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Form invariance of differential equations in general relativity

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Einstein equations for several matter sources in Robertson–Walker and Bianchi I type metrics, are shown to reduce to a kind of second-order nonlinear ordinary differential equation $\ddot{y}+\alpha f(y)\dot{y}+\beta f(y)\int f(y)dy+\gamma f(y)=0$. Also, it appears in the generalized statistical mechanics for the most interesting value $q=-1$. The invariant form of this equation is imposed and the corresponding nonlocal transformation is obtained. The linearization of that equation for any α, β, and γ is presented and for the important case $f=by^n+k$ with $\beta=\alpha^2(n+1)/(n+2)^2$ its explicit general solution is found. Moreover, the form invariance is applied to yield exact solutions of some other differential equations. © *1997 American Institute of Physics.* [S0022-2488(97)02603-0]

I. INTRODUCTION

Exact solutions of the Einstein equations are difficult to obtain due to their nonlinear nature. There exist several interesting physical problems where the Einstein field equations for homogeneous, isotropic and spatially flat cosmological models with no cosmological constant[1–6] and for a time decaying cosmological constant,[7] or Bianchi I type metric[8] with a variety of matter sources, reduce to particular cases of the second-order nonlinear ordinary differential equation

$$\ddot{y}+\alpha f(y)\dot{y}+\beta f(y)\int f(y)dy+\gamma f(y)=0, \tag{1}$$

where $y=y(x)$, $f(y)$ is a real function and the dot means differentiation with respect to x. α, β, and γ are constant parameters.

Recently, it was shown that some galactic models of astrophysical relevance, when investigated with the ''generalized'' statistical mechanics,[9] can be exactly described by solutions to the Boltzmann equations that maximize the generalized Tsallis entropy for $q=-1$,[10] and it was found that the corresponding probability distribution function satisfies Eq. (1).[11]

It is believed that quantum effects played a fundamental role in the early Universe. For instance, vacuum polarization and particle production arise from a quantum description of matter. It is known that both of them can be modeled in terms of a classical bulk viscosity.[12] Using the relativistic second-order theory of nonequilibrium thermodynamics—called extended irreversible thermodynamics developed in Refs. 13 and 14—it was considered a homogeneous isotropic spatially flat universe, filled with a causal viscous fluid whose equilibrium pressure obeys a γ-law equation of state, while the transport equation of the viscous pressure is

$$\sigma+\tau\dot{\sigma}=-3\zeta H-\frac{1}{2}\epsilon\tau\sigma\left(3H+\frac{\dot{\tau}}{\tau}-\frac{\dot{\zeta}}{\zeta}-\frac{\dot{T}}{T}\right), \tag{2}$$

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with $\epsilon=0$.[15] Following Ref. 16 for $m=1/2$, it was shown in Ref. 1 that the expansion rate satisfies a modified Painlevé–Ince equation that has the form of Eq. (1) with $f(y)=y$ and $\gamma=0$.

Cosmological models with a viscous fluid source have been studied using the full causal irreversible thermodynamics with the full version of the transport equation for the bulk viscous pressure.[17,5,6] Relating the equilibrium temperature T with the energy density in the simplest way to guarantee a positive heat capacity, it was shown that the expansion rate satisfies Eq. (1) for $m=1/2$, with $f(y)=y^{-1/r}$ and $\gamma=0$.[5] Also, the early time evolution of a dissipative universe leads to an equation for the expansion rate that has the form (1),[4,18] in the relaxation dominated regime.

Another interesting example appears when an anisotropic universe, described by a Bianchi type I metric, is driven by a minimally coupled scalar field with an exponential potential. The Klein–Gordon equation for the scalar field and the Einstein equations for the metric are expressed in terms of the semiconformal factor G and their derivatives.[19] Then, the solutions of this equation set can be obtained if one is able to solve the following Einstein equation for G,

$$G\frac{\ddot{G}}{\dot{G}}+(c-1)\dot{G}+\frac{c_1}{\dot{G}}=c_2, \tag{3}$$

which, making the substitution $G=y^{1/c}$ Eq. (3) becomes (1).[8] A similar result is obtained in the particular case when the Bianchi type I metric reduces to a flat Robertson–Walker space-time.[2]

From the generalized Tsallis entropy, defined as[9]

$$S_q=k(q-1)^{-1}\sum_i (p_i-p_i^q), \tag{4}$$

the generalized statistical mechanics can be constructed where k is a positive constant, q is a real number that characterizes the statistic and the sum is made over all the microscopic configurations whose probabilities are p_i. It leads to the conventional Boltzmann–Shannon statistic in the limit $q\rightarrow 1$ and it is found to be a good framework to study astrophysical problems, as are the generalized Freeman disk[20] and Kalnajs oscillations of a slab of stars.[21] Taking the generalized Fisher information for Tsallis statistics[22]

$$I_q=\left\langle\left(\frac{(d/dx)\,f_d}{f_d(x)}\right)^2\right\rangle, \tag{5}$$

where $f_d(x)$ is the probability distribution function, and solving the variational problem in order to find the distribution function that maximizes the Fisher information, a differential equation of type (1) is obtained for $y=\dot{f}_d/f_d$, where $f(y)=y$, $\alpha=(2q-1)$, $\beta=\frac{1}{2}q(q-1)$, and $\gamma=0$.[11] For relevant physical applications the most interesting value of the statistic parameter is $q=-1$,[10] in this case the above equations can be solved explicitly and the general solution will be given in Sec. III.

