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FORCING POSITIVE PARTITION RELATIONS

BY

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ABSTRACT. We show how to force two strong positive partition relations on ω_1 and use them in considering several well-known open problems.

In [32] Sierpiński proved that the well-known Ramsey Theorem [27] does not generalize to the first uncountable cardinal by constructing a partition $[\omega_1]^2 = K_0 \cup K_1$ with no uncountable homogeneous sets. Sierpiński's partition has been analyzed in several directions. One direction was to improve this relation so as to get much stronger negative partition relations on ω_1. The direction taken in this paper is to prove stronger and stronger positive relations on ω_1 which do not appear to be refutable by Sierpiński's partition. The first result of this kind is due to Dushnik and Miller [9] who proved

$$\omega_1 \to (\omega_1, \omega)^2.$$

This was later improved by Erdös and Rado [11] to

$$\omega_1 \to (\omega_1, \omega + 1)^2.$$

In [17] Hajnal proved the following result which shows that the Erdös-Rado theorem is, in a sense, a best possible result of this sort in ZFC:

$$\text{CH implies } \omega_1 \nrightarrow (\omega_1, \omega + 2)^2.$$

Problem 8 of Erdös and Hajnal [12, 13] asks whether $\omega_1 \nrightarrow (\omega_1, \omega + 2)^2$ can be proved without the continuum hypothesis, i.e., whether $\omega_1 \to (\omega_1, \omega + 2)^2$ is consistent with ZFC. The first result on this problem is due to Laver [24] who proved that

$$\text{MA}_{\aleph_1} \text{ implies } \omega_1 \to \left(\omega_1, \binom{\omega_1}{\omega}\right)^2.$$

This result was improved by Hajnal (see [24]) to

$$\text{MA}_{\aleph_1} \text{ implies } \omega_1 \to \left(\omega_1, \binom{\omega_1}{\alpha}\right)^2 \text{ for all } \alpha < \omega_1.$$

Clearly these results leave open the problem whether $\omega_1 \to (\omega_1, \omega + 2)^2$ is consistent. In this paper we shall prove the consistency of

$$\omega_1 \to (\omega_1, \alpha)^2 \quad \text{for all } \alpha < \omega_1.$$

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Let us now consider the following relation introduced by Fred Galvin (it can be considered as a dual of the usual $\to$ relation). Let ϕ and ψ be order types and let r and κ be cardinals. Then the symbol

$$\phi \overset{*}{\to} (\psi)^r_{<\kappa}$$

means: If $\phi = \operatorname{tp} A$, and if $[A]^r = \bigcup_{i \in I} K_i$ is a disjoint partition such that $|K_i| < \kappa$ for all $i \in I$, then there is a $B \subseteq A$ such that $\operatorname{tp} B = \psi$ and $|[B]^r \cap K_i| \leq 1$ for all $i \in I$. Let $\phi \overset{*}{\to} (\psi)^r_\kappa$ iff $\phi \overset{*}{\to} (\psi)^r_{<\kappa^+}$.

It is easily seen that

$$\phi \to (\psi)^r_\kappa \text{ implies } \phi \overset{*}{\to} (\psi)^r_\kappa.$$

Hence, $\omega_1 \overset{*}{\to} (\omega_1)^2_2$ is a weakening of $\omega_1 \to (\omega_1)^2_2$, and it is not "obviously" refuted by Sierpiński's partition. However, Galvin (unpublished) proved that

$$\text{CH implies } \omega_1 \overset{*}{\nrightarrow} (\omega_1)^2_2.$$

He asked whether $\omega_1 \overset{*}{\nrightarrow} (\omega_1)^2_2$ is a theorem of ZFC or not. We answer this question by proving the consistency of

$$\omega_1 \overset{*}{\to} (\omega_1)^2_{<\aleph_0},$$

which is, in a sense, best possible since $\omega_1 \overset{*}{\nrightarrow} (3)^2_{\aleph_0}$.

The next problem we consider is the well-known S-space problem from general topology [**22**, **28**, **30**]. It essentially asks for a strong partition property of ω_1. To state this problem we need some definitions. A topological space X is hereditarily separable iff every subspace of X has a countable dense subset. X is called hereditarily Lindelöf iff for every family $\mathfrak{U}$ of open subsets of X, there is a countable $\mathfrak{U}_0 \subseteq \mathfrak{U}$ such that $\bigcup \mathfrak{U}_0 = \bigcup \mathfrak{U}$. The S-space problem asks whether every regular hereditarily separable topological space is hereditarily Lindelöf. A counterexample to this problem is called an S-space. The problem has been intensively studied since the late 1960's, and its present formulation is due to several mathematicians [**22**, **28**, **30**]. The first example of an S-space was constructed by M. E. Rudin [**29**] using a Suslin tree. Since then a number of constructions have appeared using various assumptions such as $\diamondsuit$, CH,.... Also a number of partial nonexistence results have appeared using mainly MA + $\neg$CH (see [**22**, **28**, **30**]). In this paper we shall prove the consistency of:

> *Every regular hereditarily separable topological space is hereditarily Lindelöf.*

Hence the S-space problem is undecidable on the basis of the usual axioms of set theory. Working independently and somewhat later, J. Baumgartner proved the consistency of ZFC + "there are no weak-HFD's". (HFD's form an important class of subspaces of $\{0,1\}^{\aleph_1}$ used by Hajnal and Juhász and others in constructing various sorts of S-spaces [**22**, **28**].)

Next we are going to consider the problem of bounds on the cardinalities of Hausdorff spaces with no uncountable discrete subspaces. (A set $D \subseteq X$ is a discrete subspace of X if for every $d \in D$ there exists an open subset U_d of X such that $D \cap U_d = \{d\}$.) That the cardinality of such a space has a bound was first independently noticed by Isbell **[20]** (for completely regular spaces), Efimov **[10]** and de Groot **[16]**. The bound they found was $2^{2^{2^{\aleph_0}}}$. This bound was improved to $2^{2^{\aleph_0}}$ first by de Groot **[16]** for the class of all regular spaces, and then by Hajnal and Juhász **[18]** for the class of all Hausdorff spaces. The natural question which remained unanswered is due to de Groot, Efimov and Isbell [**12**, Problem 77] and asks whether there exists a Hausdorff space of cardinality $(2^{\aleph_0})^+$ with no uncountable discrete subspaces. The first result on this problem is due to Hajnal and Juhász **[19]** who constructed such a space using a forcing argument. A compact example has since been constructed by Fedorčuk **[14]** using $\diamondsuit$. In this note we shall prove the consistency of:

Every Hausdorff space with no uncountable discrete subspaces has cardinality $\leqslant 2^{\aleph_0}$.

We shall deduce this result from the consistency of the following statement, which is of independent interest:

If X is a Hausdorff space with no uncountable discrete subspace, then every point of X is the intersection of countably many open subsets of X.

The results of this paper were proved while I was visiting the Department of Mathematics at Dartmouth College during the academic year 1980–81. I would like to express my gratitude to Professor James Baumgartner for making this visit possible. I would also like to thank Professor Fred Galvin for a very stimulating correspondence concerning a class of problems about strong partition relations on ω_1, a small part of which is considered in this paper. The results of this paper were announced in [**34**, **35**, and **36**].

1. In this section we construct a model of ZFC + $\mathrm{MA}_{\aleph_1}$ in which ω_1 satisfies two strong partition relations which will be used in the rest of the paper. Our forcing terminology is standard (see [**5**, **21**, **23**]). All undefined terms concerning the partition calculus can be found in **[37]**. If $A, B \subseteq \omega_1$, then by $A \otimes B$ we denote $\{\{\alpha, \beta\}: \alpha \in A, \beta \in B, \alpha \neq \beta\}$. If $K \subseteq [\omega_1]^2$ and $\alpha \in \omega_1$, then $K(\alpha)$ denotes the set $\{\beta < \omega_1: \{\alpha, \beta\} \in K\}$. The symbol

$$\omega_1 \to (\omega_1, (\omega_1; \text{fin}\, \omega_1))^2$$

means: For any partition $[\omega_1]^2 = K_0 \cup K_1$ either

(1) there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$, or else

(2) there is an $A \in [\omega_1]^{\aleph_1}$ and a family $\mathfrak{B}$ of $\aleph_1$ disjoint finite subsets of ω_1 such that $(\{\alpha\} \otimes F) \cap K_1 \neq \varnothing$ for all $\alpha \in A$ and $F \in \mathfrak{B}$ with $\alpha < \min F$.

The set A which satisfies condition (2) is called a bad set. The purpose of this section is to prove the following theorem.

THEOREM 1. *If ZF is consistent, then so is ZFC plus the following statements simultaneously*:

(i) $MA + 2^{\aleph_0} = \aleph_2$,

(ii) $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\, \omega_1))^2$,

(iii) $\omega_1 \overset{*}{\to} (\omega_1)^2_{<\aleph_0}$.

If A is a set, then $\mathcal{C}_A$ denotes the set of all finite partial functions from A into 2 ordered by $\supseteq$. Thus $\mathcal{C}_{\omega_1}$ is the standard poset for adding $\aleph_1$ Cohen reals. If $\alpha < \beta \leqslant \omega_1$, then we let

$$\mathcal{C}_{\alpha,\beta} = \{p \in \mathcal{C}_{\omega_1} : p \subseteq [\alpha, \beta)\}.$$

Let $\mathcal{C}_\beta = \mathcal{C}_{0,\beta}$.

Let $\mathcal{S}$ denote the set of all pairs $\langle a, A\rangle$ where a is a countable closed subset of ω_1 and A is a closed and unbounded subset of ω_1. We order $\mathcal{S}$ by

$$\langle a, A\rangle \leqslant \langle b, B\rangle \quad \text{iff} \quad b = a \cap (\max(b) + 1)\ \&\ A \subseteq B\ \&\ a \setminus b \subseteq B.$$

Then $\mathcal{S}$ is the Jensen closed unbounded set poset [**8**]. It is clear that $\mathcal{S}$ is a σ-closed poset. Moreover, every countable set $\mathcal{S}_0 \subseteq \mathcal{S}$ of pairwise compatible elements has a greatest lower bound $\langle a, A\rangle \in \mathcal{S}$ defined by

$$a = \overline{\bigcup \{b : \exists B(\langle b, B\rangle \in \mathcal{S}_0)\}} \quad \text{and} \quad A = \bigcap \{B : \exists b(\langle b, B\rangle \in \mathcal{S}_0)\}.$$

Let $G_{\mathcal{S}}$ be a generic subset of $\mathcal{S}$. Then

$$C_{\mathcal{S}} = \bigcup \{a : \exists A(\langle a, A\rangle \in G_{\mathcal{S}})\}$$

is a closed unbounded subset of ω_1 which is almost included in every club subset of ω_1 from the ground model [**8**].

