FORMAL GROUPS, POWER SYSTEMS AND ADAMS OPERATORS

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Abstract. This paper provides a systematic presentation of the connection between the theory of one-dimensional formal groups and the theory of unitary cobordism. Two new algebraic concepts are introduced: formal power systems and two-valued formal groups. A presentation of the general theory of formal power systems is given, and it is shown that cobordism theory gives a nontrivial example of a system which is not a formal group. A two-valued formal group is constructed whose ring of coefficients is closely related to the bordism ring of a symplectic manifold. Finally, applications of formal groups and power systems are made to the theory of fixed points of periodic transformations of quasicomplex manifolds.

Bibliography: 17 citations.

The theory of one-dimensional commutative formal groups at the present time consists of three parts:

1) The general theory at the basis of which lies Lazard's theorem [8] on the existence of a universal formal group whose coefficient ring is the ring of polynomials over the integers.

2) Formal groups over arithmetic rings and fields of finite characteristic — for a survey of this theory see [4].

3) Commutative formal groups in cobordism theory and in the theory of cohomology operations and characteristic classes [12], [13], [5], [9], [14].

Quillen has recently shown that the formal group $f(u, v)$ which occurs in the topology of "geometrical cobordism" is universal [14]. His proof makes use of Lazard's theorem on the existence of a universal formal group whose ring of coefficients is a torsion-free polynomial ring.

In the first section of this paper we prove the universality of the group of "geometrical cobordism" directly by starting from its structure, as investigated in Theorem 4.8 of [2], without recourse to Lazard's theorem. Moreover, in §1 we give formulas for calculating the cohomology operations in cobordism by means of the Hirzebruch index.

In connection with the theory of Adams operations in cobordism the operation of "raising to powers" in formal groups is of particular importance (see [12], [13]). This operation can be axiomatized and studied for its own sake; in addition there are topologically important power systems which do lie within formal groups (see §2a). In §2b we examine a distinctive "two-valued formal group" which is closely connected with simplicial cobordism theory. §3 and the Appendix are devoted to

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the systematization and development of the application of formal groups to fixed point theory.

§ 1. Formal groups

First of all we introduce some definitions and general facts concerning the theory of formal groups. All rings considered in this paper are presumed to be commutative with unit.

Definition 1.1. A *one-dimensional formal commutative group* F over a ring R is a formal power series $F(x,y) \in R[[x,y]]$ which satisfies the following conditions:

a) $F(x,0) = F(0,x) = x$,

b) $F(F(x,y),z) = F(x,F(y,z))$,

c) $F(x,y) = F(y,x)$.

In the following a formal series $F(x,y)$ which satisfies axioms a), b) and c) will simply be called a formal group.

Definition 1.2. A *homomorphism* $\phi\colon F \to G$ *of formal groups* over a ring R is a formal series $\phi(x) \in R[[x]]$ such that $\phi(0) = 0$ and $\phi(F(x,y)) = G(\phi(x),\phi(y))$.

If the formal series $\phi_1(x)$ determines the homomorphism $\phi_1\colon F \to G$ and the formal series $\phi_2(x)$ determines the homomorphism $\phi_2\colon G \to H$, it follows immediately from Definition 1.2 that the formal series $\phi_2(\phi_1(x))$ determines the composite homomorphism $\phi_2 \cdot \phi_1\colon F \to H$.

For formal groups F and G over R we denote by $\mathrm{Hom}_R(F,G)$ the set of all homomorphisms from F into G. With respect to the operation

$$(\phi_1 + \phi_2)(x) = G(\phi_1(x),\phi_2(x)), \quad \phi_1,\phi_2 \in \mathrm{Hom}_R(F,G),$$

the set $\mathrm{Hom}_R(F,G)$ is an Abelian group.

By $T(R)$ we denote for any ring R the category of all formal groups over R and their homomorphisms. It is not difficult to verify that the category $T(R)$ is semiadditive, i.e. for any $F_1, F_2, F_3 \in T(R)$ the mapping

$$\mathrm{Hom}_R(F_1,F_2) \times \mathrm{Hom}_R(F_2,F_3) \to \mathrm{Hom}_R(F_1,F_3),$$

defined by composition of homomorphisms, is bilinear.

Let $F(x,y) = x + y + \sum \alpha_{i,j} x^i y^j$ be a formal group over R_1, and let $r\colon R_1 \to R_2$ be a ring homomorphism. Let $r[F]$ be the formal series

$$r[F](x,y) = x + y + \sum r(\alpha_{i,j}) x^i y^j,$$

which is clearly a formal group over R_2. If the series $\phi(x) = \sum \alpha_i x^i$ gives the homomorphism $\phi\colon F \to G$ of formal groups over R_1, then the formal series $r[\phi](x) = \sum \phi(a_i) x^i$ gives the homomorphism $r[\phi]\colon r[F] \to r[G]$ of formal groups over R_2. Thus any ring homomorphism $r\colon R_1 \to R_2$ provides a functor from the category $T(R_1)$ into the category $T(R_2)$. Summing up, we may say that over the category of all commutative rings with unit we have a functor defined which associates with each ring R the semiadditive category $T(R)$ of all one-dimensional commutative formal groups over R.

Let R be a torsion-free ring and $F(x,y)$ a formal group over it. As was shown in [8] (see also [4]), there exists a unique power series $f(x) = x + \sum (a_n/(n+1)) x^{n+1}$, $a_n \in R$, over the ring $R \otimes Q$ such that

$$F(x,y) = f^{-1}(f(x) + f(y)). \tag{1.3}$$

Definition 1.4. The power series $f(x) = x + \sum(a_n/(n+1))x^{n+1}$, which obeys (1.3), is called the *logarithm of the group* $F(x,y)$, and is denoted by the symbol $g_F(x)$.

In [4] the notion of an invariant differential on a formal group $F(x,y)$ over a ring R was introduced, and it is shown there that the collection of all invariant differentials is the free R-module of rank 1 generated by the form $\omega = \psi(x)dx$, where $\psi(x) = ([\partial F(x,y)/\partial y]_{y=0})^{-1}$. By the invariant differential on the group $F(x,y)$ we shall mean the form ω.

If the ring R is torsion free, then $\omega = dg_F(x)$. We point out that it was demonstrated in [12] that the logarithm of the formal group $f(u,v)$ of "geometrical cobordism" is the series $g(u) = u + \sum[CP^n]u^{n+1}/(n+1)$. Consequently for the group $f(u,v)$ we have

$$\omega = dg(u) = \left(\sum_{n=0}^{\infty}[CP^n]u^n\right)du = CP(u)du.$$

Let

$$f(u,v) = u + v + \sum e_{i,j}u^iv^j, \quad e_{i,j} \in \Omega_U^{-2(i+j-1)},$$

be the formal group of geometrical cobordism.

Lemma 1.5. *The elements $e_{i,j}$, $1 \le i < \infty$, $1 \le j < \infty$, generate the whole cobordism ring Ω_U.*

Proof. From the formula for the series $f(u,v)$ given in [2] (Theorem 4.8), we obtain

$$e_{1,1} = [H_{1,1}] - 2[CP^1], \quad e_{1,i} \approx [H_{1,i}] - [CP^{i-1}],\ i > 1, \quad e_{i,j} \approx [H_{i,j}],\ i > 1, j > 1,$$

where the sign $\approx$ denotes equality modulo factorizable elements in the ring Ω_U. Since $s_1([H_{1,1}]) = 2$, we have $e_{1,1} = -[CP^1]$; since $s_{i-1}([H_{1,i}]) = 0$ for any $i > 1$, we have $e_{i,1} \approx -[CP^{i-1}]$. According to the results in [10] and [11], the elements $[H_{i,j}]$, $i,j > 1$, and $[CP^i]$ generate the ring Ω_U. This proves the lemma. □

Theorem 1.6 (Lazard–Quillen)**.** *The formal group of geometrical cobordism $f(u,v)$ over the cobordism ring Ω_U is a universal formal group, i.e. for any formal group $F(x,y)$ over any ring R there is a unique ring homomorphism $r\colon \Omega_U \to R$ such that $F(x,y) = r[f(u,v)]$.*

We show first that for a torsion-free ring R Theorem 1.6 is an easy consequence of Lemma 1.5. Let R be a torsion-free ring and let F be an arbitrary formal group over it; let

$$g_F(x) = x + \sum \frac{a_n}{n+1}x^{n+1}, \quad a_n \in R.$$

Consider the ring homomorphism $r\colon \Omega_U \to R \otimes Q$ such that $r([CP^n]) = a_n$. We have $r[g_f] = g_F$, and, since

$$F(x,y) = g_F^{-1}(g_F(x) + g_F(y)) \quad \text{and} \quad f(x,y) = g_f^{-1}(g_f(x) + g_f(y)),$$

also $r[f(x,y)] = F(x,y)$. Consequently $r(e_{i,j}) \in R$. By now applying Lemma 1.5, we find that $\operatorname{Im} r \subset R \subset R \otimes Q$. This proves Theorem 1.6 for torsion-free rings.

Proof of Theorem 1.6. Recall that by $s_n(e)$, $e \in \Omega_U^{-2n}$, we denote the characteristic number corresponding to the characteristic class $\sum t_i^n$. It follows from the proof of Lemma 1.5 that for any $i,j > 1$ we have the formula $s_{i+j-1}(e_{i,j}) = -C_{i+j}^i$. It is known that the greatest common divisor of the numbers $\{C_n^i\}_{n=1,\dots,(n-1)}$ is equal to 1 if $n \neq p^l$ for any prime $p \geq 2$, and is equal to p if $n = p^l$. Consequently a number $\lambda_{i,n}$ exists such that

$$\sum \lambda_{i,n} C_n^i = \begin{cases} 1, & \text{if } n \neq p^l, \\ p, & \text{if } n = p^l, \end{cases} \tag{$*$}$$

For each n, let us consider a fixed set of numbers $(\lambda_{i,n})$ which satisfy $(*)$. From [10] and [11] we have that the elements $y_n = \sum \lambda_{i,n} e_{i,n-i} \in \Omega_U^{-2(n-1)}$, $n = 2, 3, \ldots$, form a multiplicative basis for the ring Ω_U.

Let R be an arbitrary ring and let $F(x, y) = x + y + \sum \alpha_{i,j} x^i y^j$ be a formal group over this ring. We define the ring homomorphism $r\colon \Omega_U \to R$ by the formula $r(y_n) = \sum \lambda_{i,n} \alpha_{i,n-i}$; we shall show that $r(e_{i,n-i}) = \alpha_{i,n-i}$ for any i, n. From the commutative property of the formal group F it follows that $\alpha_{i,n-i} = \alpha_{n-i,i}$; from associativity we have

$$C_{i+j}^i \alpha_{i+j,k} - C_{j+k}^j \alpha_{j+k,i} = P(\alpha_{m,l}),$$

where $P(\alpha_{m,l})$ is a polynomial in the elements $\alpha_{m,l}$, $m + l < i + j + k$. It is clear that the form of the polynomial P does not depend on the formal group $F(x, y)$. Since $e_{1,1} = y_1$, we have $r(e_{1,1}) = \alpha_{1,1}$. We assume that for any number $n < n_0$ the equation $r(e_{i,n-i}) = \alpha_{i,n-i}$ is already proved. We have

$$C_{i+j}^i e_{i+j,k} - C_{j+k}^j e_{j+k,i} = P(e_{m,l}), \quad m + l < n_0 = i + j + k,$$
$$r(P(e_{m,i})) = P(r(e_{m,l})) = P(\alpha_{m,l}) = C_{i+j}^i \alpha_{i+j,k} - C_{j+k}^j \alpha_{j+k,i}.$$

It follows from the number-theoretical properties of the C_p^q that for any $i_0 \geq 1$ and $n_0 = i_0 + j_0 + k_0$ the element α_{i_0,j_0+k_0} can be represented as an integer linear combination of the elements $r(y_{n_0}) = \sum \lambda_{i,n_0} \alpha_{i,j+k}$, and $r(P(e_{m,l})) = C_{i+j}^i \alpha_{i+j,k} - C_{j+k}^j \alpha_{j+k,i}$. Since the form of this linear combination depends neither on the ring R nor on the formal group F, we find that $r(e_{i_0,n_0-i_0}) = \alpha_{i_0,n_0-i_0}$. This concludes the induction, and Theorem 1.6 is proved. □

It will be useful to indicate several important simple consequences of Theorem 1.6:

1. *In the class of rings R over $\mathbf{Z}_p$ the formal group $f(u, u) \otimes_{\mathbf{Z}} \mathbf{Z}_p$ is universal over the ring $\Omega_U \otimes_{\mathbf{Z}} \mathbf{Z}_p$.*

2. *In the class of formal groups over graded rings the formal group of geometrical cobordism $f(u, v)$, considered as having the natural grading of cobordism theory, is universal.*

In this case $\dim u = \dim v = \dim f(u, v) = 2$. Therefore the above refers to the class of formal groups F over commutative even-graded rings R, where $R = \sum_{i \geq 0} R^{-2i}$, and all components of the series $F(u, v)$ have dimension 2. Of course, the general case of a graded ring reduces to the latter through the multiplication of the grading by a number.

3. The semigroup of endomorphisms of the functor T, which assigns to a commutative ring R the set $T(R)$ of all commutative one-dimensional formal groups over R, is denoted by A^T. *This semigroup A^T coincides with the semigroup of all ring automorphisms $\Omega_U \to \Omega_U$.* In the graded case we refer to the functor as T_{gr} and the semigroup as A^T_{gr}, which coincides with the semigroup of all dimension preserving homomorphisms $\Omega_U \to \Omega_U$. *The "Adams operators" $\Psi^k \in A^T_{gr}$ form*

the center of the semigroup A^T_{gr}. The application of these operators Ψ^k to a formal group $F(x,y)$ over any ring proceeds according to the formula

$$\Psi^k F(x,y) = x + y + \sum k^{i+j-1}\alpha_{i,j}x^i y^j,$$

where $F(x,y) = x + y + \sum \alpha_{i,j}x^i y^j$.

We note that the semigroup A^0 of all multiplicative operations in U^*-theory is naturally imbedded in the semigroup A^T_{gr} (see [12], Appendix 2) by means of the representation $(*)$ of the ring A^U over Ω_U; the elements of A^T are given in the theory of characteristic classes by rational "Hirzebruch series"

$$K(1+u) = Q(u), \quad Q(u) = \frac{u}{a(u)}, \quad a(u) = u + \sum_{i\geq 1}\lambda_i u^i, \quad \lambda_i \in \Omega_U \otimes Q.$$

What sort of Hirzebruch series give integer homomorphisms $\Omega_U \to \Omega_U$, i.e. belong to A^T? How does one distinguish $A^0 \subset A^T_{gr}$?

