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*Research Article*

# Forcing Strong Convergence of a Mann-Based Iteration for Nonexpansive and Monotone Operators in a Hilbert Space

**Songtao Lv**

*School of Mathematics and Statistics, Shangqiu Normal University, Shangqiu, China*

Correspondence should be addressed to Songtao Lv; sqlvst@yeah.net

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Mann iteration is weakly convergent in infinite dimensional spaces. We, in this paper, use the nearest point projection to force the strong convergence of a Mann-based iteration for nonexpansive and monotone operators. A strong convergence theorem of common elements is obtained in an infinite dimensional Hilbert space. No compact conditions are needed.

## 1. Introduction: Preliminaries

In the real world, there are a lot of nonlinear phenomena, which can be modelled into variational inequalities and variational inclusions, such as signal processing, image recovery, and machine learning; see, e.g., [1–7] and the references therein. Fixed point methods are powerful and popular for dealing various nonlinear operator equations and inequalities in abstract spaces, in particular, for variational inequalities and variational inclusions. Recently, various efficient fixed point methods have been introduced and investigated; see, e.g., [8–13] and the references therein. Let $T$ be a nonlinear operator on a Hilbert space $H$, which is endowed with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\| \cdot \|$. The fixed point set of $T$ is presented by $\mathrm{Fix}(T)$. Recall that $T$ is said to be contractive iff there is a real number $a \in (0,1)$ such that

$$\|Tx - Ty\| \le a\|x - y\|, \quad \forall x, y \in H. \tag{1}$$

Recall that $T$ is said to be nonexpansive iff

$$\|Tx - Ty\| \le \|x - y\|, \quad \forall x, y \in H. \tag{2}$$

Recall that $T$ is said to be firmly nonexpansive iff

$$\|Tx - Ty\|^2 \le \langle x - y, Tx - Ty \rangle, \quad \forall x, y \in H. \tag{3}$$

It is clear that the class of firmly nonexpansive mappings is a special class of nonexpansive mappings. One knows the projection operator (see below) is firmly nonexpansive. The class of nonexpansive operators is significant in various nonlinear equations and mathematical programming computation. It also has wide real applications in applied and industrial fields. For various iterative methods, Mann iteration is popular for dealing with fixed points of nonexpansive operators. It reads

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n x_n, \tag{4}$$

where $\{\alpha_n\}$ is a real number sequence in the interval $(0,1)$. However, the Mann iteration is weakly convergent only in infinite dimensional spaces; see, e.g., [14] and the references therein. To force the strong convergence without possible compact assumptions, various regularized methods have been investigated in Hilbert spaces and Banach spaces recently; see, e.g., [15–19] and the references therein. One of the efficient regularized methods is the Halpern iteration, which reads

$$x_{n+1} = (1 - \alpha_n)Tx_n + \alpha_n x, \tag{5}$$

where $\{\alpha_n\}$ is a real number sequence in the interval $(0,1)$ and $x$ is a fixed anchor. With some conditions on $\{\alpha_n\}$, it was proved that $\{x_n\}$ converges to $x$, which is a special fixed point of $T$, that is, the nearest point in $\mathrm{Fix}(T)$ to $x$. Halpern [20] pointed out that conditions (c1) $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$ and (c2) $\sum_{n=1}^{\infty} \alpha_n = \infty$ are necessary if the Halpern iteration scheme converges in norm. In view of (c2), the Halpern iteration may not be a fast iteration. Recently, a number of researchers investigated the problem of removing (c2) with the aid of projections; see, e.g., [21–24] and the references therein. In 2000, Moudafi [25] further proposed the viscosity approximation iteration, which reads as follows:

$$x_{n+1} = (1 - \alpha_n) T x_n + \alpha_n S x_n, \tag{6}$$

where $S$ is a contraction. This approximation method, which improves the property of the class of nonexpansive mappings, is popular from the viewpoint of variational inequalities. Indeed, the fixed point also solves a monotone variational inequality with $S$. Another popular regularized method is the hybrid projection method, which was considered by Nakajo and Takahashi [18] for fixed points of nonexpansive mappings first. Indeed, they studied the following algorithm:

