Original Article

Journal of Time Series Analysis

First version received July 2012 | Published online in Wiley Online Library: 27 September 2012

(wileyonlinelibrary.com) DOI: 10.1111/j.1467-9892.2012.12000.x

Forecasting with prediction intervals for periodic autoregressive moving average models

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Periodic autoregressive moving average (PARMA) models are indicated for time series whose mean, variance and covariance function vary with the season. In this study, we develop and implement forecasting procedures for PARMA models. Forecasts are developed using the innovations algorithm, along with an idea of Ansley. A formula for the asymptotic error variance is provided, so that Gaussian prediction intervals can be computed. Finally, an application to monthly river flow forecasting is given, to illustrate the method.

Keywords: Periodic correlation; autoregressive moving average; forecasting.

1. INTRODUCTION

Mathematical modelling and simulation of seasonal time series are critical issues in many areas of application, including surface water hydrology. Most river flow time series are periodically stationary, that is, their mean and covariance functions are periodic with respect to time. To account for the periodic correlation structure, a periodic autoregressive moving average (PARMA) model can be useful. PARMA models are also appropriate for a wide variety of time series applications in geophysics and climatology. In a PARMA model, the parameters in a classical ARMA model are allowed to vary with the season. Since PARMA models explicitly describe seasonal fluctuations in mean, standard deviation and autocorrelation, they can generate more faithful models and simulations of natural river flows.

A stochastic process $\{\tilde{X}_t\}_{t\in\mathbb{Z}}$ is periodically stationary if its mean $E\tilde{X}_t$ and covariance $\mathrm{Cov}(\tilde{X}_t, \tilde{X}_{t+h})$ for $h \in \mathbb{Z}$ are periodic functions of time t with the same period S. A periodically stationary process $\{\tilde{X}_t\}$ is called a PARMA$_S(p, q)$ process if the mean-centred process $X_t = \tilde{X}_t - \mu_t$ is of the form

$$X_t - \sum_{k=1}^{p} \phi_t(k) X_{t-k} = \varepsilon_t + \sum_{j=1}^{q} \theta_t(j) \varepsilon_{t-j} \tag{1}$$

where $\{\varepsilon_t\}$ is a sequence of random variables with mean zero and standard deviation $\sigma_t > 0$ such that $\{\delta_t = \sigma_t^{-1}\varepsilon_t\}$ is i.i.d. The autoregressive parameters $\phi_t(j)$, the moving average parameters $\theta_t(j)$ and the residual standard deviations σ_t are all periodic with the same period $S \geq 1$. Periodic ARMA modelling has a long history, starting with the work of Gladyshev (1961) and Jones and Brelsford (1967). The book of Franses and Paap (2004) gives a nice overview of the subject. Applications to river flows are developed in the book of Hipel and McLeod (1994). Since river flows typically exhibit seasonal variations in mean, standard deviation and autocorrelation structure, PARMA models are appropriate, see for example Thompstone *et al.* (1985), Salas and Obeysekera (1992), McLeod (1994), Anderson and Meerschaert (1998), Anderson *et al.* (2007) and Bowers *et al.* (2012).

In this study, a recursive forecasting algorithm for PARMA time series is developed, based on minimizing mean squared error. We detail the computation of h-step ahead forecasts, based on the innovations algorithm, and an idea of Ansley (Ansley 1979; Lund and Basawa 2000). R codes to implement these forecasts and compute the asymptotic variance of the forecast errors are available upon request from the authors. The article is laid out as follows. Section 2 details the algorithms for computing h-step-ahead forecasts for any $h \geq 1$, based on work of Lund and Basawa (2000). Their paper developed maximum likelihood estimates for the PARMA model parameters, using projection arguments based on the innovations algorithm, but their results can easily be adapted to our purposes. Section computes the associated forecast error variances. Theorem 1 is the main theoretical result of this study. Section 4 illustrates the methods of this study by forecasting two years of average monthly flows for the Fraser River.