From the point of view of Hirzebruch series the action of a series $a = a(u) = u + \sum \lambda_i u^{i+1}$, $\lambda_i \in \Omega_U \otimes Q$, on the ring Ω_U is determined by the formula

$$a([CP^n]) = \left[\left(\frac{u}{a(u)}\right)^{n+1}\right]_n,$$

where $[f(u)]_n$ denotes the nth coefficient of the series $f(u)$. Note that $a^{-1}(u) = u + \sum [a([CP^n])/(n+1)]u^{n+1}$, where $a^{-1}(a(u)) = u$. This formula is proved in [13] (see also [2]) for series $a(u)$ giving homomorphisms $\Omega_U \to \mathbf{Z}$, and carries over with no difficulty to all series which give homomorphisms $\Omega_U \to \Omega_U$.

We must check that the indicated operation (in the "Hirzebruch genus" $Q(u) = u/a(u)$ sense) of a series $a(u)$ on Ω_U does not coincide with the operation $(*)$ of the series $a(u) \in \Omega \otimes Q[[u]]$ on the ring $\Omega_U = U^*(\text{point})$, which defines a multiplicative cohomology operation in U^*-theory (see [12]). For example, for $a(u) = u$ we have $a([CP^n]) = 0$, $n \geq 1$, and $a^*([CP^n]) = [CP^n]$; as is proved in cobordism theory (see [12] or [2]), for the series $a(u) = g(u) = \sum [CP^n]u^{n+1}/(n+1)$ we have the formula $a^*([CP^n]) = 0$, $n \geq 1$, and for the series $a(u) = g^{-1}(u)$ the formula $a([CP^n]) = [CP^n]$.

There arises the transformation of series of the ring $\Omega \otimes Q[[u]]$

$$\phi\colon a(u) \to \phi a(u),$$

defined by the requirement $a[x] = (\phi a)^*[x]$ for all $x \in \Omega_U$, where we already know that $\phi u = g(u)$ and $\phi(g^{-1}(u)) = u$. We have

Theorem 1.7. *The transformation of series of the ring $\Omega_U \otimes Q[[u]]$*

$$g\colon a(u) \mapsto a(g(u))$$

has the same properties as $a[x] = a(g)^*[x]$ for any element $x \in \Omega_U$ where

$$g(u) = \sum \frac{[CP^n]}{n+1}u^{n+1}, \quad a[CP^n] = \left[\left(\frac{u}{a(u)}\right)^{n+1}\right]_n$$

and $b^*[x]$ is the result of the application to the element $x \in U^*(\text{point}) = \Omega_U$ of the multiplicative operation b of $A^U \otimes Q$, given by its value $b(u) = u + \sum \lambda_i u^i$, $\lambda_i \in \Omega_U \otimes Q$, on the geometrical cobordism $u \in U^2(CP^\infty)$.

Proof. Let $b \in A^U \otimes Q$ be a multiplicative operation and let $\tilde{b}(\xi)$ be the exponential characteristic class of the fiber ξ with values in U^*-theory, which on the Hopf fiber η over CP^n, $n \le \infty$, is given by the series $\tilde{b}(\eta) = (b(u)/a)^{-1}$; let $u \in U^2(CP^n)$ be a geometrical cobordism. As was shown in [12], for any U-manifold X^n we have the formula

$$b^*([X]) = \epsilon D\tilde{b}(-\tau(X^n)),$$

where $[X^n] \in \Omega_U^{-2n}$ is the bordism class of the manifold X^n, $\epsilon\colon X \to$ (point), D is the Poincaré–Atiyah duality operator, and τ is the tangent fiber.

By using the formulas $Du^k = (CP^{n-k}) \in U_{n-k}(CP^n)$, $n < \infty$, and $\tau(CP^n)+1 = (n+1)\eta$, we obtain

$$b^*([CP^n]) = \sum_{k=0}^{n} [CP^k] \left[\left(\frac{u}{b(u)} \right)^{n+1} \right]_{n-k} = \sum_{k=0}^{\infty} \frac{[CP^k]}{2\pi i} \int_{|u|=\epsilon} \left(\frac{u}{b(u)} \right)^{n+1} \frac{du}{u^{n+1-k}},$$

$$\begin{aligned} \sum_{n=0}^{\infty} \frac{b^*([CP^n])}{n+1} t^{n+1} &= \int_0^t \sum_{n=0}^{\infty} b^*([CP^n]) t^n \, dt = \\ &= \int_0^t \left(\sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{[CP^k]}{2\pi i} \int_{|u|=\epsilon} \frac{u^k \, du}{b(u)^{n+1}} \right) t^n \, dt \\ &= \frac{1}{2\pi i} \int_{|u|=\epsilon} \sum_{k=0}^{\infty} [CP^k] u^k \left(\int_0^t \sum_{n=0}^{\infty} \left(\frac{t}{b(u)} \right)^n \frac{dt}{b(u)} \right) du \\ &= \frac{1}{2\pi i} \int_{\substack{|u|=\epsilon \\ |t|<|b(u)|}} - \ln \left(1 - \frac{t}{b(u)} \right) \left(\sum_{k=0}^{\infty} [CP^k] u^k \, du \right) \\ &= \frac{1}{2\pi i} \int_{\substack{|u|=\epsilon \\ |t|<|b(u)|}} - \ln \left(1 - \frac{t}{b(u)} \right) dg(u), \end{aligned}$$

where $dh(u)$ is the invariant differential of the formal group $f(u, v)$. By setting $g(u) = v$, we obtain from the formula for the inversion of series

$$\int_{\substack{|u|=\epsilon \\ |t|<|b(g^{-1}(v))|}} - \ln \left(1 - \frac{t}{b(g^{-1}(v))} \right) dv = (b(g^{-1}(v)))^{-1}(t) = g(b^{-1}(t)).$$

Thus

$$\sum_{n=0}^{\infty} \frac{b^*([CP^n])}{n+1} t^{n+1} = g(b^{-1}(t)).$$

On the other hand, as was indicated above, we have the formula

$$\sum \frac{a([CP^n])}{n+1} t^{n+1} = a^{-1}(t).$$

Consequently, if $b(u) = a(g(u))$, then $b^*([CP^n]) = a([CP^n])$ for any n. Since the elements $\{[CP^n]\}$ generate the entire ring $\Omega_U \otimes Q$; the theorem is proved. □

Another proof of Theorem 1.7 can be obtained from the properties of the Chern–Dold character ch_U (see [2]). Let $\phi\colon \Omega_U \to \Omega_U$ be a ring homomorphism and let $a(u) = u = \sum \lambda_i u^i$ be the corresponding Hirzebruch genus. We shall show that if the multiplicative operation $b \in A^U \otimes Q$ acts on the ring $U^*(\text{point}) = \Omega_U$ as a homomorphism ϕ, then its value on the geometrical cobordism $u \in U^2(CP^\infty)$ is equal to the series $a(g(u))$, where $g(u) = u + \sum [CP^n]u^{n+1}/(n+1)$. We have

$$\mathrm{ch}_U(u) = t + \sum \alpha_i t^{i+1} \in \mathcal{H}^*(CP^\infty, \Omega_U \otimes Q),$$
$$\alpha_i \in \Omega_U^{-2i} \otimes Q,\ t \in H^2(CP^\infty, \mathbf{Z}), \quad \mathrm{ch}_U(g(u)) = t.$$

Since $a^{-1}(t) = \sum \phi([CP^n])t^{n+1}/(n+1)$, we have

$$a(t) = t + \sum \phi(\alpha_i)t^{i+1} = t + \sum b^*(\alpha_i)t^{i+1}.$$

Thus

$$a(g(u)) = g(u) + \sum b^*(\alpha_i)g(u^{i+1}),$$
$$\mathrm{ch}_U(a(g(u))) = t + \sum b^*(\alpha_i)t^{i+1} = b^*(\mathrm{ch}_U(u)) = \mathrm{ch}_U(b(u)).$$

Since the homomorphism $\mathrm{ch}_U \colon U^*(CP^\infty) \otimes Q \to \mathcal{H}^*(CP^\infty, \Omega_U \otimes Q)$ is a monomorphism, we find that $a(g(u)) = b(u)$. This proves the theorem.

By Theorem 1.6 any integer Hirzebruch genus, or, equivalently, any homomorphism $Q \colon \Omega_U \to \mathbf{Z}$, defines a formal group over $\mathbf{Z}$, and conversely (similarly for the ring $\mathbf{Z}_p$). In this connection the Hirzebruch genus, which defines this homomorphism, can be rational. Equivalent (or strongly isomorphic in the terminology of [4]) formal groups are defined by the Hirzebruch series $Q(z)$ and $Q'(z)$, which are connected by the formula $z/Q(z) = \phi^{-1}(z/Q'(z))$, where $\phi^{-1}(u) = u + \sum_{i\geq 1} \lambda_i u^{i+1}$, $\lambda_i \in \mathbf{Z}$. This follows from the fact that the logarithms of the formal groups are equal to $g_Q(z) = (z/Q(z))^{-1}$, and by definition we have $g_Q(z) = g_{Q'}(\phi(z))$.

Let us consider the integer Q-genus given by the rational series $g_Q(u)$. Then the Q'-genus such that $g_Q(u) = g_{Q'}(\phi(u))$, $\phi(u) = u + \sum_{i\geq 1} \lambda_i u^{i+1}$, $\lambda_i \in \mathbf{Z}$, also has integer values on Ω_U. In this connection the meaning of equivalence for Hirzebruch genera is the same as for formal groups. What sort of examples of formal groups are considered in topology in connection with the well-known multiplicative classes c, T, L, A?

1. **The Euler characteristic** $c \colon \Omega_U \to \mathbf{Z}$. We have

$$f_c(u,v) = \frac{u+v-2uv}{1-uv}, \quad g_c(u) = \frac{u}{1-u}.$$

As a formal group, this genus is equivalent to the trivial one.

2. **The Todd genus** $T \colon \Omega_U \to \mathbf{Z}$. Here we have the law of multiplication

$$f_T(u,v) = u+v-uv, \quad g_T(u) = -\ln(1-u), \quad T(z) = \frac{-z}{1-e^{-z}} = \frac{-z}{g_T^{-1}(z)}.$$

3. **The L-genus** $\tau \colon \Omega_U \to \mathbf{Z}$ **and the A-genus** $A \colon \Omega_U \to \mathbf{Z}$, where $g^{-1}(z) = \operatorname{th} z$ and $g_A^{-1}(z) = \frac{1}{2}\operatorname{sh}(2z)$. It is easily seen that these are strongly isomorphic to formal groups; both of them are strongly isomorphic over $\mathbf{Z}_2$ to a linear group, and over $\mathbf{Z}[1/2]$ to a multiplicative one (the Todd genus).

4. **The T_y-genus (see [3])** $T_y([CP^n]) = \sum_{i=0}^n (-y^i)$**.** Here the law of multiplication is defined over the ring $\mathbf{Z}[[y]]$ and has the form

$$f_{T_y}(u,v) = \frac{u+v+(y-1)uv}{1+uvy}, \quad g_{T_y} = \frac{1}{(y+1)}\ln\left(1+(1+y)\frac{u}{1-u}\right).$$

We have for $y = -1, 0, 1$ the genera c, T and L, respectively. The simple integral change of variables $u = \phi(u')$ allows us to put f_{T_y} into the form

$$\phi^{-1} f_{T_y}(\phi(u'), \phi(v')) = u' + v'(y-1)u'v'.$$

For all values of y this group reduces either to a linear one or to a multiplicative one over the p-adic integers $\mathbf{Z}_p$.

Thus we see that in topology the multiplicative genera connected with other non-trivial formal groups over $\mathbf{Z}, \mathbf{Z}_p$ or $\mathbf{Z}/p\mathbf{Z}$ have not been considered previously.

§ 2. Formal power systems and Adams operators

Definition 2.1. A *formal power system* over a ring R is a collection of power series $\{f_k(u), k = \pm 1, \pm 2, \dots, f_k(u) \in R[[u]]\}$ such that $f_k(f_l(u)) = f_{kl}(u)$.

Consider on the ring $R[[u]]$ the operation of inserting one formal power series into another. With respect to this operation $R[[u]]$ is an associative (noncommutative) semigroup with unit. The role of the unit is played by the element u. Let $\mathbf{Z}^*$ denote the multiplicative semigroup of nonzero integers.

Definition 2.2. Any homomorphism $f\colon \mathbf{Z}^* \to R[[u]]$ will be called a *formal power system.*

Definition 2.3. We shall say that a formal power system is of *type s* if for any number k the series $f_k(u)$ has the form

$$f_k(u) = k^s u + \sum_{i \geq 1} \mu_i(k) u^{i+1}, \quad \mu_i(k) \in R.$$

We shall always assume the number s to be positive. Not every power system has type $s \geq 1$. For example, $f_k(u) = u^{k^s}$. More generally, the case

$$f_k(u) = \lambda_0(k) u^{k^s} + O(u^{k^s+1}) = \sum_{i \geq 0} \lambda_i(k) u^{k^s+i}$$

is possible.

Here it is especially important to distinguish two cases: 1) $\lambda_0(k) = 1$, 2) $\lambda_0(k) \not\equiv 1$, but R does not have zero divisors. In the first case there exists a substitution $v = B(u) \in R[[u]] \otimes Q$, $v = u + O(u^2)$, in the ring such that $B(f_k(B^{-1}(v))) = v^{k^s}$ (the argument is similar to the proof of Lemma 2.4 below). 2) is the more general case, where $\lambda_0(k) \not\equiv 1$. Here a similar substitution exists and is correct over a field of characteristic zero which contains the ring R. Examples of such power systems may be found readily in the theory of cohomology operations in U^*-theory, by composing them out of series of operations $s_\omega \in A^U$ with coefficients in Ω_U. We are interested principally in Adams operations, and shall therefore consider only systems of type $s \geq 1$.

As in the theory of formal groups, an important lemma concerning "rational linearization" also plays a role in the theory of power systems. We note that the proof of this lemma presented below is similar to the considerations of Atiyah and Adams in K-theory (see [1]).

Lemma 2.4. *For any formal power system of type s there exists a series, not depending on k, such that the equation $f_k(u) = B^{-1}(k^s B(u))$, where $B^{-1}(B(u)) = u$, is valid in the ring $R[[u]] \otimes Q$.*

The series $B(u)$ is uniquely defined by the power system, and is called its logarithm.[1]

[1] We point out that a formula for the logarithm of a formal group was given in [5] for the cobordism theory of power systems of type $s = 1$ for these groups. However, there it is necessary to make use of important additional information concerning the coefficients of the power systems of u^k as functions of k.

