by ensuring the existence of lower bounds for any generic filter over any κ-sized elementary submodel. The role of clause 3(a)iv is to ensure that no cofinal branch is added to $\mathcal{T}(G_0)$.

Definition 4.8 ([13]) Let G_0 be $\mathbb{Q}_0$-generic over V and work in an outer model of V$[G_0]$[13] with the same bounded subsets of κ^+ as V. Let S be a subset of $S^{\kappa^+}_\kappa$.

(1) An element t of $\mathcal{T}(G_0)$ is an ***S*-node** if $t[\delta] \not\subseteq \delta$ holds for all $\delta \in S^{\kappa^+}_\kappa \setminus S$.
(2) Given a partial function $h : T(S) \xrightarrow{part} \mathcal{T}(G_0)$, we define

$$o(h) \;=\; \sup\{\mathrm{dom}(t) \mid t \in \mathrm{ran}(h)\}.$$

[13] I.e. a model of ZFC in which V$[G]$ is a transitive class.

(3) We let $\mathbb{P}(S, G_0)$ denote the unique poset defined by the following clauses:

(a) A condition in $\mathbb{P}(S, G_0)$ is a pair (h, X) satisfying the following statements:

(i) h is an order-preserving partial function of cardinality at most κ from the tree $T(S)$ to the tree $\mathcal{T}(G_0)$ with the property that $\mathrm{dom}(h)$ is closed under initial segments.

(ii) X is a partial function from κ^+ to $\bigcup\{{}^{\beta+1}\kappa^+ \mid \beta < \kappa^+\}$ of cardinality at most κ such that

$$o(h) \cap S^{\kappa^+}_\kappa \subseteq \mathrm{dom}(X)$$

and

$$X(\alpha) \subseteq (\bigcup G_0)(\alpha)$$

for all $\alpha \in \mathrm{dom}(X) \cap S^{\kappa^+}_\kappa$.

(iii) $\mathrm{dom}(h(t)) = \sup(\mathrm{ran}(t))$ for all $t \in \mathrm{dom}(h)$.

(iv) $h(t)$ is an S-node for all $t \in \mathrm{dom}(h)$.

(v) $X(\alpha) \nsubseteq h(t)$ for all $t \in \mathrm{dom}(h)$ and $\alpha \in \mathrm{dom}(X)$.

(vi) If $\langle t_\zeta \mid \zeta < \kappa\rangle$ is a strictly increasing sequence of elements of $\mathrm{dom}(h)$, then $\bigcup_{\zeta<\kappa} h(t_\zeta) \in \mathcal{T}(G_0)$.

(b) A condition (h, X) is stronger than a condition (k, Y) if and only if $k \subseteq h$ and $Y \subseteq X$ hold.

(4) The **order** of a condition $p = (h, X)$ in $\mathbb{P}(S, G_0)$, denoted by $o(p)$, is defined to be the ordinal

$$\max\{\bigcup \mathrm{dom}(X),\ \bigcup\{\mathrm{dom}(h(t)) \mid t \in \mathrm{dom}(h)\}\}.$$

Remark 4.9 In the above definition, the requirements on $X(\alpha)$ differ depending on whether or not $\mathrm{cof}(\alpha) = \kappa$. If $\alpha \in S^{\kappa^+}_\kappa \cap \mathrm{dom}(X)$, then $X(\alpha)$ is a proper initial segment of the function $(\bigcup G_0)(\alpha)$.[14] In combination with requirement (3(a)v) in the above definition, this shows that for all $\alpha \in \mathrm{dom}(X) \cap S^{\kappa^+}_\kappa$, there is an ordinal $\eta_\alpha < \alpha$ such that no node in the range of h can extend $(\bigcup G_0)(\alpha) \restriction \eta_\alpha$. On the other hand, if $\alpha \in \mathrm{dom}(X)$ with $\mathrm{cof}(\alpha) < \kappa$, then the only requirement on $X(\alpha)$ is that nothing in the range of h is allowed to extend $X(\alpha)$.

As pointed out near the bottom of page 1684 of [13], the poset $\mathbb{P}(S, G_0)$ is $<\kappa$-closed. Requirements (3(a)iii) and (3(a)vi) of Definition 4.8 are mainly needed for the proof of Lemma 4.10 below.

Lemma 4.10 *Let G_0 be $\mathbb{Q}_0$-generic over* V*, let* V_1 *be an outer model of* $\mathrm{V}[G_0]$ *with the same bounded subsets of κ^+ as* V*, and let K be $\mathbb{P}(S, G_0)^{\mathrm{V}_1}$-generic over* V_1 *for some set S that is bistationary in $S^{\kappa^+}_\kappa$ in* V_1*. In* $\mathrm{V}_1[K]$*, define*

$$h_K = \bigcup\{h \mid (h, X) \in K\}$$

[14] The domain of $X(\alpha)$ is required to be a successor ordinal, so $X(\alpha)$ cannot be the entire function $(\bigcup G_0)(\alpha)$.

and

$$X_K = \bigcup\{X \mid (g, X) \in K\}.$$

Let $\delta = (\kappa^+)^{\mathrm{V}}$. *Then the following statements hold:*

(1) h_K *is a total, order-preserving function from* $(T(S))^{\mathrm{V}_1}$ *to* $(\mathcal{T}(G_0))^{\mathrm{V}[G_0]}$ *whose range consists entirely of S-nodes.*
(2) X_K *is a total function from* δ *to* $({}^{<\delta}\delta)^{\mathrm{V}}$.
(3) No element of $\mathrm{ran}(h_K)$ *extends an element of* $\mathrm{ran}(X_K)$.
(4) Suppose M *is an outer model of* $\mathrm{V}_1[K]$ *and* $\vec{y} = \langle y_\zeta \mid \zeta < \lambda\rangle$ *is an increasing sequence of nodes in* $\mathrm{ran}(h_K)$ *in* M. *If we set* $f_{\vec{y}} = \bigcup_{\zeta<\lambda} y_\zeta$, *then*

$$f_{\vec{y}} \restriction \gamma \neq (\bigcup G_0)(\gamma)$$

for all $\gamma \in (S^\delta_\kappa)^{\mathrm{V}}$.

Proof Statement (1) is [13, Claim 3.11].[15]

Statement (2) is an easy density argument, and statement (3) follows directly from requirement 3(a)v of Definition 4.8.

For Statement (4), let M and $\vec{y} \in M$ be as stated, and suppose for a contradiction there exists $\gamma < \delta$ with $\mathrm{cof}(\gamma)^{\mathrm{V}} = \kappa$ and $f_{\vec{y}} \restriction \gamma = (\bigcup G_0)(\gamma)$.

Work in M. The statements (1), (2), and (3) are obviously upward absolute from $\mathrm{V}_1[K]$ to M. By Statement (2), we know that γ is in the domain of X_K, and, by applying Remark 4.9 to some condition in K whose second coordinate has γ in its domain, we can find $\rho_\gamma < \gamma$ with $X_K(\gamma) = (\bigcup G_0)(\gamma) \restriction \rho_\gamma$. Note that, by the definition of $\mathbb{Q}_0$, we know that $(\bigcup G_0)(\gamma)$ is a total function on γ, and therefore our assumption implies that $\gamma \leq \mathrm{dom}(f_{\vec{y}})$. Then $\rho_\gamma \in \mathrm{dom}(f_{\vec{y}})$, and there is some $\zeta_* < \lambda$ with $\rho_\gamma \in \mathrm{dom}(y_{\zeta_*})$. In particular, we know that

$$y_{\zeta_*} \restriction \rho_\gamma = f_{\vec{y}} \restriction \rho_\gamma = (\bigcup G_0)(\gamma) \restriction \rho_\gamma = X_K(\gamma).$$

But this implies that $y_{\zeta_*} \in \mathrm{ran}(h_K)$ extends $X_K(\gamma) \in \mathrm{ran}(X_K)$, contradicting Statement (3). □

We now describe the iteration that will witness the poset from Theorem 4.2. This is a slight variant of the iteration described at the bottom of page 1684 of [13].

The main differences are:

- The length of our iteration is at least 2^{2^κ}. This is to allow for the case when, in the ground model, the cardinal 2^{κ^+} is very large.
- More significantly, at a given stage α of our iteration, when considering the set $\dot{S}_\alpha$ given to us by the bookkeeping device, we only force with the poset $\mathbb{P}(\dot{S}_\alpha, G_0)$ if the statement

$$\text{“} S^{\kappa^+}_\kappa \setminus \dot{S}_\alpha\ contains\ a\ stationary\ set\ in\ I[\kappa^+]\text{”} \tag{2}$$

[15] This was the only place in the argument where requirement (3(a)iii) from Definition 4.8 played a role. This requirement was used to fix the error from [23]. It ensures that the function h_K is total.

holds in the corresponding generic extension of the ground model. This will ensure (via an application of Lemma 2.12) that the complements of the $\dot{S}_\alpha$'s remain stationary throughout the iteration, which in turn will be the key to showing that the tree $\mathcal{T}(G_0)$ has no cofinal branch in the final model.

Remark 4.11 In the GCH setting of [13], requiring (2) to hold is no restriction at all, since in that scenario, this statement holds for **every** set bistationary in $S^{\kappa^+}_\kappa$.

But, if $\kappa^{<\kappa} > \kappa$ holds, then the requirement (2) seems to be needed in order to prove that the iteration adds no cofinal branch to the tree $\mathcal{T}(G_0)$.

In the following, we fix a cardinal ε satisfying $\varepsilon^{2^\kappa} = \varepsilon$.

Let $\mathcal{C}$ denote the set of all partial functions from ε to $\mathrm{H}(\kappa^+)$ of cardinality at most κ.

Then our assumptions on ε imply that $|\mathcal{C}| = \varepsilon$.

Next, let $\mathcal{N}$ denote the set of all partial functions from $S^{\kappa^+}_\kappa \times 2^\kappa$ to $\mathcal{C}$. Again, our assumptions imply that $|\mathcal{N}| = \varepsilon$ and we can pick an ε-to-one surjection $b : \varepsilon \longrightarrow \mathcal{N}$.

Definition 4.12 We define

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\xi \mid \alpha \leq \varepsilon,\ \xi < \varepsilon \rangle$$

to be a $<\kappa^+$-support iteration satisfying the following clauses:

(1) $\dot{\mathbb{Q}}_0$ is chosen in a canonical way that ensures that the map

$$i : \mathbb{Q}_0 \longrightarrow \mathbb{P}_1;\ q \longmapsto \langle \check{q} \rangle$$

is an isomorphism.

(2) Assume that $\alpha \in [1, \varepsilon)$ has the property that the poset $\mathbb{P}_\alpha$ is $<\kappa^+$-distributive and there exists a sequence $\langle q_{\gamma,\xi} \mid (\gamma, \xi) \in \mathrm{dom}(b(\alpha)) \rangle$ of conditions in $\mathbb{P}_\alpha$ such that the following statements hold:

- If $(\gamma, \xi) \in \mathrm{dom}(b(\alpha))$, then $\mathrm{sprt}(q_{\gamma,\xi}) = \mathrm{dom}(b(\alpha)(\gamma, \xi))$.
- If $(\gamma, \xi) \in \mathrm{dom}(b(\alpha))$, $\ell \in \mathrm{sprt}(q_{\gamma,\xi})$ and $b(\alpha)(\gamma, \xi)(\ell) = x$, then $q_{\gamma,\xi}(\ell) = \check{x}$.

If we define

$$\dot{B}_\alpha = \{(\check{\gamma}, q_{\gamma,\xi}) \mid (\gamma, \xi) \in \mathrm{dom}(b(\alpha))\},$$

then there exists a $\mathbb{P}_\alpha$-name $\dot{S}_\alpha$ for a subset of $S^{\kappa^+}_\kappa$ such that the following statements hold in $\mathrm{V}[G]$ whenever G is $\mathbb{P}_\alpha$-generic over V and G_0 is the induced $\mathbb{Q}_0$-generic filter over V:

(a) $\dot{\mathbb{Q}}^G_\alpha = \mathbb{P}(\dot{S}^G_\alpha, G_0)^{\mathrm{V}[G]}$.
(b) If the set $S^{\kappa^+}_\kappa \setminus \dot{B}^G_\alpha$ contains a stationary set in $I[\kappa^+]$, then $\dot{S}^G_\alpha = \dot{B}^G_\alpha$.
(c) If the set $S^{\kappa^+}_\kappa \setminus \dot{B}^G_\alpha$ does not contain a stationary set in $I[\kappa^+]$, then $\dot{S}^G_\alpha = \emptyset$.