LEMMA 1. *Let $\mathcal{C} = \mathcal{C}_{\omega_1}$ be the standard poset for adding $\aleph_1$ Cohen reals. Let $G_{\mathcal{C}}$ be a V-generic subset of $\mathcal{C}$. Let $\mathcal{S}$ be the Jensen club set poset in $V[G_{\mathcal{C}}]$. Let (in V) $\mathcal{Q}$ be a c.c.c. poset and let $G_{\mathcal{Q}}$ be a $V[G_{\mathcal{C}}]$-generic subset of $\mathcal{Q}$. Let, in $V[G_{\mathcal{Q}}]$, $[\omega_1]^2 = K_0 \cup K_1$ and $[\omega_1]^2 = \bigcup_{\zeta<\omega_1} J_\zeta$ be two disjoint partitions such that the first partition has no bad sets, while each color of the second partition is finite. Let $G_{\mathcal{S}}$ be a $V[G_{\mathcal{C}}][G_{\mathcal{Q}}]$-generic subset of $\mathcal{S}$. In $V[G_{\mathcal{C}}][G_{\mathcal{Q}}][G_{\mathcal{S}}]$, we define*

$$\mathbb{S}_0 = \{s \in [\omega_1]^{<\aleph_0} : s \text{ is separated by } C_{\mathcal{S}}\ \&\ [s]^2 \subseteq K_0\},$$

$$\mathbb{S}_1 = \left\{s \in [\omega_1]^{<\aleph_0} : s \text{ is separated by } C_{\mathcal{S}}\ \&\ \forall \zeta < \omega_1 \left|[s]^2 \cap J_\zeta\right| \leqslant 1\right\}.$$

Let $\mathbb{S}_0$ and $\mathbb{S}_1$ be partially ordered by $\supseteq$. Then both $\mathbb{S}_0$ and $\mathbb{S}_1$ are c.c.c. posets in $V[G_{\mathcal{C}}][G_{\mathcal{Q}}][G_{\mathcal{S}}]$.

PROOF. We first prove the lemma for the poset $\mathbb{S}_0$. So let, in $V[G_{\mathcal{C}}][G_{\mathcal{Q}}][G_{\mathcal{S}}]$, $\langle s_\xi : \xi < \omega_1\rangle$ be an ω_1-sequence of elements of $\mathbb{S}_0$. By the standard Δ-system argument we may assume that s_ξ's are disjoint, increasing and of the same cardinality n, where $1 \leqslant n < \omega$.

From now on we work in $V[G_{\mathcal{C}}][G_{\mathcal{Q}}]$ and fix an $\mathfrak{S}$-name $\langle \dot{s}_\xi \colon \xi < \omega_1 \rangle$ for the sequence $\langle s_\xi \colon \xi < \omega_1 \rangle$ and a condition $\langle a_0, A_0 \rangle \in \mathfrak{S}$ which forces that n and $\langle \dot{s}_\xi \colon \xi < \omega_1 \rangle$ have the above properties. We shall find $\langle b, B \rangle \leqslant \langle a_0, A_0 \rangle$ such that

$$\langle b, B \rangle \Vdash_{\mathfrak{S}} \exists \xi < \eta < \omega_1 \left(\dot{s}_\xi \otimes \dot{s}_\eta \subseteq K_0 \right).$$

This will finish the proof of Lemma 1 for the poset $\mathbb{S}_0$.

Note that $\mathfrak{S}$ is defined in $V[G_{\mathcal{C}}]$, hence it is not necessarily σ-closed in $V[G_{\mathcal{C}}][G_{\mathcal{Q}}]$. Instead of $\mathfrak{S}$ we shall work with the set of all $\langle a, A \rangle \in \mathfrak{S}$ such that $A \in V$. The restriction causes no loss of generality since this set is dense in $\mathfrak{S}$. By an abuse of notation we denote it also by $\mathfrak{S}$.

Let θ be a big enough regular cardinal, and let N_0 be a countable elementary submodel of H_θ such that $N_0 \cap V[G_{\mathcal{C}}] \in V[G_{\mathcal{C}}]$, and such that $\mathfrak{S}$, $\langle a_0, A_0 \rangle$, $\langle \dot{s}_\xi \colon \xi < \omega_1 \rangle$, K_0, $K_1 \in N_0$. Since $V[G_{\mathcal{C}}][G_{\mathcal{Q}}]$ is a c.c.c. extension of $V[G_{\mathcal{C}}]$, such a submodel exists. Let $\delta_0 = N_0 \cap \omega_1$, and let $F \subseteq \omega_1 \backslash \delta_0$ be a fixed set of size n. Let

$$\mathcal{W}_F = \Big\{ \langle a, A \rangle \in \mathfrak{S} \cap N_0 \colon \left(\langle a, A \rangle \perp \langle a_0, A_0 \rangle \right) \vee \\ \left(\langle a, A \rangle \leqslant \langle a_0, A_0 \rangle \,\&\, \exists s \subseteq \delta_0 \exists \xi < \delta_0 \left(s \otimes F \subseteq K_0 \,\&\, \langle a, A \rangle \Vdash_{\mathfrak{S}} \dot{s}_\xi = s \right) \right) \Big\}.$$

CLAIM 1. $\mathcal{W}_F$ is a dense open subset of $\mathfrak{S} \cap N_0$.

PROOF. Let $\langle b, B \rangle \leqslant \langle a_0, A_0 \rangle$ be a given element of $\mathfrak{S} \cap N_0$. By induction on $0 \leqslant i \leqslant n$, for each strictly increasing sequence $\langle x_1, \ldots, x_{n-i} \rangle$ of ordinals $< \omega_1$, we define the statements $\Phi_{n-i}(x_1, \ldots, x_{n-i})$ as follows:

$$\Phi_n(x_1, \ldots, x_n) \quad \text{iff } \exists \xi < \omega_1 \exists \langle a, A \rangle \leqslant \langle b, A \rangle \left(\langle a, A \rangle \Vdash_{\mathfrak{S}} \dot{s}_\xi = \{x_1, \ldots, x_n\} \right);$$

$$\Phi_{n-i}(x_1, \ldots, x_{n-i}) \quad \text{iff } \left| \{ y < \omega_1 \colon \Phi_{n-i+1}(x_1, \ldots, x_{n-i}, y) \} \right| = \aleph_1 \quad \text{for } 0 < i \leqslant n.$$

We shall prove that Φ_0 holds.

Starting from N_0 we build a strictly increasing, continuous sequence $\langle N_\alpha \colon \alpha < \omega_1 \rangle$ of countable elementary submodels of H_θ. Let $\delta_\alpha = N_\alpha \cap \omega_1$ for $\alpha < \omega_1$, and let $D = \{\delta_\alpha \colon \alpha < \omega_1\}$. Then D is a closed unbounded subset of ω_1. Since $\mathcal{C} \times \mathcal{Q}$ is a c.c.c. poset, we can find $\langle a', A' \rangle \leqslant \langle b, B \rangle$ and $\gamma < \omega_1$ such that $A' \backslash \gamma \subseteq D$. Choose $\langle a, A \rangle \leqslant \langle a', A' \rangle$, $\xi < \omega_1$, and $\{z_1, \ldots, z_n\} \subseteq \omega_1 \backslash \gamma$ such that

$$\langle a, A \rangle \Vdash_{\mathfrak{S}} \dot{s}_\xi = \{z_1, \ldots, z_n\}.$$

This shows that $\Phi_n(z_1, \ldots, z_n)$ holds. By induction on i we shall now show that $\Phi_{n-i}(z_1, \ldots, z_{n-i})$ holds for all $0 \leqslant i < n$. Note that $\{z_1, \ldots, z_n\}$ is separated by D, so we can find $\alpha_1 < \cdots < \alpha_n < \omega_1$ such that $\delta_{\alpha_i} \leqslant z_i < \delta_{\alpha_i + 1}$ for all $i = 1, \ldots, n$. So let us assume that $\Phi_{n-i+1}(z_1, \ldots, z_{n-i+1})$ holds for some $0 < i < n$. Let

$$Z_{n-i+1} = \{ z < \omega_1 \colon \Phi_{n-i+1}(z_1, \ldots, z_{n-i}, z) \}.$$

Then $z_{n-i+1} \in Z_{n-i+1}$ by the assumption. Note that the parameters in the definition of Z_{n-i+1} are all members of $N_{\alpha_{n-i+1}}$, which implies $Z_{n-i+1} \in N_{\alpha_{n-i+1}}$. Hence we must have $N_{\alpha_{n-i+1}} \models Z_{n-i+1}$ is uncountable. Hence Z_{n-i+1} is really uncountable. This shows that $\Phi_{n-i}(z_1, \ldots, z_{n-i})$ holds and finishes the induction step.

Thus, in particular, $\Phi_1(z_1)$ holds, and by repeating the above argument we conclude that $\{z < \omega_1 \colon \Phi_1(z)\}$ is uncountable. Hence Φ_0 holds.