Proof. We shall show that for a given power system $f_k = \{f_k(u)\}$ of type s we are able to reconstruct, by an inductive process, the series $B(u) = u + \lambda_1 u^2 + \dots$. Assume that we have already constructed the series $v_n = B_n(u) \in R[[u]] \otimes Q$ such that for the formal power system $\{f_k^{(n)}(v_n)\} = \{B_n f_k(B_n^{-1}(v_n))\}$ we have the formula $f_k^{(n)}(v_n) = k^s v_n + \mu(k) v_n^{n+1} + O(v_{n+2})$, $\mu(k) \in R$. By using the relation $f_l^{(n)}(f_k^{(n)}(v_n)) = f_k^{(n)}(f_l^{(n)}(v_n))$, we obtain for all k and l

$$(kl)^s v_n + (l^s \mu(k) + \mu(l) k^{(n+1)s}) v_n^{n+1} = (kl)^s v_n + (k^s \mu(l) + \mu(k) l^{(n+1)s}) v_n^{n+1}.$$

Consequently

$$\frac{\mu(k)}{k^s(k^{ns}-1)} = \frac{\mu(l)}{l^s(l^{ns}-1)} = \lambda \in R \otimes Q,$$

where λ does not depend on k or l. Let us set $B_{n+1}(u) = v_n - \lambda v_n^{n+1}$. Direct substitution now shows that

$$B_{n+1}(f_k(B_{n+1}^{-1}(v_{n+1}))) = k^s v_{n+1} + O(v_{n+1}^{n+2}).$$

This completes the inductive step. We set $B(u) = \varinjlim B_n(u)$. Thus $B(f_k(B^{-1}(B(u)))) = k^s B(u)$, i.e. $f_k(u) = B^{-1}(k^s B(u))$, which completes the proof. □

An important example of a formal power system is the operation of raising to the power s in the universal formal group $f(u,v)$ over the ring Ω_U. The operations of raising to a power in $f(u,v)$ have the form $k^s \Psi^{k^s}(u)$, $\Psi^{k^s} \in A^U$. Let us denote by $\Lambda(s)$ the subring in Ω_U generated by all the coefficients of the formal series

$$k^s \Psi^{k^s}(u) = k^s u + \sum \mu_i^{(s)}(k) u^{i+1} \in \Omega_U[[u]] = U^*(CP^\infty)$$

for all k. In Theorem 4.11 of [2] the coefficients of the series $k^s\Psi^{k^s}$ are described in terms of the manifolds $M_{k^s}^{n-1} \subset CP^n$, $k = \pm1, \pm2, \dots$, which are the zero cross-sections of k^sth tensor degree of the Hopf fiber η over CP^n. In particular, from this theorem it follows that modulo factorizable elements in the ring Ω_U we have the equation

$$\mu_i^{(s)}(k) \approx [M_{k^s}^i] - k^s[CP^i].$$

Since $\tau(M_{k^s}^i) = \phi^*((i+1)\eta - \eta^{k^s})$, where $\phi\colon M_{k^s}^i \subset CP^{i+1}$ is an imbedding mapping, we have $s_i([M_{k^s}^i]) - s_i(k^s[CP^i]) = k^s(1 - k^{si})$. The calculation of the Chern numbers s_i (t-characteristic in the terminology of [13]) of the elements $\mu_i^{(s)}(k)$ is easily performed by the method of [13].

Lemma 2.5. *Let $\Lambda(s) = \sum \Lambda_n$ be the ring generated by the elements $\mu_i^{(s)}(k)$ for all k and i. The smallest value of the t-characteristic on the group Λ is equal to the greatest common divisor of the numbers $k^s(k^{ns}-1)$, $k = 2, 3, \dots$. In particular, the ring $\Lambda(s)$ does not coincide with the ring Ω_U for any s, but the rings $\Lambda(s) \otimes Q$ and $\Omega_U \otimes Q$ are isomorphic.*

Theorem 2.6. *The formal power system of type s generated by the Adams operations $f_U(u) = \{k^s\Psi^{k^s}\}$, $k = \pm1, \pm2, \dots$, and considered over the ring $\Lambda(s)$, is a universal formal system of type s on the category of torsion-free rings, i.e. for any formal power system $f = \{f_k(u)\}$ of type s over any torsion-free ring R there exists a unique ring homomorphism $\phi\colon \Lambda(s) \to R$ such that $f = \phi[f_U]$.*

Proof. Let $B(u) = u + \sum \lambda_i u^{i+1}$, $\lambda_i \in R \otimes Q$, be the logarithm of the formal power system $f = \{f_k\}$. Consider the ring homomorphism $\phi\colon \Omega_U \otimes Q \to R \otimes Q$, $\phi([CP^n])/(n+1) = \lambda_n$. Since the coefficients of the formal power system $\{k^s \Psi^{k^s}\}$ generate the entire ring $\Lambda(s)$, we see that the homomorphism ϕ, restricted to the ring $\Lambda(s) \subset \Omega_U \otimes Q$, is integral, i.e. $\operatorname{Im} \phi(\Lambda(s)) \subset R \subset R \otimes Q$. This proves the theorem. $\square$

To each formal power system $f = \{f_k(u)\}$ of type s over a torsion-free ring R we may associate a formal one-parameter group $B^{-1}(B(u) + B(v))$ over the ring $R \otimes Q$, where $B(u)$ is the logarithm of the power system. From Theorem 2.6 we obtain

Corollary 2.7. *Let $\phi\colon \Lambda(s) \to R$ be the homomorphism corresponding to the formal power system $f = \{f_k(u)\}$. In order that the group $B^{-1}(B(u) + B(v))$ be defined over the ring R, it is necessary and sufficient that the homomorphism ϕ the extendable to a homomorphism $\hat{\phi}\colon \Omega_U \to R$.*

Thus the question of the relation of the concepts of a formal power system and a formal one-parameter group over a torsion-free ring R is closely related to the problem of describing the subrings $\Lambda(s)$ in Ω_U.

We shall demonstrate that the series $B^{-1}(k^s B(u))$ has the form $B^{-1}(k^s B(u)) = k^s u + k^s(k^s - 1)\lambda u^2 + \dots$, $\lambda \in R \otimes Q$, where $B(u) = u \pm \lambda u^2 + \dots$. Since the expression $k^s(k^s - 1)\lambda$ is integer valued for all integers k, it follows that an element $\lambda \in R \otimes Q$ can have in its denominator the Milnor–Kervaire–Adams constant $M(s)$, equal to the greatest common divisor of the numbers $\{k^s(k^s - 1)\}$. For example, $M(1) = 2$, $M(2) = 12$. For the series $B(u)$ obtained from a formal group over R the second coefficient λ can have only 2 in the denominator. For all $s > 1$ a realization of the universal system indicated in Theorem 2.6 does not, of course, occur naturally. A natural realization would be one over a subring of the ring Ω_U, where the second coefficient λ of the logarithm $B(u) = u + \lambda u^2 + \dots$ for a system of type $s = 2l$ would coincide with the well-known Milnor–Kervaire [6] manifold $V^s \in \Omega_U^{-4s}$, where $\lambda = \pm V^s / M(s)$, as follows from our considerations on the integer valuedness of $\lambda \cdot M(s)$. For $s = 2$ such a system will be given below.

It is simplest to describe the connection between the notions of a formal power system of type s and of a formal one-parameter group for $s = 1$. We consider the category of torsion-free rings which are modules over the p-adic integers. The system $f_U = \{k^s \Psi^{k^s}(u)\}$, considered over the ring $\Lambda(s) \otimes \mathbf{Z}_p$, is a universal formal system of type s for systems over such rings. Consider in the ring Ω_U some fixed multiplicative system of generators $\{y_i\}$, $\dim y_i = -2i$; let us denote by $\Lambda_p \subset \Omega_U$ the subring generated by the elements $y_{(p^j - 1)}$, $j = 0, 1, \dots$, and by $\pi_p\colon \Omega_U \to \Omega_U$ the projection such that

$$\pi_p(y_i) = \begin{cases} y_i, & \text{if } i = p^j - 1, \\ 0 & \text{otherwise.} \end{cases}$$

According to Lemma 2.5 the minimum value of the t-characteristic on the group $\Delta(1)_n \subset \Lambda(1)$ is equal to the greatest common divisor of the numbers $k(k^n - 1)$, $k = 2, 3, \dots$. In the canonical factorization of the number $\{k(k^n - 1)\}$ into prime factors only first powers can appear, and since $t(y_{(p^j - 1)}) = p$, $j > 0$ (see [11]), it follows that the homomorphism $\pi_p\colon \Lambda(1) \to \Lambda_p$ is an epimorphism. Let us define $f^{(p)} = \{\pi_p^*[k\Psi^k]\}$.

Corollary 2.8. *For any projection of the type π_p the coefficients of the series $f_U^{(p)}$ generate the entire ring Λ_p, which coincides with the ring of coefficients of the formal group $\pi_p^*(f_U(u,v)) = f_U^{(p)}(u,v)$.*

We consider now the special projection $\bar\pi_p \colon \Omega_U \otimes \mathbf{Z}_p \to \Omega_U \otimes \mathbf{Z}_p$ such that $\bar\pi_p^*([CP^i]) = 0$ if $i+1 \neq p^h$, and $\bar\pi_p^*([CP^i]) = [CP^i]$ if $i+1 = p^h$. This projection was given in [14] starting from the Cartier operation over formal groups. As was indicated in §1, the projection $\bar\pi_p$ can be considered as a "cohomological" operation on the set of all formal one-parameter groups over any commutative $\mathbf{Z}_p$-ring R.

We shall say that the formal group $F(u,v)$ over the $\mathbf{Z}_p$-ring R belongs to the class P if $\bar\pi_p^*(F(u,v)) = F(u,v)$. Note that the group $\bar\pi_p^* f_U(u,v)$ is a universal formal group for groups of class P over the ring $\Lambda_p = \operatorname{Im} \bar\pi_p^*(\Omega_U)$.

From the description of the operator $\bar\pi_p^*$ and the definition of the action of the projection $\bar\pi_p^*$ on the collection of groups it follows easily that for a torsion-free $\mathbf{Z}_p$-ring R the group $F(u,v)$ belongs to the class P it and only if its logarithm has the form $g_F(u) = u + \sum \lambda_i u^{p^i}$.

We shall say that a formal power system $f(u)$ over a torsion-free $\mathbf{Z}_p$-ring R belongs to the class P it its logarithm has the form $B(u) = u + \sum \lambda_i u^{p^i}$.

Lemma 2.9. *The power system $\bar\pi_p^*[k\Psi^k(u)]$ is a universal formal power system of type 1 for the class P over the ring $\Lambda = \operatorname{Im} \bar\pi_p^*(\Omega_U)$.*

The proof of this lemma follows easily from Lemma 2.4 and Corollary 2.8.

From Lemmas 2.4, 2.5 and 2.9 we have

Theorem 2.10. *Let R be a torsion-free $\mathbf{Z}_p$-ring, $f(u)$ a formal power system of type 1 of the class P over R, and $B(u)$ the logarithm of $f(u)$. Then a formal one-parameter group $F(u,v) = B^{-1}(B(u)+B(v))$ in class P is defined over the ring R, and, moreover, the mapping $f(u) \mapsto F(u,v) = B^{-1}(B(u)+B(v))$ sets up a one-to-one correspondence between the collection of all formal power systems of type 1 of class P over R and the collection of all one-parameter formal groups of class P over R.*

We shall now show that for a power system over a ring with torsion, as distinct from the case of formal groups, the theorem that any system can be lifted to a system over a torsion-free ring is not true. It will follow from this, in particular, that the formal system $\{k^s\Psi^{k^s}(u)\}$ over the ring $\Lambda(s)$ is not universal on the category of all rings.

Example. Consider the ring $R = \mathbf{Z}_p = \mathbf{Z}/p\mathbf{Z}$; we shall display a power system which cannot be lifted to a system over the ring $\mathbf{Z}_p$ of p-adic integers. Let $f(u) = \{f_k(u) = ku + \sum_{i\geq 1} \mu_i(k) u^{p^i}\}_{k=\mu_0(k)}$ be a formal power system. Note that in R we have the identity $x^p = x$. Since $f_k(f_l(u)) = f_{kl}(u)$ we have

$$\mu_1(kl) = k\mu_1(l) + l\mu_1(k), \quad \dots, \quad \mu_i(kl) = \sum_{\substack{j+q=i \\ j\geq 0, q\geq 0}} \mu_j(k)\mu_q(l).$$

Consequently the value of the function $\mu_i(k)$ for all $i \geq 1$ and prime numbers k can be given arbitrarily. For example, the values of the function $\mu_1(k)$ for the primes $k = 2, 3, 5, \dots$ are arbitrary. Such functions $\mu_1(k)$ form a continuum. For formal systems of type $s = 1$, obtained from a system over $\mathbf{Z}_p$ by means of the homomorphism of reduction modulo p, by Lemma 2.4 the function $\mu_1(k)$ has the form $(k(k^{p-1}-1)/p)\cdot\gamma = \mu_1(k)$, where γ is the p-adic unit. Reduction of μ_1 (mod p) gives a monomial over $\mathbf{Z}_p$. From this we have

Theorem 2.11. *There exists a continuum of formal power systems over the ring $R = \mathbf{Z}_p$ which are not homomorphic images of any power system over the p-adic integers* (*and in general over any torsion-free ring*).

§ 2a

We shall display another geometrical realization of a universal power system of type $s = 2$ which has an interesting topological meaning. In the universal formal group $f(u,v)$ the operation $u \to \bar{u} = -\Psi^{-1}(u)$, $f(u,\bar{u}) = 0$, is the lifting of the operation of complex conjugation into the cobordism of K-theory. Therefore the combination of the form $u\bar{u} = -u\Psi^{-1}(u)$ for geometrical cobordism has the sense of a "square modulus" $|u|^2 = u\bar{u}$.

Let $F(u,v)$ be a formal group over the ring R and let $\bar{u}$ be the element inverse to u, i.e. $F(u,\bar{u}) = 0$. Consider the element $x = u\bar{u} \in R[[u]]$, and let $[u]_k = F(u,\dots,u)$ (k places), where $F(u,\dots,u) = F(u,F(u,\dots))$. Define $\phi_k(x) = [u]_k \cdot [\bar{u}]_k$, where the product is the ordinary product in the ring $R[[u]]$. We have

Lemma 2.12. *For a formal group $F = F(u,v)$ the values of the series $\phi_k(x) = [u]_k[\bar{u}]_k$ lie in the ring $R[[x]] = R[[u\bar{u}]]$ and define a power system of type $s = 2$ over the ring R.*

Proof. Let $f_U = f(u,v)$ be a universal group over the ring $R = \Omega_U$, and let $f(u,v) = g^{-1}(g(u)+g(v))$. Define $B^{-1}(-y) = g^{-1}(-\sqrt{y})g^{-1}(\sqrt{y})$. Since $[u]_k = g^{-1}(kg(u))$, we have $\phi_k(x) = g^{-1}(kg(u))g^{-1}(-kg(u)) = B^{-1}(-k^2g(u)^2)$. Furthermore, $x = g^{-1}(g(u))g^{-1}(-g(u)) = B^{-1}(-g(u)^2)$. Therefore $B(x) = g(u)^2$ and $\phi_k(x) = B^{-1}(-k^2g(u)^2) = B^{-1}(k^2B(x))$. Consequently $\phi_k(x)$ is a formal power system of type $s = 2$ over the ring Ω_U, with logarithm $B(x)$. In view of the universality of the group f_U over Ω_U this completes the proof of the lemma in the general case. □

We shall give a topological interpretation of Lemma 2.12. Consider the Thom spectrum $MSp = (MSp(n))$ of the symplectic group Sp. In particular $MSp(1) = KP^\infty$ is infinite-dimensional quaternion projective space. The canonical imbedding $S^1 \to Sp(1) \to SU(2)$ defines a mapping $\phi\colon CP^\infty \to KP^\infty$, and consequently a mapping $\phi^*\colon U^*(KP^\infty) \to U^*(CP^\infty)$, where $U^*(KP^\infty) = \Omega_U[[x]]$, $\dim_R(x) = 4$, $U^*(CP^\infty) = \Omega_U[[u]]$, $\dim_R u = 2$ and $\phi^*(x) = u\bar{u}$. This follows from the fact that the canonical $Sp(1)$-bundle γ over KP^∞ restricted to CP^∞ goes into $\eta + \bar{\eta}$, and $x = \sigma_2(\gamma) \to \sigma_1(\eta)\sigma_1(\bar{\eta}) = u\bar{u}$, where σ_i is the Chern class in cobordism theory.