(3) Assume that $\alpha \in [1, \varepsilon)$ has the property that the poset $\mathbb{P}_\alpha$ is $<\kappa^+$-distributive and there exists no sequence of conditions in $\mathbb{P}_\alpha$ with the properties listed in (2). Then $\dot{S}_\alpha = \emptyset$ and $\dot{\mathbb{Q}}_\alpha^G = \mathbb{P}(\dot{S}_\alpha^G, G_0)^{V[G]}$ whenever G is $\mathbb{P}_\alpha$-generic over V and G_0 is the induced $\mathbb{Q}_0$-generic filter over V.
(4) If $\alpha \in [1, \varepsilon)$ has the property that the poset $\mathbb{P}_\alpha$ is not $<\kappa^+$-distributive, then $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for a trivial poset.

Remark 4.13 We include the cases (2c) and (3) in the above definition of the name $\dot{S}_\alpha$ to simplify notation later on. Note that, since we have $T(\emptyset) = \emptyset$, conditions in $\mathbb{P}(\emptyset, G_0)$ always have trivial first coordinate, and the poset $\mathbb{P}(\emptyset, G_0)$ is forcing equivalent to $\mathrm{Add}(\kappa^+, 1)$.

Throughout the rest of this paper, $\mathbb{P}$ refers to the poset $\mathbb{P}_\varepsilon$. Moreover, in order to conform to the notation from [13], if G is $\mathbb{P}_\alpha$-generic over V for some $\alpha \leq \varepsilon$, then we let G_0 denote the induced $\mathbb{Q}_0$-generic filter over V.

Definition 4.14 A condition p in $\mathbb{P}$ is called **flat** if there exists a sequence $\langle x_\alpha \mid i \in \mathrm{sprt}(p)\rangle$ with the property that $x_\alpha \in \mathrm{H}(\kappa^+)$ and

$$p \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} \text{“} p(\alpha) = \check{x}_\alpha \text{”}$$

hold for all $\alpha \in \mathrm{sprt}(p)$ with the property that $\mathbb{P}_\alpha$ is $<\kappa^+$-distributive.

Just as in [13], the flat conditions turn out to be dense in $\mathbb{P}$, as we will see in Lemma 4.20 below.

Although the density of the flat conditions is not needed to prove the κ^+-Jensen-completeness of $\mathbb{P}$ in Lemma 4.20, it will be crucial for the proofs of the following statements:

- The “tails” of the above iteration are proper with respect to IA_κ (see Lemma 4.27), which in turn is important for the proof that the tree $\mathcal{T}(G_0)$ has no cofinal branches in $\mathbb{P}$-generic extensions of V (see Lemma 4.28).
- If $2^\kappa = \kappa^+$, then $\mathbb{P}$ satisfies the κ^{++}-chain condition.

The function $p(f_0, g)$ defined in Definition 4.15 below is a natural attempt to form a flat condition out of a $(W, \mathbb{P})$-generic filter for some elementary substructure W of size κ.

Definition 4.15 Suppose $W \prec (\mathrm{H}(\theta), \in, \mathbb{P})$ with $|W| = \kappa \subseteq W$, and $g \subseteq \mathbb{P} \cap W$ is a $(W, \mathbb{P})$-generic filter in V.

(1) Set $g_0 = \{q \in \mathbb{Q}_0 \mid \exists p \in g \ p(0) = \check{q}\}$.[16]
(2) Given $0 < \alpha \leq \varepsilon$, we define

$$g_\alpha = \{p \upharpoonright \alpha \mid p \in g\}.$$

[16] This notation is chosen to keep in line with the notational convention from [13] of identifying $\mathbb{P}_1$ with $\mathbb{Q}_0$ and referring to the induced $\mathbb{Q}_0$-generic filter by G_0. Notice that $\bigcup g_0$ is easily a condition in $\mathbb{Q}_0$.

(3) Given $0 < \alpha < \varepsilon$ with the property that the poset $\mathbb{P}_\alpha$ is $<\kappa^+$-distributive, let $\dot{c}_{g,\alpha}$, $\dot{h}_{g,\alpha}$ and $\dot{X}_{g,\alpha}$ denote the canonical $\mathbb{P}_\alpha$-names with the property that,[17] whenever G is $\mathbb{P}_\alpha$-generic over V, then

- $\dot{c}^G_{g,\alpha} = \{p(\alpha)^G \mid p \in g\}$,
- $\dot{h}^G_{g,\alpha} = \bigcup\{h \mid \exists X\ (h, X) \in \dot{c}^G_{g,\alpha}\}$, and
- $\dot{X}^G_{g,\alpha} = \bigcup\{X \mid \exists h\ (h, X) \in \dot{c}^G_{g,\alpha}\}$.

(4) If f_0 is a condition in $\mathbb{Q}_0$ extending $\bigcup g_0$, then we define a function $p(f_0, g)$ with domain $W \cap \varepsilon$ by setting

$$p(f_0, g) = \langle f_0\rangle^\frown\langle \mathsf{pair}_{\mathbb{P}_\alpha}(\dot{h}_{g,\alpha}, \dot{X}_{g,\alpha}) \mid \alpha\rangle \in W \cap [1, \varepsilon)\ with\ \mathbb{P}_\alpha{<}\kappa^+ - distributive,$$

where $\mathsf{pair}_{\mathbb{P}_\alpha}(\dot{h}_{g,\alpha}, \dot{X}_{g,\alpha})$ denotes the canonical $\mathbb{P}_\alpha$-name for the ordered pair of $\dot{h}_{g,\alpha}$ and $\dot{X}_{g,\alpha}$.

Note that, in the last part of the above definition, the function $p(f_0, g)$ may or may not be a condition in $\mathbb{P}$. The following lemma shows how we can ensure that $p(f_0, g)$ is a flat condition below every condition in g.

Lemma 4.16 *Suppose W, g, and f_0 are as in Definition 4.15. Then one of the following statements holds:*

(1) $p(f_0, g)$ is a flat condition in $\mathbb{P}$ that extends every element of g.
(2) There is an $\alpha \in W \cap [1, \varepsilon)$ such that the following statements hold:

(a) $p(f_0, g) \restriction \alpha$ is a condition in $\mathbb{P}_\alpha$ that extends every element of g_α.
(b) If $\tau \in W$ is a $\mathbb{P}_\alpha$-name for a function from κ to the ordinals, then

$$p(f_0, g) \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“}\tau \in \check{W}\text{”}.$$

(c) There is a condition q in $\mathbb{P}_\alpha$ below $p(f_0, g) \restriction \alpha$ with the property that the following statements hold true in $\mathrm{V}[G]$, whenever G is $\mathbb{P}_\alpha$-generic over V with $q \in G$:
(i) $\mathrm{cof}(W \cap \kappa^+)^{\mathrm{V}} = \kappa$. In particular, we have $W \cap \kappa^+ \in \mathrm{dom}(\bigcup G_0)$.
(ii) Every proper initial segment of $(\bigcup G_0)(W \cap \kappa^+)$ is an element of W, and is an $\dot{S}^G_\alpha$-node.
(iii) No proper initial segment of $(\bigcup G_0)(W \cap \kappa^+)$ is an element of $\mathrm{ran}(\dot{X}^G_{g,\alpha})$.

Proof Set $g_\varepsilon = g$ and, given $\beta \leq \varepsilon$, let Φ_β denote the statement asserting that $p(f_0, g) \restriction \beta$ is a flat condition in $\mathbb{P}_\beta$ that lies below every element of g_β.

Suppose that Φ_ε fails, i.e. that part (1) of the disjunctive conclusion of the statement of the lemma fails. Let $\beta \leq \varepsilon$ be the least ordinal such that Φ_β fails.

[17] The idea behind this definition is that $\dot{c}_{g,\alpha}$ names the evaluation of the α*-th component* of g after forcing with $\mathbb{P}_\alpha$ over V, and $\dot{h}_{g,\alpha}$ and $\dot{X}_{g,\alpha}$ name the unions of the left and right components (respectively) of that α-th component of g.

Claim 4.17 *β is a successor ordinal and an element of W.*

Proof of Claim 4.17 First, we have $\beta > 0$, because the 0th component of $p(f_0, g)$ is f_0, which is assumed to be a condition stronger than $\bigcup g_0$, and g_0 is a $(W, \mathbb{Q}_0)$-generic filter.

Now, assume, towards a contradiction, that β is a limit ordinal.

Since Φ_α holds for all $\alpha < \beta$ and since the support of $p(f_0, g)$ is contained in the κ-sized set $W \cap \varepsilon$, it follows easily that $p(f_0, g) \restriction \beta$ is a condition and is below every element of g_β.

Furthermore, for each $\alpha \in W \cap \beta$, let

$$\vec{x}_\alpha = \langle x^\alpha_\xi \mid \xi \in W \cap \alpha \rangle$$

witness flatness of $p(f_0, g) \restriction \alpha$. Then for all $\alpha_0 < \alpha_1 < \beta$, it follows easily that $x^{\alpha_0}_\xi = x^{\alpha_1}_\xi$ holds for all $\xi \in W \cap \alpha_0$. So the $\vec{x}_\alpha$'s are coherent, and their union witnesses flatness of $p(f_0, g) \restriction \beta$. This shows that Φ_β holds, a contradiction.

The above computations yield an ordinal α with $\beta = \alpha + 1$. Assume, towards a contradiction, that $\alpha \notin W$.

Note that, if $r \in g$, then $r \in W$ and, since $\mathrm{sprt}(r)$ is a κ-sized element of W and $\kappa \subseteq W$, it follows that $\mathrm{sprt}(r) \subseteq W$.

Since α is not an element of W, this shows that $g_\alpha = g_\beta$ and $p(f_0, g) \restriction \beta = p\big(f_0, g) \restriction \alpha$. But, since Φ_α holds, this immediately implies that Φ_β holds too, a contradiction. □

The above claim shows that there is an $\alpha \in W \cap [1, \varepsilon)$ with $\beta = \alpha + 1$. We claim that this α witnesses part (2) of the conclusion of the lemma holds true. By the minimality of β, we know that (2a) holds and $\mathbb{P}_\alpha$ is $<\kappa^+$-distributive. Moreover, since $p(f_0, g) \restriction \alpha$ is a condition that extends the $(W, \mathbb{P}_\alpha)$-generic filter g_α, part (2b) holds by Lemma 2.2.

Claim 4.18 *There is an $x \in \mathrm{H}(\kappa^+)$ with*

$$p(f_0, g) \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“} p(f_0, g)(\alpha) = \check{x} \text{”}. \tag{3}$$

Furthermore, if G is $\mathbb{P}_\alpha$-generic over V with $p(f_0, g) \restriction \alpha \in G$, then the following statements hold in $\mathrm{V}[G]$:

*(1) The pair $(\dot{h}^G_{g,\alpha}, \dot{X}^G_{g,\alpha})$ satisfies all requirements to be a condition in the poset $\mathbb{P}(\dot{S}^G_\alpha, G_0)$, **with the possible exception of** requirement (3(a)vi) of Definition 4.8. In particular, the following statements hold:*

(a) Every element of $\mathrm{ran}(\dot{h}^G_{g,\alpha})$ is an $\dot{S}^G_\alpha$-node (i.e. requirement (3(a)iv) of Definition 4.8 is satisfied).

(b) No element of $\mathrm{ran}(\dot{h}^G_{g,\alpha})$ extends an element of $\mathrm{ran}(\dot{X}^G_{g,\alpha})$ (i.e. requirement (3(a)v) of Definition 4.8 is satisfied).

*(2) **If** the pair $(\dot{h}^G_{g,\alpha}, \dot{X}^G_{g,\alpha})$ is **not** a condition in $\mathbb{P}(\dot{S}^G_\alpha, G_0)$, then the following statements hold:*

(a) $\mathrm{cof}(W \cap \kappa^+)^{\mathrm{V}} = \kappa$. *In particular, we have* $W \cap \kappa^+ \in \mathrm{dom}(\bigcup G_0)$*).*
(b) Every proper initial segment of $(\bigcup G_0)(W \cap \kappa^+)$ *is an element of* W*, and is an* $\dot{S}^G_\alpha$*-node.*
(c) No proper initial segment of $(\bigcup G_0)(W \cap \kappa^+)$ *is an element of* $\mathrm{ran}(\dot{X}^G_{g,\alpha})$*.*

Proof of Claim 4.18 In order to make use of Lemma 4.10, it will be more convenient to work with the transitive collapse of W instead of W itself. Let H_W be the transitive collapse of W, and let $\sigma : H_W \longrightarrow W \prec \mathrm{H}(\theta)$ be the inverse of the collapsing map.