Thus, it turns out to be of great interest to analyze Eq. (1) from the physical and mathematical point of view. The paper is organized as follows, in Section II we introduce an invariant form and use it to reduce Eq. (1) to a linear, inhomogeneous ordinary second-order differential equation with constant coefficients, by means of a nonlocal transformation. Then, its parametric general solution is given. In Section III we extend the nonlocal transformation and find the explicit general solution of a modified Painlevé–Ince equation for $\beta=1/9$.[23] In Section IV we use the nonlocal invariance to obtain a new class of differential equations for which the general solution is found. In Section V the conclusions are stated.

II. FORM INVARIANCE

The differential equation (1), which appears in several interesting physical problems, has been solved and studied in particular cases using nonlocal transformations, as was previously stated. To investigate Eq. (1) we write it in invariant form

$$\frac{\ddot{y}}{f(y)}+\alpha\dot{y}+\beta\int f(y)dy+\gamma=\frac{\overline{y}''}{\overline{f}(\overline{y})}+\overline{\alpha}\overline{y}'+\overline{\beta}\int \overline{f}(\overline{y})d\overline{y}+\overline{\gamma}, \tag{6}$$

under the nonlocal transformation group defined by the transformation

$$\beta f(y)dy=\overline{\beta}\overline{f}(\overline{y})d\overline{y}, \tag{7}$$

$$\frac{\beta}{\alpha}f(y)dx=\frac{\overline{\beta}}{\overline{\alpha}}\overline{f}(\overline{y})d\overline{x}, \tag{8}$$

$$\frac{\beta}{\alpha^2}=\frac{\overline{\beta}}{\overline{\alpha}^2}, \tag{9}$$

$$\beta c+\gamma=\overline{\beta}\overline{c}+\overline{\gamma}, \tag{10}$$

where $\overline{f}(\overline{y})$ is a real function of $\overline{y}=\overline{y}(\overline{x})$, the prime indicates differentiation with respect to $\overline{x}$, $\overline{\alpha}$, $\overline{\beta}$, $\overline{\gamma}$ are constant parameters, and $c(\overline{c})$ is an integration constant provided by the integral on the left(right) hand side of Eq. (6). By invariant form we mean that the left-hand side of Eq. (6) transforms into the right-hand side under the nonlocal transformation defined by Eqs. (7–10) for any functions $f,\overline{f}$. The parameters α, β, γ, $\overline{\alpha}$, and $\overline{\beta}$ satisfy Eqs. (9 and 10).

The form invariance group can be used to linearize Eq. (1). In fact, taking the function $\overline{f}(\overline{y})=1$, $\overline{\alpha}=\alpha$, $\overline{\beta}=\beta$, and $\overline{\gamma}=\gamma$ (this means $\overline{c}=c$) in the invariant form (6) and the transformation (7–10), they become

$$\frac{\ddot{y}}{f(y)}+\alpha\dot{y}+\beta\int f(y)dy+\gamma=\overline{y}''+\alpha\overline{y}'+\beta\overline{y}+\beta c+\gamma, \tag{11}$$

$$\overline{y}=\int f(y)dy, \quad \overline{x}=\int f(y)dx. \tag{12}$$

Without loss of generality we choose $\overline{c}=c=0$. So, if the invariant (11) vanishes, then, Eq. (1) transforms into

$$\overline{y}''+\alpha\overline{y}'+\beta\overline{y}+\gamma=0, \tag{13}$$

under the transformation of variables (12). This is a linear, second-order ordinary differential equation with constant coefficients. Its general solution is

(a) $\beta\neq\frac{\alpha^2}{4}$

$$\overline{y}=c_1\exp(\lambda_1\overline{x})+c_2\exp(\lambda_2\overline{x})-\frac{\gamma}{\beta}, \tag{14}$$

where λ_1 and λ_2 are the roots of the characteristic polynomial of Eq. (13). We indicate the integration constants with $c, c_1, \ldots, c_n$ and $\overline{c}, \overline{c_1}, \ldots, \overline{c_n}$.