$$\begin{cases} x_0 \in C, \\ y_n = (1 - \alpha_n) T x_n + \alpha_n x_n, \\ Q_n = \{x \in C\colon \langle x_n - x, x_n - x_0 \rangle \le 0\}, \\ C_n = \{x \in C\colon \|x - y_n\| \le \|x - x_n\|\}, \\ x_{n+1} = \mathrm{Proj}_{Q_n \cap C_n} x_0, \end{cases} \tag{7}$$

where $C$ is a closed, convex, and nonempty subset of $H$ and $\mathrm{Proj}_{Q_n \cap C_n}$ is the nearest point projection onto the intersection set. They obtained a strong convergence theorem for nonexpansive mappings in a real Hilbert spaces without compact assumption on $T$. For more general nonlinear mappings though the projection-based method, we refer to [26–30] and the references therein.

Let $C$ be a convex and closed subset of a real Hilbert space $H$. From now on, $\mathrm{Proj}_C$ is borrowed to denote the nearest projection onto subset $C$, i.e., $\mathrm{Proj}_C(x) \coloneqq \arg\min\{\|x - y\|, y \in C\}$. Let $A$ be a nonlinear mapping on $H$. Recall that $A$ is said to be

(1) *Strongly monotone* iff there exists a positive constant $\xi$ such that $\langle Ax - Ay, x - y\rangle \ge \xi \|x - y\|^2$, $\forall x, y \in H$

(2) *Monotone* iff $\langle Ax - Ay, x - y\rangle \ge 0$, $\forall x, y \in H$

(3) *Cocoercive* iff there exists a positive constant $\forall x, y \in H$ such that $\langle Ax - Ay, x - y\rangle \ge \xi \|Ax - Ay\|^2$, $\forall x, y \in H$

Let $B\colon H \rightrightarrows H$ be a multivalued nonlinear mapping. Next, we turn our attention to the class of multivalued mappings. $B$ is said to be a monotone mapping if and only if for all $x, y \in H$, $f \in By$, and $e \in Bx \Rightarrow \langle e - f, x - x\rangle > 0$. The symbol $B^{-1}(0)$ is used to stand for the set of zero points of $B$. Mapping $B$ is said to be a maximally monotone mapping iff the graph of $B$, $\mathrm{Graph}(B)$, is not contained in the graph of any other monotone mapping properly. Let $J_\beta^B = (I\ d + \beta B)^{-1}$, where $I\ d$ is the identity mapping and $\beta$ is a constant. This operator is called the resolvent of $B$. Its domain is denoted by $\mathrm{Dom}(B)$ in this paper. It is clear $B^{-1}(0) = \mathrm{Fix}(J_\beta^B)$.

Consider the following variational inclusion problem, which finds a point $x \in C$ such that $x \in (B + A)^{-1}(0)$, where $B$ is a multivalued maximally monotone mapping and $A$ is a $\xi$-cocoercive mapping. For the inclusion problem, splitting methods (FB, PR, and DR) are popular for zero points of the sum of the monotone mappings. Splitting methods were considered by many authors for image recovery, signal processing, and machine learning. The FB-type splitting method means an iterative method for which each iteration involves only with the individual operators not the sum. In this paper, with the condition that the solution set is nonempty, we consider finding a $\theta \in C$ such that $\theta \in F(T) \cap (B + A)^{-1}(0)$, where $T$ is a nonexpansive mapping with a nonempty fixed point set, $B$ is a multivalued maximally monotone mapping, and $A$ is a $\xi$-cocoercive mapping. We establish a strong convergence with the aid of hybrid projection and FB splitting in a Hilbert space. Our strong convergence theorem requires less restriction on parameter sequences and the operators.

To show our main findings, we also need the following necessary tools.