In the following, if b is a set, then we will write

$$\bar{b} = \sigma^{-1}[b] \subseteq H_W.$$

Note that $\bar{b} = \sigma^{-1}(b)$ holds for all $b \in W$ and we will frequently use this abbreviation in the following arguments.

Since g is a $(W, \mathbb{P})$-generic filter, we know that $\bar{g}$ is a $\bar{\mathbb{P}}$-generic over H_W, and, in particular, it follows that $\bar{g}_\gamma$ is $\bar{\mathbb{P}}_\gamma$-generic over H_W for all $\gamma \in W \cap \varepsilon$.

If we define

$$k = \{q(\bar{\alpha})^{\bar{g}_\alpha} \mid q \in \bar{g}_{\alpha+1}\} \subseteq H_W[\bar{g}_\alpha],$$

then k is $\sigma^{-1}(\dot{\mathbb{Q}}_\alpha)^{\bar{g}_\alpha}$-generic over $H_W[\bar{g}_\alpha]$ with $H_W[\bar{g}_{\alpha+1}] = H_W[\bar{g}_\alpha][k]$.

Set $\delta = (\kappa^+)^{H_W}$, and note that $\delta = \mathrm{crit}\,(\sigma)$, because $|W| = \kappa \subseteq W$.

Since Φ_α holds, we know that $p(f_0, g) \restriction \alpha$ is a condition in $\mathbb{P}_\alpha$ that extends every element of the $(W, \mathbb{P}_\alpha)$-generic filter g_α. In particular, $p(f_0, g) \restriction \alpha$ it is a total master condition for W. By Lemma 2.2, every $\mathbb{P}_\alpha$-name for a function from κ to the ordinals in W is forced by $p(f_0, g) \restriction \alpha$ to be evaluated to an element of W. It follows that

$$H_W \cap {}^\kappa\mathrm{Ord} = H_W[\bar{g}_\alpha] \cap {}^\kappa\mathrm{Ord}. \tag{4}$$

Let h_k be the union of the left coordinates of k and let X_k be the union of the right coordinates of k.

By (4), we can apply Lemma 4.10 with the ground model H_W and the outer model $H_W[\bar{g}_\alpha]$ and derive the following statements:

- The function

$$h_k : T(\sigma^{-1}(\dot{S}_\alpha)^{\bar{g}_\alpha})^{H_W[\bar{g}_\alpha]} \longrightarrow \sigma^{-1}(\dot{\mathcal{T}}(\dot{G}_0))^{\bar{g}_0}$$

  is order preserving and every element of its range is a $\sigma^{-1}(\dot{S}_\alpha)^{\bar{g}_\alpha}$-node in $H_W[\bar{g}_\alpha]$.
- No element of $\mathrm{ran}(h_k)$ extends an element of the range of the function

$$X_k : \delta \longrightarrow ({}^{<\delta}\delta)^{H_W}.$$

Set $x = (h_k, X_k)$. In the following, we will show that $p(f_0, g) \restriction \alpha$ and x satisfy (3), and $p(f_0, g) \restriction \alpha$ forces the other statements of the claim to hold true.

Let G be $\mathbb{P}_\alpha$-generic over V with $p(f_0, g) \restriction \alpha \in G$. Work in $V[G]$. Since $\alpha \in W$ and $p(f_0, g) \restriction \alpha$ is a W-total master condition that, in particular, extends every element of g_α, it follows that $W[G] \cap \mathrm{V} = W$, $G \cap W = g_\alpha$, and σ can be canonically lifted to an elementary embedding

$$\hat{\sigma} : H_W[\bar{g}_\alpha] \longrightarrow W[G] \prec \mathrm{H}(\theta)[G].$$

satisfying

$$\hat{\sigma}(\tau^{\bar{g}_\alpha}) = \sigma(\tau)^G$$

for every $\bar{\mathbb{P}}_\alpha$-name τ in H_W. Note that $\mathrm{ran}(\hat{\sigma}) = W[G]$ holds.

Now, pick an element q of $\bar{g}_{\alpha+1} \subseteq H_W$. By the definition of $\hat{\sigma}$, we then have

$$\hat{\sigma}(q(\bar{\alpha})^{\bar{g}_\alpha}) = \sigma(q(\bar{\alpha}))^G = (\sigma(q)(\sigma(\bar{\alpha})))^G = (\sigma(q)(\alpha))^G.$$

Since $q \in \bar{g}_{\alpha+1}$ implies that $\sigma(q) \in g_{\alpha+1}$, we can now conclude that $\hat{\sigma}(q(\bar{\alpha})^{\bar{g}_\alpha}) \in \dot{c}^G_{g,\alpha}$. These computations show that $\hat{\sigma}[k] \subseteq \dot{c}^G_{g,\alpha}$.

Next, fix a condition p in g. Since $g \cup \{\alpha\} \subseteq W = \mathrm{ran}(\sigma)$, we then have $\bar{p} \restriction (\bar{\alpha}+1) \in \bar{g}_{\alpha+1}$ and $\bar{p}(\bar{\alpha})^{\bar{g}_\alpha} \in k$.

By the definition of $\hat{\sigma}$, we now know that

$$p(\alpha)^G = \sigma(\bar{p}(\bar{\alpha}))^G = \hat{\sigma}(\bar{p}(\bar{\alpha})^{\bar{g}_\alpha}) \in \hat{\sigma}[k].$$

This shows that $\dot{c}^G_{g,\alpha} \subseteq \hat{\sigma}[k]$ and, together with the above computations, we can conclude that

$$\hat{\sigma}[k] = \dot{c}^G_{g,\alpha}. \tag{5}$$

By (5) and the elementarity of $\hat{\sigma}$, we also have $\hat{\sigma}[h_k] = \dot{h}^G_{g,\alpha}$ and $\hat{\sigma}[X_k] = \dot{X}^G_{g,\alpha}$.

Note that conditions in $\sigma^{-1}(\dot{\mathbb{Q}}_\alpha)^{\bar{g}_\alpha}$ are elements of $\mathrm{H}(\delta)^{H_W[\bar{g}_\alpha]}$ and hence k is a subset of $\mathrm{H}(\delta)^{H_W[\bar{g}_\alpha]}$.

Since the critical point of $\hat{\sigma}$ is δ, it follows that k, h_k and X_k are all pointwise fixed by $\hat{\sigma}$. In particular, we have

$$x = (h_k, X_k) = (\dot{h}^G_{g,\alpha}, \dot{X}^G_{g,\alpha}). \tag{6}$$

Since $p(f_0, g)(\alpha)$ is, by definition, the $\mathbb{P}_\alpha$-name $\mathsf{pair}_{\mathbb{P}_\alpha}(\dot{h}_{g,\alpha}, \dot{X}_{g,\alpha})$, this completes the proof of (3).

Part (1) of the claim follows by the properties of (h_k, X_k) over $H_W[\bar{g}_\alpha]$ discussed above, together with the equality (6), elementarity of $\hat{\sigma}$, and the fact that $\hat{\sigma}$ fixes bounded subsets of δ that lie in $H_W[\bar{g}_\alpha]$. For example, to verify requirement (4.8(a)iv) of Definition 4.8, suppose t is in the range of h_k. Then t is in the range of the left coordinate of some condition in $k \subseteq \hat{\sigma}^{-1}(\mathbb{P}(\dot{S}^G_\alpha, G_0)^{\mathrm{V}[G]})$, and hence t is an $\hat{\sigma}^{-1}(\dot{S}^G_\alpha)$-node in $H_W[\bar{g}_\alpha]$. By elementarity of $\hat{\sigma}$ and the fact that $\hat{\sigma}$ fixes y, it follows that t is an $\dot{S}^G_\alpha$-node in $\mathrm{V}[G]$.

The remaining requirements of Definition 4.8, except for requirement (3(a)vi), are easily verified for the pair displayed in (6) in a similar manner.

Now, we prove that $p(f_0, g) \upharpoonright \alpha$ forces the statements in part (2) of the claim. Recall G is an arbitrary $\mathbb{P}_\alpha$-generic filter over V with $p(f_0, g) \upharpoonright \alpha \in G$. Assume that the ordered pair (6) is **not** a condition in $\mathbb{P}(\dot{S}^G_\alpha, G_0)$ in $V[G_i]$. In the following, we show that the statements (2a), (2b), and (2c) of the claim hold true in V$[G]$. By part (1) of the claim, it must be requirement (3(a)vi) of Definition 4.8 that fails. In particular, there is an increasing sequence $\vec{y} = \langle y_\zeta \mid \zeta < \kappa \rangle$ of nodes in the range of h_k in V$[G]$ such that the function $f_{\vec{y}} = \bigcup_{\zeta<\kappa} y_\zeta$ is not an element of $\mathcal{T}(G_0)$. Since the elements of the range of h_k are κ-sized objects in $H_W[\bar{g}_\alpha]$ and hence in H_W by (4), this implies that the domain of $f_{\vec{y}}$ is at most $\delta = (\kappa^+)^{H_W}$. In summary, there is some ordinal $\gamma \in (S^{\kappa^+}_\kappa)^{\mathrm{V}}$ such that

$$V[G] \models \text{“}\gamma \leq \delta = (\kappa^+)^{H_W} \textit{ and } f_{\vec{y}} \upharpoonright \gamma = (\bigcup G_0)(\gamma)\text{”}. \tag{7}$$

By part (4) of Lemma 4.10 – viewing H_W as the ground model V, $H_W[\bar{g}_\alpha]$ as the outer model V_1, k as the generic filter K, and $V[G]$ as the outer model M from the statement of that lemma – the ordinal γ cannot be strictly smaller than δ, and hence we can conclude that $\gamma = \delta$. But this implies that $\mathrm{cof}(W \cap \kappa^+)^{\mathrm{V}} = \mathrm{cof}(\delta)^{\mathrm{V}} = \mathrm{cof}(\gamma)^{\mathrm{V}} = \kappa$, proving part (2a) of the claim.

In summary, we have shown that $W \cap \kappa^+ = \mathrm{dom}(f_{\vec{y}})$ and

$$f_{\vec{y}} = (\bigcup G_0)(W \cap \kappa^+). \tag{8}$$

Since $f_{\vec{y}}$ is a union of functions in the range of h_k, and (4) together with the fact that the critical point of σ is δ imply that $h_k \subseteq W$, every proper initial segment of $f_{\vec{y}}$ is an element of W. Furthermore, every proper initial segment of $f_{\vec{y}}$ is extended by some $y \in \mathrm{ran}(h_k)$, which is an $\dot{S}^G_\alpha$-node in V$[G]$ by part (1) of the claim. Hence, every proper initial segment of $f_{\vec{y}}$ is an $\dot{S}^G_\alpha$-node in V$[G]$, and is an element of W. Together with (8), this proves part (2b) of the claim. Finally, to prove part (2c) of the claim, suppose $\gamma \in \mathrm{dom}(X_k)$, define $\eta = \mathrm{dom}(X_k(\gamma))$ and assume, towards a contradiction, that $f_{\vec{y}} \upharpoonright \eta = \bar{X}(\alpha)$. Note that $\eta < \delta$, because $X_k \subseteq H_W[\bar{g}_\alpha]$. In particular, we have $y_\zeta \upharpoonright \eta = X_k(\alpha)$ for some $\zeta < \kappa$. But this contradicts the fact from part (1) that nothing in the range of h_k extends any function from the range of X_k. □

It remains to prove part (2c) of the lemma, which will essentially follow from part (2) of Claim 4.18, though we first must dispense with a technicality. Recall that $\Phi_{\alpha+1}$ fails, but Φ_α holds. Next, we observe that the failure of $\Phi_{\alpha+1}$ is due to the function $p(f_0, g) \upharpoonright (\alpha+1)$ not being a condition at all (rather than being a condition but failing to extend $g_{\alpha+1}$, or being a condition but failing to be flat):

Claim 4.19 *Some condition in $\mathbb{P}_\alpha$ below $p(f_0, g) \upharpoonright \alpha$ forces that*

$$p(f_0, g)(\alpha) = \mathsf{pair}_{\mathbb{P}_\alpha}(\dot{h}_{g,\alpha}, \dot{X}_{g,\alpha}) \tag{9}$$

*is **not** a condition in $\dot{\mathbb{Q}}_\alpha$.*

Proof of Claim 4.19 Assume not, i.e. suppose that $p(f_0, g) \restriction \alpha$ forces that the pair in (9) to be a condition in $\dot{\mathbb{Q}}_\alpha$. Since the components of the pair in (9) are given by the union of the left and right coordinates of $\dot{c}_{g,\alpha}$, the fact that the ordering of $\dot{\mathbb{Q}}_\alpha$ is given reversed inclusion now implies that the condition $p(f_0, g) \restriction \alpha$ forces $p(f_0, g)(\alpha)$ to be stronger than every condition in $\dot{c}_{g,\alpha}$. Since the validity of Φ_α implies that $p(f_0, g) \restriction \alpha$ is stronger than every condition in g_α, it follows that

$$p(f_0, g) \restriction (\alpha + 1) \;=\; (p(f_0, g) \restriction \alpha)^\frown(\alpha, \mathsf{pair}_{\mathbb{P}_\alpha}(\dot{h}_{g,\alpha}, \dot{X}_{g,\alpha}))$$

is stronger than every condition in $g_{\alpha+1}$.