We set

$$\phi_k(x) = (k^2\Psi^k)x = k^2x + \sum_{i=1}^{\infty}\mu_i(k)x^{i+1}, \quad x \in U^4(KP^\infty),\ \mu_i(k) \in \Omega_U^{-4i}.$$

From the properties of the operation Ψ^k (see [12]) we obtain

$$\begin{aligned}\phi_k(\phi_l(x)) &= k^2\phi_l(x) + \sum_{i=1}^{\infty}\mu_i(k)(\phi_l(x))^{i+1}\\ &= k^2(l^2\Psi^l(x)) + \sum_{i=1}^{\infty}\mu_i(k)l^{2i+2}\Psi^l(x^{i+1})\\ &= l^2\Psi^l(k^2\Psi^k(x)) = l^2k^2\Psi^{lk}(x) = \phi_{kl}(x).\end{aligned}$$

Here we have used the formula $\Psi^l(\mu_i(k)) = l^{2i}\mu_i(k)$. Consequently the set of functions $\phi(x) = \{\phi_k(x)\}$ is a formal power system of type $s = 2$ over the ring Ω_U. Since $k^2\Psi^k(x) = k^2\Psi^k(u\bar{u}) = [u]_k[\bar{u}]_k$, by this means we obtain a topological proof of Lemma 2.12.

Remark. We note that in §VII of the paper of Novikov [13] in the proof of Theorem 1b in Example 3 the case of groups of generalized quaternions was analyzed and the "square modulus" system arose there; the properties of this system are required for carrying out a rigorous proof for this example, without which Theorem 1b cannot be proved. Indeed, we used the fact that $k^2\Psi^k(w)$ is a series in the variable w with coefficients in Ω_U, where $w = \sigma_2(\Delta_1)$. Moreover, for carrying out the proof of Theorem 1b, in analogy with Theorem 1 we require the fact that the Δ_i are all obtained from Δ_1 by means of the Adams operators, where the Δ_i are the 2-dimensional irreducible representations of the group of generalized quaternions.

Let us consider in more detail the logarithm $B(x) = -g(u)^2$ of the formal type 2 power system introduced in Lemma 2.12. Let t and z be the generators of the cohomology groups $H^2(CP^\infty;\mathbf{Z})$ and $H^4(KP^\infty;\mathbf{Z})$, respectively. Since $\phi^*(c_2(\gamma)) = c_1(\eta)c_1(\bar{\eta})$, $\phi^*(z) = -t^2$. We have $\mathrm{ch}_U(g(u)) = t$ and $\mathrm{ch}_U(B(x)) = -t^2 = z$. Consequently

$$B^{-1}(x) = \mathrm{ch}_U(x)|_{z=x} \in \mathcal{H}^*(KP^\infty;\Omega_U\otimes Q) = \Omega_U\otimes Q[[z]].$$

Let Ψ^0 be the multiplicative operation in $U^*\otimes Q$ theory, given by the series

$$\Psi^0(u) = \lim_{k\to 0}\left(\frac{1}{k}g^{-1}(kg(u))\right) = g(u) = u + \sum\frac{[CP^n]}{n+1}u^{n+1}.$$

Recall that in [2] and [12] the operation Ψ^0 was denoted by the symbol Φ. We have $\mathrm{ch}_U\,\Psi^0(x) = \Psi^0\,\mathrm{ch}_U(x) = z$; here we have used the fact that $\Psi^0(y) = 0$, where $y\in\Omega_U^{-2n}$, $n>0$. Since the homomorphism $\mathrm{ch}_U\colon U^*(KP^\infty)\to\mathcal{H}^*(KP^\infty;\Omega_U\otimes Q)$ is a monomorphism, it follows from the equation $\mathrm{ch}_U(B(x)) = z = \mathrm{ch}_U(\Psi(x))$ that $B(x) = \Psi^0(x)$. According to Theorem 2.3 of [2], we have the formula

$$\mathrm{ch}_U(u) = t + \sum_{n=1}^{\infty}[M^{2n}]\frac{t^{n+1}}{(n+1)!}$$

for an element $u \in U^2(CP^\infty)$, where $s_\omega(-\tau(M^{2n})) = 0$, $\omega \neq (n)$ and $s_{(n)}(M^{2n}) = -(n+1)!$. Consequently

$$\mathrm{ch}_U(u\bar{u}) = \left(t + \sum_{n=1}^{\infty}[M^{2n}]\frac{t^{n+1}}{(n+1)!}\right)\left(-t + \sum_{n=1}^{\infty}(-1)^{n+1}[M^{2n}]\frac{t^{n+1}}{(n+1)!}\right)$$
$$= -t^2\sum_{n=2}^{\infty}\left(\sum_{\substack{i+j=2n\\ i\geq 1, j\geq 1}}(-1)^i\frac{[M^{2i-2}][M^{2j-2}]}{i!\,j!}\right)t^{2n},$$

and we obtain the formula

$$B^{-1}(x) = x + \sum_{n=2}^{\infty}[N^{4n-4}]\frac{x^n}{(2n)!}, \quad [N^{4n-4}] \in \Omega_U^{-4n+4},$$

where

$$[N^{4n-4}] = \sum_{\substack{i+j=2n\\ i\geq 1, j\geq 1}}(-1)^{n+i}C_{2n}^i[M^{2i-2}][M^{2j-2}]$$

and $[M^{2m}] \in \Omega_U^{-2n}$ are bordism classes which are uniquely defined by the conditions

$$s_\omega(-\tau(M^{2m})) = 0, \quad \omega \neq (mn) \quad \text{and} \quad s_{(m)}(\tau(M^{2m})) = -(m+1)!.$$

We have

Theorem 2.13. *The type 2 power system constructed in Lemma 2.12 for the group $f(u,v)$ of geometrical cobordism is universal in the class of torsion-free rings if considered over the minimal ring of its coefficients $\Lambda \subset \Omega_U$.*

The proof follows easily from the fact that the coefficients of the series $B^{-1}(x)$ and $B(x)$ are not all zero and are algebraically independent in $\Omega_U \otimes Q$.

From the preceding lemma we have

Corollary 2.14. *For any complex X the image of the mapping $[X, KP^\infty] \stackrel{\alpha}{\longmapsto} U^4(X)$, which associates with the mapping $\phi\colon X \to KP^\infty$ its fundamental class $\phi^*(\sigma_2(\gamma))$ in U^*-theory, is the region of definition of a power system of type 2 having the form $B^{-1}(k^2B(x))$, where $B^{-1}(x) = g^{-1}(\sqrt{x})g^{-1}(-\sqrt{x})$. The Adams operators on this image are given by $k^2\Psi^k(x) = B^{-1}(k^2B(x))$, $\Psi^0(x) = B(x) \in U^*(X)\otimes Q$.*

Questions. Is a type 2 power system defined directly on quaternion Sp-cobordisms $[X, KP^\infty] \to Sp^4(X)$? Is the image $\operatorname{Im}\alpha$ closed with respect to power operations?

What are the inter-relations between the ring of coefficients of the power system $B^{-1}(k^2B(x))$ with the image of $\Omega_{Sp} \to \Omega_U$?

Note that the restriction $U^*(MSp(n)) \to U^*(MU(n)) \to U^*(CP_1^\infty \times \cdots \times CP_n^\infty)$ consists of all elements of the form $F(|u_1|^2, \ldots, |u_n|^2) \cdot \prod_{i=1}^n |u_i|^2$, where F is any symmetric polynomial (in distinction from classical cohomology, where we have symmetrical functions of squares).

As was pointed out above, for the series $B^{-1}(x)$ we have

$$B^{-1}(x) = x + \sum_{n=2}^{\infty}[N^{4n-4}]\frac{x^n}{(2n)!}, \quad [N^{4n-4}] \in \Omega_U^{-4n+4},$$

where
$$[N^{4n-4}] = \sum_{\substack{i+j=2n\\ i\geq 1, j\geq 1}} (-1)^{n+i} C^i_{2n} [M^{2i-2}][M^{2j-2}].$$
In particular,
$$[N^4] = -8[M^4] + 6[M^2]^2 = (2K) \in \operatorname{Im}(\Omega_{Sp} \to \Omega_U),$$
where $K = 8[CP^2] - 9[CP^1]^2$.

Theorem 2.15. *For $n \geq 2$ the bordism classes $[N^{4n-4}]$ belong to the image of the homomorphism $\Omega^{-4n+4}_{Sp} \to \Omega^{-4n+4}_U$. In addition, for $n \equiv 1 \mod 2$ the elements $[N^{4n-4}]/2 \in \Omega^{-4n+4}_U$ already belong to the group $\operatorname{Im}(\Omega_{Sp} \to \Omega_U)$.*

Proof. Let $v \in Sp^*(Kp^\infty)$ be the canonical element. As is well known, $p_1(\gamma) = v$ (see [13]) and $\omega^*(p_1(\gamma)) = \sigma_2(\gamma) \in U^4(KP^\infty)$, where p_1 is the first Pontrjagin class in the symplectic cobordism of the canonical $Sp(1)$-fiber γ over KP^∞ and $\omega\colon Sp^* \to U^*$ is the natural transformation of cobordism theory. We shall calculate the coefficients of the series
$$\operatorname{ch}_{Sp}(p_1(\gamma)) = z + \sum_{n=1}^{\infty} \frac{C_n}{\lambda_n} z^{n+1} \in \mathcal{H}^4(KP^\infty, \Omega^*_{Sp} \otimes Q) = \Omega^*_{Sp} \otimes Q[[z]],$$
where ch_{Sp} is the Chern–Dold characteristic in Sp-theory (see [2]), $C_n \in \Omega^{-4n}_{Sp}$ are indivisible elements in the group Ω^{-4n}_{Sp}, and $\lambda_n \in \mathbf{Z}$. Since $\operatorname{ch}\gamma = \operatorname{ch}(\eta + \bar{\eta}) = e^t + e^{-t} = 2 + t^2 + \cdots + 2t^{2n}/(2n)! + \cdots$ and $z \to -t^2$, $t \in H^2(CP^\infty, \mathbf{Z})$, we have
$$\operatorname{ch}_{Sp}(p_1(\gamma)) = -\operatorname{ch}_2 \gamma + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n+2)!}{2\lambda_n} C_n \operatorname{ch}_{2n+2}(\gamma).$$
By making use of the decomposition principle for quaternion fibers and the additivity of the first Pontrjagin class we now find that for any $Sp(m)$-fiber ζ over any complex X we have
$$\operatorname{ch}_{Sp}(p_1(\zeta)) = -\operatorname{ch}_2(\zeta) + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n+2)!}{2\lambda_n} C_n \operatorname{ch}_{2n+2}(\zeta).$$
By Bott's theorem we have the isomorphism
$$\beta\colon \tilde{K}Sp(S^{4n}) \xrightarrow{\simeq} KO^4(S^{4n}), \quad \beta(\zeta) = (1-\gamma_1) \otimes_H \zeta,$$
where γ_1 is the $Sp(1)$-Hopf fiber over S^4.

We shall next identify the elements $\zeta \in \tilde{K}Sp(S^{4n})$ with their images in the group $K(S^{4n})$. The formula $\operatorname{ch}(c\beta(\zeta)) = \operatorname{ch}(\zeta)$ is easily verified, where $c\colon KO^4 \to K^4$ is the homomorphism of complexification.

Let ξ_n and z_n denote the generators of the groups $\tilde{K}Sp(S^{4n}) = \mathbf{Z}$ and $H^{4n}(S^{4n}; \mathbf{Z}) = \mathbf{Z}$, respectively. From Bott's results concerning the homomorphism of complexification it follows that $\operatorname{ch}\xi_n = a_n z_n$, where
$$a_n = \begin{cases} 1, & \text{if } n \equiv 1 \mod 2,\\ 2, & \text{if } n \equiv 0 \mod 2.\end{cases}$$
Thus
$$\operatorname{ch}_{Sp}(p_1(\xi_n)) = (-1)^n \frac{(2n)!}{2\lambda_{n-1}} C_{n-1} \operatorname{ch}_{2n} \xi_n = (-1)^n \frac{(2n)!}{2\lambda_{n-1}} a_n C_{n-1} \cdot z_n.$$

Since
$$\mathrm{ch}_{Sp}(p_1(\xi_n)) \in \mathcal{H}^4(S^{4n}; \Omega^*_{Sp}) \subset \mathcal{H}^4(S^{4n}; \Omega^*_{Sp} \otimes Q),$$
we find that the number $((2n)!/2\lambda_{n-1})a_n$ is an integer for any n.

It follows from [7] that under the composition of homomorphisms
$$\tilde{K}Sp(X) \xrightarrow{p_1} \Omega^4_{Sp}(X) \xrightarrow{\omega} \Omega^4_U(X) \xrightarrow{\mu} \tilde{K}(X)$$
the element $\zeta \in \tilde{K}Sp(X)$ goes into the element $-\zeta \in \tilde{K}(X)$, where μ is the "Riemann–Roch" homomorphism. We have
$$\begin{aligned}-a_n z_n &= \mathrm{ch}(-\xi_n) = \mathrm{ch}(\mu\omega p_1(\xi_n)) = \mu\omega\,\mathrm{ch}_{Sp}(p_1(\xi_n))\\ &= \mu\omega\left((-1)^n \frac{(2n)!}{2\lambda_{n-1}} a_n C_{n-1} z_n\right) = (-1)^n \frac{(2n)!}{2\lambda_{n-1}} a_n Td(\omega(C_{n-1}))z_n,\end{aligned}$$
where $Td(\omega(C_{n-1}))$ is the Todd genus of the quasicomplex manifold $\omega(C_{n-1})$. Since the Todd genus of any $(8m+4)$-dimensional SU-manifold is even, we find that $nTd(\omega(C_{n-1})) = a_n\delta_n$ for any n, where δ_n is an integer. We have
$$(-1)^n \frac{(2n)!}{2\lambda_{n-1}} a_n \delta_n = 1.$$
Thus the number $2\lambda_{n-1}/(2n)!\, a_n$ is an integer. On the other hand, it was shown earlier that the number $(2n)!\, a_n/2\lambda_{n-1}$ is an integer also. Consequently $2\lambda_{n-1}/(2n)!\, a_n = \pm 1$. Without limiting generality, we may assume that $2\lambda_{n-1}/(2n)!\, a_n = 1$. Since $a_{n-1} \cdot a_n = 2$ for any $n > 0$, it follows that $a_{n-1} \cdot \lambda_{n-1}/(2n)! = 1$, and we find
$$\lambda_{n-1} = \frac{(2n)!}{a_{n-1}}, \quad C_{n-1} = (-1)^n p_1(\xi_n) \in \Omega^{-4n+4}_{Sp} \cong Sp^4(S^{4n}),$$
$$Td(\omega(C_{n-1})) = (-1)^{n-1} a_n.$$
We have therefore proved the following lemma.