(b) $\beta = \frac{\alpha^2}{4}$

$$\overline{y} = (c_1 + c_2 \overline{x}) \exp\left(-\frac{\overline{x}}{2}\right) - \frac{\gamma}{\beta}. \tag{15}$$

The real solutions can be classified as follows (we also assume that α, β, and γ are real). For $\alpha > 0$ and $\beta < \alpha^2/4$ we have two real, negative roots for a strong damped solution. For $\beta = \alpha^2/4$ we have a double-negative root for a critically damped solution. For $\alpha > 0$ and $\beta > \alpha^2/4$ we have two complex roots with negative real parts for a weakly damped solution. For the case $\alpha < 0$ growing solutions occur.

The transformation of variables (12), relates the general solution of Eq. (1) with $\overline{y(x)}$ through Eq. (14). We find that

$$y = y(\overline{y(x)}), \tag{16}$$

$$x = \int \frac{1}{f(y(\overline{y(x)}))} d\overline{x} \tag{17}$$

are the parametric equations for x and y in terms of $\overline{x}$. In the particular case $f(y) = y$ we have shown that a class of nonlinear modified Painlevé–Ince equations can be transformed into a linear second-order ordinary differential equation by a nonlocal transformation.

The theory introduced by Lie considers the invariance of the differential equations under point transformations. He showed that the one-dimensional free particle equation has the eight-dimensional SL(3,*R*) group of point transformations. This is the maximum number of symmetry generators for a second-order differential equation of the form[24]

$$\ddot{y} + h(\dot{y}, y, x) = 0. \tag{18}$$

In our case Eq. (1) has the form of Eq. (18). Then, it has eight or less point symmetries. However, it becomes Eq. (13) under the transformation of variables (12) and can be cast into the free particle equation by a local point transformation. So, Eq. (13) always has eight symmetry generators. We conclude this section by observing that the nonlocal transformation (7–10) changes the number of symmetry generators for the class of differential equations (1) and the physics contained in the original problem.

The nonconstant parameters case: Here we allow the parameters in Eq. (1) and in the transformation (7–10) to be functions of the independent variable, that is, $\alpha = \alpha(x)$, $\beta = \beta(x)$, and $\gamma = \gamma(x)$. In order to preserve the form (1) we choose $\overline{\alpha(x)} = \alpha(\overline{x})$ and $\overline{\beta(x)} = \beta(\overline{x})$. In this case, the invariant form (6) reads

$$\frac{\ddot{y}}{f(y)} + \alpha(x)\dot{y} + \beta(x)\int f(y)dy + \gamma(x) = \frac{\overline{y}''}{\overline{f(\overline{y})}} + \alpha(\overline{x})\overline{y}' + \beta(\overline{x})\int \overline{f(\overline{y})}d\overline{y} + \gamma(\overline{x}), \tag{19}$$

where $\overline{x}$ is the transformed of the point x. Therefore, taking $\overline{\gamma} = \gamma$ and $\overline{f(\overline{y})} = 1$ we can linearize the equation

$$\ddot{y} + \alpha(x)f(y)\dot{y} + \beta(x)f(y)\int f(y)dy + \gamma(x)f(y) = 0, \tag{20}$$

which transforms into

$$\overline{y''}+\alpha(\overline{x})\overline{y'}+\beta(\overline{x})\overline{y}+\gamma(\overline{x})=0. \tag{21}$$

An important physical problem of general relativity, concerning the motion of expanding shear-free perfect fluids,[25] is governed by the ordinary differential equation

$$\ddot{y}=F(x)y^2, \tag{22}$$

where $F(x)$ is an arbitrary function from which the equation of state can be computed. A complete symmetry analysis of this differential equation was given in Ref. 26. Here we see that it is contained in the set of equations (20) when $\alpha(x)=0$, $\beta(x)=-3F(x)/2$, $\gamma(x)=0$, and $f(y)=y^{1/2}$. Then, choosing $\overline{f}(\overline{y}) = (\overline{y})^{-1/2}$ in Eqs. (7–10), the transformation of variables is

$$\overline{y}=\frac{y^3}{9}, \quad \overline{x}=\int \frac{y^2}{3}dx, \tag{23}$$

and Eq. (22) becomes

$$\overline{y''}=3F(\overline{x}), \tag{24}$$

thus

$$\overline{y}=\int\left[\int F(\overline{x})d\overline{x}\right]d\overline{x}+c_1\overline{x}+c_2, \tag{25}$$

is the general solution of the simple linear equation (24).

