Foreign Exchange Rates

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Abstract
We establish a simple, yet completely general model for foreign exchange rates (FXR), in the context of multidimensional, possibly incomplete, Itô SDE market/econometric models. A very simple example is presented as well.

1 Introduction

We adopt a point of view, which can be contrasted to, e.g., [1], that the foreign exchange rates (FXR) between two economies are driven by the totality of *optimized* investment opportunities in the corresponding economies, risk-less and risky, and therefore also *very much* affected by the value of the prevailing market (relative) risk aversion parameter γ (see below). Other way around, FXR is hereby singled out as a particularly efficient statistics for estimating γ, the much needed parameter value for pricing of any financial derivative in incomplete markets (see [4,5,6,7,8]). Methodologically, we employ the general optimal portfolio theory (see [5]; see also [2,3]) to study FXR. In the follow-up note we also introduce a totally consistent theory of foreign exchange derivatives. Here and in the follow-up note we announce shortly results to be elaborated in [8].

REMARK 1. From *Financial Times* (June 5, 2007): "... earlier this year, the yen rose amid a general increase in risk aversion."

2 General FXR SDE

Consider two (simple) economies (see [5]), *domestic* $\mathcal{E}_d$, and *foreign* $\mathcal{E}_f$, with corresponding interest rates $r_d(t)$ and $r_f(t)$. The two economies are described quantitatively via a finite set of dynamic *factors* and *tradables*. Factors are considered cumulatively across both economies and denoted by $A(t) = \{A_1(t), \ldots, A_m(t)\}$, while (risky) tradables are considered separately, and denoted by $S_d(t) = \{S_{d,1}(t), \ldots, S_{d,k_d}(t)\}$, and $S_f(t) = \{S_{f,1}(t), \ldots, S_{f,k_f}(t)\}$, respectively. Simpler cases $S_d(t) = \emptyset$, $S_f(t) = \emptyset$, or $S_d(t) = S_f(t) = \emptyset$, can also be considered. Factors and tradables are assumed to obey Itô SDE dynamics

$$dA(t) = b(t, A(t))\, dt + c(t, A(t)).dB(t) \tag{2.1}$$

$$\begin{aligned} dS_d(t) &= S_d(t)\,(a_{s,d}(t, A(t)) - \mathbb{D}_d(t, A(t)))\, dt + S_d(t)\, \sigma_{s,d}(t, A(t)).dB(t) \\ dS_f(t) &= S_f(t)\,(a_{s,f}(t, A(t)) - \mathbb{D}_f(t, A(t)))\, dt + S_f(t)\, \sigma_{s,f}(t, A(t)).dB(t) \end{aligned} \tag{2.2}$$

where $B(t) = \{B_1(t), \ldots, B_n(t)\}$ is a vector of n independent standard Brownian motions, $b(t, A)$ is the m-vector-valued-function of factor-drifts, $c(t, A)$ is $m \times n$ factor-diffusion-matrix, $a_{s,d}(t, A)$ is the k_d-vector of appreciation rates (before dividends) for the tradables in the domestic economy, $a_{s,f}(t, A)$ is the k_f-vector of appreciation rates (before dividends) for the tradables in the foreign economy, $\mathbb{D}_d(t, A)$ and $\mathbb{D}_f(t, A)$ are the k_d- and k_f-vectors of dividend rates of the corresponding assets, $\sigma_{s,d}(t, A)$ and $\sigma_{s,f}(t, A)$ are the volatility $k_d \times n$- and $k_f \times n$-matrices. All of those functions (as well as r_d and r_f, the interest rates, if not included in A) are called *market coefficients.*