Lemma 2.16. *For the canonical element $v = p_1(\gamma) \in Sp^4(KP^\infty)$ and the Chern–Dold characteristic in symmetric cobordism we have the formula*
$$\mathrm{ch}_{Sp}(p_1(\gamma)) = z + \sum_{n=2}^{\infty} a_{n-1} C_{n-1} \frac{z_n}{(2n)!}.$$

From the formula $\omega\,\mathrm{ch}_{Sp}\, p_1(\gamma) = \mathrm{ch}_U\, \sigma_2(\gamma)$ we obtain
$$\begin{aligned}B^{-1}(z) = \mathrm{ch}_U\, \sigma_2(\gamma) &= z + \sum_{n=2}^{\infty} [N^{4n-4}] \frac{z^n}{(2n)!}\\ \|\\ \omega\,\mathrm{ch}_{Sp}\, p_1(\gamma) &= z + \sum_{n=2}^{\infty} a_{n-1}\omega(C_{n-1}) \frac{z^n}{(2n)!}\end{aligned}$$
Consequently in the group Ω^{-4n+4}_U we have the identity
$$a_{n-1} \cdot \omega(C_{n-1}) = [N^{4n-4}]$$
for any n. This proves the theorem. ∎

Corollary 2.17. *The rational envelope of the ring of coefficients of the power system $B^{-1}(k^2B(x))$ of type $s = 2$ coincides with the group $\mathrm{Hom}^*_{A^U}(U^*(MSp), \Omega_U)$ which is the rational envelope of the image $\Omega_{Sp} \to \Omega_U$, where A^U is the ring of operations of U^* cobordism theory.*

We note that the element $(u + \bar{u}) = \sigma_1(\xi + \bar{\xi}) \in U^2(CP^\infty)$ can be expressed in terms of $x = u\bar{u} = \sigma_2(\xi + \bar{\xi})$. We have

$$(u + \bar{u}) = g^{-1}(g(u)) + g^{-1}(-g(u)) = F(g(u)^2) = F(-B(x)) = G(x),$$

where

$$F(\alpha^2) = g^{-1}(\alpha) + g^{-1}(-\alpha) = -[CP^1]\alpha^2 + \sum 2\frac{[M^{4n+2}]}{(2n+2)!}\alpha^{2n+2}.$$

Lemma 2.18. *For any k the series $G_k(x) = F(-k^2B(x))$ lie in $\Omega_U[[x]]$ and determine over the ring $\Omega_U[1/[CP^1]]$ a formal type 2 power system by means of the formula $\phi_k(w) = F(k^2F^{-1}(w))$, where $w = u + \bar{u} = G(x)$.*

The first assertion of the lemma follows from the fact that $G_K(x) = [u]_k + [\bar{u}]_k = \sigma_1(\xi^k + \bar{\xi}^k)$. The second assertion follows from the invertibility of the series $F(\alpha^2)$ in the ring $\Omega_U[[1/[CP^1]\alpha^2]]$, as a consequence of which $-B(x) = F^{-1}(G(x))$ and $G_k(x) = F(k^2F^{-1}(G(x)))$.

Corollary 2.19. *Let $F(u, v) = u + v + \alpha_{1,1}uv + \dots$ be a formal group over the ring R. If we map the element $\alpha_{1,1}$ into R, then the formal power system of type $s = 2$, defined by the series $\phi_k(w) = [u]_k + [u]_k \in R[[w]]$, $w = u + u$, is defined over the ring R.*

Let $\phi(x) = \{\phi_k(x)\}$ be a type $s = 2$ power system over a torsion-free ring. It is natural to pose the problem:

$(*)$ Describe all rings R such that 1) $\Lambda \subset R$; 2) there exists over the ring R a one-dimensional formal group $F(u, v)$ from which the original formal power system $\{\phi_k(w)\}$ is obtained as a system of the form $\{[u]_k[\bar{u}]_k\}$, $x = u\bar{u}$.

We note that the set of all such pairs $(R, F(u, v))$ forms a category in which the morphisms $(R_1, F_1) \to (R_2, F_2)$ are the ring homomorphisms $R_1 \to R_2$ which preserve the ring Λ and take the group F_1 into the group F_2. Next we shall display a universal formal group in this category, and by this means we shall obtain a complete solution to the problem posed above.

We consider first the case where $\Lambda = \Omega_U$ and $\phi(x) = \{\phi_k(x) = [u]_k[\bar{u}]_k = B^{-1}(k^2B(x))\}$, $x = u\bar{u}$.

Lemma 2.20. *The power system $\phi(x) = \{B^{-1}(k^2B(x))\}$ together with the series $G(x) = F(-B(x)) = u + \bar{u}$ completely determines the original formal group $f(u, v) = g^{-1}(g(u) + g(v))$.*

Proof. By knowing the series $G(x)$ we can calculate the series $\bar{u} = \theta(u)$ from the equation $u + \theta(u) = G(u \cdot \theta(u))$. Then, knowing the series $B(x)$, we can calculate the series $g(u)$ from the equation $B(u \cdot \theta(u)) = -g(u)^2$. □

Remark. The proof of Lemma 2.20 actually uses the fact that the elements u and $\bar{u}$ are the roots of the equation

$$y^2 - (u + \bar{u})y + u\bar{u} = y^2 - G(x)y + x = 0$$

over $\Omega_U[[x]]$.

From the formula introduced above it follows that the coefficients of the series $F(x)$ and $B(x)$ are algebraically independent and generate the entire ring $\Omega_U \otimes Q$. We have

$$F(x) = \sum_{i\geq 0} y_i x^{i+1}, \quad y_i \in \Omega_U^{-4i-2} \otimes Q; \qquad B(x) = \sum_{i\geq 0} z_i x^{i+1}, \quad z_i \in \Omega_U^{-4i} \otimes Q$$

and

$$\Omega_U \otimes Q = Q[y_i] \otimes Q[z_i].$$

Now let $\phi_k(x) = \{\phi_k(x)\}$ be an arbitrary type $s = 2$ formal power system over a torsion-free ring Λ and let $\mathcal{B}(x) = \sum \beta_i x^{i+1}$ be its logarithm. Consider the ring homomorphism $\chi\colon \Omega_U \otimes Q \to \Lambda \otimes Q[y_i]$, defined by the equation $\chi(z_i) = \beta_i$, $\chi(y_i) = y_i$, and let R denote the subring of $\Lambda \otimes Q[y_i]$ which is generated by the ring Λ and the image of the ring $\Omega_U \to \Omega_U \otimes Q$ under the homomorphism χ. The one-dimensional formal group $F(u, v)$, which is the image of the group $f(u, v)$ over Ω_U, is defined over R. From the universality of the group $f(u, v)$ and from Lemma 2.20 it follows easily that the group $F(u, v)$ over R is a universal solution of problem $(*)$ for the system $\{\phi_k(x)\}$ over $\Lambda \subset R$.

We note that from the proof of Lemma 2.20 there follows a direct construction for the formal group $F(u, v)$ over R from the system $\{\phi_k(x)\} = \{\mathcal{B}(k^2\mathcal{B}(x))\}$ over Λ. Indeed, it is necessary to carry out the following procedure. Consider the ring $\Lambda \otimes Q[y_i]$ and over it the series $F(x) = \sum y_i x^{i+1}$ and the corresponding series $G(x) = F(-\mathcal{B}(x))$; then, as in Lemma 2.20, with respect to the series $\mathcal{B}(x)$ and $G(x)$, find the series $g_F(u) \in \Lambda \otimes Q[[u, y_i]]$. The ring R is then the minimal extension of the ring Λ in $\Lambda \otimes Q[y_i]$, which contains the ring of coefficients of the group $F(u, v) = {g_F}^{-}1(g_F(u) + g_F(v))$.

§2b

We next turn our attention to the case where the power system $B^{-1}(k^2B(x)) = k^2\Psi^k(u\bar{u})$ is related to a distinctive "two-valued formal group"

$$F^{\pm}(x, y) = B^{-1}((\sqrt{B(x)} \pm \sqrt{B(y)})^2),$$

in which the operation of raising to a power is single valued, and indeed

$$B^{-1}(k^2B(x)) = F^{\pm}(\underbrace{x, \dots, x}_{k \text{ places}}).$$

If $x = u\bar{u}$, $y = v\bar{v}$, then $F^{\pm}(x, y) = \{|f(u, v)|^2; |f(u, \bar{v})|^2\}$ and for the $U(1)$-fibers ξ, η over $CF^\infty \times CP^\infty$, where $u = \sigma_1(\xi)$, $v = \sigma_1(\eta)$, we have

$$F^{\pm}(x, y) = \{\sigma_2(\xi\eta + \bar{\xi}\bar{\eta}); \sigma_2(\xi\bar{\eta} + \bar{\xi}\eta)\}, \quad x = \sigma_2(\xi + \bar{\xi}),\ y = \sigma_2(\eta + \bar{\eta}).$$

Lemma 2.21. *The sum $F^+(x, y) + F^-(x, y)$ and the product $F^+(x, y) \cdot F^-(x, y)$ of values for the two-valued group do not contain roots and lie in the ring $\Omega_U[[x, y]]$.*

Proof. Consider the mapping $CP^\infty \times CP^\infty \to KP^\infty \times KP^\infty$ whose image $U^*(KP^\infty \times KP^\infty) \to U^*(CP^\infty \times CP^\infty)$ is precisely $\Omega_U[[x, y]] \subset \Omega_U[[u, v]]$, $x = u\bar{u}$, $y = v\bar{v}$. Since $x = \sigma_2(\xi + \bar{\xi})$ and $y = \sigma_2(\eta + \bar{\eta})$, we have that $\sigma_2((\xi + \bar{\xi})(\eta + \bar{\eta})) = a$ lies in $\Omega_U[[x, y]]$; moreover

$$a = \sigma_2(\xi\eta + \bar{\xi}\bar{\eta})\sigma_2(\xi\bar{\eta} + \bar{\xi}\eta) + \sigma_1(\xi\eta + \bar{\xi}\bar{\eta})\sigma_1(\xi\bar{\eta} + \bar{\xi}\eta) = F^+(x, y) + F^-(x, y) + \sigma_1\sigma_1'.$$

Next,

$$\sigma_1(\xi\eta + \bar{\xi}\bar{\eta}) = g^{-1}(g(u) + g(v)) + g^{-1}(-g(u) - g(v)),$$
$$\sigma_1(\xi\bar{\eta} + \bar{\xi}\eta) = g^{-1}(g(u) - g(v)) + g^{-1}(g(v) - g(u)).$$

Let $g(u) = \gamma$, $g(v) = \delta$. Therefore

$$\sigma_1 \cdot \sigma_1' = \sigma_1(\xi\eta + \bar{\xi}\bar{\eta})\sigma_1(\xi\bar{\eta} + \bar{\xi}\eta) = [g^{-1}(\gamma+\delta) + g^{-1}(-\gamma-\delta)][g^{-1}(\gamma-\delta) + g^{-1}(\delta-\gamma)],$$

i.e. $\sigma_1 \cdot \sigma_1'$ is a function of γ^2 and δ^2. Also, since $\gamma^2 = g(u)^2 = -B(x)$ and $\delta^2 = g(v)^2 = -B(y)$, the product $\sigma_1(\xi\eta + \bar{\xi}\bar{\eta})\sigma_1(\xi\bar{\eta} + \bar{\xi}\eta)$ is a function of x and y. Since

$$F^+(x, y) + F^-(x, y) = \sigma_2((\xi + \bar{\xi})(\eta + \bar{\eta}))-) - \sigma_1(\xi\eta + \bar{\xi}\bar{\eta})\sigma_1(\xi\bar{\eta} + \bar{\xi}\eta),$$

it follows that $F^+(x, y) + F^-(x, y) \in \Omega_U[[x, y]]$.

We conclude the proof by noting that

$$F^+(x, y) \cdot F^-(x, y) = \sigma_2(\xi\eta + \bar{\xi}\bar{\eta})\sigma_2(\xi\bar{\eta} + \bar{\xi}\eta) = \sigma_4((\xi + \bar{\xi})(\eta + \bar{\eta})) \in \Omega_U[[x, y]]. \quad \Box$$

Let us set

$$F^+(x, y) + F^-(x, y) = \Theta_1(x, y), \quad F^+(x, y) \cdot F^-(x, y) = \Theta_2(x, y).$$

It now follows from Lemma 2.21 that the law of multiplication in the two-valued formal group $F^{\pm}(x, y) = B^{-1}((\sqrt{B(x)} \pm \sqrt{B(y)})^2)$ is given by solving the quadratic equation

$$Z^2 - \Theta_1(x, y)Z + \Theta_2(x, y) = 0$$

over the ring $\Omega_U[[x, y]]$. Let $\Lambda \subset \Omega_U$ denote the minimal subring in Ω_U generated by the coefficients of the series $\Theta_1(x, y)$ and $\Theta_2(x, y)$. We have $\Lambda = \sum_{n \geq 0} \Lambda_{4n}$, $\Lambda_{4n} \subset \Omega_U^{-4n}$.

Our next problem is to describe the ring Λ, which it is natural to look upon as the ring of coefficients of the two-valued formal group $F^{\pm}(x, y)$. In the ring Λ it is useful to distinguish the two subrings Λ' and Λ'' which are generated by the coefficients of the series $\Theta_1(x, y)$ and $\Theta_2(x, y)$ respectively. As will be shown next, neither of the rings Λ' and Λ'' coincides with Λ. It is interesting to note that the ring of coefficients of the formal power system $\phi(x) = \{\phi_k(x)\} = \{B^{-1}(k^2 B(x))\}$ lies in, but does not coincide with, the ring Λ'. This follows from the facts that $\phi_1(x) = x$ and $\phi_2(x) = \Theta_1(x, x)$, and that for any $k \geq 3$ the formula

$$\phi_k(x) = \Theta_1(\phi_{k-1}(x), x) - \phi_{k-2}(x)$$

is valid.