The nearest point projection operator $\mathrm{Proj}_C$ has the following property:

$$\|\mathrm{Proj}_C y - \mathrm{Proj}_C x\|^2 \le \langle y - x, \mathrm{Proj}_C y - \mathrm{Proj}_C(x)\rangle, \quad \forall x, y \in H. \tag{8}$$

**Lemma 1** (see [31]). *Let $H$ be a Hilbert space, and let $C$ be a convex, closed, and nonempty subset of $H$. Let $T$ be a nonexpansive mapping on $C$. Then, $\mathit{Fix}(T)$ is convex and closed.*

*Remark 1.* Let $H$ be a Hilbert space, and let $C$ be a convex, closed, and nonempty subset of $H$. Let $A\colon C \longrightarrow H$ be a $\xi$-cocoercive mapping, and let $B\colon H \rightrightarrows H$ be a multivalued maximally monotone operator. Then, $\mathrm{Fix}(J_\beta^B(I\ d - \beta A)) = (B + A)^{-1}(0)$, where $\beta$ is some constant and $I\ d$ is the identity mapping. Besides, the resolvent is firmly nonexpansive. From Lemma 1, we have that $(B + A)^{-1}(0)$ is convex and closed.

**Lemma 2** (see [31]). *Let $H$ be a Hilbert space, and let $C$ be a convex, closed, and nonempty subset of $H$. Let $T$ be a nonexpansive mapping on $C$. Then, $I\ d - T$ is demiclosed (let $\{x_n\}$ be a sequence weakly converging to $x$, and let $Tx_n - x_n \longrightarrow \infty$ be $n \longrightarrow \infty$. Then, $x$ is a fixed point of $T$).*

## 2. Main Results

**Theorem 1.** *Assume that $H$ is a Hilbert space and $C$ is a convex and closed subset in space $H$. Assume that $A$ is a single-valued $\xi$-cocoercive mapping from set $C$ to space $H$ and $B$ is a set-valued maximally monotone mapping from $H$ to $H$. Assume that $T$ is a nonexpansive mapping from $C$ to $C$, and $\mathit{CSS}(B, A, T) = (B + A)^{-1}(0) \cap \mathit{Fix}(T)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are positive real number sequences. Let $\{x_n\}$ be a sequence in set $C$ generated in the following iterative process:*

$$\begin{cases} x_0 \in C, \\ y_n = (1-\alpha_n)Tx_n + \alpha_n J_{\beta_n}^B (x_n - \beta_n A x_n), \\ Q_n = \{x \in C: \langle x_n - x, x_n - x_0 \rangle \le 0\}, \\ C_n = \{x \in C: \|x - y_n\| \le \|x - x_n\|\}, \\ x_{n+1} = \mathrm{Proj}_{Q_n \cap C_n} x_0, \end{cases} \tag{9}$$

*where $J_{\beta_n}^B$ is the resolvent mapping $(Id + \beta_n B)^{-1}$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (i) $1 > \alpha_n \ge \alpha > 0$ with $\alpha$ being a fixed real number and (ii) $0 < \beta \le \beta_n \le \beta' < 2\xi$ with $\beta$ and $\beta'$ being two fixed real numbers. Then, the sequence $\{x_n\}$ converges strongly to $Proj_{CSS(A,B,T)} x_0$.*

*Proof.* From Lemma 1, we have that $\mathrm{Fix}(T)$ is convex and closed. From Remark 1, we have that $(B + A)^{-1}(0)$ is convex and closed. Hence, $\mathrm{CSS}(B, A, T)$ is convex and closed. This shows that the metric (nearest point) projection onto the set is well-defined.