By HARA (or CRRA) utility function we shall mean $\psi_\gamma(X) = X^{1-\gamma}/(1-\gamma)$ for $\gamma \neq 1$, and $\psi_1(X) = \log(X)$, where X denotes the available (investor's) wealth, and $\gamma > 0$ is called the *relative risk aversion parameter.* Recall (see [5]) the solution of the HARA optimal portfolio problem: Solve (the risk premium PDE; $\nabla := \nabla_A$)

$$\begin{aligned}&\frac{\partial g_{\gamma,T}}{\partial t} + \frac{1}{2}\,\mathrm{Tr}(c.c^T.\nabla\nabla g_{\gamma,T}) + \left(b - \frac{\gamma-1}{\gamma}\,(a_s - r).(\sigma_s.\sigma_s{}^T)^{-1}.\sigma_s.c^T\right).\nabla g_{\gamma,T}\\ &+\frac{1}{2}\,\nabla g_{\gamma,T}.c.\left(\mathbb{I} - \frac{\gamma-1}{\gamma}\,\sigma_s{}^T.(\sigma_s.\sigma_s{}^T)^{-1}.\sigma_s\right).c^T.\nabla g_{\gamma,T}\\ &= \frac{\gamma-1}{\gamma}\left(\frac{1}{2}\,(a_s - r).(\sigma_s.\sigma_s{}^T)^{-1}.(a_s - r) + r\,\gamma\right)\end{aligned} \tag{2.3}$$

$$g_{\gamma,T}(T, A) = 0 \tag{2.4}$$

for $t < T$, where we have emphasized the dependence on the time horizon T, and then the optimal portfolio rule is equal to (the vector of cash values of investments into each of the risky assets):

$$\Pi^{H,\star}_{\gamma,T}(t, X, A) = \frac{X}{\gamma}\,(a_s - r + \nabla g_{\gamma,T}.c.\sigma_s{}^T).(\sigma_s.\sigma_s{}^T)^{-1}. \tag{2.5}$$

Although the limit $\lim_{T\to\infty} g_{\gamma,T}(t, A)$ does not exist (as it is easy to see), the limit $\lim_{T\to\infty} \nabla g_{\gamma,T}(t, A)$, generally speaking, does exist (see [5,6,8]). So, we define

$$\mathfrak{R}_s(t, A) = a_s(t, A) - r(t, A) + \left(\lim_{T\to\infty} \nabla g_{\gamma,T}(t, A)\right).c(t, A).\sigma_s(t, A)^T \tag{2.6}$$

the (infinite horizon) risk premium, so that the optimal portfolio rule with the infinite time horizon is given by

$$\Pi^{H,\star}_{\gamma,\infty}(t, X, A) = \frac{X}{\gamma}\,\mathfrak{R}_s.(\sigma_s.\sigma_s{}^T)^{-1}. \tag{2.7}$$

Substituting the formula for the HARA infinite-horizon optimal portfolio (2.7), we get the optimal wealth evolution equation for both economies, domestic and foreign:

$$\begin{aligned}\frac{dX_d(t)}{X_d(t)} &= \left(\frac{1}{\gamma}\,\mathfrak{R}_{s,d}.(\sigma_{s,d}.\sigma_{s,d}{}^T)^{-1}.(a_{s,d} - r_d) + r_d\right)dt\\ &\qquad + \frac{1}{\gamma}\,\mathfrak{R}_{s,d}.(\sigma_{s,d}.\sigma_{s,d}{}^T)^{-1}.\sigma_{s,d}.dB(t)\end{aligned} \tag{2.8}$$

in the case of the domestic economy, and analogously in the case of the foreign economy.

DEFINITION 1. The (bilateral) FXR process, denoted by $Y_{d,f}(t)$, representing the price of the foreign currency in units of the domestic currency, is such a process that, by definition,

$$\frac{dX_d(t)}{X_d(t)} = \frac{d(Y_{d,f}(t)\,X_f(t))}{Y_{d,f}(t)\,X_f(t)} \tag{2.9}$$

i.e., the FXR process $Y_{d,f}(t)$ makes it irrelevant whether investments are made in the domestic or in the foreign economy.