The canonical mapping of the spectra $MSp \to MU$, which corresponds to the inclusion mapping $Sp(n) \subset U(2n)$, defines an epimorphism $A^U \to U^*(MSp)$, and consequently the inclusion of the ring $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$ in Ω_U. We shall next identify the ring $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$ with its image in Ω_U.

Theorem 2.22. *The quadratic equation*

$$Z^2 - \Theta_1(x, y)Z + \Theta_2(x, y) = 0,$$

which determines the law of multiplication in the two-valued formal group $F^{\pm}(x, y)$, *is defined over the ring* $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$, *and, moreover,*

$$\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U) \otimes Z\left[\frac{1}{2}\right] \cong \Lambda \otimes Z\left[\frac{1}{2}\right],$$

where Λ *is the ring of coefficients of the group* $F^{\pm}(x, y)$.

Remark 2.23. Apparently the rings $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$ and Λ are isomorphic, but at the present time the authors do not have a rigorous proof of this fact.

Let $\Omega_U(Z)$ be the subring of $\Omega_U \otimes Q$ which is generated by the elements all of whose Chern numbers are integers. As was shown in [2], the ring $\Omega_U(Z)$ is isomorphic to the ring of coefficients of the logarithm of the universal formal group $f(u, v)$, i.e. $\Omega_U(Z) = Z[\frac{1}{2}[CP^1], \dots, [CP^n]/(n+1), \dots]$. The Chern–Dold characteristic ch_U for any complex X defines a natural transformation

$$\mathrm{ch}_U \colon H_*(X) \to \mathrm{Hom}_{A^U}(U^*(X), \Omega_U(Z))$$

(see [2], Theorem 1.9), which, as is easily shown, is an isomorphism for torsion-free complexes in the homology. We have

$$\mathrm{ch}_U \colon H_*(MSp) \xrightarrow{\approx} \mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U(Z)).$$

The inclusion mapping $\Omega_U \subset \Omega_U(Z)$ and the canonical homomorphism $A^U \to U^*(MSp)$ lead to the commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U) \arrow[r] \arrow[d] & \Omega_U \arrow[d] \\
H_*(MSp) \arrow[r, "\lambda"] & \Omega_U(Z),
\end{tikzcd}$$

in which all the homomorphisms are monomorphisms.

Since $\lambda(h) = (\mathrm{ch}_U\, v, h)$ and $\mathrm{ch}_U\, x = B^{-1}(x)$, where $h \in H_*(MSp)$, v is the generator of the A^U-module $U^*(MSp)$, and x is the generator of the group $U^4(KP^\infty)$, it follows that the ring $\mathrm{Im}\,\lambda \subset \Omega_U(Z)$ coincides with the ring of coefficients of the logarithm of the power system $\{B^{-1}(k^2 B(x))\}$. Thus it follows from the diagram that the ring $\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U)$ coincides with the subring of Ω_U whose elements are monomials in the elements $y_i \in \Omega_U(Z)$ with integral coefficients, where $B(x) = x + \sum y_i x^{i+1}$.

As an immediate check it is easy to see that the coefficients of the series

$$\Theta_1(x, y) = F^+(x, y) + F^-(x, y) \quad \text{and} \quad \Theta_2(x, y) = F^+(x, y) \cdot F^-(x, y),$$

where

$$F^{\pm}(x, y) = B^{-1}((\sqrt{B(x)} \pm \sqrt{B(y)})^2) = B^{-1}\left(\left(x\sqrt{\frac{B(x)}{x}} \pm \sqrt{\frac{B(y)}{y}}\right)^2\right)$$

are polynomials with integral coefficients from among the coefficients of the series $B(x)$. The proof of the first part of the theorem is therefore complete.

For the proof of the second part of the theorem we require a lemma, which is itself of some interest.

Lemma 2.24. *Let $\Lambda = \sum \Lambda_{4n}$ be the ring of coefficients of a two-valued formal group. The minimum positive value of the t-characteristic on the group Λ_{4n} is equal to $2^{s(n)}p$ if $2n = p^i - 1$, where p is a prime, and is equal to $2^{s(n)}$ if $2n \neq p^i - 1$ for all p, where*

$$s(n) = \begin{cases} 3, & \text{if } n = 2^j - 1, \\ 2, & \text{if } n \neq 2^j - 1. \end{cases}$$

Now, since

$$\mathrm{Hom}_{A^U}(U^*(MSp), \Omega_U) \otimes Z\left[\frac{1}{2}\right] \subset \Omega_U\left[\frac{1}{2}\right]$$

is a polynomial ring, the proof of the second part of the theorem is easily obtained, via a standard argument concerning the t-characteristic, from the results of [11] and Lemma 2.24.

Proof of Lemma 2.24. Let x and y be the generators of the group $U^4(KP^\infty \times KP^\infty)$. We have

$$\Theta_1(x,y) = 2x + 2y + \sum \beta_{i,j} x^i y^j, \quad \beta_{i,j} = \beta_{j,i} \in \Omega_U^{-4(i+j-1)},$$
$$\Theta_2(x,y) = x^2 - 2xy + y^2 \sum \alpha_{i,j} x^i y^j, \quad \alpha_{i,j} = \alpha_{j,i} \in \Omega_U^{-4(i+j-2)}.$$

Let z_1 and z_2 be the generators of the group $H^4(KP^\infty \times KP^\infty)$. By using Corollary 2.4 of [2] we obtain immediately from the definitions of the series $\Theta_1(x,y)$ and $\Theta_2(x,y)$ that

$$\mathrm{ch}_U\,\Theta_1(x,y) = 2z_1 + 2z_2 + 4\sum_{m\geq 1}(-1)^m \frac{[M^{4m}]}{(2m+1)!}\sum_{l=0}^{m+1} C_{2m+2}^{2l} z_1^l z_2^{m-l+1},$$
$$\mathrm{ch}_U\,\Theta_2(x,y) = z_1^2 - 2z_1 z_2 + z_2^2$$
$$+ 4\sum_{m\geq 1}(-1)^m \frac{[M^{4m}]}{(2m+1)!}\sum_{l=0}^{m+1}(C_{2m}^{2l} - 2C_{2m}^{2l-2} + C_{2m}^{2l-4}) \cdot z_1^l z_2^{m-l+2},$$

where $s_{2m}([M^{4m}]) = -(2m+1)!$. On the other hand,

$$\mathrm{ch}_U\, x = B^{-1}(x) = z_1 + \sum_{m\geq 1}[N^{4m}]\frac{z_1^{m+1}}{(2m+2)!},$$

where $s_{2m}([N^{4m}]) = (-1)^{m+1} \cdot 2(2m+2)!$. By combining these formulas we find

a) $s_{2m}(\beta_{m+1,0}) = 0, \quad s_{2m}(\beta_{l,m-l+1}) = (-1)^{m+1} 4C_{2m+2}^{2l}, \quad 0 < l < m+1,$

b) $s_{2m}(\alpha_{m+2,0}) = 0, \quad s_{2m}(\alpha_{m+1,1}) = (-1)^{m+1} 4(C_{2m}^2 - C_{2m}^0),$

$s_{2m}(\alpha_{l,m-l+2}) = (-1)^{m+1} 4(C_{2m}^{2l} - 2C_{2m}^{2l-2} + C_{2m}^{2l-4}), \quad 1 < l < m+1.$

We set $\phi_{n,i} = C_{2n}^{2i} - C_{2n}^{2i-2}$. From equations a) and b) we obtain that the smallest value of the t-characteristic on the group Λ_{4n} is equal to the greatest common divisor of the numbers $\{4C_{2n+2}^{2l}, 4\phi_{n,1}\}_{l=1,\dots,n}$. Since the greatest common divisor of the numbers $\{C_{2n+2}^{2l}\}_{l\neq 0,n+1}$ is even for $n+1 = 2^j$, and odd for the remaining n, by using the formula $\phi_{n,l} + C_{2n+2}^{2l} = 2C_{2n+1}^{2l}$, we complete the proof of the lemma. □

Remark. It follows from a) that the coefficients of the series $\Theta_1(x,y) = F^+(x,y) + F^-(x,y)$ do not generate the entire ring of coefficients of the two-valued formal group. From b) there follows a similar assertion for the series $\Theta_2(x,y) = F^+(x,y) \cdot F^-(x,y)$.

Let $F(u,v)$ be a formal group over the ring R, and let $g_F(u)$ be its logarithm. Consider the complete set $(\xi_0 = 1, \xi_1, \dots, \xi_{m-1})$ of mth roots of unity. Let

$$B_m^{-1}(-y) = \prod_{j=0}^{m-1} g_F^{-1}(\xi_j \sqrt[m]{y}), \quad x = \prod_{j=0}^{m-1} g_F^{-1}(\xi_j g_F(u)) \otimes Q[[u]].$$

Then $-B_m(x) = g_F(u)^m$ and we obtain the formal power system

$$F_k^{(m)}(x) = B_m^{-1}(k^m B_m(x)) = \prod_{j=0}^{m-1} g_F^{-1}(k\xi_j g_F(u))$$

of type m. The coefficients of the series $F_k^{(m)}(x) = B_m^{-1}(k^m B_m(x))$ automatically lie in the ring R for a formal group $F(u,v)$ with complex multiplication by ξ_j (raising to the power ξ_j). The particular case $m = 2$ of this construction was examined in detail in Lemma 2.12.[2]

Example. Consider the formal group $f^{(p)}(u,v) = \bar{\Pi}_p^*(f(u,v))$, where $\bar{\Pi}_p^*$ is Quillen's p-adic projection of geometric cobordism and $f(u,v)$ is the universal formal group over Ω_U. As we have already noted, the logarithm $g^{(p)}(u)$ of the group $f^{(p)}(u,v)$ has the form

$$g^{(p)}(u) = \bar{\pi}_p^* g(u) = \sum_{h \geq 0} \frac{[CP^{p^h-1}]}{p^h} u^{p^h}.$$

Let $m = (p-1)$; then

$$\xi_j^{p^h} = \xi_j \quad \text{and} \quad g^{(p)}(\xi_j u) = \xi_j g^{(p)}(u), \quad (g^{(p)})^{-1}(\xi_j g^{(p)}(u)) = \xi_j u.$$

We have

$$x = -u^{p-1} = \prod_{j=0}^{p-2} (g^{(p)})^{-1}(\xi_j g^{(p)}(u)), \quad B_{p-1}(x) = -(g^{(p)}(u))^{p-1} = B_{p-1}(-u^{p-1}).$$

Thus formal raising to a power $F_k^{(p-1)}(x) = B_{p-1}^{-1}(k^{p-1}B_{p-1}(x))$ for the group $f^{(p)}(u,v)$ is "integer valued", and $F_k^{(p-1)}(x) = k^{p-1}\Psi^k(-u^{p-1})$. Consequently in U_p^*-theory the $(p-1)$th powers of geometrical cobordism are the region of definition of a power system of type $s = p-1$.

We now note that the roots of unity of degree $p-1$ lie in the ring of p-adic integers $\mathbf{Z}_p$. Therefore $g^{-1}(\xi_j g(u)) \in \Omega_U[[u]] \otimes \mathbf{Z}_p$ and

$$\prod_{j=0}^{p-2} g^{-1}(\xi_j g(u)) = x \in \Omega_U[[u]] \otimes \mathbf{Z}_p, \quad \prod_{j=0}^{p-2} g^{-1}(k\xi_j g(u)) \in \Omega_U[[u]] \otimes \mathbf{Z}_p,$$

and the series $B_m(x)$ defines a power system of type $m = p-1$, whose p-adic projection was displayed in the example.

The Adams operators are evaluated for an element x by the formula $k^{p-1}\Psi^k(x) = B_{p-1}^{-1}(k^{p-1}B_{p-1}(x))$ in $U^* \otimes \mathbf{Z}_p$-theory.

In analogy with Theorem 2.13 we have

Theorem 2.25. *The power system $B_{p-1}^{-1}(k^{p-1}B_{p-1}(x))$ of type $s = p-1$, considered over the minimal ring of its coefficients, is universal in the class of all power systems of type $(p-1)$ over torsion-free $\mathbf{Z}_p$-rings.*

The proof, as did that for Theorem 2.13, follows from the fact that the coefficients of the series $B_{p-1}(x)$ are all different from zero and algebraically independent in $\Omega_U \otimes Q_p$, where Q_p is the field of p-adic numbers.

[2]Here it is also appropriate to speak of the "manifold of the formal group"

$$F(x,y) = -B_m^{-1}[(\sqrt[m]{B_m(x)} + \sqrt[m]{B_m(y)})m].$$

It would be interesting to know the nature of the ring of coefficients in this case.

§ 3. Fixed points of transformations of order p

We turn now to a different question which is also connected with the formal group of geometrical cobordism and a type $s = 1$ system associated with it; namely, to the theory of fixed points of transformations T ($T^0 = 1$) of quasicomplex manifolds (see [5]; [9]; [13]), which act so that the manifolds of fixed points have trivial normal bundle (or, for example, only isolated fixed points $\mathcal{P}_1, \dots, \mathcal{P}_q \in M^n$, $T(\mathcal{P}_j) = \mathcal{P}_j$.). If the transformation $dT|\mathcal{P}_j$ has eigenvalues $\lambda_k^{(j)} = \exp\{2\pi i x_k^{(j)}/p\}$, $k = 1, \dots, n$, $j = 1, \dots, q$, then the "Conner–Floyd invariants" $\alpha_{2n-1}(x_1^{(j)}, \dots, x_n^{(j)}) \in U_{2n-1}(BZ_p)$ and it is known that

$$U^*(BZ_p) = \Omega_U[[u]]/p\Psi^p(u) = 0 \quad \text{(see [12])},$$

$$\alpha_{2n-1}(x_1, \dots, x_n) = \prod_{j=1}^{n} \frac{u}{g^{-1}(x_j g(u))} \cap \alpha_{2n-1}(1, \dots, 1),$$

where $u^k \cap \alpha_{2n-1}(1, \dots, 1) = \alpha_{2(n-k)-1}(1, \dots, 1)$ (see [5], [9], [13]) and $g^{-1}(xg(u)) = x\Psi^x(u)$. Here it is already clear that only the coefficients of the power system enter into the expression for $U^*(BZ_p)$ and $\alpha_{2n-1}(x_1, \dots, x_n)$. There is still one further question: on which classes of Ω_U can the group $Z_p = \mathbf{Z}/p\mathbf{Z}$ act? As is shown in [5] and [9], the basis relations

$$0 = \alpha_{2n-1}(x_1, \dots, x_n) - \prod_{j=1}^{n} \frac{u}{x_j \Psi^{x_j}(u)} \cap \alpha_{2n-1}(1, \dots, 1)$$

and

$$0 = p\frac{\Psi^p(u)}{u} \cap \alpha_{2n-1}(1, \dots, 1),$$

are realized on the manifolds $M^n(x_1, \dots, x_n)$ and $M^n(p)$, and determine the elements

$$\left[\prod_{j=1}^{n} \frac{u}{x_j \Psi^{x_j}(u)}\right]_n \in \Omega_U^{2n} \pmod{p\Omega_U} \quad \text{and} \quad \left[p\frac{\Psi^p(u)}{u}\right]_n \in \Omega_U^{2n} \pmod{p\Omega_U},$$

whence it follows that the cobordism class of manifolds with action Z_p of this sort coincides (mod $p\Omega_U$) with the Ω_U-module $\tilde{\Lambda}(1) = \Omega_U \cdot \Lambda^+(1)$, where $\Lambda^+(1)$ is the positive part of the ring $\Lambda(1)$ of coefficients of the power system $g^{-1}(kg(u))$. On the other hand, from Atiyah and Bott's results [15] for the complex d'' on forms of type $(0, q)$ and holomorphic transformations $T\colon M^n \to M^n$ we may introduce the following formula for the Todd genus $T(M^n) \mod p$, for example.