We have already noted that the parameters of each statement Φ_i $(0 \leqslant i \leqslant n)$ are members of N_0. Hence, if we let $Y_1 = \{y < \omega_1: \Phi_1(y)\}$ then $Y_1 \in N_0$, and by the fact that Φ_0 holds, Y_1 is uncountable. We claim that, for some $y \in Y_1 \cap \delta_0$, $\{y\} \otimes F \subseteq K_0$. Otherwise the following holds:

$$N_0 \models \forall \delta < \omega_1 \exists F_\delta \in [\omega_1 \setminus \delta]^n \forall y \in Y_1 \cap \delta((\{y\} \otimes F_\delta) \cap K_1 \neq \varnothing).$$

Since $N_0 \prec H_\theta$, this sentence also holds in H_θ. However, this easily gives a bad set with respect to $[\omega_1]^2 = K_0 \cup K_1$, contradicting the assumption that $V[G_{\mathcal{C}}][G_{\mathcal{Q}}]$ is a property K extension of $V[G_{\mathcal{Q}}]$ which contains no bad sets. So pick $y_1 \in Y_1 \cap \delta_0$ such that $\{y_1\} \otimes F \subseteq K_0$. Let $Y_2 = \{y < \omega_1: \Phi_2(y_1, y)\}$. Then by the assumption that $\Phi_1(y_1)$ holds, Y_2 is uncountable. Clearly $Y_2 \in N_0$. By repeating the above argument, we can find a $y_2 \in Y_2 \cap \delta_0$ such that $\{y_2\} \otimes F \subseteq K_0$. Proceeding in this way we construct $y_1 < y_2 < \cdots < y_n$ such that

$$\{y_1, \ldots, y_n\} \otimes F \subseteq K_0, \quad \text{and} \quad \Phi_n(y_1, \ldots, y_n) \text{ holds.}$$

Hence $N_0 \models \Phi(y_1, \ldots, y_n)$. This means that we can find $\xi < \delta_0$ and $\langle a, A \rangle \leqslant \langle b, B \rangle$ such that $\langle a, A \rangle \in \mathcal{E} \cap N_0$ and

$$\langle a, A \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi = \{y_1, \ldots, y_n\}.$$

This shows that $\langle a, A \rangle \in \mathcal{W}_F$ and completes the proof of Claim 1.

Define $R \subseteq (\mathcal{E} \cap N_0) \times \delta_0 \times [\delta_0]^n$ by

$$R(\langle a, A \rangle, \xi, s) \quad \text{iff} \quad \langle a, A \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi = s.$$

Since $\mathcal{E} \cap N_0$ and R can be coded using only a countable amount of information, we can find some $\alpha < \omega_1$ such that

$$\mathcal{E} \cap N_0 \in V[G_{\mathcal{C}_\alpha}] \quad \text{and} \quad R \in V[G_{\mathcal{C}_\alpha}][G_{\mathcal{Q}}].$$

In $V[G_{\mathcal{C}_\alpha}]$ we choose a mapping π which maps $\mathcal{C}_{\alpha, \alpha+\omega}$ isomorphically into a dense subset of $\{\langle a, A \rangle \in \mathcal{E} \cap N_0: \langle a, A \rangle \leqslant \langle a_0, A_0 \rangle\}$. Let

$$G = G_{\mathcal{C}_{\alpha, \alpha+\omega}} = G_{\mathcal{C}} \cap \mathcal{C}_{\alpha, \alpha+\omega}.$$

Then G is a $V[G_{\mathcal{C}_\alpha}]$-generic subset of $\mathcal{C}_{\alpha, \alpha+\omega}$. So $\pi''G$ is a countable pairwise compatible subset of $\mathcal{E}$. Hence $\pi''G$ has a lower bound $\langle \bar{a}, \bar{A} \rangle$ in $\mathcal{E}$.

Note that, for each $F \in [\omega_1 \setminus \delta_0]^n$, $\mathcal{W}_F$ is definable from F, K_0, K_1, $\mathcal{E} \cap N_0$, $\langle a_0, A_0 \rangle$, δ_0, and R. Hence, for each $F \in [\omega_1 \setminus \delta]^n$, $\mathcal{W}_F \in V[G_{\mathcal{C}_\alpha}][G_{\mathcal{Q}}]$. Since G is also a $V[G_{\mathcal{C}_\alpha}][G_{\mathcal{Q}}]$-generic subset of $\mathcal{C}_{\alpha, \alpha+\omega}$, it follows that $\pi''G$ intersects each $\mathcal{W}_F$ for $F \in [\omega_1 \setminus \delta_0]^n$. Hence, for each $F \in [\omega_1 \setminus \delta_0]^n$, there is an $\langle a, A \rangle \in \mathcal{W}_F$ such that $\langle \bar{a}, \bar{A} \rangle \leqslant \langle a, A \rangle$.

Pick $\langle b, B \rangle \leqslant \langle \bar{a}, \bar{A} \rangle$, $\eta < \omega_1$, and $F \in [\omega_1 \setminus \delta_0]^n$ such that $\langle b, B \rangle \Vdash_{\mathcal{E}} \dot{s}_\eta = F$. Let $\langle a, A \rangle \in \mathcal{W}_F$ be such that $\langle b, B \rangle \leqslant \langle a, A \rangle$. Thus, for some $s \in [\delta_0]^n$ and $\xi < \delta_0$, we have

$$s \otimes F \subseteq K_0 \quad \text{and} \quad \langle a, A \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi = s.$$

Hence

$$\langle b, B \rangle \Vdash_{\mathcal{E}} \dot{s}_\xi \otimes \dot{s}_\eta \subseteq K_0.$$

This completes the proof of Lemma 1 for the case of poset $\mathcal{S}_0$.

The proof that $\mathbb{S}_1$ is a c.c.c. poset in $V[G_{\mathcal{C}}][G_{\mathcal{Q}}][G_{\mathcal{S}}]$ is similar, so we mention only the main differences. Again we start with a given sequence $\langle s_\xi: \xi < \omega_1 \rangle$ of elements of $\mathbb{S}_1$. We may assume the s_ξ's form a Δ-system with root s, and if we let $t_\xi = s_\xi \setminus s$ for $\xi < \omega_1$, then the t_ξ's are strictly increasing and of the same cardinality n, where $1 \leq n < \omega$.

Working in $V[G_{\mathcal{C}}][G_{\mathcal{Q}}]$, we fix $\mathcal{S}$-names $\langle \dot{s}_\xi: \xi < \omega_1 \rangle$ and $\langle \dot{t}_\xi: \xi < \omega_1 \rangle$ for the sequences $\langle s_\xi: \xi < \omega_1 \rangle$ and $\langle t_\xi: \xi < \omega_1 \rangle$, and a condition $\langle a_0, A_0 \rangle \in \mathcal{S}$ which forces that $n, s, \langle \dot{s}_\xi: \xi < \omega_1 \rangle$ and $\langle \dot{t}_\xi: \xi < \omega_1 \rangle$ have the above properties. We need to find $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ such that

$$\langle b, B \rangle \Vdash_{\mathcal{S}} \exists \xi < \eta < \omega_1 \, \forall \zeta < \omega_1 \, \left| \left[\dot{s}_\xi \cup \dot{s}_\eta \right]^2 \cap J_\zeta \right| \leq 1.$$

We fix a cardinal θ and a countable elementary symbol N_0 of H_θ as before. Let

$$D = \left\{ \delta < \omega_1 : \lim(\delta) \,\&\, \forall \alpha < \delta \, \forall \zeta < \omega_1 \left(J_\zeta \cap [\delta]^2 \neq \varnothing \Rightarrow J_\zeta(\alpha) \subseteq \delta \right) \right\}.$$

Then D is a closed unbounded subset of ω_1 such that $D \in N_0$. Now we fix an $F \in [\omega_1 \setminus \delta_0]^n$ which is separated by D such that

$$\forall \zeta < \omega_1 \, \left| [s \cup F]^2 \cap J_\zeta \right| \leq 1,$$

and define

$$\begin{aligned} \mathcal{W}_F = \Big\{ \langle a, A \rangle \in \mathcal{S} \cap N: & (\langle a, A \rangle \perp \langle a_0, A_0 \rangle) \\ & \vee \Big(\langle a, A \rangle \leq \langle a_0, A_0 \rangle \& \, \exists t \subseteq \delta_0 \, \exists \xi < \delta_0 \\ & \Big(\forall \zeta < \omega_1 \left| [s \cup t \cup F]^2 \cap J_\zeta \right| \leq 1 \,\&\, \langle a, A \rangle \Vdash_{\mathcal{S}} \dot{t}_\xi = t \Big) \Big) \Big\}. \end{aligned}$$

As before we claim that $\mathcal{W}_F$ is a dense open subset of $\mathcal{S} \cap N_0$. So let $\langle b, B \rangle \leq \langle a_0, A_0 \rangle$ be a given element of $\mathcal{S} \cap N_0$. The statements $\Phi_{n-i}(x_1, \ldots, x_{n-i})$ $(0 \leq i \leq n)$ are defined as before. The proof that Φ_0 holds is also the same.

Let $Y_1 = \{ y < \omega_1 : \Phi_1(y) \}$. Then Y_1 is uncountable and $Y_1 \in N_0$. Let $\{\zeta_1, \ldots, \zeta_{k_1}\}$ be a list of all $\zeta < \omega_1$ such that $[s \cup F]^2 \cap J_\zeta \neq \varnothing$. Since each J_ζ is finite, and since $Y_1 \cap \delta_0$ is infinite, we can find $y_1 \in Y_1 \cap \delta_0$ such that

$$(\{y_1\} \otimes (s \cup F)) \cap J_\zeta = \varnothing \quad \text{for all } \zeta \in \{\zeta_1, \ldots, \zeta_{k_1}\}.$$

We claim that

$$\left| \left[s \cup \{y_1\} \cup F \right]^2 \cap J_\zeta \right| \leq 1 \quad \text{for all } \zeta < \omega_1.$$

Otherwise, let $\zeta < \omega_1$ be such that $[s \cup \{y_1\} \cup F]^2 \cap J_\zeta$ contains two different edges l_0 and l_1. It is clear that l_0 and l_1 cannot both be subsets of $s \cup \{y_1\}$ since some condition from $\mathcal{S}$ forces this set to be a subset of a member of $\dot{\mathbb{S}}_1$. From the definition of D it easily follows that $\max l_0 = \max l_1$. Hence, $\max l_0 = \max l_1 \in F$. Hence, for some $i < 2$, $l_i \in s \otimes F$, which means that $\zeta \in \{\zeta_1, \ldots, \zeta_{k_1}\}$. It follows that

$$(\{y_1\} \otimes (s \cup F)) \cap J_\zeta = \varnothing.$$

Consequently, $\min l_0 \neq y_1$ and $\min l_1 \neq y_1$, which yields the contradiction $l_1, l_2 \in [s \cup F]^2$.