Note that $\|x - y_n\|^2 \le \|x - x_n\|^2$ is equivalent to $2\langle x, x_n - y_n\rangle \le \|x_n\|^2 - \|y_n\|^2$. Let $x$ and $x'$ be the points in $C_n$. Then,

$$\begin{aligned} 2r\langle x, x_n - y_n\rangle &\le r\left(\|x_n\|^2 - \|y_n\|^2\right), \\ 2(1-r)\langle x', x_n - y_n\rangle &\le (1-r)\left(\|x_n\|^2 - \|y_n\|^2\right), \end{aligned} \tag{10}$$

where $r$ is a real number in $(0, 1)$. Adding the two inequalities above, we have

$$2\langle rx + (1-r)x', x_n - y_n\rangle \le \|x_n\|^2 - \|y_n\|^2, \tag{11}$$

that is,

$$\|rx + (1-r)x' - y_n\| \le \|rx + (1-r)x' - x_n\|. \tag{12}$$

It shows that $rx + (1-r)x' \in C_n$. $C_n$ is convex. The closedness of $C_n$ is obvious. The definition of $\xi$-cocoercive mappings send us to the situation $\mathrm{Id} - \beta_n A$ is a nonexpansive mapping for each $n$. Indeed, for any $w, v \in C$,

$$\begin{aligned} &\|(Id - \beta_n A)w - (Id - \beta_n A)v\|^2 \\ &= \beta_n^2\|Aw - Av\|^2 - 2\beta_n\langle Aw - Av, w - v\rangle + \|w - v\|^2 \\ &\le \beta_n(\beta_n - 2\xi)\|Aw - Av\|^2 + \|w - v\|^2. \end{aligned} \tag{13}$$

This indicates $\mathrm{Id} - \beta_n A$ is a mapping of nonexpansive. Observe that $\mathrm{CSS}(B, A, T) \subset C_n$. Indeed, from the nonexpansivity of the resolvent, we have

$$\begin{aligned} \|y_n - p\| &\le (1-\alpha_n)\|Tx_n - p\| + \alpha_n\|J_{\beta_n}^B(x_n - \beta_n A x_n) - p\| \\ &= (1-\alpha_n)\|Tx_n - Tp\| + \alpha_n\|J_{\beta_n}^B(x_n - \beta_n A x_n) - J_{\beta_n}^B(p - \beta_n A p)\| \\ &\le (1-\alpha_n)\|x_n - p\| + \alpha_n\|(\mathrm{Id} - \beta_n A)x_n - (\mathrm{Id} - \beta_n A)p\| \\ &\le \|x_n - p\|, \quad \forall p \in \mathrm{CSS}(A, B, T). \end{aligned} \tag{14}$$

So, we complete the proof $\mathrm{CSS}(A, B, T) \subset C_n$.

On the contrary, it is obvious that $Q_n$ is convex and closed. Next, one shows that $\mathrm{CSS}(B, A, T) \subset Q_n \cap C_n$. Borrowing $C_0 = C$, we have $\mathrm{CSS}(B, A, T) \subset Q_0 \cap C_0$. Let $x_m$ be a given vector, and $CSS(B, A, T) \subset Q_m \cap C_m$ for some positive integer $m$. There is a vector $x_{m+1} \in Q_m \cap C_m$ with $x_{m+1} = \mathrm{Proj}_{Q_m \cap C_m} x_0$. There holds $\langle x_0 - x_{m+1}, x_{m+1} - j\rangle \ge 0$ for all $j \in Q_m \cap C_m$. Borrowing $\mathrm{CSS}(B, A, T) \subset Q_m \cap C_m$, we get $\mathrm{CSS}(B, A, T) \subset Q_{m+1}$. Thus, $\mathrm{CSS}(B, A, T) \subset Q_{m+1} \cap C_{m+1}$. Hence, $\mathrm{CSS}(B, A, T) \subset Q_n \cap C_n$ for all $n$.