Lemma 3.1. *Let $\lambda_k^{(j)} = \exp\{2\pi i x_k^{(j)}/p\}$ be the eigenvalues of the transformation dT on the fixed points $\mathcal{P}_j$, $j = 1, \dots, q$, $k = 1, \dots, n$. Then*

$$-T(M^n) \equiv$$
$$\sum_{j=1}^{q} \frac{p-1}{\prod\limits_{k=1}^{n} -x_k^{(j)}} \sum_{l=-\left[\frac{n}{p-1}\right]}^{\infty} (-p)^l \left[\prod_{k=1}^{n} \frac{-x_k^{(j)} z}{1 - \exp\left\{-x_k^{(j)}\left(z + \frac{z^p}{p}\right)\right\}}\right]_{n+l(p-1)} \mod p.$$

This formula and its proof were communicated by D. K. Faddeev.

Proof. For the Euler characteristic $\chi(T)$ of the indicated elliptic complex we have the Atiyah–Bott formula:

$$\chi(T)=\sum_{j=1}^{q}\frac{1}{\det(1-\overline{dT})\mathcal{P}_j}=\sum_{j=1}^{q}\prod_{k=1}^{n}\frac{1}{1-\exp\left\{-\frac{2\pi i x_k^{(j)}}{p}\right\}}.$$

Since $(1/p)\sum_{l\in Z_p}\chi(T^1)=\phi$ is the alternating sum of the dimensions of the invariant spaces of the action T on the homologies of the complex and $\chi(1)=T(M^n)$, we have

$$\chi(1)=T(M^n)=\sum_{j=1}^{q}\sum_{l=1}^{p-1}\prod_{k=1}^{n}\frac{1}{1-\exp\left\{-\frac{2\pi i x_k^{(j)}l}{p}\right\}}+p\phi.$$

If $\mathrm{Tr}\colon Q(\sqrt[p]{1})\to Q$ is the number-theoretic trace, then by definition we have

$$-T(M^n)\equiv\sum_{j=1}^{q}\mathrm{Tr}\left(\prod_{k=1}^{n}\frac{1}{1-\exp\left\{-\frac{2\pi i x_k^{(j)}}{p}\right\}}\right)\mod p.$$

The field $Q(\sqrt[p]{1})$ and the field Q axe imbedded in their p-adic completions $k=Q_p(\epsilon)$, $\epsilon=\sqrt[p]{1}$, and Q_p. There exists in the field k an element λ such that $\lambda^{p-1}=-p$ and $k=Q_p(\lambda)$. Next, $\mathrm{Tr}(\lambda^s)=0$ for $s\not\equiv 0 \pmod{(p-1)}$ and $\mathrm{Tr}(\lambda^{k(p-1)})=(-1)^k p^k(p-1)$. Since $\epsilon=\exp(z+z^p/p)|_{z=\lambda}$, we have

$$\exp\left\{-\frac{2\pi i x_k}{p}\right\}=\epsilon^{-x_k}=\exp\left\{-\left(x_kz+x_k\frac{z^p}{p}\right)\right\}$$

(in k). Therefore

$$\prod_{k=1}^{n}\frac{1}{1-\epsilon^{-x_k}}=\prod_{k=1}^{n}\frac{1}{1-\exp\left\{-\left(x_kz+x_k\frac{z^p}{p}\right)\right\}}\Bigg|_{z=\lambda}$$

$$=\frac{1}{z^n\prod_{k=1}^{n}-x_k}\prod_{k=1}^{n}\frac{x_kz}{1-\exp\left\{-x_k\left(z+\frac{z^p}{p}\right)\right\}}=\frac{(-1)^n}{z^n\prod_{k=1}^{n}x_k}\left(1+\sum_{s=1}^{\infty}P_s(x_1,\dots,x_n)\lambda^s\right).$$

Therefore

$$\mathrm{Tr}\left(\prod_{k=1}^{n}\frac{1}{1-\epsilon^{-x_k}}\right)$$

$$=\frac{p-1}{\prod_{k=1}^{n}-x_k}\sum_{l=-\left[\frac{n}{p-1}\right]}^{\infty}(-p)^k\left[\prod_{k=1}^{n}\frac{x_kz}{1-\exp\left\{-x_k\left(z+\frac{z^p}{p}\right)\right\}}\right]_{n+l(p-1)}.$$

The proof of the lemma is concluded by summing over the fixed points. □

For $p>n+1$ this gives the formula

$$T(M^n)=\sum_{j=1}^{k}\frac{(-1)^n}{x_1^{(j)}\dots x_n^{(j)}}\left[\prod\frac{-x_k^{(j)}}{1-\exp\{-x_k^{(j)}z\}}\right]_n,$$

proved in [13] as a consequence of Tamura's results.

We see that by Atiyah and Bott's procedure each fixed point is assigned a rational invariant. How does the analogous procedure look in bordism theory?

Let us define the function $\gamma_p(x_1, \dots, x_n) \in \Omega_U[1/p]$ such that under the action of T on M^n, $T^p = 1$, with isolated fixed points $\mathcal{P}_1, \dots, \mathcal{P}_q$ having weights $x_k^{(j)}$, $j = 1, \dots, q$, $k = 1, \dots, n$, the relation

$$\sum_{j=1}^{q} \gamma_p(x_1^{(j)}, \dots, x_n^{(j)}) \equiv [M^n] \mod p\Omega_U$$

is valid. Consider the $\Omega_U \otimes \mathbf{Z}_p$-free resolution of the module $U_*(BZ_p, \text{point})$:

$$0 \to F_1 \xrightarrow{d} F_0 \to U_*(BZ_p, \text{point}) \to 0,$$

where for the generators of $U_*(BZ_p, *)$ we take the elements $\alpha_{2n-1}(x_1, \dots, x_n) \in U_{2n-1}(BZ_p, \text{point})$ and the minimal module of relations is spanned by the relations

$$a(x_1, \dots, x_n) = \alpha_{2n-1}(x_1, \dots, x_n) - \left(\prod_{i=1}^{n} \frac{u}{x_i \Psi^{x_i}(u)} \right) \cap \alpha_{2n-1}(1, \dots, 1)$$

and

$$a_n = p\alpha_{2n-1}(1, \dots, 1) + \left(\frac{p\Psi^p(u)}{u} \right) \cap \alpha_{2n-1}(1, \dots, 1).$$

Let $\Phi\colon F_1 \to \Omega_U \otimes \mathbf{Z}_p$ denote the $\Omega_U \otimes \mathbf{Z}_p$-module such that

$$\Phi(a(x_1, \dots, x_n)) = \left[\prod_{i=1}^{n} \frac{u}{x_i \Psi^{x_i}(u)} \right]_n \in \Omega_U \otimes \mathbf{Z}_p \quad \text{and} \quad \Phi(a_n) = -\left[\frac{p\Psi^p(u)}{u} \right]_n.$$

As we pointed out above, for any set of weights $(x_1, \dots, x_n)$ we have the congruence $\Phi(a(x_1, \dots, x_n)) \equiv [M^n] \mod p$, where M^n is a quasicomplex manifold on which the relation $a(x_1, \dots, x_n)$ is realized. Relative to the operation of multiplying out the relations in $U_*(BZ_p)$ the group F_1 is a ring, and, as is clear, the homomorphism $\Phi \mod p\colon F_1 \to \Omega_U \pmod{\Omega_U}$ coincides with the well-known ring homomorphism which associates with each relation in F_1 a bordism class mod p of the manifold on which this relation is realized. The homomorphism Φ can be extended to a homomorphism

$$\gamma_p\colon F_0 \to \Omega_U \otimes Q_p, \quad \gamma_p(dF_1) = \Phi.$$

Lemma 3.2. *For any set of weights $(x_1, \dots, x_n)$ we have the formula*

$$\gamma_p(x_1, \dots, x_n) = \left[\frac{1}{x_1 \cdots x_n} \left(\prod_{i=1}^{n} \frac{u}{x_i \Psi^{x_i}(u)} \right) \frac{u}{\Psi^p(u)} \right]_n,$$

In particular,

$$\gamma_p(1, \dots, 1) = \left[\frac{p\Psi^p(u)}{u} \right]_n.$$

Proof. In the free $\Omega_U \otimes \mathbf{Z}_p$-module F_0 we have the identity

$$\alpha_{2n-1}(x_1, \dots, x_n) = a(x_1, \dots, x_n) + \sum_{k=0}^{n-1} \left[\prod_{j=1}^{n} \frac{u}{x_j \Psi^{x_j}(u)} \right]_k \alpha_{2n-2k-1},$$

$$\left[\frac{u}{x_i \Psi^{x_i}(u)} \right]_k \in \Omega_U^{2k} \otimes \mathbf{Z}_p, \quad \alpha_{2n-2k-1} = \alpha_{2n-2k-1}(1, \dots, 1).$$

Also, since $\gamma_p\colon F_0 \to \Omega_U \otimes Q_p$ is an $\Omega_U \otimes \mathbf{Z}_p$-module homomorphism and

$$\gamma_p(a(x_1, \dots, x_n)) = \left[\prod_{i=1}^{n} \frac{u}{x_i \Psi^{x_i}} \right]_n ,$$

it is sufficient to prove the lemma for the set of weights $(1, \dots, 1)$. We have

$$-a_n + \sum_{k=0}^{n-1} \left[\frac{p\Psi^p(u)}{u} \right]_k \alpha_{2n-2k-1} = 0, \quad n \geq 1,$$

$$\gamma_p(a_n) + \sum_{k=0}^{n-1} \left[\frac{p\Psi^p(u)}{u} \right]_k \gamma_p(\alpha_{2n-2k-1}) = 0,$$

$$\left[\frac{p\Psi^p(u)}{u} \right]_n + \sum_{k=0}^{n-1} \left[\frac{p\Psi^p(u)}{u} \right]_k \gamma_p(\alpha_{2n-2k-1}) = 0,$$

$$\left(\frac{p\Psi^p(u)}{u} \right) \left(1 + \sum_{j=1}^{\infty} \gamma_p(\alpha_{2j-1}) u^j \right) = p,$$

$$1 + \sum_{j=1}^{\infty} \gamma_p(\alpha_{2j-1}) u^j = \frac{u}{\Psi^p(u)},$$

and the lemma is proved. □

It follows immediately from the definition of the homomorphism $\Phi\colon F_1 \to \Omega_U \otimes \mathbf{Z}_p$ that $\operatorname{Im}\Phi(F_1) = \tilde{\Lambda}(1) \otimes \mathbf{Z}_p \subset \Omega_U \otimes \mathbf{Z}_p$, where $\tilde{\Lambda}(1) = \Lambda^+(1) \cdot \Omega_U$ and $\Lambda(1)$ is the ring of coefficients of the power system $\{k\Psi^k(u)\}_{k=\pm 1, \pm 2, \dots}$.

Lemma 3.3. *The group $\operatorname{Im}\gamma_p(F_0) \subset \Omega_U \otimes Q_p$ coincides with the $\Omega_U \otimes \mathbf{Z}_p$-module spanned by the system of polynomial generators $\delta_{n,p}$ of the ring $\Omega_U(\mathbf{Z}) \otimes \mathbf{Z}_p$ of coefficients of the logarithm of the formal group $f(u,v) \otimes \mathbf{Z}_p$, where $1 + \sum_{n=1}^{\infty} \delta_{n,p} t^n = t/\Psi^p(t)$.*

The proof of the lemma follows easily by evaluating the t-characteristic of the coefficients of the series $\Psi^p(t) = g^{-1}(pg(u))/p$, by means of the fact that all the Chern numbers of the coefficients of the series $p\Psi^p(u)$ are divisible by p, and from the form of the functions $\gamma_p(x_1, \dots, x_n)$ given in Lemma 3.2.

From the exactness of the sequence

$$0 \to F_1 \to F_0 \to U_*(BZ_p, \mathrm{point}) \to 0$$

we now find that a $\Omega_U \otimes \mathbf{Z}_p$-module homomorphism

$$\gamma_p\colon U_*(BZ_p, \mathrm{point}) \to \gamma(F_0)/\Phi(F_1)$$

is defined, where $\Phi(F_1) = \tilde{\Lambda}(1) \otimes \mathbf{Z}_p$ and $\gamma(F_0)/\Phi(F_1) \subset \Omega_U(\mathbf{Z})/\tilde{\Lambda}(1) \otimes \mathbf{Z}_p$, which is clearly an epimorphism. By collecting these results together, we arrive at the following theorem.

Theorem 3.4. *Functions $\gamma_p(x_1, \dots, x_n)$ of the fixed points are defined which take on values in the ring $\Omega_U(\mathbf{Z}) \otimes \mathbf{Z}_p$ of coefficients of the logarithm $g(u) = \sum([CP^n]/(n+1))u^{n+1}$ of the formal group $f(u,v) \otimes \mathbf{Z}_p$ for which*

a) *for the action of the transformation T on the quasicomplex manifold M^n, $T^p = 1$, with the fixed manifold of classes $\lambda_j \in \Omega_U$, having weights $(x_k^{(j)}) \in Z_p^*$ in the (trivial) normal bundles, we have the relations*

$$[M^n] \equiv \sum_j \lambda_j \gamma_p(x_1^{(j)}, \dots, x_{m_j}^{(j)}) \mod p\Omega_U, \quad [M^n] \in \tilde\Lambda(1),\ m_i + \dim \lambda_i = n,$$

and

$$\gamma_p(x_1, \dots, x_m) = \left[\frac{u}{\Psi^p(u)} \prod_{j=1}^m \frac{u}{x_j \Psi^{x_j}(u)} \right]_m ;$$

b) *the factor module over $\Omega_U \otimes \mathbf{Z}_p$, equal to $\Omega_U(\mathbf{Z})/\tilde\Lambda(1) \otimes \mathbf{Z}_p$, contains the nontrivial image of the module $U^*(BZ_p, \mathrm{point})$, γ_p coinciding under the homomorphism with the factor module with respect to $\tilde\Lambda(1)$ of the submodule in $\Omega_U(\mathbf{Z}) \otimes \mathbf{Z}_p$ which is spanned by the system of polynomial generators $\delta_{n,p}$.*

Here $\tilde\Lambda(1)$ is the Ω_U-module which is generated by the ring $\Lambda^+(1)$ of coefficients of the power system $\{g^{-1}(kg(u))\}$ of type $s = 1$, and $1 + \sum_{n\ge1} \delta_{n,p} t^n = t/\Psi^p(t)$.