Now let $Y_2 = \{y < \omega_1: \Phi_2(y_1, y)\}$. Then Y_2 is uncountable and $Y_2 \in N_0$. Working as above, we can find $y_2 \in Y_2 \cap \delta$ such that

$$\left|[s \cup \{y_1, y_2\} \cup F]^2 \cap J_\zeta\right| \leqslant 1 \quad \text{for all } \zeta < \omega_1.$$

Proceeding in this way we construct $y_1 < y_2 < \cdots < y_n < \delta_0$ such that

$$\forall \zeta < \omega_1 \left|[s \cup \{y_1, \ldots, y_n\} \cup F]^2 \cap J_\zeta\right| \leqslant 1,$$

and

$$\Phi_n(y_1, \ldots, y_n) \text{ holds.}$$

Hence $N_0 \models \Phi_n(y_1, \ldots, y_n)$. So we can find $\xi < \delta_0$ and $\langle a, A\rangle \leqslant \langle a_0, A_0\rangle$ such that $\langle a, A\rangle \in \mathcal{S} \cap N_0$ and

$$\langle a, A\rangle \Vdash_{\mathcal{S}} \dot{t}_\xi = \{y_1, \ldots, y_n\}$$

This shows that $\langle a, A\rangle \in \mathcal{W}_F$. Hence, $\mathcal{W}_F$ is a dense open subset of $\mathcal{S} \cap N_0$.

We leave the remainder of the proof of Lemma 1 to the reader since the rest of the proof for $\mathbb{S}_1$ is like the proof for $\mathbb{S}_0$.

Now we are going to describe a mixed iteration $\langle \mathcal{P}_\alpha: \alpha \leqslant \omega_2\rangle$ of Cohen and Jensen partially ordered sets. This will be done by induction on α, and for this purpose let E and O denote the sets of all even and odd ordinals $< \omega_2$, respectively.

If $\alpha = 0$, then $\mathcal{P}_\alpha = \varnothing$.

If $\alpha \in E$, then $\mathcal{P}_{\alpha+1}$ is the set of all functions p with domain $\alpha + 1$ such that $p \restriction \alpha \in \mathcal{P}_\alpha$ and

$$\Vdash_{\mathcal{P}_\alpha} p(\alpha) \text{ is a member of } \dot{\mathcal{C}}_{\{\alpha\} \times \omega_1}.$$

If $p, q \in \mathcal{P}_{\alpha+1}$, let $p \leqslant q$ iff $p \restriction \alpha \leqslant q \restriction \alpha$ and $p \restriction \alpha \Vdash_{\mathcal{P}_\alpha} p(\alpha) \supseteq q(\alpha)$.

If $\alpha \in O$, then $\mathcal{P}_{\alpha+1}$ is the set of all functios p with domain $\alpha + 1$ such that $p \restriction \alpha \in \mathcal{P}_\alpha$ and

$$\Vdash_{\mathcal{P}_\alpha} p(\alpha) \text{ is a member of the Jensen club set poset } \dot{\mathcal{S}}_\alpha.$$

If $p, q \in \mathcal{P}_{\alpha+1}$, let $p \leqslant q$ iff $p \restriction \alpha \leqslant q \restriction \alpha$ and $p \restriction \alpha \Vdash_{\mathcal{P}_\alpha} p(\alpha) \leqslant q(\alpha)$ in the Jensen ordering.

If α is a limit ordinal with $\operatorname{cf} \alpha = \omega$, then $\mathcal{P}_\alpha$ is the set of all functions p with domain α such that $p \restriction \beta \in \mathcal{P}_\beta$ for all $\beta < \alpha$, and for some $\gamma < \alpha$, $\Vdash_{\mathcal{P}_\beta} p(\beta) = \varnothing$ for all $\beta \in [\gamma, \alpha) \cap E$.

If α is a limit ordinal with $\operatorname{cf} \alpha > \omega$, then $\mathcal{P}_\alpha$ is the set of all functions p with domain α such that $p \restriction \beta \in \mathcal{P}_\beta$ for all $\beta < \alpha$, and for some $\gamma < \alpha$, $\Vdash_{\mathcal{P}_\beta} p(\beta) = \varnothing$ for $\beta \in [\gamma, \alpha) \cap E$ and $\Vdash_{\mathcal{P}_\beta} p(\beta) = \langle \varnothing, \omega_1\rangle$ for $\beta \in [\gamma, \alpha) \cap O$.

In both limit cases we put $p \leqslant q$ iff $p \restriction \beta \leqslant q \restriction \beta$ for all $\beta < \alpha$.

From now on let $\alpha \leqslant \omega_2$ be a fixed ordinal. For $p \in \mathcal{P}_\alpha$ we define $\operatorname{supp}(p) = \{\beta < \alpha: p(\beta) \neq \varnothing$ if $\beta \in E$ and $p(\beta) \neq \langle \varnothing, \omega_1\rangle$ if $\beta \in O\}$. Then it is easily checked that $\operatorname{supp}(p) \cap E$ is finite, and that $\operatorname{supp}(p) \cap O$ is at most countable.

We say that $p \in \mathcal{P}_\alpha$ is a determined condition if, for every $\beta \in \operatorname{supp}(p) \cap E$, there is some $s_\beta(p) \in \mathcal{C}_{\{\gamma\} \times \omega_1}$ such that $p(\beta) = s_\beta(p)$ (more precisely, $p(\beta) = (s_\beta(p))\check{}$). If $p \in \mathcal{P}_\alpha$ is a determined condition, then by $\sigma(p)$ we denote

$$\bigcup \{s_\beta(p): \beta \in \operatorname{supp}(p) \cap E\}$$

considered as a member of $\mathcal{C}(\alpha) := \mathcal{C}_{(E\cap\alpha)\times\omega_1}$. By induction on α it is easily seen that the set of all determined conditions is dense in $\mathcal{P}_\alpha$. So from now on we shall always work with determined conditons, and use $\mathcal{P}_\alpha$ informally to denote also the set of all determined numbers of $\mathcal{P}_\alpha$.

Note that the function σ: $\mathcal{P}_\alpha \to \mathcal{C}(\alpha)$ is order preserving, and has the property that if $\sigma(p) = r$ and $r' \leqslant r$, then for some $p' \leqslant p$, $\sigma(p') = r'$. Hence, forcing with $\mathcal{P}_\alpha$ can be considered as forcing first with $\mathcal{C}(\alpha)$ and then with $\{p \in \mathcal{P}_\alpha : \sigma(p) \in G_{\mathcal{C}(\alpha)}\}$.

LEMMA 2. *For every $\dot{f} \in V^{\mathcal{P}_\alpha}$ and $p_0 \in \mathcal{P}_\alpha$ with the property $p_0 \Vdash_{\mathcal{P}_\alpha} \dot{f}\colon \omega \to On$, there are $\dot{g} \in V^{\mathcal{C}(\alpha)}$ and $p \leqslant p_0$ such that $\sigma(p) = \sigma(p_0)$ and $p \Vdash_{\mathcal{P}_\alpha} \dot{f} = \dot{g}$.*

PROOF. If $p \in \mathcal{P}_\alpha$ and if $r \in \mathcal{C}(\alpha)$ is compatible with $\sigma(p)$, then $p \wedge r$ denotes the following member q of $\mathcal{P}_\alpha$. If $\beta \in O$, then $q(\beta) = p(\beta)$. If $\beta \in E$, then $q(\beta) = p(\beta) \cup (r \restriction \{\beta\} \times \omega_1)$. It is clear that this is a well-defined condition and that $p \wedge r \leqslant p$.

CLAIM 2. Assume $p, p' \in \mathcal{P}_\alpha$ are such that $p' \leqslant p$. Then there exists a $q \in \mathcal{P}_\alpha$, with $q \leqslant p$ such that $\sigma(q) = \sigma(p)$, and $p' = q \wedge \sigma(p')$.

PROOF. We define $q \restriction \beta$ by induction on $\beta < \alpha$. Assume $q \restriction \beta$ is defined. If $\beta \in E$, let $q(\beta) = p(\beta)$. If $\beta \in O$, let $q(\beta)$ be a $\mathcal{P}_\beta$-name for a member of the Jensen poset $\dot{\mathfrak{S}}_\beta$ which is equal to $p'(\beta)$ if $p' \restriction \beta$ is a member of $G_{\mathcal{P}_\beta}$, and equal to $p(\beta)$ otherwise. If β is a limit ordinal, let $q \restriction \beta = \bigcup\{q \restriction \gamma : \gamma < \beta\}$. Now by induction on $\beta < \alpha$ one easily checks that

$$q \restriction \beta \in \mathcal{P}_\beta, \quad q \restriction \beta \leqslant p \restriction \beta, \quad \sigma(q \restriction \beta) = \sigma(p \restriction \beta),$$

and

$$p' \restriction \beta \leqslant (q \wedge \sigma(p')) \restriction \beta \leqslant p' \restriction \beta.$$

This completes the proof of Claim 2.

Starting from p_0, by induction on $n < \omega$, we define a sequence $\langle p_n : n < \omega \rangle$ of members of $\mathcal{P}_\alpha$, and for each $n < \omega$ sequences $\langle r_\xi^n : \xi < \delta_n \rangle$ and $\langle x_\xi^n : \xi < \delta_n \rangle$ of members of $\mathcal{C}(\alpha)$ and On, respectively, such that

(1) $p_{n+1} \leqslant p_n$,
(2) $\sigma(p_{n+1}) = \sigma(p_n)$,
(3) $p_{n+1} \wedge r_\xi^n \Vdash_{\mathcal{P}_\alpha} \dot{f}(n) = x_\xi^n$,
(4) $\{r_\xi^n : \xi < \delta_n\}$ is a maximal antichain below $\sigma(p_n)$.

Let us first see how to prove the lemma using such sequences. By induction on $\beta \leqslant \alpha$ we construct $p \restriction \beta \in \mathcal{P}_\beta$ such that $p \restriction \beta \leqslant p_n \restriction \beta$ and $\sigma(p \restriction \beta) = \sigma(p_n \restriction \beta)$ for all $n < \omega$. Assume $p \restriction \beta$ is constructed. If $\beta \in E$, let $p(\beta) = p_0(\beta)$. If $\beta \in O$, let $p(\beta)$ be a $\mathcal{P}_\beta$-name for the greatest lower bound of $\{p_n(\beta) : n < \omega\}$ in $\dot{\mathfrak{S}}_\beta$. If β is a limit ordinal, let $p \restriction \beta = \bigcup\{p \restriction \gamma : \gamma < \beta\}$. Then it is easily checked that $p \restriction \beta$ is a well-defined condition, and that $p = p \restriction \alpha$ has the properties $p \leqslant p_n$ and $\sigma(p) = \sigma(p_n)$ for all $n < \omega$. Since p is uniquely determined by $\langle p_n : n < \omega \rangle$, we shall denote it by $\bigwedge_{n<\omega} p_n$.