III. EXTENDED NONLOCAL TRANSFORMATION

The integral in Eq. (17) can be performed analytically and the general solution $y=y(x)$ of Eq. (1) obtained explicitly for a special set of functions $f(y)$. For this purpose we generalize the nonlocal transformation group defined by Eqs. (7–10) extending it to

$$f_{11}(y)dy+f_{12}(y)dx=\overline{f}_{11}(\overline{y})d\overline{y}+\overline{f}_{12}(\overline{y})d\overline{x}, \tag{26}$$

$$f_{21}(y)dy+f_{22}(y)dx=\overline{f}_{21}(\overline{y})d\overline{y}+\overline{f}_{22}(\overline{y})d\overline{x}. \tag{27}$$

For simplicity we begin our investigations restricting ourselves to the case $x=\overline{x}$, that is, $f_{21}=\overline{f}_{21}=0$, $f_{22}=\overline{f}_{22}=1$ and requiring the invariant form (6) to be invariant under the remaining nonlocal transformation group, defined by Eqs. (26 and 27) with the above restrictions. Under these assumptions we can write the nonlocal transformation as

$$\dot{\overline{y}}=p+q\dot{y}, \tag{28}$$

where the functions p and q are expressed in terms of the functions f_{11}, f_{12}, $\overline{f}_{11}$, and $\overline{f}_{12}$. So, they have a specific dependence on the variables y and $\overline{y}$

$$p(y,\overline{y})=\frac{f_{12}(y)}{\overline{f}_{11}(\overline{y})}-\frac{\overline{f}_{12}(\overline{y})}{\overline{f}_{11}(\overline{y})}, \tag{29}$$

$$q(y,\overline{y})=\frac{f_{11}(y)}{\overline{f}_{11}(\overline{y})}. \tag{30}$$

Inserting Eq. (28) in Eq. (6) we get

$$\frac{\ddot{y}}{f}+\alpha\dot{y}+\beta\int f\,dy+\gamma=\frac{q}{\bar{f}}\ddot{y}+\left[\frac{\partial q}{\partial y}+q\frac{\partial q}{\partial \bar{y}}\right]\frac{\dot{y}^2}{\bar{f}}+\left[\frac{\partial p}{\partial y}+q\frac{\partial p}{\partial \bar{y}}+p\frac{\partial q}{\partial \bar{y}}\right]\frac{\dot{y}}{\bar{f}}$$
$$+\frac{p}{\bar{f}}\frac{\partial p}{\partial \bar{y}}+\bar{\alpha}[p+q\dot{y}]+\bar{\beta}\int \bar{f}\,d\bar{y}+\bar{\gamma}, \tag{31}$$

and comparing the coefficients of $\dot{y}^2$, we have

$$\frac{\partial q}{\partial y}+q\frac{\partial q}{\partial \bar{y}}=0, \tag{32}$$

whose solution is

$$q(y,\bar{y})=\frac{\bar{y}}{y}. \tag{33}$$

Using Eq. (33) and comparing the coefficients of $\ddot{y}$ we easily find that $f=y$ and $\bar{f}=\bar{y}.$ But, the comparisons of the coefficients of $\dot{y}$ and the remaining terms give the equations

$$\alpha=\left[\frac{\partial p}{\partial y}+\frac{\bar{y}}{y}\frac{\partial p}{\partial \bar{y}}+\frac{p}{y}\right]\frac{1}{\bar{y}}+\bar{\alpha}\frac{\bar{y}}{y}, \tag{34}$$

$$\beta\int y\,dy+\gamma=\frac{p}{\bar{y}}\frac{\partial p}{\partial \bar{y}}+\bar{\alpha}p+\bar{\beta}\int \bar{y}\,d\bar{y}+\bar{\gamma}. \tag{35}$$

The function p that satisfies Eq. (34) is given by

$$p(y,\bar{y})=\frac{\alpha}{3}y\bar{y}-\frac{\bar{\alpha}}{3}\bar{y}^2+h(y,\bar{y}), \tag{36}$$

where the function $h(y,\bar{y})$ satisfies the partial differential equation

$$y\frac{\partial h}{\partial y}+\bar{y}\frac{\partial h}{\partial \bar{y}}+h=0. \tag{37}$$

It can be seen that the solutions of Eq. (37) are given by $h=h_0/y$, where h_0 is an arbitrary function of the quotient $\bar{y}/y$. So, the form of the solution for p is

$$p(y,\bar{y})=\frac{\alpha}{3}y\bar{y}-\frac{\bar{\alpha}}{3}\bar{y}^2+\frac{h_0(\bar{y}/y)}{y}. \tag{38}$$

Comparing Eq. (30) with Eq. (33) we have $\bar{f}_{11}(\bar{y})=1/\bar{y},$ and comparing Eq. (29) with Eq. (38), we obtain

$$h_0(\bar{y}/y)=c_1\frac{y}{\bar{y}}+c_2\frac{\bar{y}}{y}. \tag{39}$$

Inserting Eq. (39) in Eq. (35) we find that $c_1=c_2=0,\ \gamma+\beta c=\bar{\beta}\bar{c}+\bar{\gamma},$ and

$$\beta=\frac{2\alpha^2}{9}, \quad \overline{\beta}=\frac{2\overline{\alpha}^2}{9}. \tag{40}$$