One next observes that $x_n$ is a bounded sequence. As we have showed that $\mathrm{CSS}(B, A, T)$ is convex and closed set in $C$, a unique vector $\mu \in \mathrm{CSS}(A, B, T)$ with $\mu = \mathrm{Proj}_{\mathrm{CSS}(A,B,T)} x_0$ is guaranteed. We have the construction of $x_{n+1}$, that is, $\mathrm{Proj}_{Q_n \cap C_n} x_0 = x_{n+1}$. So,

$$\|x_0 - x_{n+1}\| \le \|x_0 - \nu\|, \tag{15}$$

for each $\nu \in Q_n \cap C_n$. By $\mu \in \mathrm{CSS}(A, B, T) \subset Q_n \cap C_n$, we obtain

$$\|x_0 - x_{n+1}\| \le \|x_0 - \mu\|, \tag{16}$$

that infers $x_n$ is a bounded sequence. Our next step shows $\|x_{n+1} - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Because $x_n = \mathrm{Proj}_{Q_n} x_0$ and $x_{n+1} \in Q_n \cap C_n \subset Q_n$, one infers that

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|. \tag{17}$$

Borrowing the conclusion ($x_n$ is a bounded sequence), one infers that the limit of $\{\|x_0 - x_n\|\}$ exists. We may suppose that $\lim_{n\longrightarrow\infty}\|x_0 - x_n\| = d > 0$. Observe

$$\begin{aligned} &\|x_{n+1} - x_0\|^2 - \|x_0 - x_n\|^2 \\ &\ge \|x_{n+1} - x_0\|^2 - \|x_0 - x_n\|^2 - 2\langle x_0 - x_n, x_n - x_{n+1}\rangle \\ &= \|x_n - x_0\|^2 + \|x_0 - x_{n+1}\|^2 + 2\langle x_n - x_0, x_0 - x_{n+1}\rangle \\ &= \|x_n - x_{n+1}\|^2 \ge 0, \end{aligned} \tag{18}$$

thanks to $\langle x_0 - x_n, x_n - x_{n+1}\rangle \ge 0$ ($x_{n+1} \in Q_n$ and the property of the metric projection). By the limit of the limit of $\{\|x_0 - x_n\|\}$, one infers $\lim_{n\longrightarrow\infty}\|x_{n+1} - x_n\|^2 = 0$.

Note that $x_{n+1}$ is in $C_n$. So,

$$\|x_{n+1} - y_n\| \le \|x_{n+1} - x_n\|. \tag{19}$$

That indicates that $x_{n+1} - y_n \longrightarrow 0$ as $n \longrightarrow \infty$. Furthermore, $x_n - y_n \longrightarrow 0$ as $n \longrightarrow \infty$. Let $z_n = J^B_{\beta_n}(x_n - \beta_n A x_n)$. For any $p \in \mathrm{CSS}(B, A, T)$, $\xi$-cocoercive and resolvent operators send us to

$$\begin{aligned}
&\|p - z_n\|^2 \\
&\leq \left\|J^B_{\beta_n}(p - \beta_n A p) - J^B_{\beta_n}(x_n - \beta_n A x_n)\right\|^2 \\
&\leq \|(p - \beta_n A p) - (x_n - \beta_n A x_n)\|^2 \\
&\leq \|p - x_n\|^2 - (2\xi - \beta_n)\beta_n \|Ap - Ax_n\|^2.
\end{aligned} \tag{20}$$

So,

$$\begin{aligned}
&\|p - y_n\|^2 \\
&\leq (1 - \alpha_n)\|Tx_n - p\|^2 + \alpha_n \|z_n - p\|^2 \\
&\leq (1 - \alpha_n)\|Tx_n - Tp\|^2 + \alpha_n\left(\|p - x_n\|^2 - (2\xi - \beta_n)\beta_n\|Ap - Ax_n\|^2\right) \\
&\leq \|p - x_n\|^2 - \alpha_n(2\xi - \beta_n)\beta_n\|Ap - Ax_n\|^2.
\end{aligned} \tag{21}$$

That is,

$$\alpha_n(2\xi - \beta_n)\beta_n\|Ap - Ax_n\|^2 \leq \|y_n - x_n\|\left(\|p - x_n\| + \|p - y_n\|\right). \tag{22}$$