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2. FORECASTING

This section develops the PARMA prediction equations based on orthogonal projection, to minimize the mean squared prediction error among all linear predictors. If the PARMA process has Gaussian innovations, then this will also minimize the mean squared prediction error among all predictors. Throughout this study, we assume that the model (1) admits a causal representation

$$X_t = \sum_{j=0}^{\infty} \psi_t(j)\varepsilon_{t-j} \tag{2}$$

where $\psi_t(0) = 1$ and $\sum_{j=0}^{\infty} |\psi_t(j)| < \infty$ for all t, and satisfies an invertibility condition

$$\varepsilon_t = \sum_{j=0}^{\infty} \pi_t(j)X_{t-j} \tag{3}$$

where $\pi_t(0) = 1$ and $\sum_{j=0}^{\infty} |\pi_t(j)| < \infty$ for all t. Then $\psi_t(j) = \psi_{t+kS}(j)$ and $\pi_t(j) = \pi_{t+kS}(j)$ for all integers t, j, k. The seasonal mean $\mu_i = \mathrm{E}[\tilde{X}_{kS+i}]$, autocovariance $\gamma_i(\ell) = \mathrm{E}[X_{kS+i}X_{kS+i+\ell}]$ and autocorrelation $\rho_i(\ell) = \gamma_i(\ell)/\sqrt{\gamma_i(0)\gamma_{i+\ell}(0)}$ are also periodic functions of i with the same period S. Note that $\gamma_i(\ell) = \gamma_{i+\ell}(-\ell)$.

Fix a probability space on which the $\mathrm{PARMA}_S(p, q)$ model (1) is defined, and let $\tilde{\mathcal{H}}_n = \overline{\mathrm{sp}}\{1, \tilde{X}_0, \ldots, \tilde{X}_{n-1}\} = \overline{\mathrm{sp}}\{1, X_0, \ldots, X_{n-1}\}$ denote the set of all linear combinations of these random variables in the Hilbert space of finite variance random variables on that probability space, with the inner product $\langle X, Y\rangle = \mathrm{E}(X\,Y)$. A simple projection argument shows that $P_{\tilde{\mathcal{H}}_n}\tilde{X}_n = \mu_n + P_{\mathcal{H}_n}X_n$ where $\mathcal{H}_n = \overline{\mathrm{sp}}\{X_0, \ldots, X_{n-1}\}$, hence it suffices to develop forecasts for the mean-centered process X_t, and then add back the seasonal mean. An efficient forecasting algorithm uses the transformed process (Ansley 1979; Lund and Basawa, 2000)

$$W_t = \begin{cases} X_t, & t = 0, \ldots, m-1 \\ X_t - \sum_{k=1}^{p} \phi_t(k)X_{t-k}, & t \geq m \end{cases} \tag{4}$$

where $m = \max(p,q)$. Computing the autocovariance $C(j, \ell) = E(W_jW_\ell)$ of the transformed process (4) shows that $C(j, \ell) = 0$ whenever $\ell > m$ and $\ell > j + q$ (see Lund and Basawa 2000 eqn 3.16). Define $\hat{X}_0 = \hat{W}_0 = 0$ and let

$$\hat{X}_n = P_{\mathcal{H}_n}(X_n) \quad \text{and} \quad \hat{W}_n = P_{\mathcal{H}_n}(W_n) \tag{5}$$

denote the one-step projections of X_n and W_n onto $\mathcal{H}_n$, respectively, for $n \geq 1$. Write

$$\hat{W}_n = \sum_{j=1}^{n} \theta_{n,j}\left(W_{n-j} - \hat{W}_{n-j}\right), \tag{6}$$

where $\theta_{n,1}, \ldots, \theta_{n,n}$ are the unique projection coefficients that minimize the mean squared error $v_n = \mathrm{E}[(W_n - \hat{W}_n)^2]$. Uniqueness of $\theta_{n,j}$ follows from invertibility of the covariance matrix of $X_0, X_1, \ldots, X_t$, for all $t \geq 1$, which holds under the causal model assumption with $\sigma_t > 0$ for all seasons (see Lund and Basawa, 2000, Proposition 4.1]. Apply the innovations algorithm (Brockwell and Davis, 1991, Proposition 5.2.2) to the transformed process (4) to get