Therefore, the final invariant form and the resulting nonlocal transformation are

$$\frac{\ddot{y}}{y}+\alpha\dot{y}+\frac{\alpha^2}{9}y^2+\beta c+\gamma=\frac{\overline{\ddot{y}}}{\overline{y}}+\overline{\alpha}\overline{\dot{y}}+\frac{\overline{\alpha}^2}{9}\overline{y}^2+\overline{\beta}\overline{c}+\overline{\gamma}, \tag{41}$$

$$\frac{\dot{y}}{y}+\frac{\alpha}{3}y=\frac{\overline{\dot{y}}}{\overline{y}}+\frac{\overline{\alpha}}{3}\overline{y}. \tag{42}$$

In the particular case in which the invariant form (41) vanishes, the left-hand side gives rise to a nonlinear differential equation

$$\ddot{y}+\alpha y\dot{y}+\frac{\alpha^2}{9}y^3+\gamma y=0 \tag{43}$$

(where, without loss of generality we have taken $c=\overline{c}=0$, so that, $\gamma=\overline{\gamma}$), that can be solved using the invariance properties formulated above. To do this, we make $\overline{\alpha}=0$ on the right-hand side of Eq. (41). Then, inserting its solution in Eq. (42), it can be integrated giving the general solution

$$y=\frac{3}{\alpha}\frac{2c_1x+c_2}{c_1x^2+c_2x+c_3}, \quad \gamma=0. \tag{44}$$

$$y=\frac{3\sqrt{\gamma}}{\alpha}\frac{c_1\exp(\sqrt{\gamma}x)+c_2\exp(-\sqrt{\gamma}x)}{c_1\exp(\sqrt{\gamma}x)-c_2\exp(-\sqrt{\gamma}x)+c_3}, \quad \gamma\neq 0. \tag{45}$$

It can be seen that Eq. (43) has eight Lie point symmetries and it is equivalent to a second-order linear differential equation under a point transformation.[27] On the other hand, for any other value of the coefficient $\beta \neq 2\alpha^2/9$, Eq. (43) has two point Lie symmetries and we cannot find a point transformation that cast it in a linear equation.[27] However, using the invariant form (11) and the transformation of variables (12) for $f=y$, we have proved that Eq. (43) can always be linearized whatever the value of the coefficient of y^3 is. Therefore, using the invariance properties of the form (6) we have obtained the same results that come by the Lie theory of symmetries. In addition, we have linearized Eq. (43) when it has less than eight Lie point symmetries.

IV. SOLUTION OF NEW CLASSES OF DIFFERENTIAL EQUATIONS

Now, we are going to investigate the case when the invariant expression (6) vanishes, and we shall construct several important classes of solvable second-order nonlinear ordinary differential equations. To do this, we must seek the nonlocal transformation defined by Eqs. (28) and (33) with the condition that the invariant (31) vanishes. This leads to the equations that determine it

$$\alpha f=\frac{y}{\overline{y}}\left[\frac{\partial p}{\partial y}+\frac{\overline{y}}{y}\frac{\partial p}{\partial \overline{y}}+\frac{p}{y}\right]+\overline{\alpha}\overline{f}, \tag{46}$$

$$\beta f\int f dy+\gamma f=\frac{y}{\overline{y}}\left[p\frac{\partial p}{\partial \overline{y}}+\overline{\alpha}p\overline{f}+\overline{\beta}\overline{f}\int \overline{f}d\overline{y}+\overline{\gamma}\overline{f}\right], \tag{47}$$

and we shall show a set of functions $f,\overline{f}$ for which the nonlocal transformation exists. The solution of Eq. (46) can be obtained writing

$$p(y,\overline{y})=\alpha\overline{y}\overline{p}_0(y)+p_1(\overline{y})+p_2(y,\overline{y}), \tag{48}$$

where each function satisfies

$$f=2p_0+yp_0', \tag{49}$$

$$p_1'+\frac{p_1}{\overline{y}}+\overline{\alpha}\overline{f}=0, \tag{50}$$

$$y\frac{\partial p_2}{\partial y}+\overline{y}\frac{\partial p_2}{\partial\overline{y}}+p_1=0, \tag{51}$$

where the prime indicates the derivative with respect to the argument of the function. Solving the system (49–51) and inserting their solutions in Eq. (48), we find the solution of Eq. (46), that is:

$$p(y,\overline{y})=\alpha\frac{\overline{y}}{y^2}\int yf\,dy-\frac{\overline{\alpha}}{\overline{y}}\int\overline{f}\overline{y}d\overline{y}+\frac{h_0(\overline{y}/y)}{y}. \tag{52}$$

Comparing Eq. (52) with Eq. (29), the function $h_0(\overline{y}/y)$ is given by Eq. (39), but these terms can be absorbed in a redefinition of the integration constants provided by the two integrals of Eq. (52). Then, without loss of generality we take them equal to zero.