Define $\dot{g} \in V^{\mathcal{C}(\alpha)}$ to be a function from ω into the ordinals such that

$$\|\dot{g}(n) = x_\xi^n\| = r_\xi^n \quad \text{for } n < \omega \text{ and } \xi < \delta_n.$$

Then by (1)–(4) we have

$$p \Vdash_{\mathcal{P}_\alpha} \dot{f} = \dot{g}.$$

So we are left with the construction of $\langle p_n \colon n < \omega \rangle$. Assume p_n is defined. By induction on $\xi < \delta_n$ we define sequences $\langle q_\xi^n \colon \xi < \delta_n \rangle$, $\langle r_\xi^n \colon \xi < \delta_n \rangle$ of members of $\mathcal{P}_\alpha$, $\mathcal{C}(\alpha)$ and On, respectively, such that

(5) $q_\zeta^n \leqslant q_\xi^n \leqslant p_n$ for $\xi < \zeta$,

(6) $\sigma(q_\xi^n) = \sigma(p_n)$,

(7) $r_\xi^n \leqslant \sigma(p_n)$,

(8) $r_\xi^n \perp r_\zeta^n$ for $\xi \neq \zeta$,

(9) $q_\xi^n \wedge r_\xi^n \Vdash_{\mathcal{P}_\alpha} \dot{f}(n) = x_\xi^n$.

The ordinal δ_n is a countable ordinal determined by the fact that $\{r_\xi^n \colon \xi < \delta_n\}$ is a maximal antichain below $\sigma(p_n)$. Assume q_ξ^n's, r_ξ^n's and x_ξ^n's are defined for every $\xi < \zeta < \omega_1$. If $\{r_\xi^n \colon \xi < \zeta\}$ is a maximal antichain below $\sigma(p_n)$, we let $\delta_n = \zeta$ and $p_{n+1} = \bigwedge_{\xi<\delta_n} q_\xi^n$. Clearly (1)–(4) are satisfied. So let us assume $\{r_\xi^n \colon \xi < \zeta\}$ is not a maximal antichain below $\sigma(p_n)$. Pick $r \leqslant \sigma(p_n)$ which is incompatible with each r_ξ^n $(\xi < \zeta)$. Let $q = \bigwedge_{\xi<\zeta} q_\xi^n$. Choose $q' \leqslant q \wedge r$ and x_ζ^n such that $q' \Vdash_{\mathcal{P}_\alpha} f(n) = x_\zeta^n$. By Claim 2 we can find $q_\zeta^n \leqslant q$ such that $\sigma(q_\zeta^n) = \sigma(q) = \sigma(p_n)$ and $q' = q_\zeta^n \wedge \sigma(q')$. Let $r_\zeta^n = \sigma(q')$. It is clear that (5)–(9) are satisfied. This completes the proof of Lemma 2.

Note that, in particular, Lemma 2 shows that $\mathcal{P}_\alpha$ preserves $\aleph_1$. If $\alpha < \omega_2$, let

$$\mathcal{P}_{\alpha,\omega_2} = \{ p \restriction (\omega_2 \setminus \alpha) \colon p \in \mathcal{P}_\alpha \}.$$

Let $\mathcal{P}_{\alpha,\omega_2}$ be ordered in $V^{\mathcal{P}_\alpha}$ by the ordering $\dot{\leqslant}$ defined as follows:

$$q' \dot{\leqslant} q \quad \text{iff} \quad \exists p \in G_{\mathcal{P}_\alpha} \, (p \cup q' \leqslant p \cup q).$$

Then $\mathcal{P}_{\omega_2} = \mathcal{P}_\alpha * \mathcal{P}_{\alpha,\omega_2}$, and Lemma 2 easily gives

$$\Vdash_{\mathcal{P}_\alpha} \mathcal{P}_{\alpha,\omega_2} \text{ is equivalent to } \dot{\mathcal{P}}_{\omega_2}.$$

LEMMA 3. *Assume CH. Then $\mathcal{P}_{\omega_2}$ satisfies the $\aleph_2$-chain condition.*

PROOF. Since elements of $\mathcal{P}_{\omega_2}$ have countable supports, a standard application of Fodor's Lemma shows that we may restrict ourselves to proving that, for each $\alpha < \omega_2$, $\mathcal{P}_\alpha$ satisfies the $\aleph_2$-c.c.

So let $\alpha < \omega_2$ and let $\mathcal{D}_\alpha \subseteq \mathcal{P}_\alpha$ be the set of all $p \in \mathcal{P}_\alpha$ such that for every $\beta \in O$, there is a $\mathcal{C}(\beta)$-name a_p^β for a countable closed subset of ω_1 and a $\mathcal{P}_\beta$-name A_p^β for a closed and unbounded subset of ω_1 such that $p \restriction \beta \Vdash_{\mathcal{P}_\beta} p(\beta) = \langle a_p^\beta, A_p^\beta \rangle$.

CLAIM 3. For every $q \in \mathcal{P}_\alpha$ there is a $p \in \mathcal{D}_\alpha$ such that $p \leqslant q$ and $\sigma(p) = \sigma(q)$.

PROOF. We prove the claim by induction on α.

Assume $\alpha = \beta + 1$. If $\beta \in E$, there is nothing to be proved. So assume $\beta \in O$. By Lemma 2 we can find $q' \leqslant q \restriction \beta$ and a $\mathcal{C}(\beta)$-name a_p^β such that $\sigma(q') = \sigma(q \restriction \beta) = \sigma(q)$ and

$$q' \Vdash_{\mathcal{P}_\beta} \text{the first coordinate of } q(\beta) \text{ is equal to } a_p^\beta.$$

By the induction hypothesis we can find a $p' \in \mathcal{D}_\beta$ such that $p' \leqslant q'$ and $\sigma(p') = \sigma(q')$. Let

$$p = p' \cup \left\{ \left(\beta, \left\langle a_p^\beta, A_p^\beta \right\rangle \right) \right\},$$

where A_p^β is a $\mathcal{P}_\beta$-name such that $\Vdash_{\mathcal{P}_\beta} A_p^\beta$ is the second coordinate of $q(\beta)$. Then $p \in \mathcal{D}_\alpha$, $p \leq q$ and $\sigma(p) = \sigma(q)$.

First assume $\operatorname{cf} \alpha = \omega$. Let $\langle \alpha_n \colon n < \omega \rangle$ be a strictly increasing sequence of ordinals cofinal with α such that $\alpha_0 = 0$. By induction on $n < \omega$ we construct a sequence $\langle p_n \colon n < \omega \rangle$ of elements of $\mathcal{P}_\alpha$ such that $p_0 = q$ and

(1) $p_{n+1} \leq p_n$ and $\sigma(p_{n+1}) = \sigma(p_n)$,

(2) $p_{n+1} \restriction \alpha_{n+1} \in \mathcal{D}_{\alpha_{n+1}}$.

Assume p_n is defined. By the induction hypothesis we can find $p'_{n+1} \in \mathcal{D}_{\alpha_{n+1}}$ such that $p'_{n+1} \leq p_n \restriction \alpha_{n+1}$ and $\sigma(p'_{n+1}) = \sigma(p_n \restriction \alpha_{n+1})$. Let $p_{n+1} \in \mathcal{P}_\alpha$ be defined by $p_{n+1} \restriction \alpha_{n+1} = p'_{n+1}$ and $p_{n+1}(\beta) = p_n(\beta)$ for all $\beta \in [\alpha_{n+1}, \alpha)$. Then $p_{n+1} \leq p_n$, $\sigma(p_{n+1}) = \sigma(p_n)$, and $p_{n+1} \restriction \alpha_{n+1} \in \mathcal{D}_{\alpha_{n+1}}$.

Define $p \in \mathcal{P}_\alpha$ as follows. If $\beta \in E$, let $p(\beta) = q(\beta)$. So suppose $\beta \in O$, and let $n < \omega$ be such that $\beta \in [\alpha_n, a_{n+1})$. Let a_p^β be a $\mathcal{C}(\beta)$-name for the closure of

$$\bigcup \left\{ a_{p_i}^\beta \colon n < i < \omega \right\},$$

and let A_p^β be a $\mathcal{P}_\beta$-name for $\bigcap \{ A_{p_i}^\beta \colon n < i < \omega \}$. Let $p(\beta) = \langle a_p^\beta, A_p^\beta \rangle$. Then $p \in \mathcal{D}_\alpha$, $p \leq q$ and $\sigma(p) = \sigma(q)$.

Now assume $\operatorname{cf} \alpha > \omega$. Since $\operatorname{supp}(q)$ is countable, there is a a $\gamma < \alpha$ such that $\operatorname{supp}(q) \subseteq \gamma$. Using the induction hypothesis, we can find a $p' \in \mathcal{D}_\gamma$ such that $p' \leq q \restriction \gamma$ and $\sigma(p') = \sigma(q \restriction \gamma) = \sigma(q)$. Define $p \in \mathcal{P}_\alpha$ by $p \restriction \gamma = p'$ and $p(\beta) = \varnothing$ for $\beta \in [\gamma, \alpha) \cap E$ and $p(\beta) = \langle \varnothing, \omega_1 \rangle$ for $\beta \in [\gamma, \alpha) \cap O$. Then $p \in \mathcal{D}_\alpha$, $p \leq q$ and $\sigma(p) = \sigma(q)$. This proves the claim.

Suppose $p, q \in \mathcal{D}_\alpha$ are such that $p(\beta) = q(\beta)$ for every $\beta \in E$ and $\Vdash_{\mathcal{C}(\beta)} a_p^\beta = a_q^\beta$ for every $\beta \in O$. We claim that then p and q are compatible in $\mathcal{P}_\alpha$. To see this let us define $p' \in \mathcal{D}_\alpha$ as follows. If $\beta \in E$, let $p'(\beta) = p(\beta) = q(\beta)$. If $\beta \in O$, we choose $p'(\beta)$ to satisfy $\Vdash_{\mathcal{P}_\beta} p'(\beta) = \langle a_p^\beta, A_p^\beta \cap A_q^\beta \rangle$. Then clearly $p' \in \mathcal{D}_\alpha$ and $p' \leq p, q$. Since there are only $\aleph_1$ $\mathcal{C}(\alpha)$-names of countable closed subsets of ω_1, this finishes the proof of Lemma 3.