THEOREM 1 (General FXR SDE). For a given relative risk aversion parameter $\gamma \in (0, \infty]$, the Itô SDE governing the (bilateral) HARA FXR $Y_{d,f,\gamma}(t) = Y_{d,f}(t)$ is given by

$$\frac{dY_{d,f}(t)}{Y_{d,f}(t)} = \Big(r_d - r_f + \frac{1}{\gamma}\left(\mathfrak{A}_{s,d}.(\sigma_{s,d}.\sigma_{s,d}{}^T)^{-1}.(a_{s,d} - r_d)\right.$$
$$- \mathfrak{A}_{s,f}.(\sigma_{s,f}.\sigma_{s,f}{}^T)^{-1}.(a_{s,f} - r_f)\Big) + \frac{1}{\gamma^2}\,\mathfrak{A}_{s,f}.(\sigma_{s,f}.\sigma_{s,f}{}^T)^{-1}.\mathfrak{A}_{s,f}$$
$$- \frac{1}{\gamma^2}\,\mathfrak{A}_{s,f}.(\sigma_{s,f}.\sigma_{s,f}{}^T)^{-1}.\sigma_{s,f}.\sigma_{s,d}{}^T.(\sigma_{s,d}.\sigma_{s,d}{}^T)^{-1}.\mathfrak{A}_{s,d}\Big)\,dt \tag{2.10}$$
$$+ \frac{1}{\gamma}\left(\mathfrak{A}_{s,d}.(\sigma_{s,d}.\sigma_{s,d}{}^T)^{-1}.\sigma_{s,d} - \mathfrak{A}_{s,f}.(\sigma_{s,f}.\sigma_{s,f}{}^T)^{-1}.\sigma_{s,f}\right).dB(t)$$

for $-\infty < t < \infty$, where $\mathfrak{A}_{s,d}(t, A)$ and $\mathfrak{A}_{s,f}(t, A)$ are given by (2.6), and where, for each $T < \infty$, $g_{\gamma,d,T}(t, A)$ and $g_{\gamma,f,T}(t, A)$ are characterized via (2.4)–(2.5), modified to account for domestic and foreign economies. In cases $S_d(t) = \emptyset$, $S_f(t) = \emptyset$, or $S_d(t) = S_f(t) = \emptyset$, we can still apply (2.10), by setting $\mathfrak{A}_{s,f} = 0$, $\mathfrak{A}_{s,d} = 0$, or $\mathfrak{A}_{s,d} = \mathfrak{A}_{s,f} = 0$, respectively.

Proof. See [8].

Notice that if $f = d$, (2.10) becomes $dY_{d,d}(t) / Y_{d,d}(t) = 0$.

PROPOSITION 1 (Multilateral FXR—the cross-currency rule). Let $Y_{d,f_1}(t)$, $Y_{d,f_2}(t)$, and $Y_{f_1,f_2}(t)$ be the bilateral FXR processes. Then

$$\frac{dY_{f_1,f_2}(t)}{Y_{f_1,f_2}(t)} = \frac{d(Y_{d,f_2}(t) / Y_{d,f_1}(t))}{Y_{d,f_2}(t) / Y_{d,f_1}(t)}. \tag{2.11}$$

Proof. See [8].

REMARK 2. Definition 1, Theorem 1, and Proposition 1 can all be extended somewhat to model FXR under "economic imbalances" between the considered economies (see [8]).

3 A Very Simple FXR Example

The simplest stochastic example of two economies and the corresponding FXR process would be the case when the risky assets prices in both, domestic and foreign economies are modeled simply as log-normal processes. So, let $S_1(t)$ be, for example, a market index in the domestic economy, and let $S_2(t)$ be a market index in the foreign economy. Denote by r_d and r_f the interest rates, by $\mathbb{a}_d$ and $\mathbb{a}_f$ the (pre-dividend) appreciation rates, by $\mathbb{D}_d$ and $\mathbb{D}_f$ the dividend rates, by σ_d and σ_f the volatilities, for domestic and foreign markets, respectively, and by $\rho_{2,1}$ the instantaneous correlation (all assumed known constants, or functions of time only). So, assume that the two economies are represented by two factors (and tradables) $A = \{S_1, S_2\}$ obeying:

$$\begin{aligned} dS_1(t) &= S_1(t)\,(\mathbb{a}_d - \mathbb{D}_d)\,dt + S_1(t)\,\sigma_d\,dB_1(t) \\ dS_2(t) &= S_2(t)\,(\mathbb{a}_f - \mathbb{D}_f)\,dt + S_2(t)\,\sigma_f\left(\rho_{2,1}\,dB_1(t) + \sqrt{1-\rho_{2,1}^2}\,dB_2(t)\right). \end{aligned} \tag{3.1}$$