Furthermore, by Claim 4.18, there is an $x_\alpha \in \mathrm{V}$ such that

$$p(f_0, g) \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“}\check{x}_\alpha = p(f_0, g)(\alpha)\text{”}.$$

Since Φ_α holds, we know that $p(f_0, g) \restriction \alpha$ is flat. Let $\langle x_\ell \mid \ell \in W \cap \alpha\rangle$ witness its flatness. Then the sequence $\langle x_\ell \mid \ell \in W \cap (\alpha + 1)\rangle$ witnesses the flatness of $p(f_0, g) \restriction (\alpha + 1)$. In summary, $p(f_0, g) \restriction (\alpha + 1)$ is a flat condition below every member of $g_{\alpha+1}$, contradicting the fact that $\Phi_{\alpha+1}$ fails. □

Part (2c) of the lemma now follows immediately from Claim 4.19, and part (2) of Claim 4.18. □

The above results now allow us to prove the following key lemma.

Lemma 4.20 *The poset $\mathbb{P}$ is κ^+-Jensen complete, and the flat conditions are dense in $\mathbb{P}$.*

Before we prove this result, we make a couple of remarks.

Remark 4.21 In [13, Claim 3.13], a weaker version of Lemma 4.20, stating that $\mathbb{P}$ is *κ-proper*, was proven. This concept was defined in [13, Definition 3.4] and only makes sense under the assumption that $\kappa^{<\kappa} = \kappa$. It, in particular, implies that the given poset is $<\kappa^+$-distributive.

In the non-GCH setting, in particular, when we do not assume $\kappa^{<\kappa} = \kappa$, perhaps the most natural analogue of κ-properness is our notion of IA_κ-proper (Defininition 2.8). In fact, changing just a few words in the proof of [13, Claim 3.13] would suffice to prove that (even without GCH) the poset $\mathbb{P}$ is proper for IA_κ and is $<\kappa^+$-distributive. However, that conclusion does not suffice for applications in our main theorems, since, for example, IA_{ω_1}-properness, even together with $<\omega_2$-strategic closure, does not guarantee preservation of the *Proper Forcing Axiom*.[18]

We seem to need the stronger property of κ^+-Jensen completeness (i.e. $<\kappa^+$-directed closure), which we prove in Lemma 4.20. This requires some reorganization and strengthening of the argument of [13, Claim 3.13], but the main ideas of the proof of Lemma 4.20 are very similar to the proof of [13, Claim 3.13].

[18] E.g. if $2^{\omega_1} = \omega_2$, one can code $\mathrm{IA}_{\omega_1} \cap \wp_{\omega_2}(\mathrm{H}(\omega_2))$ as a stationary subset S of S^2_1. Then, shooting an ω_1-club through S with initial segments is IA_{ω_1}-totally proper, but kills PFA.

Remark 4.22 Iterations using $<\kappa^+$-support, where each iterand is $<\kappa^+$-directed closed, are themselves $<\kappa^+$-directed closed. However, this fact seems to *not* be applicable to the iteration $\mathbb{P}_\varepsilon$ constructed in Definition 4.12. That is, it is not clear if, say, the first non-trivial poset used of the form $\mathbb{P}(S, G_0)$ is equivalent to a $<\kappa^+$-directed closed from the point of view of $V[G_0]$ (and we suspect it is not, in general). The key to Lemma 4.20 (and to the analogous, but weaker [13, Claim 3.13]) is the flexibility in having G_0 not be decided yet.

Proof of Lemma 4.20 First, we check κ^+-Jensen completeness. Since each iterand is $<\kappa$-closed and the iteration uses κ-sized supports, the entire iteration is $<\kappa$-closed. So by Corollary 2, to show that $\mathbb{P}$ is κ^+-Jensen complete, it suffices to show that whenever

- $W \prec (\mathrm{H}(\theta), \in, \mathbb{P})$ with $|W| = \kappa$ and $W \cap \kappa^+ \in \kappa^+$, and
- $g \subseteq W \cap \mathbb{P}$ is a $(W, \mathbb{P})$-generic filter,

then g has a lower bound in $\mathbb{P}$. So fix such a filter g for the remainder of the proof. Given $\alpha \in W \cap [0, \varepsilon]$, define g_α as in Definition 4.15. Set $\delta = W \cap \kappa^+$. We consider two cases:

Case 1: $\mathrm{cof}(\delta) < \kappa$. Set $f_0 = \bigcup g_0$, and consider the function $p(f_0, g)$ from Definition 4.15. We claim that $p(f_0, g)$ is flat condition and lies below all members of g.

Assume not. Then, by Lemma 4.16, there is an $\alpha \in W \cap [1, \varepsilon)$ such that $p(f_0, g) \upharpoonright \alpha$ is a condition below all conditions in g_α, and there is some $q_\alpha \leq_{\mathbb{P}_\alpha} p(f_0, g) \upharpoonright \alpha$ in $\mathbb{P}_\alpha$ that forces all the statements in part (2c) of Lemma 4.16 to hold. In particular, by part (2(c)i), we know that $\mathrm{cof}(\delta) = \mathrm{cof}(W \cap \kappa^+) = \kappa$, contrary to our case.

Case 2: $\mathrm{cof}(\delta) = \kappa$. Since $|W| = \kappa < \delta$, we can fix $t : \delta \longrightarrow \delta$ such that $t \upharpoonright \kappa$ is not an element of W. Define

$$f_0 \;=\; (\bigcup g_0) \cup \{(\delta, t)\}.$$

Given $\gamma \in \delta \cap S^{\kappa^+}_\kappa$, we have $t \upharpoonright \gamma \notin W$ and $(\bigcup g_0)(\gamma) \in W$. In particular, we have $t \upharpoonright \gamma \neq (\bigcup g_0)(\gamma)$ for all $\gamma \in \delta \cap S^{\kappa^+}_\kappa$. Since $\mathrm{cof}(\delta) = \kappa$, this shows that f_0 is a condition in $\mathbb{Q}_0$ that extends $\bigcup g_0$.

Let $p(f_0, g)$ be the function defined in Definition 4.15. We claim that $p(f_0, g)$ is a flat condition that lies below every element of g. Assume not. Then, by Lemma 4.16, there is an $\alpha \in W \cap [1, \varepsilon)$ such that $p(f_0, g) \upharpoonright \alpha$ is a condition in $\mathbb{P}_\alpha$, and that, by part (2(c)ii) of that lemma, there is some condition $q \leq_{\mathbb{P}_\alpha} p(f_0, g) \upharpoonright \alpha$ such that q forces that every proper initial segment of $(\bigcup \dot{G}_0)(\check{\delta})$ is an element of W. But the 0th component of $p(f_0, g) \upharpoonright \alpha$, and hence of q, extends the function f_0, and therefore

$$q(0) \Vdash_{\mathbb{P}_0} \text{“}(\bigcup \dot{G}_0)(\check{\delta}) = \check{t}\text{”}.$$

In particular, every proper initial segment of t is an element of W, contrary to our choice of t.

This completes the proof of κ^+-Jensen completeness. To see that the flat conditions are dense in $\mathbb{P}$, let p_0 be any condition in $\mathbb{P}$. Fix $W \prec (\mathrm{H}(\theta), \in, \mathbb{P}, p_0)$ such that $|W| =$

$\kappa \subseteq W$ and $W \in \mathrm{IA}_\kappa$. By Lemma 2.7 and the $<\kappa$-closure of $\mathbb{P}$, there exists a $(W, \mathbb{P})$-generic filter g such that $p_0 \in g$. Note that $W \in \mathrm{IA}_\kappa$ implies that $\mathrm{cof}(W \cap \kappa^+) = \kappa$. This shows that we can repeat the argument from the above Case 2, define f_0 as above and conclude that the function $p(f_0, g)$ is a flat condition that is below every member of g and therefore also below p_0. □

Lemma 4.20 and Corollary 2 now immediately yield the following corollary:

Corollary 3 *The poset $\mathbb{P}$ is forcing equivalent to a $<\kappa^+$-directed closed forcing. In particular, it adds no new sets of size κ, and, in the case $\kappa = \omega_1$, it preserves all standard forcing axioms, such as* MM^{++}. □

Remember that the order $o(p)$ of a condition in a poset of the form $\mathbb{P}(S, G_0)$ was defined in part (4) of Definition 4.8.

Lemma 4.23 *If p is a flat condition in $\mathbb{P}$, then there exists $\beta < \kappa^+$ with the property that*

$$p \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“} o(p(\alpha)) \leq \check{\beta} \text{”}$$

holds for all $1 \leq \alpha \in \mathrm{sprt}(p)$.

Proof Let $\langle x_\alpha \mid \alpha \in \mathrm{sprt}(p)\rangle$ be a sequence witnessing the flatness of p. For each $\alpha \in \mathrm{sprt}(p)$, pick $\beta_\alpha < \kappa^+$ such that β_α is not in the transitive closure of x_α. Since $|\mathrm{sprt}(p)| \leq \kappa$, we know that

$$\beta = \sup\{\beta_\alpha \mid \alpha \in \mathrm{sprt}(p)\} < \kappa^+$$

has the desired properties. □

Our next task is to prove that tails of the iteration behave nicely. But first we need *tail* versions of Definition 4.15 and Lemma 4.16. Note that in Definition 4.24 below, since $\alpha_0 \geq 1$, the entire filter G_0 has already been determined. So unlike Definition 4.15, the candidate for a condition below g will not involve any f_0.

Definition 4.24 Suppose that $\alpha_0 \in [1, \varepsilon)$ and G_{α_0} is $\mathbb{P}_{\alpha_0}$-generic over V. Working in $\mathrm{V}[G_{\alpha_0}]$, suppose that $W \prec (\mathrm{H}(\theta)[G_{\alpha_0}], \in, \mathbb{P}/G_{\alpha_0})$ with $|W| = \kappa \subseteq W$, and $g \subseteq W \cap \mathbb{P}/G_{\alpha_0}$ is a $(W, \mathbb{P}/G_{\alpha_0})$-generic filter. For each $\alpha \in W \cap [\alpha_0, \varepsilon)$, define $\mathbb{P}_\alpha/G_{\alpha_0}$-names $\dot{c}_{g,\alpha}$, $\dot{h}_{g,\alpha}$ and $\dot{X}_{g,\alpha}$ analogously to Definition 4.15, and define a function $p(g)$ with domain $W \cap [\alpha_0, \varepsilon)$ by setting

$$p(g) = \langle \mathsf{pair}_{\mathbb{P}_\alpha/G_{\alpha_0}}(\dot{h}_{g,\alpha}, \dot{X}_{g,\alpha}) \mid i \in W \cap [\alpha_0, \varepsilon)\rangle.$$

We now also have a *tail* variant of Lemma 4.16:

Lemma 4.25 *Suppose* $\alpha_0 \in [1, \varepsilon)$ *and* G_{α_0} *is* $\mathbb{P}_{\alpha_0}$*-generic over* V. *Work in* $V[G_{i_0}]$ *and suppose W and g are as in Definition 4.24. Given* $\alpha \in [\alpha_0, \varepsilon]$*, set*

$$g_\alpha = \{p \restriction \alpha \mid p \in g\}.$$

Then one of the following statements holds:

(1) $p(g)$ is a flat condition in $\mathbb{P}/G_{\alpha_0}$ that extends every element of g.
(2) There is an $\alpha \in W \cap [\alpha_0, \varepsilon)$ such that the following statements hold:

(a) $p(g) \upharpoonright \alpha$ is a flat condition in $\mathbb{P}_\alpha/G_{\alpha_0}$ that is stronger than every element of g_α.
(b) If $\tau \in W$ is a $\mathbb{P}_\alpha/G_{\alpha_0}$-name for a function from κ to the ordinals, then

$$p(g) \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha/G_{\alpha_0}} \text{“}\tau \in \check{W}\text{”}.$$

(c) There is a condition q in $\mathbb{P}_\alpha/G_{\alpha_0}$ below $p(g) \upharpoonright \alpha$ with the property that the following statements hold true in $\mathrm{V}[G_{\alpha_0}, G]$, whenever G is $\mathbb{P}_\alpha/G_{\alpha_0}$-generic over $\mathrm{V}[G_{\alpha_0}]$ with $q \in G$:
(i) $\mathrm{cof}(W \cap \kappa^+) = \kappa$.
(ii) Every proper initial segment of $(\bigcup G_0)(W \cap \kappa^+)$ is an element of W, and is an $\dot{S}^G_\alpha$-node.
(iii) No proper initial segment of $(\bigcup G_0)(W \cap \kappa^+)$ is an element of $\mathrm{ran}(\dot{X}^G_{g,\alpha})$.