$$\begin{aligned} v_0 &= C(0,0) \\ \theta_{n,n-k} &= v_k^{-1}\left[C(n,k) - \sum_{j=0}^{k-1} \theta_{k,k-j}\theta_{n,n-j}v_j\right] \\ v_n &= C(n,n) - \sum_{j=0}^{n-1} \left(\theta_{n,n-j}\right)^2 v_j \end{aligned} \tag{7}$$

solved in the order v_0, $\theta_{1,1}$, v_1, $\theta_{2,2}$, $\theta_{2,1}$, v_2, $\theta_{3,3}$, $\theta_{3,2}$, $\theta_{3,1}$, $v_3, \ldots$ and so forth. Since $C(j, \ell) = 0$ whenever $\ell > m$ and $\ell > j + q$, it is not hard to check that $\theta_{n,j} = 0$ whenever $j > q$ and $n \geq m$.

Then it follows from Lund and Basawa (2000, eqn 3.4) that the one-step predictors (5) for a $\mathrm{PARMA}_S(p, q)$ process (1) can be computed recursively using

$$\hat{X}_n = \begin{cases} \sum_{j=1}^{n} \theta_{n,j}\left(X_{n-j} - \hat{X}_{n-j}\right) & 1 \leq n < m, \\ \sum_{j=1}^{p} \phi_n(j)X_{n-j} + \sum_{j=1}^{q} \theta_{n,j}\left(X_{n-j} - \hat{X}_{n-j}\right) & n \geq m. \end{cases} \tag{8}$$

To simplify notation, denote the first season to be forecast as season 0, and suppose that the season of the oldest data point is $S - 1$. If the total number of available data is not a multiple of S, we discard a few ($< S$) of the oldest observations, to obtain the data set of $\tilde{X}_0, \tilde{X}_1, \ldots, \tilde{X}_{n-1}$, where $n = NS$ is fixed. Then it follows from Lund and Basawa (2000, eqn 3.36 that the h-step ahead predictors are given by

$$P_{\mathcal{H}_n}X_{n+h} = \sum_{j=h+1}^{n+h} \theta_{n+h,j}\left(X_{n+h-j} - \hat{X}_{n+h-j}\right) \tag{9}$$

when $n < m$ and $0 \leq h \leq m - n - 1$, and

$$P_{\mathcal{H}_n}X_{n+h} = \sum_{k=1}^{p} \phi_h(k) P_{\mathcal{H}_n}X_{n+h-k} + \sum_{j=h+1}^{q} \theta_{n+h,j}\left(X_{n+h-j} - \hat{X}_{n+h-j}\right) \tag{10}$$

otherwise, where the coefficients $\theta_{n,j}$ are computed via the innovations algorithm (7) applied to the transformed process (4). A large-sample approximation for the h-step predictor is given by

$$P_{\mathcal{H}_n}X_{n+h} \approx \sum_{k=1}^{p} \phi_h(k) P_{\mathcal{H}_n}X_{n+h-k} + \sum_{j=h+1}^{q} \theta_h(j)\left(X_{n+h-j} - \hat{X}_{n+h-j}\right), \tag{11}$$

since $\theta_{n+h,j} \rightarrow \theta_h(j)$ as $n \rightarrow \infty$ (Anderson *et al.*, 1999, Corollary 2.2.3).

3. FORECAST ERRORS

The next result explicitly computes the variance of the forecast errors, and a simpler asymptotic variance that is useful for computations. We continue to assume that the mean-centred data $X_0, X_1, \ldots, X_{n-1}$ spans exactly N years, so that $n = NS$.