From Eqs. (47 and 52) we obtain the difficult integrodifferential equation that satisfies the functions f and $\overline{f}$. It reads

$$-\frac{\alpha^2}{y^4}\left[\int fy\,dy\right]^2+\beta\frac{f}{y}\int f\,dy+\gamma\frac{f}{y}=-\frac{\overline{\alpha}^2}{\overline{y}^4}\left[\int\overline{f}\overline{y}d\overline{y}\right]^2+\overline{\beta}\frac{\overline{f}}{\overline{y}}\int\overline{f}d\overline{y}+\overline{\gamma}\frac{\overline{f}}{\overline{y}}. \tag{53}$$

In what follows we shall show a set of functions $f,\overline{f}$ that are solutions of this integrodifferential equation and construct three sets of nonlinear differential equations that can be linearized and explicitly solved.

A. Case a

An interesting solvable equation set can be obtained when we choose the functions $f,\overline{f}$ as:

$$f=by^n+k,\quad \overline{f}=\overline{b}\overline{y}^{\overline{n}}+\overline{k}. \tag{54}$$

Taking into account that the left-hand side of Eq. (53) depends on y and its right hand side depends on $\overline{y}$, it must be a constant. So, inserting the functions given by Eq. (54) in Eq.(53) and after some algebra, it provides the constraints satisfied by the parameters

$$\beta=\alpha^2\frac{n+1}{(n+2)^2},\quad \overline{\beta}=\overline{\alpha}^2\frac{\overline{n}+1}{(\overline{n}+2)^2}, \tag{55}$$

$$\beta k^2-\alpha^2\frac{k^2}{4}=\overline{\beta}\overline{k}^2-\overline{\alpha}^2\frac{\overline{k}^2}{4}. \tag{56}$$

In addition, the function $p(y,\overline{y})$ is given by

$$p(y,\bar{y}) = \alpha y\left[\frac{b}{n+2}y^n + \frac{k}{2}\right] - \bar{\alpha}\bar{y}\left[\frac{\bar{b}}{\bar{n}+2}\bar{y}^{\bar{n}} + \frac{\bar{k}}{2}\right]. \tag{57}$$

Finally inserting Eqs. (54)–(56) in the invariant form (6), we have

$$\ddot{y} + \alpha\,[by^n + k]\dot{y} + \beta\left[b^2\frac{y^{2n+1}}{n+1} + bk\frac{n+2}{n+1}y^{n+1} + k^2 y\right] = 0, \tag{58}$$

$$\ddot{\bar{y}} + \bar{\alpha}\,[\bar{b}\bar{y}^{\bar{n}} + \bar{k}]\dot{\bar{y}} + \bar{\beta}\left[\bar{b}^2\frac{\bar{y}^{2\bar{n}+1}}{\bar{n}+1} + \bar{b}\bar{k}\frac{\bar{n}+2}{\bar{n}+1}\bar{y}^{\bar{n}+1} + \bar{k}^2\bar{y}\right] = 0. \tag{59}$$

Besides, from Eqs. (28), (33), and (57) we obtain the nonlocal transformation (26) in invariant form

$$\frac{\dot{y}}{y} + \frac{\alpha b y^n}{n+2} + \frac{\alpha k}{2} = \frac{\dot{\bar{y}}}{\bar{y}} + \frac{\bar{\alpha}\bar{b}\bar{y}^{\bar{n}}}{\bar{n}+2} + \frac{\bar{\alpha}\bar{k}}{2}, \tag{60}$$

that links Eqs. (58) and (59). To integrate these equations we use their invariant property along with Eqs. (55) and (56) and analyze two different cases. In the first case, we choose $\bar{b}=0$, $\bar{\alpha}=\alpha$, $\bar{k}=k$, and $\bar{n}=n$. Then, $\bar{\beta}=\beta$ by Eqs. (56) and (59) reduces to a linear second-order differential equation for $\bar{y}=\hat{y}$ with constant coefficients

$$\ddot{\hat{y}} + \alpha k\dot{\hat{y}} + \alpha^2 k^2\frac{n+1}{(n+2)^2}\hat{y} = 0. \tag{61}$$