Now we are ready to finish the proof of Theorem 1. Assume GCH holds. Let $\langle \mathcal{P}_\alpha \colon \alpha \leq \omega_2 \rangle$ be the iteration defined above and let $\mathcal{P} = \mathcal{P}_{\omega_2}$. Then in $V^{\mathcal{P}}$, $2^{\aleph_0} = 2^{\aleph_1} = \aleph_2$ holds. Working in $V^{\mathcal{P}}$, we define a finite support iteration $\langle \dot{\mathcal{Q}}_\xi \colon \xi \leq \omega_2 \rangle$ of c.c.c. posets of size $\leq \aleph_1$ à la Solovay and Tennenbaum [33], such that if $\dot{\mathcal{Q}} = \dot{\mathcal{Q}}_{\omega_2}$, then $V^{\mathcal{P} * \dot{\mathcal{Q}}}$ satisfies (i)–(iii) of Theorem 1.

Assume $\xi < \omega_2$ and that in $V^{\mathcal{P} * \dot{\mathcal{Q}}_\xi}$ we have a partition $[\omega_1]^2 = \dot{K}_0 \cup \dot{K}_1$ with no bad sets. Pick an even ordinal $\alpha < \omega_2$ such that $\dot{\mathcal{Q}}_\xi, \dot{K}_0, \dot{K}_1 \in V^{\mathcal{P}_\alpha}$. We have already remarked that $\mathcal{P}_{\alpha,\omega_2}$ is, in $V^{\mathcal{P}_\alpha}$ (equivalent to) a mixed iteration of length ω_2 of Cohen and Jensen posets. It begins by first introducing $\aleph_1$ Cohen reals and then adding a Jensen club. So by Lemma 1, the poset $\dot{\mathcal{S}}_\xi$ of all finite 0-homogeneous subsets of ω_1 which are separated by $C_{\dot{\mathcal{G}}_{\alpha+1}}$ is a c.c.c. poset in $V^{\mathcal{P}_{\alpha+2} * \dot{\mathcal{Q}}_\xi}$. Hence, one condition $s_0 \in \mathcal{S}_\xi$ forces that the generic object is uncountable. At the next step of the iteration we force with $\{ s \in \dot{\mathcal{S}}_\xi \colon s \supseteq s_0 \}$, which we again denote by $\dot{\mathcal{S}}_\xi$. We know that $\dot{\mathcal{Q}}_\xi * \dot{\mathcal{S}}_\xi$ is a c.c.c. poset in $V^{\mathcal{P}_{\alpha+2}}$, but we have to show that it remains c.c.c after forcing with $\mathcal{P}_{\alpha+2,\omega_2}$. Let us prove the following more general fact. Let $\bar{\mathcal{Q}}$ be an arbitrary c.c.c. poset. Then $\bar{\mathcal{Q}}$ remains c.c.c. after forcing with $\mathcal{P} = \mathcal{P}_{\omega_2}$. Otherwise, pick a $\mathcal{P}$-name

$\langle \dot{q}_\gamma : \gamma < \omega_1 \rangle$ for an ω_1-sequence of incompatible members of $\bar{\mathcal{Q}}$. As in the proof of Lemma 2, by induction on γ we construct a decreasing sequence $\langle p_\gamma : \gamma < \omega_1 \rangle$ of members of $\mathcal{P}$, a sequence $\langle r_\gamma : \gamma < \omega_1 \rangle$ of members of $\mathcal{C}_{E\times\omega_1}$, and a sequence $\langle q_\gamma : \gamma < \omega_1 \rangle$ of members of $\bar{\mathcal{Q}}$ such that

(1) $\sigma(p_\gamma) = \sigma(p_\delta)$ for $\gamma < \delta < \omega_1$,

(2) $p_\gamma \wedge r_\gamma \Vdash_{\mathcal{P}} \dot{q}_\gamma = q_\gamma$.

Pick an $A \in [\omega_1]^{\aleph_1}$ such that r_γ and r_δ are compatible, whenever $\gamma, \delta \in A$. Then it is easily checked that we have reached a contradiction since $\{q_\gamma : \gamma \in A\}$ is an uncountable antichain of $\bar{\mathcal{Q}}$.

Similarly, one defines posets for getting $\omega_1 \overset{*}{\to} (\omega_1)^2_{<\aleph_0}$ as well as the posets for getting $MA_{\aleph_1}$. This completes the proof of Theorem 1.

The following result (in ZFC) is an easy consequence of Lemma 1.

THEOREM 2. *$\omega_1 \overset{*}{\to} (\text{closed } \alpha)^2_{<\aleph_0}$ for all $\alpha < \omega_1$.*

PROOF. Let $[\omega_1]^2 = \bigcup_{\xi<\omega_1} J_\xi$ be a given disjoint partition such that $|J_\xi| < \aleph_0$ for all $\xi < \omega_1$. Let $\omega \leqslant \alpha < \omega_1$ be fixed. Let $\dot{\mathcal{S}} \in V^{\mathcal{C}_{\omega_1} * \dot{\mathcal{G}}}$ be the set of all finite subsets of ω_1 separated by $C_{\dot{\mathcal{G}}}$ such that $|[s]^2 \cap J_\xi| \leqslant 1$ for all $\xi < \omega_1$. By Lemma 1, $\dot{\mathcal{S}}$ is a c.c.c. poset in $V^{\mathcal{C}_{\omega_1} * \dot{\mathcal{G}}}$. So we can find an $s_0 \in \dot{\mathcal{S}}$ such that $s_0 \Vdash_{\dot{\mathcal{S}}} \bigcup G_{\dot{\mathcal{S}}}$ is stationary in ω_1. Thus, in particular, we can find a $p_0 \in \mathcal{C}_{\omega_1} * \dot{\mathcal{G}} * \dot{\mathcal{S}}$ and a $(\mathcal{C}_{\omega_1} * \dot{\mathcal{G}} * \dot{\mathcal{S}})$-name $\dot{A}$ such that

$$p_0 \Vdash \dot{A} \text{ is a closed subset of } \omega_1 \text{ of type } \alpha + 1 \,\&\, \forall \xi < \omega_1 \left|[\dot{A}]^2 \cap J_\xi\right| \leqslant 1.$$

Pick a $(\mathcal{C}_{\omega_1} * \dot{\mathcal{G}} * \dot{\mathcal{S}})$-name $\dot{f}$ such that $p_0 \Vdash \dot{f} : \alpha + 1 \to \dot{A}$ is the unique isomorphism. Let $\langle \alpha_n : n < \omega \rangle$ be an enumeration of $\alpha + 1$. Now by induction on $n < \omega$ we define a decreasing sequence $\langle p_n : n < \omega \rangle$ of elements of $\mathcal{C}_{\omega_1} * \dot{\mathcal{G}} * \dot{\mathcal{S}}$ and a sequence $\langle \beta_n : n < \omega \rangle$ of ordinals $< \omega_1$ such that $p_{n+1} \Vdash \dot{f}(\alpha_n) = \beta_n$, making sure that $B = \{\beta_n : n < \omega\}$ is a closed subset of ω_1. Then $\operatorname{tp} B = \alpha + 1$ and $\forall \xi < \omega_1 \ |[B]^2 \cap J_\xi| \leqslant 1$. This completes the proof.

REMARKS. (1) The closed unbounded set poset was defined and first used in buildling c.c.c. posets in the extension by Jensen [**8**]. The fact that an elementary chain of submodels is useful in proving the c.c.c. property of posets with separated conditions was first realized by Shelah [**1**, **3**]. The use of the Cohen generic reals in building conditions in σ-closed posets was first made explicit by Avraham [**2**]. The first mixed iteration of Cohen posets and σ-closed posets was defined by Mitchell [**26**]. It is clear that if we want only to preserve $\aleph_1$, then in the above mixed iteration the Jensen posets can be replaced by any σ-closed poset. If we want the iteration to have the $\aleph_2$-c.c., the σ-closed posets must satisfy one of the standard strong $\aleph_2$-chain conditions.

(2) The posets involved in Lemma 1 can also be iterated in a countable support iteration $\langle \mathcal{T}_\alpha : \alpha \leqslant \omega_2 \rangle$. A Laver type argument shows that $\mathcal{T}_{\omega_2}$ preserves $\aleph_1$ [**5**, **25**]. Using GCH, one then shows that $\mathcal{T}_{\omega_2}$ satisfies the $\aleph_2$-c.c.

(3) If we are not interested in the exact equiconsistency result, we could use the Proper Forcing Axiom (PFA; [**6**, **7**, **31**]) in showing that (i)–(iii) of Theorem 1 are

consistent. Namely, in this case, in Lemma 1, we can disregard $\mathcal{Q}$ and $\mathcal{C}_{\omega_1}$ and directly show by the same proof that $\dot{\mathbb{S}}_0$ and $\dot{\mathbb{S}}_1$ are c.c.c. posets in $V^{\mathcal{E}}$. To build a condition which will meet all the $\mathcal{W}_F$'s, we need only use $MA_{\aleph_1}$, a consequence of PFA.

(4) It is clear that the proof of Lemma 1 also shows that each (finite) power of the poset $\mathbb{S}_1$ satisfies the c.c.c. Hence the model of Theorem 1 can also satisfy the following partition property of ω_1 stronger than $\omega_1 \overset{*}{\to} (\omega_1)^2_{<\aleph_0}$:

> *If* $[\omega_1]^2 = \bigcup_{i\in I} K_i$ *is a disjoint partition where each* K_i *is finite, then there is a decomposition* $\omega_1 = \bigcup_{n<\omega} A_n$ *such that*
>
> $$\forall n < \omega\, \forall i \in I \left|[A_n]^2 \cap K_i\right| \leq 1.$$

2. This section begins with a discussion of the partition relation $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\, \omega_1))^2$ and ends with the applications mentioned in the introduction.

For $G \subseteq [\omega_1]^2$, Chr(G) denotes the chromatic number of G and equals the minimal cardinal κ for which there is a partition $\omega_1 = \bigcup_{\xi<\kappa} A_\xi$ such that $[A_\xi]^2 \cap G = \varnothing$ for all $\xi < \kappa$.

THEOREM 3. *Assume* $MA_{\aleph_1}$ *and* $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\, \omega_1))^2$. *Then for every* $G \subseteq [\omega_1]^2$ *either* Chr$(G) \leq \aleph_0$, *or else there is an* $A \in [\omega_1]^{\aleph_1}$ *and a family* $\mathcal{B}$ *of* $\aleph_1$ *disjoint finite subsets of* ω *such that* $(\{\alpha\} \otimes F) \cap G \neq \varnothing$ *for all* $\alpha \in A$ *and* $F \in \mathcal{B}$ *with* $\alpha < \min F$.