Proof The proof is almost identical to the proof of Lemma 4.16, except we work in $\mathrm{V}[G_{\alpha_0}]$ instead of V. We leave the details to the reader. □

Lemma 4.26 *If $\alpha < \varepsilon$ and G is $\mathbb{P}_\alpha$-generic over* V*, then the tail of the iteration $\mathbb{P}/G$ is $<\kappa$-closed in* $\mathrm{V}[G]$.

Proof Let $\langle q_\xi \mid \xi < \mu \rangle$ be a descending sequence with $\mu < \kappa$ in $\mathbb{P}/G$ in $\mathrm{V}[G]$. Since $\mathbb{P}/G \subseteq \mathbb{P}$ and Lemma 4.20 shows that $\mathbb{P}_\alpha$ is $<\kappa$-closed in V, this sequence is an element of V. Let $\dot{G}$ denote the canonical $\mathbb{P}_\alpha$-name for the generic filter in V. Fix a condition p in G such that

$$p \Vdash_{\mathbb{P}_\alpha} \text{“}Every\ condition\ in\ \dot{G}\ is\ compatible\ with\ \check{q}_\xi\ in\ \check{\mathbb{P}}\text{”}$$

holds in V for all $\xi < \mu$. Work in V. Given $\xi < \mu$, a standard density argument now shows that

$$p \Vdash_{\mathbb{P}_\alpha} \text{“}\check{q}_\xi \upharpoonright \check{\alpha} \in \dot{G}\text{”}$$

and the separativity of $\mathbb{P}_\alpha$ allows us to conclude that $p \leq_{\mathbb{P}_\alpha} q \upharpoonright \alpha$ holds.

Fix a condition r below p in $\mathbb{P}_\alpha$ and set

$$r_\xi = r^\frown(q_\xi \upharpoonright [\alpha, \varepsilon))$$

for all $\xi < \mu$. Then $\langle r_\xi \mid \xi < \mu \rangle$ is a descending sequence of conditions in $\mathbb{P}$, and, by the proof of Lemma 4.20, this sequence has a lower bound r_μ in $\mathbb{P}$. Then $r_\mu \upharpoonright \alpha \leq_{\mathbb{P}_\alpha} r$ and $r_\mu \leq_{\mathbb{P}} q_\xi$ for all $\xi < \mu$.

By genericity, we can now find a condition q in $\mathbb{P}$ with the property that $q \upharpoonright \alpha \in G$ and $q \leq_{\mathbb{P}} q_\xi$ for all $\xi < \mu$. But then we can conclude that q is a condition in $\mathbb{P}/G$ in $\mathrm{V}[G]$ with $q \leq_{\mathbb{P}/G} q_\xi$ for all $\xi < \mu$. □

The proof of the following lemma is similar to the proof of [13, Claim 3.14], but there are some subtle differences since we do not assume that $\kappa^{<\kappa} = \kappa$. Roughly, we replace their use of *κ-properness* with IA_κ-properness (Definition 2.8) and verify that the argument still goes through.

Lemma 4.27 *If $\alpha_0 < \varepsilon$ and G is $\mathbb{P}_{\alpha_0}$-generic over* V*, then the tail of the iteration $\mathbb{P}/G$ is IA_κ-totally proper in* $\mathrm{V}[G]$.

Proof For $\alpha_0 = 0$, the statement of the lemma follows immediately from Lemmas 2.7 and 4.20. Therefore, we from now on assume that $1 \leq \alpha_0 < \varepsilon$.

Let G_{α_0} be $\mathbb{P}_{\alpha_0}$-generic over V and work in $\mathrm{V}[G_{\alpha_0}]$. Let θ be a sufficiently large regular cardinal, let $\lhd$ be a well-ordering of $\mathrm{H}(\theta) = \mathrm{H}(\theta)^{\mathrm{V}}[G_{\alpha_0}]$, let

$$W \prec (\mathrm{H}(\theta), \in, \alpha_0, \mathbb{P}/G_{\alpha_0}, \lhd)$$

with $W \in \mathrm{IA}_\kappa$, and let p_0 be a condition in $W \cap (\mathbb{P}/G_{\alpha_0})$. In the following, we will find a $(W, \mathbb{P}/G_{\alpha_0})$-total master condition below p_0. Define

$$t \ = \ (\bigcup G_0)(W \cap \kappa^+),$$

which is well-defined because $W \in \mathrm{IA}_\kappa$ implies that $\mathrm{cof}(W \cap \kappa^+) = \kappa$.

Case 1: *There exists $\zeta \in W \cap \kappa^+$ with $t \restriction \zeta \notin W$*. Since Lemma 4.26 implies that $\mathbb{P}/G_{\alpha_0}$ is $<\kappa$-closed, we can apply Lemma 2.7 to find a $(W, \mathbb{P}/G_{\alpha_0})$-generic filter g that includes p_0. Let $p(g)$ be the function defined in Definition 4.24 and assume, towards a contradiction, that $p(g)$ is not a condition in $\mathbb{P}/G_{\alpha_0}$ that is stronger than every element of g. Then, by part (2(c)ii) of Lemma 4.25, there is an $\alpha \in W \cap [\alpha_0, \varepsilon)$ and some condition in $\mathbb{P}_\alpha/G_{\alpha_0}$ below $p(g) \restriction \alpha$ forcing that every proper initial segment of $(\bigcup G_0)(W \cap \kappa^+)$ is an element of W. But this implies that every proper initial segment of t is an element of W, contrary to our case. This allows us to conclude that $p(g)$ is a $(W, \mathbb{P}/G_{\alpha_0})$-total master condition below p_0.

Case 2: *If $\zeta \in W \cap \kappa^+$, then $t \restriction \zeta \in W$*. Since $W \in \mathrm{IA}_\kappa$, there is a sequence

$$\vec{D} \ = \ \langle D_\xi \mid \xi < \kappa \rangle$$

listing all open dense subsets of $\mathbb{P}/G_{\alpha_0}$ that are elements of W, and such that every proper initial segment of $\vec{D}$ is an element of W. Recursively define a descending sequence $\vec{p} = \langle p_\xi \mid \xi < \kappa \rangle$ of conditions in $\mathbb{P}/G_{\alpha_0}$ as follows:

- Given $\xi < \kappa$, let p^*_ξ be the $\lhd$-least flat condition in $\mathbb{P}/G_{\alpha_0}$ below p_ξ that is an element of D_ξ. Such a condition exists by Lemma 4.20 and the open density of D_ξ. By Lemma 4.23, there exists $\beta < \kappa^+$ with

$$p^*_\xi \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \text{“} o(p^*_\xi(\alpha)) \leq \check{\beta} \text{”}$$

for all $1 \leq \alpha \in \mathrm{sprt}(p^*_\xi)$. Given $\alpha \in [\alpha_0, \varepsilon)$, let $\dot{f}_\alpha$ and $\dot{Y}_\alpha$ denote the canonical $\mathbb{P}_\alpha/G_{\alpha_0}$-names with the property that

$$p^*_\xi(\alpha)^G = (\dot{f}^G_\alpha, \dot{Y}^G_\alpha)$$

holds whenever G is $(\mathbb{P}_\alpha/G_{\alpha_0})$-generic over $\mathrm{V}[G_{\alpha_0}]$ with $p^*_\xi \restriction \alpha \in G$. Note that, by the choice of β, for each $\alpha \in \mathrm{sprt}(p^*_\xi)$, the condition $p^*_\xi \restriction \alpha$ forces that β is larger than all domains of elements of the range of $\dot{f}_\alpha$, and larger than all elements in the domain of $\dot{Y}_\alpha$. In particular, if $\alpha_0 \leq \alpha \in \mathrm{sprt}(p^*_\xi)$ and G is $(\mathbb{P}_\alpha/G_{\alpha_0})$-generic over $\mathrm{V}[G_{\alpha_0}]$ with $p^*_\xi \restriction \alpha \in G$, then

$$(\dot{f}^G_\alpha,\ \dot{Y}^G_\alpha \cup \{(\beta + 1,\ t \restriction (\beta + 1))\}) \tag{10}$$

is a condition in $\dot{\mathbb{Q}}^G_\alpha$ below $p^*_\xi(\alpha)^G$.[19] This shows that there is a condition $p_{\xi+1}$ below p^*_ξ in $\mathbb{P}$ with $\mathrm{sprt}(p_{\xi+1}) = \mathrm{sprt}(p^*_\xi)$ and the property that whenever $\alpha_0 \leq \alpha \in \mathrm{sprt}(p_{\xi+1})$ and G is $(\mathbb{P}_\alpha/G_{\alpha_0})$-generic over $\mathrm{V}[G_{\alpha_0}]$ with $p^*_\xi \restriction \alpha \in G$, then $p_{\xi+1}(\alpha)^G$ is equal to the condition in (10).
Now, assume that p_ξ is an element of W. Then p^*_ξ is obviously definable in $(\mathrm{H}(\theta)[G_{\alpha_0}], \in, \mathbb{P}, G_{\alpha_0}, \lhd)$ using the parameters p_ξ and D_ξ, which are both contained in W. Since $p^*_\xi \in W$, then β can also be taken to be an element of W. Finally, the condition $p_{\xi+1}$ is definable from p^*_ξ, β, and $t \restriction (\beta + 1)$, all of which are elements of W because of the case we are in. These arguments show that $p_\xi \in W$ implies that $p_{\xi+1} \in W$.
- If $\xi < \kappa$ is a limit ordinal, then we define p_ξ be the $\lhd$-least lower bound of the sequence $\langle p_\zeta \mid \zeta < \xi \rangle$ in $\mathbb{P}$.[20]

Note that every proper initial segment of $\vec{p}$ is an element of W, because W contains all proper initial segments of t and each proper initial segment of $\vec{p}$ is definable in the structure $(\mathrm{H}(\theta)[G_{\alpha_0}], \in, \lhd, \mathbb{P})$ using the parameter p_0 and some sufficiently long[21] proper initial segment of t. Hence, not only is each $p_{\xi+1}$ an element of D_ξ, but is in fact an element of $D_\xi \cap W$. In particular, the set $\{p_\xi \mid \xi < \kappa\}$ generates a $(W, \mathbb{P}/G_{\alpha_0})$-generic filter. Let g denote this filter, and let $p(g)$ be the function defined in Definition 4.24.

Now, assume, towards a contradiction, that $p(g)$ is not a condition below every member of g. Then by part (2(c)iii) of Lemma 4.25, there is an $\alpha \in W \cap [\alpha_0, \varepsilon)$ and a condition q in $\mathbb{P}_\alpha/G_{\alpha_0}$ below $p(g) \restriction \alpha$ with the property that whenever G is $(\mathbb{P}_\alpha/G_{\alpha_0})$-generic over $\mathrm{V}[G_{\alpha_0}]$ with $q \in G$, then no proper initial segment of $t = (\bigcup G_0)(W \cap \kappa^+)$ is an element of $\mathrm{ran}(\dot{X}^G_{g,\alpha})$. Since $\alpha \in W$ and the set $\{p_\xi \mid \xi < \kappa\}$ generates the $(W, \mathbb{P}/G_{\alpha_0})$-generic filter g, we can find $\xi_\alpha < \kappa$ with the property that $\alpha \in \mathrm{sprt}(p_{\xi_\alpha+1})$. Then $q \leq p_{\xi_\alpha+1} \restriction \alpha$. Let G be $(\mathbb{P}_\alpha/G_{\alpha_0})$-generic over $V[G_{\alpha_0}]$ with $q \in G$. Work in $V[G_{\alpha_0}, G]$. Since $p_{\xi_\alpha+1} \in g$, we know that $p_{\xi_\alpha+1}(\alpha)^G \in \dot{c}^G_{g,\alpha}$ and

[19] Recall Remark 4.9 showing that, for successor ordinals, the right coordinate of a condition does not have to agree with G_0).