By the fact that $\|y_n - x_n\| \longrightarrow 0$ as $n \longrightarrow \infty$, we have $Ax_n - Ap \longrightarrow 0$ as $n \longrightarrow \infty$. By the firm nonexpansivitity of the resolvent operator, we also have

$$\begin{aligned}
&\|p - z_n^2\| \\
&\leq \langle p - z_n, (p - \beta_n A p) - (x_n - \beta_n A x_n)\rangle \\
&= \frac{1}{2}\left(\|p - x_n\|^2 + \|p - z_n\|^2 - \|x_n - z_n - \beta_n(Ap - Ax_n)\|^2\right) \\
&\leq \frac{1}{2}\left(\|p - x_n\|^2 + \|p - z_n\|^2 - \|x_n - z_n\|^2 - \beta_n^2\|Ap - Ax_n\|^2 + 2\beta_n\|x_n - z_n\|\|Ap - Ax_n\|\right) \\
&\leq \frac{1}{2}\left(\|p - x_n\|^2 + \|p - z_n\|^2 - \|x_n - z_n\|^2 + 2\beta_n\|x_n - z_n\|\|Ap - Ax_n\|\right),
\end{aligned} \tag{23}$$

which holds that

$$\begin{aligned}
\|p - z_n\|^2 &\leq \|p - x_n\|^2 - \|x_n - z_n\|^2 + 2\beta_n\|x_n - z_n\|\|Ap - Ax_n\|, \\
\|p - y_n\|^2 &\leq (1 - \alpha_n)\|Tx_n - Tp\|^2 + \alpha_n\left\|J^B_{\beta_n}(x_n - \beta_n A x_n) - p\right\|^2 \\
&\leq (1 - \alpha_n)\|x_n - p\|^2 + \alpha_n\|z_n - p\|^2 \\
&\leq \|x_n - p\|^2 - \alpha_n\|x_n - z_n\|^2 + 2\beta_n\alpha_n\|x_n - z_n\|\|Ap - Ax_n\|.
\end{aligned} \tag{24}$$

So, $\alpha_n\|x_n - z_n\|^2 \le \|x_n - y_n\|C + 2\beta_n\alpha_n\|x_n - z_n\|\|Ap - Ax_n\|$, where $C$ is some constant. By the requirement on the control parameter and the result that $Ap - Ax_n \longrightarrow \infty$ as $n \longrightarrow \infty$ and $x_n - z_n \longrightarrow \infty$ as $n \longrightarrow \infty$. With a simple calculation, we have $x_n - Tx_n \longrightarrow \infty$ as $n \longrightarrow \infty$. We have the fact that $(\mathrm{Id} - \beta_n A)x_n \in (\mathrm{Id} + \beta_n B)z_n$. It holds

$$\frac{x_n - z_n}{\beta_n} - Ax_n \in Bz_n. \tag{25}$$

By the assumption that $B$ is maximally monotone,

$$\langle \frac{x_n - z_n}{\beta_n} - Ax_n - u, z_n - v\rangle \ge 0, \tag{26}$$

for any $u \in Bv$. By the result that $\{x_n\}$ is a bounded sequence, there is a subsequence $\{x_{n_m}\}$ converges to $\theta$ weakly. The $\xi$-cocoercive mappings yield $Ax_{n_m} \longrightarrow A\theta$. It holds $\langle -A\theta - u, \theta - v\rangle \ge 0$. It shows $0 \in (B + A)(\theta)$. Note that $\mathrm{Id} - T$ is demiclosed (Lemma 2). One asserts $\theta \in \mathrm{Fix}(T)$. One next shows that $\theta = \mathrm{Proj}_{\mathrm{CSS}(B,A,T)}x_0$ and $x_n$ converges to it strongly. Set $\overline{x} = \mathrm{Proj}_{\mathrm{CSS}(B,A,T)}x_0$. Since the functional $\|\cdot\|$ is weakly lower semicontinuous, one has

$$\|\overline{x} - x_0\| \le \|\theta - x_0\| \le \liminf_{m\longrightarrow\infty}\|x_0 - x_{n_m}\| \le \limsup_{m\longrightarrow\infty}\|x_0 - x_{n_m}\| \le \|\overline{x} - x_0\|. \tag{27}$$