PROOF. Given $G \subseteq [\omega_1]^2$, let $\mathcal{P}$ be the set of all finite $p \subseteq \omega_1$ such that $[p]^2 \cap G = \varnothing$. The ordering on $\mathcal{P}$ is $\supseteq$.

If $\mathcal{P}$ is a c.c.c. poset, then by $MA_{\aleph_1}$, $\mathcal{P}$ is σ-centered, so Chr$(G) \leq \aleph_0$.

Hence, we may assume $\mathcal{P}$ is not a c.c.c. poset. Let $\{p_\alpha: \alpha < \omega_1\}$ be an uncountable antichain of $\mathcal{P}$. A standard Δ-system argument shows we may assume the p_α's are disjoint, strictly increasing and of the same cardinality n ($1 \leq n < \omega$). Let $\langle p_\alpha(i): i < n\rangle$ be the strictly increasing enumeration of p_α, ($\alpha < \omega_1$). For each $\alpha < \beta < \omega_1$, there exist $i, j < n$ such that $\{p_\alpha(i), p_\beta(j)\} \in G$. This gives a coloring of $[\omega_1]^2$ into n^2 colors. Now an easy application of $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\, \omega_1))^2$ completes the proof of Theorem 3.

A consequence of Theorem 3 is that, in the model of §1,

$$\omega_1 \to (\text{stationary}, (\omega_1; \text{fin}\, \omega_1))^2$$

holds. However, an examination of the proof of Theorem 1 shows that, in fact, in this model, the stronger relation

$$\omega_1 \to (\text{stationary}, (\text{stationary}; \text{fin}\, \omega_1))^2$$

holds. Let us also mention the following strengthening (*) of $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\, \omega_1))^2$ in a dual direction. The consistency of this strengthening will appear in a later paper.

(*) For every partition $[\omega_1]^2 = K_0 \cup K_1$ either there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$, or else there exist $\langle A_n: n < \omega\rangle$ and $\langle \mathcal{B}_n: n < \omega\rangle$ such that:

(i) $\omega_1 \setminus \bigcup_{n<\omega} A_n$ is countable;

(ii) $\mathcal{B}_n$ is a family of $\aleph_1$ disjoint finite subsets of ω_1;

(iii) $(\{\alpha\} \otimes F) \cap K_1 \neq \varnothing$ for all $\alpha \in A_n$ and $F \in \mathcal{B}_n$ with $\alpha < \min F$.

Let us note that it is not possible to strengthen $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ at the same time in both the direction of Theorem 3 and that of (*), i.e., there is a partition $[\omega_1]^2 = K_0 \cup K_1$ with no stationary 0-homogeneous sets, but ω_1 is not a countable union of bad sets. A proof of this simple fact will also appear elsewhere.

THEOREM 4. *Assume $MA_{\aleph_1}$ and $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$. Then for every partition $[\omega_1]^2 = K_0 \cup K_1$ either there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$, or else for every $\alpha < \omega_1$ there are $B, C \subseteq \omega_1$ such that* $\text{tp}\, B = \alpha$, $|C| = \aleph_1$ *and* $[B]^2 \cup (B \otimes C) \subseteq K_1$.

In particular we have the following consequence mentioned in the introduction.

THEOREM 5. *Assume $MA_{\aleph_1}$. Then $\omega_1 \to (\omega_1, (\omega_1; \text{fin } \omega_1))^2$ implies $\omega_1 \to (\omega_1, \alpha)^2$ for all $\alpha < \omega_1$.*

PROOF OF THEOREM 4: Let $[\omega_1]^2 = K_0 \cup K_1$ be a given partition, and assume $[A]^2 \not\subseteq K_0$ for all $A \in [\omega_1]^{\aleph_1}$. For each $\alpha < \omega_1$ we shall construct $B, C \subseteq \omega_1$ such that $\text{tp}\, B = \omega^\alpha$, $|C| = \aleph_1$ and $[B]^2 \cup (B \otimes C) \subseteq K_1$. First we need some technical definitions and facts.

For each $1 \leqslant \alpha < \omega_1$ we fix a nondecreasing sequence $\langle \alpha(n) \colon n < \omega \rangle$ of smaller ordinals such that $\omega^\alpha = \Sigma_{n<\omega} \omega^{\alpha(n)}$, and if $\alpha > 1$, then $\alpha(0) \geqslant 1$. Also for every set $B \subseteq \omega_1$ of type ω^α we fix a decomposition $B = \bigcup_{n<\omega} B(n)$ such that

$$B(0) < \cdots < B(n) < \cdots \quad \text{and} \quad \text{tp}\, B(n) = \omega^{\alpha(n)}.$$

Let $\mathcal{V}$ be a fixed nonprincipal ultrafilter on ω_1. By induction on $1 \leqslant \alpha < \omega_1$ we define a nonprincipal ultrafilter $\mathcal{U}_\alpha(B)$ on every set $B \subseteq \omega_1$ of type ω^α. If $\alpha = 1$, then the isomorphism of ω and B induces $\mathcal{U}_\alpha(B)$. So now assume $1 < \alpha < \omega_1$ and define

$$D \in \mathcal{U}_\alpha(B) \quad \text{iff} \quad \{n < \omega \colon D \cap B(n) \in \mathcal{U}_{\alpha(n)}(B(n))\} \in \mathcal{V}.$$

By induction on α it easily follows that $\text{tp}\, D = \omega^\alpha$ for every $D \in \mathcal{U}_\alpha(B)$. The following lemma is due to Hajnal [**24**, p. 1031]. For the sake of completeness we sketch the proof.

CLAIM 4. Let $1 \leqslant \alpha < \omega_1$ and let $B \subseteq \omega_1$ have type ω^α. Let $\langle D_\xi \colon \xi < \omega_1 \rangle$ be a sequence of elements of $\mathcal{U}_\alpha(B)$. Then there exists a $D \subseteq B$, with $\text{tp}\, D = \omega^\alpha$ such that $D \setminus D_\xi$ is a bounded subset of D for every $\xi < \omega_1$.

PROOF. The proof is by induction on α. The case $\alpha = 1$ is a well-known consequence of $\text{MA}_{\aleph_1}$. So let $1 < \alpha < \omega_1$. By the induction hypothesis, for each $n < \omega$, there is an $E_n \subseteq B(n)$ of type $\omega^{\alpha(n)}$ such that $E_n \setminus D_\xi$ is bounded in E_n for all $\xi < \omega_1$ with the property

$$n \in N_\xi = \{m < \omega \colon D_\xi \cap B(m) \in \mathcal{U}_{\alpha(m)}(B(m))\}.$$

Now for each $\xi < \omega_1$ we fix $f_\xi \in {}^\omega\omega$ with the property that for every $n \in N_\xi$, the $f_\xi(n)$-end-section of E_n is a subset of D_ξ. Let $N \subseteq \omega$ be an infinite set almost included in each N_ξ, and let $f \in {}^\omega\omega$ eventually dominate each f_ξ. For $n \in N$, let D_n be the $f(n)$-end-section of E_n. Let $D = \bigcup_{n \in N} D_n$. Then D is as required.

Now we are ready for the proof of Theorem 4. By induction on $\alpha < \omega_1$, for each $A \in [\omega_1]^{\aleph_1}$ we shall construct $B, C \subseteq A$ such that $\text{tp}\, B = \omega^\alpha$, $|C| = \aleph_1$ and $[B]^2 \cup (B \otimes C) \subseteq K_1$. Since the case $\alpha = 0$ is trivial we assume $1 \leqslant \alpha < \omega_1$. Let $A \in [\omega_1]^{\aleph_1}$

be given. By $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\,\omega_1))^2$ there is an $A_0 \in [A]^{\aleph_1}$ and a family $\mathfrak{B}$ of $\aleph_1$ disjoint finite subsets of A such that $(\{\beta\} \otimes F) \cap K_1 \neq \varnothing$ for all $\alpha \in A_0$ and $F \in \mathfrak{B}$ with $\alpha < \min F$.

Using the induction hypothesis we recursively construct sets B_n, $C_n \subseteq A_0$ $(n < \omega)$ such that:

(1) $\text{tp}\, B_n = \omega^{\alpha(n)}$ & $|C_n| = \aleph_1$;
(2) $B_n < B_{n+1}$ & $C_n \supsetneq C_{n+1}$;
(3) $B_{n+1} \subseteq C_n$;
(4) $[B_n]^2 \cup (B_n \otimes C_n) \subseteq K_1$.

Let $B = \bigcup_{n<\omega} B$. Then $\text{tp}\, B = \omega^\alpha$ and $[B]^2 \subseteq K_1$. Pick $F \in \mathfrak{B}$ such that $\sup B \leqslant \min F$. By the assumptions on A_0 and $\mathfrak{B}$, we have that $B \subseteq \bigcup_{\gamma \in F} K_1(\gamma)$. Hence for some $\gamma = \gamma(F) \in F$, we have $K_1(\gamma) \cap B \in \mathfrak{U}_\alpha(B)$. By Claim 4 there is a $D \subseteq B$ with $\text{tp}\, D = \omega^\alpha$ such that $D \setminus K_1(\gamma(F))$ is bounded in D for every $F \in \mathfrak{B}$, with $\sup B \leqslant \min F$. Thus, for some uncountable $\mathfrak{B}_0 \subseteq \mathfrak{B}$ and $\delta \in D$ we have $D \setminus \delta \subseteq K_1(\gamma(F))$ and $\sup B \leqslant \min F$ for all $F \in \mathfrak{B}_0$. Let $B^* = D \setminus \delta$ and $C^* = \{\gamma(F): F \in \mathfrak{B}_0\}$. Then B^*, $C^* \subseteq A$, $\text{tp}\, B^* = \omega^\alpha$, $|C^*| = \aleph_1$, and $[B^*]^2 \cup (B^* \otimes C^*) \subseteq K_1$. This completes the proof.