Integrating Eq. (60) for the above value of the parameter, we obtain the general solution of Eq. (58)

$$y^n = \frac{n+2}{\alpha b n}\,\frac{\hat{y}^n}{\int \hat{y}^n dx}, \tag{62}$$

where $\hat{y}$ is any solution of Eq. (61). In the second case, when we choose $b=0$, $\alpha = \bar{\alpha}$, $k=\bar{k}$, and $n=\bar{n}$, Eq. (58) reduces to Eq. (61) for $y=\hat{y}$ and the general solution of Eq. (59) is

$$\bar{y}^{\bar{n}} = \frac{\bar{n}+2}{\bar{\alpha}\bar{b}\bar{n}}\,\frac{\bar{\hat{y}}^{\bar{n}}}{\int \bar{\hat{y}}^{\bar{n}} dx}, \tag{63}$$

where $\bar{\hat{y}}$ is any other solution of Eq. (61). Inserting the general solution of the Eqs. (58) and (59), given by Eqs. (62) and (63), in the nonlocal transformation (60), it can be integrated and the final relation between the variables y and $\bar{y}$, that transforms Eqs. (58) and (59) one on each other, is

$$y\left[\int \hat{y}^n dx\right]^{1/n}\exp\left(\frac{\alpha k}{2}x\right) = \bar{y}\left[\int \bar{\hat{y}}^{\bar{n}} dx\right]^{1/\bar{n}}\exp\left(\frac{\bar{\alpha}\bar{k}}{2}x\right). \tag{64}$$

For the particular case $n=\bar{n}=-1$, we obtain $\gamma=\alpha^2 b$ and $\bar{\gamma}=\bar{\alpha}^2\bar{b}$. All the remaining equations (60)–(64) can be applied for $n=-1$ and $\bar{n}=-1$ because they do not depend explicitly of the parameters β, $\bar{\beta}$, γ, and $\bar{\gamma}$.

In the next subsections we investigate other generalizations of Eqs. (58) and (59), that can be linearized and solved.

B. Case b

Writing the equations set (58) and (59) as

$$F(\ddot{y},\dot{y},y)=0, \quad \overline{F}(\ddot{\overline{y}},\dot{\overline{y}},\overline{y})=0, \tag{65}$$

a generalization of both equations can be done expressing them in the following way,

$$\frac{1}{y}F(\ddot{y},\dot{y},y)=\frac{1}{\overline{y}}\overline{F}(\ddot{\overline{y}},\dot{\overline{y}},\overline{y}), \tag{66}$$

which is invariant under the nonlocal transformation given by Eq. (60). It is easy to prove that the new functions

$$\tilde{F}(\ddot{y},\dot{y},y)=F(\ddot{y},\dot{y},y)+\delta y, \quad \tilde{\overline{F}}(\ddot{\overline{y}},\dot{\overline{y}},\overline{y})=\overline{F}(\ddot{\overline{y}},\dot{\overline{y}},\overline{y})+\delta\overline{y}, \tag{67}$$

where δ is a constant parameter, also satisfy the invariant condition (66)

$$\frac{1}{y}\tilde{F}(\ddot{y},\dot{y},y)=\frac{1}{\overline{y}}\tilde{\overline{F}}(\ddot{\overline{y}},\dot{\overline{y}},\overline{y}). \tag{68}$$

This *gauge symmetry* generates a new nonlinear equation that can be linearized and solved. In fact, when the invariant in Eq. (68) vanishes, it gives rise to a set of equations that transform one on each other under the same nonlocal transformation, these are:

$$\ddot{y}+\alpha[by^n+k]\dot{y}+\beta\left[b^2\frac{y^{2n+1}}{n+1}+bk\,\frac{n+2}{n+1}y^{n+1}+k^2y\right]+\delta y=0, \tag{69}$$

$$\ddot{\overline{y}}+\overline{\alpha}\,[\overline{b}\,\overline{y}^{\,\overline{n}}+\overline{k}]\dot{\overline{y}}+\overline{\beta}\left[\overline{b}^{\,2}\frac{\overline{y}^{\,2\overline{n}+1}}{\overline{n}+1}+\overline{b}\overline{k}\,\frac{\overline{n}+2}{\overline{n}+1}\overline{y}^{\,\overline{n}+1}+\overline{k}^{\,2}\,\overline{y}\right]+\delta\overline{y}=0. \tag{70}$$

In particular, to solve Eq. (69) we choose $\overline{b}=0$, $\overline{\alpha}=\alpha$, $\overline{k}=k$, and $\overline{n}=n$ ($\overline{\beta}=\beta$ by Eq. (56)) in Eq. (70). Then, it reduces to

$$\ddot{\overline{y}}+\alpha k\dot{\overline{y}}+\left[\alpha^2k^2\frac{n+1}{(n+2)^2}+\delta\right]\overline{y}=0. \tag{71}$$

Inserting the solutions of Eq. (71) in Eq. (60) and integrating it for the selected parameters, we reduce Eq. (69) to quadratures

$$y=\left[\frac{n+2}{\alpha bn}\,\frac{\overline{y}^{\,n}}{\int\overline{y}^{\,n}dx}\right]^{1/n}. \tag{72}$$

For the particular case $\overline{b}=b=1$, $k=\overline{k}=0$, $n=\overline{n}=1$, and $\delta=\gamma$, Eqs. (69) and (70) reduce to Eq. (43), the variable transformation (60) reduces to Eq. (42), and Eq. (64) gives the relation between the variables y and $\overline{y}$ that leaves invariant (41).