Theorem 1. *Define $\chi_j(0) = 1$ for all $j \geq 0$, $\chi_j(j - \ell) = 0$ for all $j \geq 0$ and $\ell > j$, and recursively*

$$\chi_j(\ell) = \sum_{k=1}^{\min(p,\,\ell)} \phi_j(k)\chi_{j-k}(\ell - k) \text{ for all } j \geq 0 \text{ and } 0 \leq \ell < j. \tag{12}$$

Then the mean-squared error $\sigma_n^2(h) = \mathrm{E}\left[\left(X_{n+h} - P_{\mathcal{H}_n}X_{n+h}\right)^2\right]$ of the h-step predictors $P_{\mathcal{H}_n}X_{n+h}$ for the PARMA$_S(p, q)$ process (1) can be computed recursively using

$$\sigma_n^2(h) = \sum_{j=0}^{h}\left(\sum_{k=0}^{j} \chi_h(k)\theta_{n+h-k,j-k}\right)^2 v_{n+h-j} \tag{13}$$

when $n \geq m$: $=$ max(p, q), and the coefficients $\theta_{n+h-k,j-k}$ and v_{n+h-j} are computed via the innovations algorithm (7) applied to the transformed process (4). Furthermore, the asymptotic mean squared error is given by

$$\sigma_n^2(h) \rightarrow \sum_{j=0}^{h} \psi_h^2(j)\sigma_{h-j}^2 \text{as } n = NS \rightarrow \infty, \tag{14}$$

where $\psi_h(j) = \sum_{k=0}^{j} \chi_h(k)\theta_{h-k}(j-k)$.

Proof. Note that the mean squared prediction error $\sigma_n^2(h) = \mathrm{E}(X_{n+h} - P_{\mathcal{H}_n}X_{n+h})^2$ for the mean-centred PARMA process (1) is not the same as the mean squared prediction error $E(W_{n+h} - P_{\mathcal{H}_n}W_{n+h})^2$ for the transformed process (4). When $n \geq m = \max(p, q)$, eqn (10) holds for any $h \geq 0$, and the second term in (10) vanishes when $h + 1 > q$. Write $X_{n+h} = \hat{X}_{n+h} + (X_{n+h} - \hat{X}_{n+h})$, and note that $\phi_{n+h}(j) = \phi_h(j)$ since $n = NS$. Substitute (8) with $n \geq m$ to get

$$\begin{aligned} X_{n+h} = \phi_h(1)X_{n+h-1} + \ldots + \phi_h(p)X_{n+h-p} \\ \sum_{j=0}^{q} \theta_{n+h,j}\left(X_{n+h-j} - \hat{X}_{n+h-j}\right), \end{aligned} \tag{15}$$

with $\theta_{n,0} = 1$ for all n. Subtract (10) from (15) and rearrange terms to get

$$\begin{aligned} X_{n+h} - P_{\mathcal{H}_n}X_{n+h} - \sum_{k=1}^{p} \phi_h(k)(X_{n+h-k} - P_{\mathcal{H}_n}X_{n+h-k}) \\ = \sum_{j=0}^{h} \theta_{n+h,j}\left(X_{n+h-j} - \hat{X}_{n+h-j}\right). \end{aligned} \tag{16}$$

Define the random vectors

$$M_n = \begin{pmatrix} X_n - \hat{X}_n \\ \vdots \\ X_{n+h} - \hat{X}_{n+h} \end{pmatrix} \quad \text{and} \quad F_n = \begin{pmatrix} X_n - P_{\mathcal{H}_n}X_n \\ \vdots \\ X_{n+h} - P_{\mathcal{H}_n}X_{n+h}. \end{pmatrix} \tag{17}$$

Write $\Phi_h = [-\phi_j(j-\ell)]_{j,\ell=0}^h$ where we define $\phi_j(0) = -1$ for all j, and $\phi_j(k) = 0$ for $k > p$ or $k < 0$. Note that $\phi_j(k)$ is periodic in S, so that $\phi_j(k) = \phi_{\langle j \rangle}(k)$, where $\langle j \rangle = j$ mod S is the season corresponding to index j. Write $\Theta_n = [\theta_{n+j,j-\ell}]_{j,\ell=0}^h$ where we define $\theta_{n,0} = 1$, and $\theta_{n,k} := 0$ for $k > q$ or $k < 0$. Then we can use (16) to write

$$\Phi_h F_n = \Theta_n M_n, \tag{18}$$

where Φ_h and Θ_n are lower triangular matrices. The entries of the innovations vector M_n are uncorrelated, with covariance matrix $V_n = \text{diag}(v_n, v_{n+1}, \ldots, v_{n+h})$. Then the covariance matrix of the vector $F_n = \Phi_h^{-1}\Theta_n M_n$ of prediction errors is

$$C_n := \mathrm{E}[F_n F_n'] = \Psi_n V_n \Psi_n' \quad \text{where } \Psi_n = \Phi_h^{-1}\Theta_n \tag{19}$$

and $'$ denotes the matrix transpose.