[20] Such a lower bound exists by Lemma 4.26.

[21] The length of this initial segment of t might depend on the given initial segment of $\vec{p}$.

hence $\dot{X}^G_{g,\alpha}$ extends the right coordinate of $p_{\xi_\alpha+1}(\alpha)^G$. By construction, the range of the right coordinate of $p_{\xi_\alpha+1}(\alpha)^G$ contains a proper initial segment of t, contradicting the properties of α and q.

Again, we can conclude that $p(g)$ is a $(W, \mathbb{P}/G_{\alpha_0})$-total master condition below the condition p_0. □

Corollary 4 *Let G be $\mathbb{P}$-generic over* V*, let $\alpha \in (0, \varepsilon)$, let G_α be the filter on $\mathbb{P}_\alpha$ induced by G, and let $S = \dot{S}_\alpha^{G_\alpha}$. Then $(S^{\kappa^+}_\kappa \setminus S)^{\mathrm{V}[G_\alpha]}$ is stationary in $V[G]$.*

Proof First, if $S = \emptyset$, then the conclusion of the lemma holds trivially, because Corollary 3 implies that $(S^{\kappa^+}_\kappa)^{\mathrm{V}[G_\alpha]} = (S^{\kappa^+}_\kappa)^{\mathrm{V}[G]}$. In the other case, we know that $S^{\kappa^+}_\kappa \setminus S$ contains a stationary set in $I[\kappa^+]$ in $\mathrm{V}[G_\alpha]$, and hence a combination of Lemma 2.12, Corollary 3 and Lemma 4.27 ensures that $S^{\kappa^+}_\kappa \setminus S$ remains stationary in $\mathrm{V}[G]$. □

Lemma 4.28 *If G is $\mathbb{P}$-generic over* V*, then the tree $\mathcal{T}(G_0)$ has no cofinal branches in* $\mathrm{V}[G]$.

Our proof of this lemma is similar to the proof of [13, Claim 3.15], but we must make the following changes:

- Whereas the proof of [13, Claim 3.15] makes use of $<\kappa$-closed elementary submodels of size κ (whose existence requires the assumption $\kappa^{<\kappa} = \kappa$), we instead use elementary submodels in IA_κ.
- We use Corollary 4 to ensure that the complement of each $\dot{S}_\alpha$ is stationary in the final model (this is used to get the right analogue of statement (8) on page 1691 of [13]).

Proof of Lemma 4.28 Let $\dot{b}$ be a $\mathbb{P}$-name for a function from κ^+ to κ^+. Assume, towards a contradiction, that there is a condition p in $\mathbb{P}$ that forces $\dot{b}$ to be a cofinal branch through $\mathcal{T}(\dot{G})$.

Fix $W \prec (\mathrm{H}(\theta), \in, \mathbb{P}, \dot{b}, p)$ with $W \in \mathrm{IA}_\kappa$. Set $\delta_W = W \cap \kappa^+$. By the $<\kappa$-closure of $\mathbb{P}$ and Lemma 2.7, there exists a $(W, \mathbb{P})$-generic filter g containing p.

By the $(W, \mathbb{P})$-genericity of g, the $<\kappa^+$-distributivity of $\mathbb{P}$, and the fact that $\dot{b} \in W$, it follows that for every $\gamma < \delta_W$, some condition p_γ in g decides the value of $\dot{b} \restriction \gamma$. Define

$$t = \bigcup \{s \in {}^{<\kappa^+}\kappa^+ \mid \exists \gamma < \delta_W \ p_\gamma \Vdash_{\mathbb{P}} \text{“}\check{s} = \dot{b} \restriction \check{\gamma}\text{”}\}. \tag{11}$$

Then t is a function from δ_W to δ_W. Moreover, by the $(W, \mathbb{P})$-genericity of g and the fact that $\dot{b}$ is forced to be a branch through $\dot{\mathcal{T}}(\dot{G}_0)$, we know that

$$\forall \gamma \in \delta_W \cap S^{\kappa^+}_\kappa \ \exists p \in g \ p \Vdash_{\mathbb{P}} \text{“}\dot{b} \restriction \gamma \neq (\bigcup \dot{G}_0)(\gamma)\text{”}. \tag{12}$$

Hence, if we let g_0 denote the 0-th component of the $(W, \mathbb{P})$-generic filter g, then we have $t \restriction \gamma \neq (\bigcup g_0)(\gamma)$ for all $\gamma < \delta_W$. It follows that

$$f_0 = (\bigcup g_0) \cup \{(\delta_W, t)\}$$

is a condition in $\mathbb{Q}_0$.

Let $p(f_0, g)$ be the function defined in Definition 4.15. Then $p(f_0, g)$ is not a condition in $\mathbb{P}$ that extends every element of g, because otherwise it would force that $\dot{b} \upharpoonright \delta_W = t = (\bigcup \dot{G}_0)(\delta_W)$ and hence it would also force that $\dot{b} \upharpoonright \delta_W \notin \dot{\mathcal{T}}(\dot{G}_0)$, contradicting our assumptions on p.

In this situation, Lemma 4.16 yields an $\alpha \in W \cap [1, \varepsilon)$ such that $p(f_0, g) \upharpoonright \alpha$ is a condition in $\mathbb{P}_\alpha$ below every member of g_α and there is a condition q below $p(f_0, g) \upharpoonright \alpha$ in $\mathbb{P}_\alpha$ with the property that whenever G is $\mathbb{P}_\alpha$-generic over V with $q \in G$, then every proper initial segment of $(\bigcup G_0)(\delta_W)$ is an element of W, and is an $\dot{S}^G_\alpha$-node. Since the 0-th coordinate of q extends f_0, we know that

$$q \Vdash_{\mathbb{P}_\alpha} \text{“}(\bigcup \dot{G}_0)(\check{\delta}_W) = \check{t}\text{”}. \tag{13}$$

Let H_W denote the transitive collapse of W, and let $\sigma : H_W \longrightarrow W \prec \mathrm{H}(\theta)$ denote the inverse of the transitive collapsing map of W. Set $\bar{g} = \sigma^{-1}[g]$, $\bar{g}_\alpha = \sigma^{-1}[g_\alpha]$, $\bar{\mathbb{P}} = \sigma^{-1}(\mathbb{P})$ and $\bar{\mathbb{P}}_\alpha = \sigma^{-1}(\mathbb{P}_\alpha)$. Let G be $\mathbb{P}_\alpha$-generic over V with $q \in G$. Since q extends every element of the $(W, \mathbb{P}_\alpha)$-generic filter g_α, it follows that $G \cap W = g_\alpha$, $\bar{g}_\alpha$ is $\bar{\mathbb{P}}_\alpha$-generic over H_W, and the map σ can be lifted to an elementary

$$\hat{\sigma} : H_W[\bar{g}_\alpha] \longrightarrow \mathrm{H}(\theta)[G]$$

by setting $\hat{\sigma}(\bar{\tau}^{\bar{g}_\alpha}) = (\sigma(\bar{\tau}))^G$ for all $\bar{\mathbb{P}}_\alpha$-names $\bar{\tau}$ in H_W. Moreover, we know that $\bar{g}$ is $\bar{\mathbb{P}}$-generic over H_W and the function

$$\bar{b} = \sigma^{-1}(\dot{b})^{\bar{g}} : \delta_W \longrightarrow \delta_W$$

is an element of $H_W[\bar{g}]$, but not necessarily an element of its inner model $H_W[\bar{g}_\alpha]$.

Claim 4.29 $t = \bar{b}$.

Proof of Claim 4.29 Fix $\gamma < \delta_W$, and set $s = t \upharpoonright \gamma$. By earlier remarks, we know that $s \in W$ and, by the definition of t in (11), there is $p \in g \subseteq W$ with $p \Vdash_{\mathbb{P}} \text{“}\dot{b} \upharpoonright \check{\gamma} = \check{s}\text{”}$. We now know that $p, \dot{b}, \gamma$, and s are elements of $W = \operatorname{ran}(\sigma)$, and since $\operatorname{crit}(\sigma) = \delta_W$, it follows that σ fixes γ and s. The elementarity of σ now implies that

$$\sigma^{-1}(p) \Vdash_{\bar{\mathbb{P}}} \text{“}\sigma^{-1}(\dot{b}) \upharpoonright \check{\gamma} = \check{s}\text{”}$$

holds in H_W. Moreover, since $p \in g$, we have $\sigma^{-1}(p) \in \bar{g}$ and hence $\bar{b} \upharpoonright \gamma = s$ holds in $H_W[\bar{g}]$. □

Let $\bar{S} = \sigma^{-1}(\dot{S}_\alpha)^{\bar{g}_\alpha}$. By Corollary 4 and the elementarity of $\sigma : H_W \longrightarrow \mathrm{H}(\theta)$, we know that $(S^{\delta_W}_\kappa \setminus \bar{S})^{H_W[\bar{g}_\alpha]}$ remains stationary when going from $H_W[\bar{g}_\alpha]$ to $H_W[\bar{g}]$. Since $\bar{b}$ maps from δ_W to δ_W and $(S^{\delta_W}_\kappa \setminus \bar{S})^{H_W[\bar{g}_\alpha]}$ is stationary in $H_W[\bar{g}]$, there is an $\ell < \delta_W$ such that the following statements hold in $H_W[\bar{g}]$:

- $\operatorname{cof}(\ell) = \kappa$.
- ℓ is closed under $\bar{b}$.
- $\ell \notin \bar{S}$.

Then $b_* = \bar{b} \upharpoonright \ell$ maps from ℓ to ℓ, and, by the $<\delta_W$-distributivity of $\bar{\mathbb{P}}$ over H_W, we know that b_* is an element of H_W. Moreover, since $\operatorname{crit}(\hat{\sigma}) = \delta_W$, we have $\hat{\sigma}(b_*) = b_*$. In addition, since $\ell = \operatorname{dom}(b_*) \in (S^{\delta_W}_\kappa \setminus \bar{S})^{H_W[\bar{g}_\alpha]}$ and ℓ is closed under b^*, we can conclude that $H_W[\bar{g}_\alpha]$ believes that b_* is not a $\bar{S}$-node.[22] Then the elementarity of $\hat{\sigma} : H_W[\bar{g}_\alpha] \longrightarrow \mathrm{H}(\theta)[G]$ implies that $\hat{\sigma}(b_*) = b_*$ is not a $\dot{S}^G_\alpha$-node in $\mathrm{H}(\theta)[G]$. By Claim 4.29, we now have $b_* = t \upharpoonright \ell$. So $t \upharpoonright \ell$ is not a $\dot{S}^G_\alpha$-node, contradicting our earlier arguments. □

We are now ready to complete the proof of the main technical result of this paper.

Proof of Theorem 4.2 Let κ be an infinite regular cardinal and let $\mathbb{P} = \mathbb{P}_\varepsilon$ be the poset constructed in Definition 4.12. Then Lemma 4.20 shows that part (1) of the theorem holds.

Next, we prove part (2) of the theorem. We will prove that the poset $\mathbb{P}$ is $(2^\kappa)^+$-*stationarily layered* (see [3, Definition 29]) in V, which, by [3, Lemma 4], implies that $\mathbb{P}$ is $(2^\kappa)^+$-Knaster. A poset $\mathbb{R}$ is λ-stationarily layered if for some sufficiently large regular cardinal θ, there are stationarily-many $M \in \wp^*_\lambda(\mathrm{H}(\theta))$ such that $M \cap \mathbb{R}$ is a regular suborder of $\mathbb{R}$. Equivalently, we can demand that every condition p in $\mathbb{R}$ has a *reduction* into $M \cap \mathbb{R}$, i.e. there exists $q \in M \cap \mathbb{R}$ such that all extensions of q in $M \cap \mathbb{R}$ are compatible with p in $\mathbb{P}$.

For all sufficiently large regular cardinals θ, the set

$$R = \{M \in \wp^*_{(2^\kappa)^+}(\mathrm{H}(\theta)) \mid M \prec \mathrm{H}(\theta),\ {}^\kappa M \subseteq M,\ \mathbb{P} \in M\}$$

is stationary in $\wp_{(2^\kappa)^+}(\mathrm{H}(\theta))$.