C. Case c

There is an important result that can be deduced from Eq. (60) when $\bar{\alpha}=\alpha$ and $\bar{k}=k$, in this case the nonlocal transformation (60) is k-independent,

$$\frac{\dot{y}}{y}+\frac{\alpha b y^n}{n+2}=\frac{\dot{\bar{y}}}{\bar{y}}+\frac{\alpha \bar{b}\bar{y}^{\bar{n}}}{\bar{n}+2}, \tag{73}$$

and by Eqs. (55) and (56)

$$\bar{n}=n, \quad \bar{n}=\frac{-n}{n+1}. \tag{74}$$

So, if we take $k(x)$ and $\delta(x)$ as functions of the independent variable x instead of constant parameters, then, there is no change in the deduction of the variable transformation (73), that comes from Eqs. (46) and (47). This means that the set of equations Eqs. (69) and (70) give rise to new solvable equations that transform between them by the nonlocal transformation (73)

$$\ddot{y}+\alpha[by^n+k(x)]\dot{y}+\beta\left[b^2\frac{y^{2n+1}}{n+1}+bk(x)\frac{n+2}{n+1}y^{n+1}+k^2(x)y\right]+\delta(x)y=0, \tag{75}$$

$$\ddot{\bar{y}}+\alpha[\bar{b}\bar{y}^{\bar{n}}+k(x)]\dot{\bar{y}}+\beta\left[\bar{b}^2\frac{\bar{y}^{2\bar{n}+1}}{\bar{n}+1}+\bar{b}k(x)\frac{\bar{n}+2}{\bar{n}+1}\bar{y}^{\bar{n}+1}+k^2(x)\bar{y}\right]+\delta(x)\bar{y}=0. \tag{76}$$

For instance, to obtain the solutions of Eq. (75) we take $\bar{b}=0$ and $\bar{n}=n$ in Eq. (76) and it becomes a general homogeneous linear second-order differential equation

$$\ddot{\bar{y}}+\alpha k(x)\dot{\bar{y}}+\left[\alpha^2k^2(x)\frac{n+1}{(n+2)^2}+\delta(x)\right]\bar{y}=0, \tag{77}$$

then, inserting the solutions of this equation in Eq. (72), we reduce Eq. (75) to quadratures.

V. CONCLUSIONS

We have introduced a new invariance concept that leads to classes of second-order nonlinear ordinary differential equations which are equivalent under nonlocal transformations. These classes contain a second-order linear ordinary differential equation with constant coefficients. The parametric expression of the solutions for an arbitrary function $f(y)$ and any values of the parameters α, β, and γ, has been found. Also, the case in which these parameters are functions of the independent variable has been investigated. Several important physical problems are mathematically described by these equation classes. Many of these arise in general relativity when the Einstein field equations are investigated for homogeneous, isotropic, and spatially flat cosmological models with no cosmological constant, or Bianchi I type metric with a variety of matter sources. Also, the probability distribution function, which maximizes the Fisher's information measure in the generalized statistical mechanics, was found to satisfy Eq. (43) for the most interesting value $q=-1$.[11]

Taking $x=\bar{x}$ in the nonlocal transformation, and imposing the form invariance of the general expression (6), we have obtained a modified Painlevé–Ince equation (43). The nonlocal transformation of variables and the general solution of these equations has been found. In this case the equation has the eight-dimensional group of Lie point group symmetries SL(3,R) and this is the maximum number of point symmetries that a second-order differential equation can have. Other sets of new nonlinear second-order differential equations are generated, that can be linearized and solved explicitly (58,69,75). It is also to be remarked that the use and application of the form invariance have led to exact solution of differential equations whose solution were unknown, in particular for modified Painlevé–Ince equations and polynomical differential equations, which usually appear in problems related with quantum effects in the very early Universe, originated by the vacuum polarization terms and particle production arising from a quantum description of matter, or when both of them are modeled in terms of a classical bulk viscosity.

In general, the problem of finding solutions of nonlinear ordinary differential equations remains open. One direction along which one can proceed is to reduce them to a linear ordinary differential equation. For instance, when Eq. (1) possesses eight-parameter Lie group it is linearizable by a point transformation. On the other hand, the nonlocal transformation (7–10) linearizes Eq. (1) even when it has less symmetries. Thus, it could mean that it has more nonlocal symmetries. We conclude that it is very interesting to study this kind of nonlocal transformations of variables and their associated nonlocal symmetries, which have received up to now little attention. We shall continue exploring this subject in future papers.

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