Compute the inverse matrix $\Phi_h^{-1} = [\chi_j(j-\ell)]_{j,\ell=0}^h$ by multiplying out

$$\left(\sum_{k=0}^{\infty} \chi_{j+k}(k) z^k\right)\left(1 - \sum_{k=1}^{p} \phi_j(k) z^k\right) = 1$$

and equating coefficients. This leads to $\chi_j(0) = 1$ and (12). Define $\Psi_n = \Phi_h^{-1}\Theta_n = [\psi_{n+j,j-\ell}]_{j,\ell=0}^h$ and note that

$$\psi_{n+j,j-\ell} = \sum_{k=0}^{j-\ell} \chi_j(k)\theta_{n+j-k,j-\ell-k}. \tag{20}$$

Since $|v_r - \sigma_r^2| \to 0$ as $r \to \infty$ (Anderson *et al*., 1999, Corollary 2.2.1 and $|\theta_{s,\ell} - \theta_s(\ell)| \to 0$ as $s \to \infty$ for all $\ell > 0$ (Anderson *et al.* 1993, Corollary 2.2.3), it follows using (13) that (14) holds.

Remark 1. It is a simple consequence of periodic stationarity that the forecast errors converge monotonically in (14), that is, $\sigma_n^2(h) \geq \sigma_{n+S}^2(h)$ for all n and h. Hence the asymptotic limit provides a lower bound on the exact forecast error.

Corollary 1. *If $\{X_t\}$ is a 0-mean Gaussian process, then the probability that X_{n+h} lies between the bounds $P_{\mathcal{H}_n}X_{n+h} \pm z_{\alpha/2}(\sum_{j=0}^{h} \psi_h^2(j)\sigma_{h-j}^2)^{\frac{1}{2}}$ approaches $(1-\alpha)$ as $n \to \infty$, where z_α is the $(1-\alpha)$-quantile of the standard normal distribution.*

Proof. Since $(X_0, X_1, \ldots, X_{n+h})'$ has a multivariate normal distribution, Problem 2.20 in Brockwell and Davis (1991) implies that $P_{\mathcal{H}_n}X_{n+h} = \mathrm{E}_{\overline{\text{sp}}\{X_0,\ldots,X_{n-1}\}}X_{n+h} = \mathrm{E}(X_{n+h}|X_0, \ldots, X_{n-1})$. Then the result follows using (14).

Remark 2. Formula (19) for the covariance matrix of the forecast errors was established by Lund and Basawa (2000, eqn 3.41) in a different notation. However, that study not develop an explicit formula for the forecast error variance.

4. APPLICATION

In this section, we apply formula (10) to forecast future values for a time series of monthly river flows. Then we apply Theorem 1 and Corollary 1 to get Gaussian 95% confidence bounds for these forecasts. All computations were carried out using the R programming language (R Development Core Team, 2008). Codes are available from the authors upon request. We consider monthly average flow for the Fraser River at Hope, British Columbia between October 1912 and September 1984, see Tesfaye *et al.* (2006) for more details. A partial series, consisting of the first 15 years of observations (Figure 1), clearly indicates the seasonal variations in monthly average flow. The entire series contains $n = NS$ observations, covering $N = 72$ years. The seasonal sample mean

$$\hat{\mu}_i = N^{-1} \sum_{k=0}^{N-1} \tilde{X}_{kS+i}, \tag{21}$$

sample standard deviation $\sqrt{\hat{\gamma}_i(0)}$ computed using the sample autocovariance