Let us now consider the following combinatorial principle introduced by Fred Galvin:

(∗∗) There are ideals $\mathfrak{I}, \mathfrak{J} \subseteq \mathcal{P}(\omega_1)$ such that:
(i) $\mathfrak{I} \cap \mathfrak{J} = [\omega_1]^{<\aleph_0}$;
(ii) $\mathfrak{I} \vee \mathfrak{J} = \{A \cup B: A \in \mathfrak{I}\, \& \, B \in \mathfrak{J}\} = [\omega_1]^{\leqslant\aleph_0}$;
(iii) $\forall A \in [\omega_1]^{\aleph_1} ([A]^{\aleph_0} \cap \mathfrak{I} \neq \varnothing \,\&\, [A]^{\aleph_0} \cap \mathfrak{J} \neq \varnothing)$.

Galvin proved that ♣ implies (∗∗) and that (∗∗) has some topological applications [**15**, Theorem 4]. He also asked for the consistency of $\neg$(∗∗). The next result shows that $\neg$(∗∗) is consistent.

THEOREM 6. *$\omega_1 \to (\omega_1; \text{fin}\,\omega_1)^2_2$ implies $\neg$(∗∗).*

PROOF. Let $\mathfrak{I}$ and $\mathfrak{J}$ be ideals satisfying (i) and (ii) of (∗∗). For each $\alpha < \omega_1$ we can find disjoint $A_\alpha \in \mathfrak{I}$ and $B_\alpha \in \mathfrak{J}$ such that $A_\alpha \cup B_\alpha = \alpha$. Define $[\omega_1]^2 = K_0 \cup K_1$ by

$$\{\beta, \alpha\}_< \in K_0 \quad \text{iff} \quad \beta \in A_\alpha.$$

Since $\omega_1 \to (\omega_1; \text{fin}\,\omega_1)^2_2$ holds, we consider the following two cases:

Case I. There is an $A \in [\omega_1]^{\aleph_1}$ and a family $\mathfrak{B}$ of $\aleph_1$ disjoint finite subsets of ω_1 such that $(\{\alpha\} \otimes F) \cap K_0 \neq \varnothing$ for all $\alpha \in A$ and $F \in \mathfrak{B}$ with $\alpha < \min F$.

For $F \in \mathfrak{B}$ we define $A(F) = \bigcup\{A_\alpha: \alpha \in F\}$. Then $A(F) \in \mathfrak{I}$ and $A \cap \min F \subseteq A(F)$ for each $F \in \mathfrak{B}$. Hence $[A]^{\aleph_0} \subseteq \mathfrak{I}$, contradicting the conjunction of (i) and (iii). This shows that (∗∗) fails in this case.

Case II. There is an $A \in [\omega_1]^{\aleph_1}$ and a family $\mathfrak{B}$ of $\aleph_1$ disjoint finite subsets of ω_1 such that $(\{\alpha\} \otimes F) \cap K_1 \neq \varnothing$ for all $\alpha \in A$ and $F \in \mathfrak{B}$ with $\alpha < \min F$.

Proceeding as in Case I we show that here $[A]^{\aleph_0} \subseteq \mathfrak{J}$, which again contradicts (∗∗). This completes the proof.

The remainder of this section is devoted to the topological applications mentioned in the introduction.

THEOREM 7. *Assume $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\,\omega_1))^2$. Let X be a toplogical space with no uncountable discrete subspaces. Let $\mathcal{U}$ be a family of open subsets of X such that $\bigcup \mathcal{U} = X$. Then there is a countable $\mathcal{U}_0 \subseteq \mathcal{U}$ such that $X = \bigcup\{\overline{U}: U \in \mathcal{U}_0\}$.*

PROOF. Assume by way of contradiction that, for every countable $\mathcal{U}_0 \subseteq \mathcal{U}$, $X \neq \bigcup\{\overline{U}: U \in \mathcal{U}_0\}$. Then by induction on $\alpha < \omega_1$, we can easily construct sequences $\langle U_\alpha: \alpha < \omega_1\rangle$ and $\langle x_\alpha: \alpha < \omega_1\rangle$ of members of $\mathcal{U}$ and X, respectively, such that

(1) $x_\alpha \in U_\alpha$,

(2) $x_\alpha \notin \bigcup\{\overline{U_\beta}: \beta < \alpha\}$.

Define $[\omega_1]^2 = K_0 \cup K_1$ by

$$\{\beta, \alpha\}_< \in K_0 \quad \text{iff} \quad x_\beta \notin U_\alpha.$$

Since $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\,\omega_1))^2$ holds, we consider the following two cases:

Case I. There is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$. Then for every $\alpha \in A$, $U_\alpha \cap \{x_\beta: \beta \in A\} = \{x_\alpha\}$. Hence $\{x_\alpha: \alpha \in A\}$ is an uncountable discrete subspace of X, a contradiction.

Case II. There is an $A \in [\omega_1]^{\aleph_1}$ and a family $\mathcal{B}$ of $\aleph_1$ disjoint subsets of ω_1 such that $(\{\alpha\} \otimes F) \cap K_1 \neq \varnothing$ for all $\alpha \in A$ and $F \in \mathcal{B}$ with $\alpha < \min F$. For $F \in \mathcal{B}$ we define

$$U(F) = \bigcup_{\gamma \in F} U_\gamma.$$

Then for each $F \in \mathcal{B}$,

$$\{x_\alpha: \alpha \in A \cap \min F\} \subseteq U(F).$$

Choose inductively an $A_0 \in [A]^{\aleph_1}$ and, for each $\alpha \in A_0$, an $F_\alpha \in \mathcal{B}$ such that if $\beta < \alpha$ are in A_0, then

$$\max F_\beta < \beta < \min F_\alpha \leqslant \max F_\alpha < \alpha.$$

Then by (1) and (2), for each $\alpha \in A_0$,

$$\left(U_\alpha \setminus \overline{U(F_\alpha)}\right) \cap \{x_\beta: \beta \in A_0\} = \{x_\alpha\}.$$

Hence $\{x_\alpha: \alpha \in A_0\}$ is an uncountable discrete subspace of X, a contradiction. This completes the proof.

THEOREM 8. *Assume $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\,\omega_1))^2$. Then every regular toplogical space with no uncountable discrete subspace is hereditarily Lindelöf.*

PROOF. Clearly it suffices to show that X is Lindelöf. So let $\mathcal{U}$ be a family of open sets such that $\bigcup \mathcal{U} = X$. Since X is a regular space, there is a family $\mathcal{V}$ of open subsets of X such that $\bigcup \mathcal{V} = X$, and such that for every $W \in \mathcal{V}$ there is a $U(W) \in \mathcal{U}$ such that $U(W) \supseteq \overline{W}$. By Theorem 7 there is a countable $\mathcal{V}_0 \subseteq \mathcal{V}$ such that

$$X = \bigcup\{\overline{W}: W \in \mathcal{V}_0\}.$$

Hence $\mathcal{U}_0 = \{U(W): W \in \mathcal{V}_0\}$ is a countable subfamily of $\mathcal{U}$ such that $\bigcup \mathcal{U}_0 = X$. This completes the proof.

COROLLARY 9. *Assume $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\,\omega_1))^2$. Then every regular hereditarily separable topological space is hereditarily Lindelöf.*

THEOREM 10. *Assume $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\,\omega_1))^2$. Let X be a Hausdorff space with no uncountable discrete subspaces. Then every point of X is the intersection of countably many open subsets of X.*

PROOF. Fix $x \in X$. Since X is a Hausdorff space, for every $y \in X\backslash\{x\}$ there is an open set U_y such that $y \in U_y$ and $x \notin \overline{U}_y$. By Theorem 7, applied to the space $X\backslash\{x\}$, there is a countable $Y \subseteq X\backslash\{x\}$ such that $X\backslash\{x\} = \bigcup\{\overline{U}_y\colon y \in Y\}$. This shows that $\{x\}$ is a G_δ subset of X.

The following theorem is a simple consequence of Theorem 10 using a result of **[18]**. However, since the result we need is a relatively simple application of $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$, we shall give some details.

THEOREM 11. *Assume $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\,\omega_1))^2$. Then every Hausdorff space with no uncountable discrete subspaces has cardinality $\leqslant 2^{\aleph_0}$.*

PROOF. Assume by way of contradiction that X is a Hausdorff space of cardinality $> 2^{\aleph_0}$ with no uncountable discrete subspaces.

Let $<$ be a well-ordering of X. By Theorem 10, for each $x \in X$ we can fix a family $\{U_x^n\colon n < \omega\}$ of open subsets of X such that $\{x\} = \bigcap_{n<\omega} U_x^n$. For $m, n < \omega$ and $\{x, y\}_< \in [X]^2$ we let

$$\{x, y\} \in K_{m,n} \quad \text{iff} \quad x \notin U_y^n \;\&\; y \notin U_x^m.$$

Clearly, $[X]^2 = \bigcup_{m,n<\omega} K_{m,n}$. By $(2^{\aleph_0})^+ \to (\aleph_1)^2_{\aleph_0}$, there are $m', n' < \omega$ and $D \in [X]^{\aleph_1}$ such that $[D]^2 \subseteq K_{m',n'}$. For $x \in D$, let $W_x = U_x^{m'} \cap U_x^{n'}$. Then for each $x \in D$, $W_x \cap D = \{x\}$. Hence, D is a discrete subspace of X, a contradiction. This completes the proof.

We conclude the paper with a remark on the following partition relation (it is dual to $\omega_1 \to (\omega_1, (\omega_1; \text{fin}\,\omega_1))^2$), denoted by

$$\omega_1 \to (\omega_1, (\text{fin}\,\omega_1; \omega_1))^2.$$

This relation means: For every partition $[\omega_1]^2 = K_0 \cup K_1$ either

(1) there is an $A \in [\omega_1]^{\aleph_1}$ such that $[A]^2 \subseteq K_0$, or

(2) there is a family $\mathscr{Q}$ of $\aleph_1$ disjoint finite subsets of ω_1 and a set $B \in [\omega_1]^{\aleph_1}$ such that $(F \otimes \{\beta\}) \cap K_1 \neq \varnothing$ for all $F \in \mathscr{Q}$ and $\beta \in B$ with $\max F < \beta$.

The consistency of $\omega_1 \to (\omega_1, (\text{fin}\,\omega_1; \omega_1))^2$ is an open problem. It is easily seen that $\omega_1 \to (\omega_1, (\text{fin}\,\omega_1; \omega_1))^2$ implies the dual statement of Theorem 8, i.e., that every regular space with no uncountable discrete subspaces is hereditarily separable.

The reader interested in the role of $\mathrm{MA}_{\aleph_1}$ in the problems we have considered here can find some information in **[4]**.

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