The market coefficients (see [5,8]), in addition to r_d and r_f, therefore, are assumed to be

$$\begin{aligned} &a_{s,d} = \{\mathbb{a}_d\},\ a_{s,f} = \{\mathbb{a}_f\},\ \sigma_{s,d} = (\,\sigma_d \quad 0\,),\ \sigma_{s,f} = \left(\,\sigma_f\,\rho_{2,1} \quad \sigma_f\sqrt{1-\rho_{2,1}^2}\,\right) \\ &b = \{S_1\,(\mathbb{a}_d - \mathbb{D}_d),\ S_2\,(\mathbb{a}_f - \mathbb{D}_f)\},\ c = \begin{pmatrix} S_1\,\sigma_d & 0 \\ S_2\,\sigma_f\,\rho_{2,1} & S_2\,\sigma_f\sqrt{1-\rho_{2,1}^2} \end{pmatrix}. \end{aligned} \tag{3.2}$$

Applying Theorem 1, after some work (see [8]), we derive the FXR SDE

$$\frac{dY_{d,f}(t)}{Y_{d,f}(t)} = \left(r_d - r_f + \frac{1}{\gamma}\left(\left(\frac{\mathrm{a}_d - r_d}{\sigma_d}\right)^2 - \left(\frac{\mathrm{a}_f - r_f}{\sigma_f}\right)^2\right)\right.$$
$$\left. - \frac{1}{\gamma^2}\,\frac{\mathrm{a}_d - r_d}{\sigma_d}\,\frac{\mathrm{a}_f - r_f}{\sigma_f}\,\rho_{2,1} + \frac{1}{\gamma^2}\left(\frac{\mathrm{a}_f - r_f}{\sigma_f}\right)^2\right)dt + \frac{1}{\gamma}\,\frac{\mathrm{a}_d - r_d}{\sigma_d}\,dB_1(t) \tag{3.3}$$
$$- \frac{1}{\gamma}\,\frac{\mathrm{a}_f - r_f}{\sigma_f}\left(\rho_{2,1}\,dB_1(t) + \sqrt{1 - \rho_{2,1}^2}\,dB_2(t)\right).$$

Denote by σ_{FXR} the FXR volatility, we calculate from (3.3)

$$\sigma_{\mathrm{FXR}} = \frac{1}{\gamma}\sqrt{\left(\frac{\mathrm{a}_d - r_d}{\sigma_d}\right)^2 - 2\,\frac{\mathrm{a}_d - r_d}{\sigma_d}\,\frac{\mathrm{a}_f - r_f}{\sigma_f}\,\rho_{2,1} + \left(\frac{\mathrm{a}_f - r_f}{\sigma_f}\right)^2}\,. \tag{3.4}$$

Solving (3.4) for γ, we obtain a simple (statistical) estimate for the (*aggregate market*) *relative risk aversion* (*for the two economies*). For example, for USD and GBP, between Dec. 31, 1985 and Aug. 31, 2005, if only monthly data is used (DJI and FTSE were taken as market representatives, and zero dividens are assumed), and if interest rates were averaged for the whole duration, the relative risk aversion parameter γ is estimated using (3.6) to be

$$\gamma_{\mathrm{USD/GBP}} = 4.17. \tag{3.5}$$

For much more elaborate FXR example-models, including stochastic interest rates, stochastic volatility, etc., the reader is referred to [8]. For the related theory of foreign exchange derivatives see [9].

REMARK 1 (continuation). As $\gamma > 1$, ignoring $1/\gamma^2$ terms, the rise in yen can be explained by the remaining drift terms in (3.3): $dY_{d,f}(t) \approx Y_{d,f}(t)\,(r_d - r_f + (((\mathrm{a}_d - r_d)/\sigma_d)^2 - ((\mathrm{a}_f - r_f)/\sigma_f)^2)/\gamma)\,dt$: the "before-risk-aversion-increase" "rate-equilibrium" was maintained by the balance between $r_d > r_f$ (low Japanese interest rate r_f), and $((\mathrm{a}_d - r_d)/\sigma_d)^2 < ((\mathrm{a}_f - r_f)/\sigma_f)^2$. With γ rising, the balance is lost, causing $dY_{d,f}(t) > 0$, the raise in yen.

4 References

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