$$\hat{\gamma}_i(\ell) = N^{-1} \sum_{j=0}^{N-1-h_i} (\tilde{X}_{jS+i} - \hat{\mu}_i)(\tilde{X}_{jS+i+\ell} - \hat{\mu}_{i+\ell}) \tag{22}$$

with $\ell \geq 0$, $h_i = \lfloor (i+\ell)/S \rfloor$, and sample autocorrelation

$$\hat{\rho}_i(\ell) = \frac{\hat{\gamma}_i(\ell)}{\sqrt{\hat{\gamma}_i(0)\hat{\gamma}_{i+\ell}(0)}} \tag{23}$$

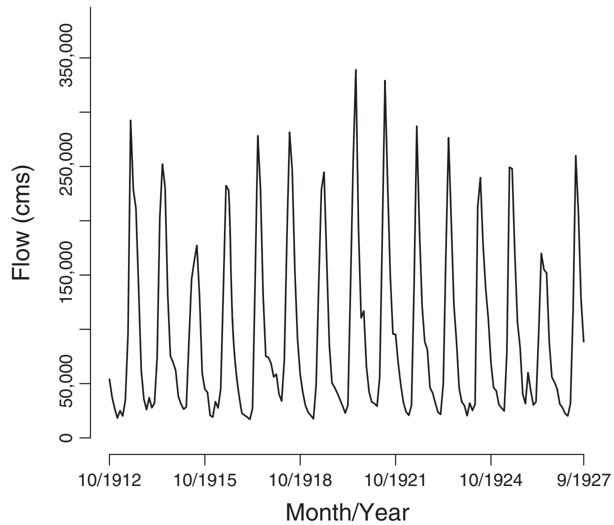

Figure 1. Part of average monthly flows in cubic metres per second (cms) for the Fraser River at Hope, BC indicate a seasonal pattern