We prove that R witnesses the $(2^\kappa)^+$-stationary layeredness of $\mathbb{P}$. Fix $M \in R$ and a condition p in $\mathbb{P}$. By the density of flat conditions in $\mathbb{P}$, we we may assume that there exists a sequence $\langle x_\alpha \mid \alpha \in \operatorname{sprt}(p)\rangle$ that witnesses that p is flat.

Furthermore, we may assume that

$$p(\alpha) = \check{x}_\alpha \iff p(\alpha) \neq \check{\emptyset} \iff p \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} \text{“}\check{x}_\alpha \in \dot{\mathbb{Q}}_\alpha\text{”}$$

holds for all $\alpha \in \operatorname{sprt}(p)$, because redefining p in this way results in a condition equivalent to p.

Set $s = M \cap \operatorname{sprt}(p)$ and define $q = p \upharpoonright s$.

Claim 4.30 *The condition q is a reduction of p into $M \cap \mathbb{P}$.*

Proof of Claim 4.30 First, we verify that q is an element of M. Since we have $x_\alpha \in \mathrm{H}(\kappa^+) \subseteq M$ for all $\alpha \in \operatorname{sprt}(p)$, the closure properties of M imply that the sequence $\langle x_\alpha \mid \alpha \in s\rangle$ is an element of M. Since the condition q is definable from the sequence $\langle x_\alpha \mid \alpha \in s\rangle$, it follows that q is also an element of M.

Next, assume r is a condition in $M \cap \mathbb{P}$ below q. Let $p \wedge r$ denote the natural amalgamation of p and r, i.e. we have $(p \wedge r)(\beta) = r(\beta)$ for all $\beta \in \operatorname{sprt}(r)$, and $(p \wedge r)(j) = p(\beta)$ for all $\beta \in \operatorname{sprt}(p) \setminus \operatorname{sprt}(r)$. Since $p \wedge r$ is clearly a function whose support has size at most κ, it is a condition in $\mathbb{P}$. We verify that $p \wedge r$ is below

[22] Recall from Definition 4.8 that a function s is an S-node if no element of $S^{\kappa^+}_\kappa \setminus S$ is closed under s

both p and r in $\mathbb{P}$ by checking inductively that $(p \wedge r) \upharpoonright \beta$ is below both $p \upharpoonright \beta$ and $p \upharpoonright \beta$ for all $\beta \leq \varepsilon$. Suppose this statement holds at all $\alpha < \beta \leq \varepsilon$. Clearly, if β is a limit ordinal, then it holds at β as well. Hence, we may assume that $\beta = \alpha + 1$ and that $(p \wedge r) \upharpoonright \alpha$ lies below both $p \upharpoonright \alpha$ and $r \upharpoonright \alpha$. If α is not in the support of either p or r, then the above statement trivially holds at β as well. Hence, we have to consider the following two cases:

Case 1: $\alpha \in \mathrm{sprt}(r)$. By definition of the condition $p \wedge r$, we have $(p \wedge r)(\alpha) = r(\alpha)$ and $r \leq_{\mathbb{P}} q$ implies that $r \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha}$ "$r(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} q(\alpha)$". Since $(p \wedge r) \upharpoonright \alpha \leq_{\mathbb{P}_\alpha} r \upharpoonright \alpha$ by our induction hypothesis, this shows that

$$(p \wedge r) \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} \text{“}(p \wedge r)(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} q(\alpha)\text{”}. \tag{14}$$

Moreover, since $r \in M$ and $|\mathrm{sprt}(r)| \leq \kappa \subseteq M$, we have $\alpha \in \mathrm{sprt}(r) \subseteq M$. In particular, if $\alpha \in \mathrm{sprt}(p)$, then $\alpha \in s$ and hence $p(\alpha) = q(\alpha)$. In combination with (14), this yields

$$(p \wedge r) \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} \text{“}(p \wedge r)(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} p(\alpha)\text{”}. \tag{15}$$

In the other case, if $\alpha \notin \mathrm{sprt}(p)$, then (15) holds trivially.

Case 2: $\alpha \in \mathrm{sprt}(p) \setminus \mathrm{sprt}(r)$. By the definition of $(p \wedge r)$, we have $(p \wedge r)(\alpha) = p(\alpha)$. Since $\alpha \notin \mathrm{sprt}(r)$, it follows trivially that

$$(p \wedge r) \upharpoonright \alpha \Vdash_{\mathbb{P}_\alpha} \text{“}(p \wedge r)(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} p(\alpha) \leq_{\dot{\mathbb{Q}}_\alpha} r(\alpha)\text{”}.$$

These computations show that q is a reduction of p into $M \cap \mathbb{P}$. □

This concludes the proof that the poset is $\mathbb{P}$ is $(2^\kappa)^+$-stationarily layered, and hence $(2^\kappa)^+$-Knaster.

We now verify part (3a) of the theorem. Let G be $\mathbb{P}$-generic over V. Suppose S is a bistationary subset of $S^{\kappa^+}_\kappa$ in $\mathrm{V}[G]$ such that $S^{\kappa^+}_\kappa \setminus S$ contains a stationary set T in $I[\kappa^+]$ in $\mathrm{V}[G]$. Pick a club D in κ^+ and a κ^+-sequence $\vec{z}$ of sets from $[\kappa^+]^{<\kappa}$ witnessing that T is an element of $I[\kappa^+]$ in $\mathrm{V}[G]$. A combination of Lemma 4.20 and part (2) of this theorem now shows that there is a subset $P \in \mathrm{V}$ of $S^{\kappa^+}_\kappa \times 2^\kappa$ and a sequence $\langle q_{\gamma,\xi} \mid (\gamma, \xi) \in P\rangle \in \mathrm{V}$ of flat conditions in $\mathbb{P}$ such that $\dot{B}^G = S$, where $\dot{B}$ is the $\mathbb{P}$-name $\{(\check{\gamma}, q_{\gamma,\xi}) \mid (\gamma, \xi) \in P\}$. Pick an element s of the set $\mathcal{N}$ defined before Definition 4.12 such that $\mathrm{dom}(s) = P$ and, if $(\gamma, \xi) \in P$, then $\mathrm{dom}(s(\gamma, \xi)) = \mathrm{sprt}(q_{\gamma,\xi})$ and $q_{\gamma,\xi}(\ell) = \check{x}$ for all $\ell \in \mathrm{dom}(s(\gamma, \xi))$ with $b(\gamma, \xi)(\ell) = x$. By our assumptions on ε and b, part (2) of this theorem allows us to find $0 < \alpha < \varepsilon$ with the property that $\dot{B}$ is a $\mathbb{P}_\alpha$-name, $b(\alpha) = s$ and $\vec{z}, D, T \in \mathrm{V}[G_\alpha]$, where G_α is the filter on $\mathbb{P}_\alpha$ induced by G. Clearly, the fact that T is bistationary in $\mathrm{V}[G]$ implies that T is bistationary in $\mathrm{V}[G_\alpha]$. Moreover, since $\mathrm{V}[G]$ and $\mathrm{V}[G_\alpha]$ have the same κ-sequences of ordinals, every element of $D \cap T$ is also approachable with respect to $\vec{z}$ in $\mathrm{V}[G_\alpha]$. Hence, we know that $S^{\kappa^+}_\kappa \setminus S$ contains a stationary set in $I[\kappa^+]$ in $\mathrm{V}[G_\alpha]$. Since our choice of α ensures that $\dot{B}_\alpha = \dot{B}$, we can conclude that $\dot{S}^{G_\alpha}_\alpha = \dot{B}^{G_\alpha}_\alpha = \dot{B}^G = S$ and hence forcing with $\dot{\mathbb{Q}}^{G_\alpha}_\alpha$ over $\mathrm{V}[G_\alpha]$ adds an order-preserving function from $T(S)$ to $\mathcal{T}(G_0)$.

Finally, we prove part (3b) of the theorem. Hence, assume that $\kappa^{<\kappa} \leq \kappa^+$ holds in V and fix an enumeration $\vec{z} = \langle z_\xi \mid \xi < \kappa^+ \rangle$ of all elements of $[\kappa^+]^{<\kappa}$ in V. By Lemma 2.10, the set M of all $\gamma \in S^{\kappa^+}_\kappa$ that are approachable with respect to $\vec{z}$ is a maximal element of $I[\kappa^+] \cap \wp(S^{\kappa^+}_\kappa)$ mod NS in V. Since $\mathbb{P}$ is $<\kappa^+$-distributive and therefore $\vec{z}$ still enumerates all of $[\kappa^+]^{<\kappa}$ in V$[G]$, it follows that M is still the set of all $\gamma \in S^{\kappa^+}_\kappa$ that are approachable with respect to $\vec{z}$ in $V[G]$, and hence M is still a maximal element of $I[\kappa^+] \cap \wp(S^{\kappa^+}_\kappa)$ mod NS in V$[G]$.

Now, suppose that $S \in$ V$[G]$ is bistationary in $S^{\kappa^+}_\kappa$ and $M \setminus S$ is stationary. Since $M \in I[\kappa^+]$ and $I[\kappa^+]$ is an ideal, it follows that $M \setminus S \in I[\kappa^+]$. So $M \setminus S$ is a stationary set in $I[\kappa^+]$. Hence by part (3a) of the theorem, there is an order-preserving function from $T(S)$ to $\mathcal{T}(G_0)$ in $V[G]$. □

5 Applications

We now apply Theorem 4.2 to prove the results presented in the introduction of the paper.

Corollary 5 *Let κ be an infinite regular cardinal satisfying $\kappa^{<\kappa} \leq \kappa^+$, let $\mathbb{P}$ be the partial order given by Theorem 4.2 and let M be a maximum element of $I[\kappa^+] \cap \wp(S^{\kappa^+}_\kappa)$ mod NS. If G is $\mathbb{P}$-generic over* V*, then the set $NS \restriction M$ is $\mathbf{\Delta}_1(\mathrm{H}((2^\kappa)^+))$-definable in* V$[G]$.

Proof Work in V$[G]$ and let T be the subtree of ${}^{<\kappa^+}\kappa^+$ given by Theorem 4.2. Then $T \subseteq {}^{<\kappa^+}\kappa^+ \in \mathrm{H}((2^\kappa)^+)$. Define $\mathcal{S}$ to be the collection of all subsets A of M such that either there exists a closed unbounded subset C of κ^+ with $C \cap M \subseteq A$ or there exists an order-preserving function from the tree $T(S^{\kappa^+}_\kappa \setminus A)$ into the tree T.

Then the set $\mathcal{S}$ is definable by a Σ_1-formula with parameters M, T and ${}^{<\kappa^+}\kappa^+$.

Claim 5.1 *The set $\mathcal{S}$ is equal to the collection of all subsets of M that are stationary in κ^+.*

Proof of Claim 5.1 First, let $A \subseteq M$ be stationary in κ^+ with the property that there is no club C in κ^+ with $C \cap M \subseteq A$. Since M is stationary in κ^+, this shows that A is bistationary in $S^{\kappa^+}_\kappa$, $M \setminus A$ is stationary, and hence Theorem 4.2 yields an order-preserving function from $T(S^{\kappa^+}_\kappa \setminus A)$ into T that witnesses that A is contained in $\mathcal{S}$. This argument shows that $\mathcal{S}$ contains all stationary subsets of M.

Now, assume, towards a contradiction, that there is a non-stationary subset A of κ^+ that is contained in $\mathcal{S}$. Then there is an order-preserving embedding of $T(S^{\kappa^+}_\kappa \setminus A)$ into T and a closed unbounded subset C of κ^+ with $A \cap C = \emptyset$. But then $C \cap S^{\kappa^+}_\kappa$ is a κ-club that is a subset of $S^{\kappa^+}_\kappa \setminus A$ and, by earlier remarks, the tree $T(S^{\kappa^+}_\kappa \setminus A)$ contains a cofinal branch. But then the tree T also contains a cofinal branch, a contradiction. □

By the above claim, the set $NS \restriction M = \wp(M) \setminus \mathcal{S}$ is definable by a Π_1-formula with parameters in $\mathrm{H}((2^\kappa)^+)$. □

In particular, the above corollary directly shows how the definability results of [13] and [23] can be derived from Theorem 4.2.

Corollary 6 *Let κ be an infinite regular cardinal satisfying $\kappa^{<\kappa} = \kappa$ and let $\mathbb{P}$ be the poset given by Theorem 4.2. If G is $\mathbb{P}$-generic over* V*, then $NS \restriction S^{\kappa^+}_\kappa$ is $\mathbf{\Delta}_1(\mathrm{H}((2^\kappa)^+))$-definable in* V$[G]$.