One gets $\|\overline{x} - x_0\| = \lim_{m\longrightarrow\infty}\|x_0 - x_{n_m}\| = \|\theta - x_0\|$. Since the framework is a Hilbert space, one gets $x_n \longrightarrow \theta$ as $n \longrightarrow \infty$. This finishes this theorem. □

Let

$$\partial f(x) = \{z \in H\colon f(x) + \langle y - x, z\rangle \le f(y), \forall y \in H\}, \quad \forall, x \in H, \tag{28}$$

where $f\colon H \longrightarrow (-\infty, \infty]$ is a proper, convex, and lower semicontinuous function. Rockfellar [32] proved that $\partial f$ is a multivalued maximally monotone operator. Let $C$ be a closed, convex, and nonempty subset of $H$ and $i_C$ be the indicator function of $C$, that is,

$$i_C x = \begin{cases} 0, & x \in C, \\ \infty, & x \notin C. \end{cases} \tag{29}$$

Furthermore, we define the normal cone $N_C(v)$ of $C$ at $v$ as follows:

$$N_C v = \{z \in H\colon \langle z, y - v\rangle \le 0, \forall y \in H\}, \tag{30}$$

for any $v \in C$. Then, $i_C\colon H \longrightarrow (-\infty, \infty]$ is proper, convex, and lower semicontinuous on $H$. $\partial i_C$ is a maximally monotone operator. Let $\mathrm{Res}_\lambda x = (\mathrm{Id} + \lambda\, \partial i_C)^{-1}x$. So, $\partial i_C x = N_C x$ and $x \in C$; we obtain

$$v = J_\lambda^{\partial i_C} x \Longleftrightarrow v = Proj_C x, \tag{31}$$

where $Proj_C^{\partial i_C}$ is the metric projection onto $C$. This yields $x \in (A + \partial i_C)^{-1}(0) \Longleftrightarrow x \in \mathrm{VI}(A, C)$, where $\mathrm{VI}(A, C)$ denotes the classical variational inequality, that is, find a point $x \in C$ such that $\langle Ax, y - x\rangle \ge 0$ for all $y \in C$.

**Corollary 1.** *Assume that $H$ is a Hilbert space and $C$ is a convex and closed subset in space $H$. Assume that $A$ is a single-valued $\xi$-cocoercive mapping from set $C$ to space $H$. Assume that $T$ is a nonexpansive mapping from $C$ to $C$ and $VI(A, C) \cap Fix(T)$ is nonempty. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are positive real number sequences. Let $\{x_n\}$ be a sequence in set $C$ generated in the following iterative process:*

$$\begin{cases} x_0 \in C, \\ y_n = (1 - \alpha_n)Tx_n + \alpha_n \mathrm{Proj}_C(x_n - \beta_n Ax_n), \\ Q_n = \{x \in C\colon \langle x_n - x, x_n - x_0\rangle \le 0\}, \\ C_n = \{x \in C\colon \|x - y_n\| \le \|x - x_n\|\}, \\ x_{n+1} = \mathrm{Proj}_{Q_n \cap C_n} x_0. \end{cases} \tag{32}$$

Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy the conditions (i) $1 > \alpha_n \ge \alpha > 0$ with $\alpha$ being a fixed real number and (ii) $0 < \beta \le \beta_n \le \beta' < 2\xi$ with $\beta$ and $\beta'$ being two fixed real numbers. Then, the sequence $\{x_n\}$ converges strongly to $\mathrm{Proj}_{\mathrm{VI}(A,C)\cap \mathrm{Fix}(T)} x_0$.

## Data Availability

The data used to support the findings of this study are included within the article.

## Conflicts of Interest

The author declares that he has no conflicts of interest.

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