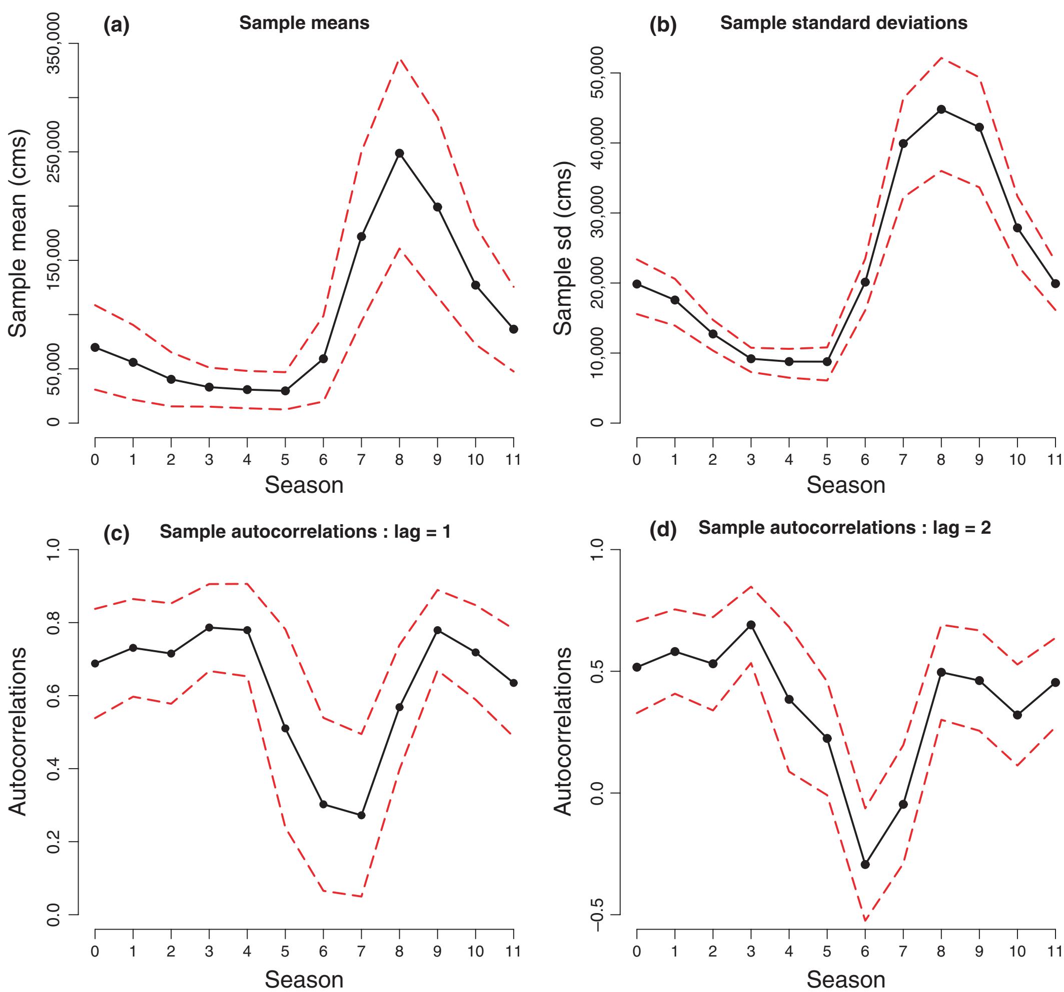

Figure 2. Statistics for the Fraser river time series: (a) seasonal mean; (b) standard deviation; (c,d) autocorrelations at lags 1 and 2. Dotted lines are 95% confidence intervals. Season = 0 corresponds to October and Season = 11 corresponds to September

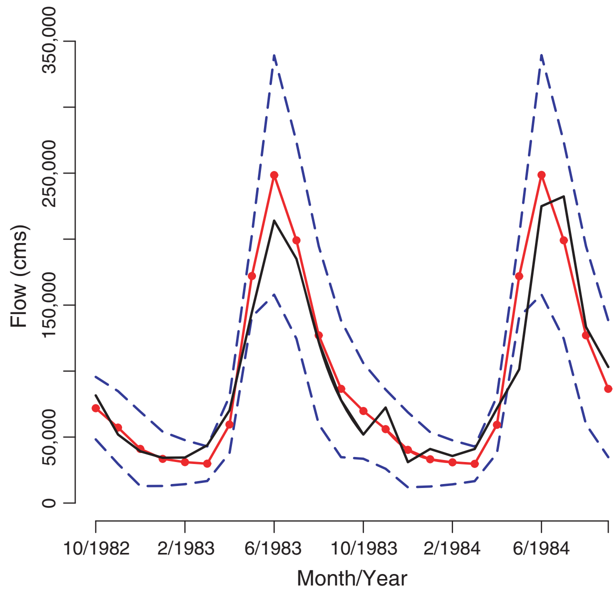

Figure 3. Twentyfour-month forecast (solid line with dots) based on 70 years of Fraser river data, with 95% prediction bounds (dotted lines). For comparison, the actual data (solid line) is also shown. The data were not used in the forecast

are plotted in Figure 2, with 95% confidence intervals obtained using the asymptotic theory in Anderson and Meerschaert (1997). Since the mean, standard deviation and correlation functions vary significantly with the season, subtracting the sample mean will not yield a stationary series, so a $\mathrm{PARMA}_S(p, q)$ model with $S = 12$ seasons is appropriate.

To validate our forecast methods, we computed a 24-month forecast using the first 70 years of data, and then comparedit with the remaining data. A $\mathrm{PARMA}_{12}(1,1)$ model $X_t - \phi_t X_{t-1} = \varepsilon_t + \theta_t \varepsilon_{t-1}$ was found adequate to capture the seasonal covariance structure in the mean-centred series $X_t = \tilde{X}_t - \mu_t$. Using the first 70 years of data, we ran 20 iterations of the innovations algorithm for periodically stationary processes (Anderson *et al.*, 1999, Proposition 2.2.1) on the sample covariance (22) to obtain estimates of the infinite order moving average coefficients $\hat{\psi}_i(j)$, then used the model equations