Proof By Lemma 2.10, if $\kappa^{<\kappa} = \kappa$ holds in V, then $S^{\kappa^+}_\kappa$ is a maximum element of $I[\kappa^+] \cap \wp(S^{\kappa^+}_\kappa)$ mod NS. Since forcing with $\mathbb{P}$ does not change cofinalities below κ^+, the desired conclusion directly follows from Corollary 5. □

The following lemma establishes a connection between principles of stationary reflection and the Π_1-definability of restrictions of the non-stationary ideals that will be crucial for proofs of our main results.

Lemma 5.2 *Let S be a stationary subset of an uncountable regular cardinal δ and let $\mathcal{E}$ be a set of stationary subsets of $S^\delta_{>\omega}$ with the property that for every stationary subset A of S, there exists $E \in \mathcal{E}$ such that A reflects at every element of E.*

If $\mathcal{E}$ is definable by a Σ_1-formula with parameter p, then the set $NS \restriction S$ is definable by a Π_1-formula with parameters p, S and $\mathrm{H}(\delta)$.

Proof Let $\mathcal{S}$ denote the collection of all subsets A of S with the property that there exists $E \in \mathcal{E}$ such that $A \cap \alpha$ is stationary in α for all $\alpha \in E$. By our assumptions on $\mathcal{E}$, the set $\mathcal{S}$ is definable by a Σ_1-formula with parameters p, S and $\mathrm{H}(\delta)$.

If $A \subseteq S$ is stationary in δ, then our assumptions on $\mathcal{E}$ ensure that A is contained in $\mathcal{S}$.

In the other direction, if $E \in \mathcal{E}$ witnesses that A is an element of $\mathcal{S}$ and C is closed unbounded in δ, then there is $\alpha \in E \cap \mathrm{Lim}(C)$ with $A \cap \alpha$ stationary in α and hence $\emptyset \neq A \cap C \cap \alpha \subseteq A \cap C$. Together, this shows that $\mathcal{S}$ is equal to the collection of all subsets of S that are stationary in δ and hence $NS \restriction S = \wp(\delta) \setminus \mathcal{S}$ is definable by a Π_1-formula with parameters p, S and $\mathrm{H}(\delta)$. □

The above lemma directly shows that strong forms of stationary reflection cause restrictions of non-stationary ideals to be $\mathbf{\Delta}_1$-definable.

Corollary 7 *Let δ be an uncountable regular cardinal, let E be a stationary subset of $S^\delta_{>\omega}$ and let S be a stationary subset of δ such that every stationary subset of S reflects almost everywhere in E (i.e. for every stationary subset A of S, there is a closed unbounded subset C of δ with the property that A reflects at every element of $C \cap E$). Then the set $NS \restriction S$ is definable by a Π_1-formula with parameters E, S and* $\mathrm{H}(\delta)$.

Proof If we define $\mathcal{E} = \{C \cap E \mid C\ \mathit{club\ in}\ \delta\}$, then $\mathcal{E}$ is definable by a Σ_1-formula with parameter E and this shows that the sets $\mathcal{E}$ and S satisfy the assumptions of Lemma 5.2. □

Note that a classical result of Magidor in [22] shows that, starting with a weakly compact cardinal, it is possible to construct a model of set theory in which every stationary subset of S^2_0 reflects almost everywhere in S^2_1. The above corollary shows that the set $NS \restriction S^2_0$ is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable in Magidor's model.

The next theorem will be used to derive Theorem 1.2.

Theorem 5.3 *Assume that $2^{\omega_1} = \omega_2$, and θ is a cardinal with $\theta^{\omega_2} = \theta$.*

Then there exists a $<\omega_2$-directed closed, cardinal-preserving poset $\mathbb{P}$ with the property that the following statements hold in $\mathrm{V}[G]$ whenever G is $\mathbb{P}$-generic over V:

(1) $2^{\omega_2} = \theta$.
(2) If for every stationary subset A of S^2_0, there is a stationary subset R of IA_{ω_1} such that $W \prec H(\omega_3)$ and A reflects at $W \cap \omega_2$ for all $W \in R$, then the set $NS \restriction S^2_0$ is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable.

Proof Let G be $\mathrm{Add}(\omega_2, \theta)$-generic over V. Since $2^{\omega_1} = \omega_2$ holds in V, we know that $\mathrm{Add}(\omega_2, \theta)$ satisfies the ω_3-chain condition in V and hence all cofinalities are preserved in $\mathrm{V}[G]$. Work in $\mathrm{V}[G]$. Then our assumptions ensure that $2^{\omega_1} = \omega_2$ and $2^{\omega_2} = \theta = \theta^{\omega_2}$. Let $\mathbb{P}$ be the poset given by Theorem 4.2 for $\kappa = \omega_1$ and $\varepsilon = \theta$, and let M be a maximum element of $I[\omega_2] \cap \wp(S^{\omega_2}_{\omega_1})$ mod NS, which exists due to the assumption that $2^{\omega} \leq \omega_2$ (see Lemma 2.10). Then Lemma 3.6 and part (1) of Theorem 4.2 show that $\mathbb{P}$ is forcing equivalent to a $<\omega_2$-directed closed poset. Moreover, since $2^{\omega_1} = \omega_2$ holds, part (2) of Theorem 4.2 shows that $\mathbb{P}$ satisfies the ω_3-chain condition. Finally, Lemma 4.20 shows that $\mathbb{P}$ has a dense subset of cardinality θ.

Now, let H be $\mathbb{P}$-generic over $\mathrm{V}[G]$ and work in $\mathrm{V}[G, H]$. By the above observations, we then have $2^{\omega_2} = \theta$. In addition, part (3b) of Theorem 4.2 shows that M is the maximum element of $I[\omega_2] \cap \wp(S^{\omega_2}_{\omega_1})$ mod NS. Moreover, Corollary 5 shows that $NS \restriction M$ is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable. In the following, assume that for every stationary subset A of S^2_0, there is a stationary subset R of IA_{ω_1} such that $W \prec H(\omega_3)$ and A reflects at $W \cap \omega_2$ for all $W \in R$. Set $\mathcal{E} = \wp(M) \setminus NS_{\omega_2}$. Then $\mathcal{E}$ is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_3)$.

Claim 5.4 *For every stationary subset A of S^2_0, there is an element E of $\mathcal{E}$ with the property that A reflects at every element of E.*

Proof of the Claim By our assumption, there is a stationary subset R of IA_{ω_1} such that $W \prec H(\omega_3)$ and A reflects at $W \cap \omega_2$ for every $W \in R$. If we now define $E_0 = \{W \cap \omega_2 \mid W \in R\}$, then E_0 is a stationary subset of $S^{\omega_2}_{\omega_1}$. Moreover, since $2^{\omega} \leq \omega_2$, each $W \in R$ has (as an element) an enumeration $\vec{z} = \langle z_\xi \mid \xi < \omega_2 \rangle$ of $[\omega_2]^{\omega}$ and therefore the internal approachability of W and the fact that $\vec{z} \in W$ imply that $W \cap \omega_2$ is approachable with respect to $\vec{z}$. Hence, the set E_0 is stationary and an element of $I[\omega_2]$. Since M is the largest such element mod NS, we have in particular that $E = E_0 \cap M$ is a stationary subset of M. □

Using Lemma 5.2, we can now conclude that $NS \restriction S^2_0$ is definable by a Σ_1-formula with parameters in $\mathrm{H}(\omega_3)$. □

Proof of Theorem 1.2 Assume that FA holds, where FA is one of the following axioms:

- $\mathrm{MM}^{+\mu}$, where μ is a cardinal and $0 \leq \mu \leq \omega_1$; or
- $\mathrm{PFA}^{+\mu}$, where μ is a cardinal and $1 \leq \mu \leq \omega_1$.

Let θ be a cardinal with $\theta^{\omega_2} = \theta$. Since PFA implies that $2^{\omega} = 2^{\omega_1} = \omega_2$ holds (see [15, Theorem 16.20 & 31.23]), our assumption allows us to apply Theorem 5.3 to obtain a $<\omega_2$-directed closed poset with the properties listed in the conclusion of theorem. Let G be $\mathbb{P}$-generic over V. Then [4, Theorem 4.7] ensures that FA holds in V[G]. Since FA holds in V[G], there exists a stationary subset R of IA_{ω_1} with the property that for all $W \in R$, we have $W \prec H(\omega_3)$ and A reflects at $W \cap \omega_2$.[23] Then Theorem 5.3 allows us to conclude that $NS \upharpoonright S^2_0$ is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable in V[$G$]. □

The next theorem will be used to derive Theorem 1.3.

Theorem 5.5 *Assume that* $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$, *and* θ *is a cardinal with* $\theta^{\omega_2} = \theta$.

Then there exists a $<\omega_2$*-directed closed, cardinal-preserving poset* $\mathbb{P}$ *with the property that the following statements hold in* V[G] *whenever* G *is* $\mathbb{P}$*-generic over* V*:*

(1) $2^{\omega_2} = \theta$.
(2) If every stationary subset of S^2_0 *reflects to a point in* S^2_1*, then the set* NS_{ω_2} *is* $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$*-definable.*

Proof Let G be $\mathrm{Add}(\omega_2, \theta)$-generic over V, let $\mathbb{P}$ be the poset produced by an application of Theorem 4.2 with $\kappa = \omega_1$ and $\varepsilon = \theta$ in V[G], and let H be $\mathbb{P}$-generic over V[G]. As above, we have $(2^{\omega_2})^{\mathrm{V}[G,H]} = \theta$ and, since $2^\omega = \omega_1$ holds in V[G], part (4) of Lemma 2.10 and part (3b) of Theorem 4.2 imply that $(S^2_1)^{\mathrm{V}[G]} = (S^2_1)^{\mathrm{V}[G,H]}$ is a maximum element of $I[\omega_2] \cap \wp(S^2_1)$ mod NS in both V[G] and V[G, H]. In particular, Corollary 5 implies that $NS \upharpoonright S^2_1$ is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable in V[$G, H$].

Now, work in V[G, H] and assume that every stationary subset of S^2_0 reflects to a point in S^2_1. Then every stationary subset of S^2_0 reflects to stationary-many points in S^2_1 and we can apply Lemma 5.2 with S^2_0 and $\wp(S^2_1) \setminus NS_{\omega_2}$ to show that $NS \upharpoonright S^2_0$ is $\mathbf{\Delta}_1(\mathrm{H}(\omega_3))$-definable. Since it is easy to see that

$$NS_{\omega_2} = \{A \subseteq \omega_2 \mid A \cap S^2_0 \in NS \upharpoonright S^2_0 \text{ and } A \cap S^2_1 \in NS \upharpoonright S^2_1\},$$

these computations allow us to conclude that NS_{ω_2} is $\mathbf{\Delta}_1$-definable. □

Proof of Theorem 1.3 Assume that $2^\omega = \omega_1$, $2^{\omega_1} = \omega_2$ and either $\mathrm{FA}^+(\sigma\text{-closed})$ or SCFA holds. Let θ be a cardinal with $\theta^{\omega_2} = \theta$, let $\mathbb{P}$ be the poset produced by an application of Theorem 5.5 and let G be $\mathbb{P}$-generic over V. Then $(2^{\omega_2})^{\mathrm{V}[G]} = \theta$ holds. Moreover, by Theorem 4.7 of [4], either $\mathrm{FA}^+(\sigma\text{-closed})$ or SCFA holds in V[G]. In the case of SCFA, the fact that CH also holds ensures (by [11, Theorem 2.7 and Observation 2.8]) that, in V[G], every stationary subset of S^2_0 reflects in a point in S^2_1.[24] In the case where $\mathrm{FA}^+(\sigma\text{-closed})$ holds, the proof of Theorem 8.3 of [2] ensures the same kind of stationary reflection.

By Theorem 5.5, this shows that NS_{ω_2} is $\mathbf{\Delta}_1$-definable in V[G]. □

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[23] For the case corresponding to MM, this follows by the proof of [6, Theorem 13]. For the case corresponding to $\mathrm{PFA}^{+\mu}$ where $\mu \geq 1$, it follows from the remark on [6, p. 20]. The ω_1-enumerations in both proofs are easily seen to be internally approachable enumerations.

[24] Note that the CH assumption seems to be required for this consequence of SCFA; see Fuchs [12] for some corrections on previous literature.

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