$$\hat{\phi}_t(1) = \hat{\psi}_t(2)/\hat{\psi}_{t-1}(1) \quad \text{and} \quad \hat{\theta}_t(1) = \hat{\psi}_t(1) - \hat{\phi}_t(1) \tag{24}$$

to get estimates of the $\mathrm{PARMA}_{12}(1,1)$ model parameters. Table 1 lists the resulting model parameters. Model adequacy was validated by visual inspection of the sample autocorrelation and partial autocorrelation plots, similar to Tesfaye *et al.* (2006), where the full time series (72 years of data) was modeled. Then we computed the transformed process (4) using these model parameters, computed the sample autocovariance of that process, and applied the innovations algorithm (7) again to get the projection coefficients $\theta_{n,j}$. Next we used (8) to compute the one-step-ahead predictors $\hat{X}_n$ for $n = 1,2,\ldots,864 = 72 \times 12$. Finally we applied (10) to get the forecasts, and used the asymptotic formula (14) to compute 95% prediction bounds, based on the assumption of Gaussian innovations. The resulting prediction, along with the 95% prediction bands, are shown in Figure . The actual data (solid line) is also shown for comparison. Note that the forecast (solid line with dots) is in reasonable agreement with the actual data (which were not used in the forecast), and that the actual data lies well within the 95% prediction bands. Since the seasonal standard deviation varies significantly, the width of the prediction intervals also varies with the season (Figure 4).

Table 1. Parameter estimates for the $\mathrm{PARMA}_{12}(1,1)$ model of average monthly flow for the Fraser River near Hope BC from October 1912 to September 1982 (first 70 years of data)

Season	Month	$\hat{\phi}$	$\hat{\theta}$	$\hat{\sigma}$
0	OCT	0.187	0.704	11761.042
1	NOV	0.592	0.050	11468.539
2	DEC	0.575	−0.038	7104.342
3	JAN	0.519	−0.041	5879.327
4	FEB	0.337	0.469	4170.111
5	MAR	0.931	−0.388	4469.202
6	APR	1.286	−0.088	15414.905
7	MAY	1.059	−0.592	30017.508
8	JUN	−2.245	2.661	32955.491
9	JUL	−1.105	0.730	30069.997
10	AUG	0.679	−0.236	15511.989
11	SEP	0.353	0.326	12111.919

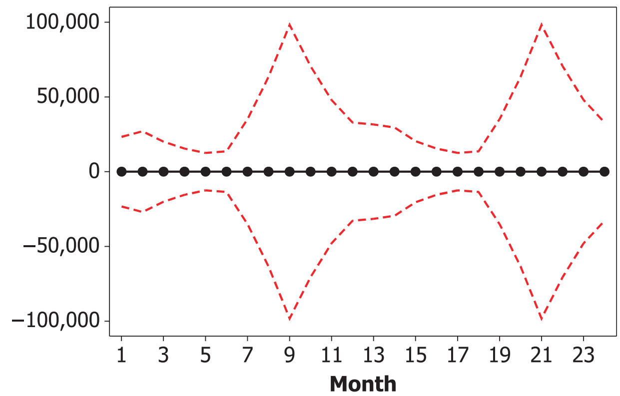

Figure 4. Width of 95% prediction bounds for the Fraser river

REMARK 3. It can be advantageous to consider reduced PARMA models, in which statistically insignificant parameter values are zeroed out to obtain a more parsimonious model (McLeod, 1993; Lund *et al.* 2005). One can also employ discrete Fourier transforms of the periodically varying parameters (Anderson *et sl.* 2007; Tesfaye *et al.* 2011), and then zero out the statistically insignificant frequencies. It would be interesting to extend the results of this study to such models.

5. CONCLUSION

Periodic ARMA models are indicated for time series whose mean, variance and correlation structure vary significantly with the season. This study has developed and implemented a practical methodology for forecasting periodic ARMA models, with Gaussian prediction intervals to provide error bounds. The procedure was demonstrated using R codes that are freely available from the authors.

Acknowledgements

This research was partially supported by NSF grants DMS-1025486, DMS-0803360, EAR-0823965 and NIH grant R01-EB012079-01.

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