Remark. If one deals with the action of a transformation T, $T^p = 1$, having isolated fixed points, then we have the group $U_{\mathrm{isol}}(Z_p) \subset U_*(BZ_p)$, spanned by all the elements $\alpha_{2n-1}(x_1, \dots, x_n)$ (without the structure of an Ω_U-module), with the resolution over Z_p:

$$0 \to G_1 \xrightarrow{d} G \to U_{\mathrm{isol}}(Z_p) \to 0.$$

where G_0 and G_1 are free and the generator of G_1 is a formal relation. As above, homomorphisms

$$\Phi\colon G_1 \to \Omega_U \otimes \mathbf{Z}_p \quad \text{and} \quad \Phi'\colon G_0 \to \Omega_U \otimes Q_p$$

are defined, where $\Phi' d = \Phi$. The factor group $\Phi'(G_0)/\Phi(G_1)$ is a p-group and there exists a homomorphism

$$\gamma\colon U_{\mathrm{isol}}(Z_p) \to \Phi'(G_0)/\Phi(G_1).$$

We now consider the mappings $U_* \to K_*$ and $U^* \to K^*$ generated by the Todd genus. For the T-genus we have

$$T(\gamma_p(x_1, \dots, x_n)) = \left[\frac{pu}{1-(1-u)^p} \prod_{k=1}^n \frac{u}{1-(1-u)^{x_k}} \right]_n \in Q_p.$$

For example, $T(\gamma_2(1, \dots, 1)) = 1/2^n$.

Under the action of the group Z_p on the manifold M^n with isolated fixed points $\mathcal{P}_1, \dots, \mathcal{P}_q$ having weights $x_k^{(j)}$, $k = 1, \dots, n$, $j = 1, \dots, q$, we have the formula

$$T(M^n) \equiv \sum_{j=1}^q T(\gamma_p(x_1^{(j)}, \dots, x_n^{(j)})) \mod p\mathbf{Z}_p,$$

where $p\mathbf{Z}_p \subset Q_p$. At first sight this formula differs from the Atiyah–Bott formula given in Lemma 3.1. The question arises of how to reconcile these two formulas.[3] Another question, similar to the subject of the Stong–Hattori theorem [7], is: does the set of relations given by Atiyah–Bott–Singer for the action of Z_p on all possible

[3]An answer to this question is given in the Appendix.

elliptic complexes define an extension $\Omega_v(\mathbf{Z}) \otimes \mathbf{Z}_p$ of the cobordism ring (more precisely, the module $\tilde{\Lambda}(1)$ and the ring $\Lambda(1)$ in Ω_U)?

We now show that the results of [17] permit us to generalize our construction to the case of the action of a transformation T, $T^p = 1$, for which the manifolds of fixed points have arbitrary normal bundle.

Let T be a transformation of order p on the manifold M^n. As was shown in [16], the normal bundle ν_j at any fixed point manifold $N_j \supset M^n$ can be represented in the form $\nu_j = \bigotimes_{k=1}^{p-1} \nu_{jk}$, where the action of the group Z_p on the fiber ν_{jk} is given by multiplication by the number $\exp(2\pi ik/p)$. Thus the set of all fixed point submanifolds of the transformation T together with their normal bundles defines an element of the group $A = \sum U_*(\prod_{k=1}^{p-1} BU(l_k))$, where the sum extends over all sets $l_1, \dots, l_{p-1}$, $l_i \geq 0$. By using the mapping $BU(n) \times BU(m) \to BU(n+m)$ (Whitney sum), a multiplication can be introduced into A. It is not difficult to show that A becomes a polynomial ring $\Omega_U[a_{j,k}]$, $j \in Z_p^*$, $k \geq 0$, where $a_{j,k}$ is the bordism class of the imbedding $CP^k \subset CP^\infty = BU(1)$, considered together with the action of the transformation $T = \exp(2\pi ij/p)$ on the Hopf fiber over CP^k. We introduce a grading into A by setting $\dim a_{j,k} = 2(k+1)$. We next describe the fixed point submanifolds N_m in terms of the generators $a_{j,k}$. For example, a fixed point with weights $(x_1, \dots, x_n)$ is described by the monomial $a_{0,x_1}, \dots, a_{0,x_n}$. Consider the canonical homomorphism $\alpha\colon A \to U_*(BZ_p, \text{point})$, corresponding to the free action of the group Z_p on the sphere bundle associated with the normal fiber at a fixed point submanifold. Let $\alpha((x_1,k_1), \dots, (x_l,k_l))$ denote the image under α of the monomial

$$a_{x_1,k_1} \dots a_{x_l,k_l}, \quad \alpha((x_1,k_1), \dots, (x_l,k_l)) \in U_{2n-1}(BZ_p, \text{point}),$$

where $n = \sum_{m=1}^{l}(k_m + 1)$. From [17] we take the following description of the elements $\alpha((x_1,k_1), \dots, (x_l,k_l))$.

Consider the formal series

$$G(u,t) = \frac{\frac{\partial}{\partial t} g(ut)}{f(u, u\bar{t})} = 1 + \sum_{n=1}^{\infty} G_n(u) t^n,$$

where $g(ut) = \sum_{n=0}^{\infty}([CP^n]/(n+1))(ut)^{n+1}$ is the logarithm of the formal group $f(u,v)$ and $u\bar{t} = g^{-1}(-g(ut))$.[4] Clearly $G_n(0) = 1$ for any $n \geq 1$. We set

$$\Psi^{x,n}(u) = \frac{\Psi^x(u)}{G_n(x\Psi^x(u))}.$$

We have $\Psi^{x,0}(u) = \Psi^x(u)$ and $\Psi^{1,n}(u) = u/G_n(u)$. From [17] we find that for any set $((x_1,k_1), \dots, (x_l,k_l))$, $\sum_{m=1}^{l}(k_m+1) = n$, we have

$$\alpha((x_1,k_1), \dots, (x_l,k_l)) = \left(\prod_{j=1}^{l} \frac{u}{x_j \Psi^{x_j,k_j}(u)} \right) \cap \alpha_{2n-1}(1, \dots, 1).$$

[4]Note that under the substitution $t \to z/u$ (u is a parameter) the differential $G(u,t)\,dt$ goes into the meromorphic differential $dg(z)/f(u,\bar{z})$ on the group, which is invariant with respect to the shift $u \to f(u,w)$, $z \to f(z,w)$.

Since $\Psi^{1,0}(u) = u$, from [16] we find that the relation

$$\alpha((x_1, k_1), \dots, (x_l, k_l)) = \left(\prod_{j=1}^{l} \frac{u}{x_j \Psi^{x_j,k_j}(u)} \right) \cap \alpha_{2n-1}(1, \dots, 1).$$

is realized in the manifold M^n determined by the element

$$\left[\prod_{j=1}^{l} \frac{u}{x_j \Psi^{x_j,k_j}(u)} \right]_n \in \Omega_v^{2n} \mod p\Omega_U.$$

By repeating the proof of Lemma 3.2, we obtain the following theorem.

Theorem 3.5. *A homomorphism $\gamma_p \colon A \otimes \mathbf{Z}_p \to \Omega_U \otimes Q_p$ is defined such that for any set $((x_1, k_1), \dots, (x_l, k_l))$ we have the formula*

$$\gamma_p((x_1, k_1), \dots, (x_l, k_l)) = \left[\frac{1}{x_1 \dots x_l} \left(\prod_{j=1}^{l} \frac{u}{x_j \Psi^{x_j,k_j}(u)} \right) \frac{u}{\Psi^p(u)} \right]_n, \ n = \sum_{m=1}^{l} (k_m + 1),$$

and if the element $a \in A$ corresponds to the union of all the fixed point submanifolds of the action of the group Z_p on M^n, then $\gamma_p(a) \equiv [M^n] \mod p$.

Appendix

The Atiyah–Bott formula, the functions $\gamma_p(x_1, \dots, x_n)$ of fixed points in bordism and the Conner–Floyd equation

Let ϵ be a primitive pth root of unity, and let $\mathrm{Tr} \colon Q(\epsilon) \to Q$ be the number-theoretic trace.

Definition 1. The Atiyah–Bott function $AB(x_1, \dots, x_n)$ of fixed points is the function which associates with each set of weights $(x_1, \dots, x_n)$, $x_j \in Z_p$, the rational number

$$AB(x_1, \dots, x_n) = -\operatorname{Tr} \left(\prod_{k=1}^{n} \frac{1}{1 - \exp\left\{ \frac{2\pi i}{p} x_k \right\}} \right).$$

As a corollary of the Atiyah–Bott formula for fixed points, we have

Theorem 2. *Let $f \colon M^n \to M^n$ be a holomorphic transverse mapping of period p of the compact complex manifold M^n, and let $\mathcal{P}_1, \dots, \mathcal{P}_q$ be its fixed points. If the mapping $df|\mathcal{P}_j$ in the tangent space at the fixed point $\mathcal{P}_j$ has the eigenvalue $\lambda_k^{(j)} = \exp(2\pi i x_k^{(j)}/p)$, $k = 1, \dots, n$, then the number*

$$\sum_{j=1}^{n} AB(x_1^{(j)}, \dots, x_n^{(j)})$$

is an integer and coincides modulo p with the Todd genus $T(M^n)$ of the manifold M^n.

Proof. According to the Atiyah–Bott theorem for an elliptic complex d'', for forms of type $(0, 1)$ we have

$$\chi(f) = \sum_{j=1}^{q} \prod_{k=1}^{n} \frac{1}{1 - \exp\left\{ \frac{2\pi i}{p} x_k^{(j)} \right\}},$$

where

$$\chi(f) = \sum_{m=0}^{\infty} (-1)^m \operatorname{Tr} f^* | M^{0,m}(M^n).$$

As is known, $\chi(1) = T(M^n)$ and $(1/p)\sum_{m \in Z_p} \chi(f^m) = \phi$ is the alternating sum of the dimensions of the invariant subspaces under the action of the transformation f^* on the cohomologies $H^{0,m}(M^n)$. Consequently

$$T(M^n) = -\sum_{j=1}^{q} \sum_{m=1}^{p-1} \prod_{k=1}^{n} \frac{1}{1 - \exp\left\{\frac{2\pi i}{p} x_k^{(j)} \cdot m\right\}} + p\phi.$$

By now making use of the definition of the number-theoretic trace and the Atiyah–Bott function, the theorem is proven. □

We shall calculate $\operatorname{Tr}(\prod_{k=1}^n 1/(1 - \zeta^x k))$, where $\zeta = \exp(2\pi i/p)$. Let us set $\theta = 1 - \zeta$. We shall perform all calculations in the field $Q_p(\theta)$. By the symbol $\simeq$ we mean equality modulo the group $p\mathbf{Z}_p \subset Q_p(\theta)$. The following lemma, like Lemma 3.1, has been provided at our request by D. K. Faddeev.

Lemma 3. *For the Atiyah–Bott function $AB(x_1, \dots, x_n)$ we have the formulas*

$$AB(x_1, \dots, x_n) \simeq \left[\frac{p\langle u\rangle_{p-1}}{\langle u\rangle_p} \prod_{k=1}^{n} \frac{u}{\langle u\rangle_{x_k}}\right]_n,$$

$$AB(x_1, \dots, x_n) \simeq \sum_{m=0}^{n} \left[\frac{p\langle u\rangle}{\langle u\rangle_p} \prod_{k=1}^{n} \frac{u}{\langle u\rangle_{x_k}}\right]_m,$$

where $\langle u\rangle_q = 1 - (1-u)^q$ is the qth power of the element u in the formal group $f(u,v) = u + v - uv$ and $[\phi(u)]_k$ is the coefficient of u^k in the power series $\phi(u)$.

Proof. First of all note that $\operatorname{Tr}(\theta^k) \simeq 0$ for all $k > 1$. We have

$$\prod_{k=1}^{n} \frac{1}{1 - \zeta^{x_k}} = \prod_{k=1}^{n} \frac{1}{1 - (1-\theta)^{x_k}} = \frac{1}{\theta^n} \prod_{k=1}^{n} \frac{\theta}{1 - (1-\theta)^{x_k}} = \frac{1}{\theta^n} \sum_{k=0}^{\infty} A_k \theta^k,$$

where $A_k \in \mathbf{Z}_p$, and

$$\operatorname{Tr}\left(\frac{1}{\theta^n} \sum_{k=0}^{\infty} A_k \theta^k\right) = \operatorname{Tr}\left(\frac{1}{\theta^n} \sum_{k=0}^{n} A_k \theta^k\right) = \sum_{k=0}^{\infty} A_k \operatorname{Tr}(\theta^{k-n}).$$

Let us set $\operatorname{Tr} \theta^{-s} = B_s$ and introduce the two formal series

$$A(u) = \sum_{k=0}^{\infty} A_k u^k \quad \text{and} \quad B(u) = \sum_{k=0}^{\infty} B_k u^k.$$

Thus we must calculate the coefficient of u^k in the series $A(u)B(u)$. We have

$$B(u) = \operatorname{Tr}\left(1 + \sum_{s=1}^{\infty} \theta^{-s} u^s\right) = \operatorname{Tr}\left(\frac{1}{1 - \theta^{-1} u}\right) = \operatorname{Tr}\left(\frac{\theta}{\theta - u}\right)$$
$$= \operatorname{Tr}\left(1 + \frac{u}{\theta - u}\right) = (p-1) + u \operatorname{Tr}\left(\frac{1}{\theta - u}\right).$$

First note that if $\phi_\alpha(u)$ is the minimal polynomial of the element α with respect to extension $Q_p(\theta)|Q_p$, then

$$\operatorname{Tr}\frac{1}{\alpha-u}=-\frac{\phi'_\alpha(u)}{\phi_\alpha(u)}.$$

Since

$$\frac{\zeta^p-1}{\zeta-1}=\frac{(1-\theta)^p-1}{-\theta}=\frac{1-(1-\theta)^p}{\theta},$$

it follows that

$$\phi_\theta(u)=\frac{1-(1-u)^p}{u}.$$

We have

$$-\operatorname{Tr}\left(\frac{1}{\theta-u}\right)=\frac{\phi'_\theta(u)}{\phi_\theta(u)}=\frac{p(1-u)^{p-1}}{1-(1-u)^p}-\frac{1}{u}.$$

Thus

$$B(u)=(p-1)+u\left(\frac{p(1-u)^{p-1}}{1-(1-u)^p}-\frac{1}{u}\right)=\frac{p(1-(1-u)^{p-1})}{1-(1-u)^p},$$

$$\operatorname{Tr}\left(\prod_{k=1}^n\frac{1}{1-\zeta^{x_k}}\right)\simeq[A(u)\cdot B(u)]_n\simeq\left[\frac{p(1-(1-u)^{p-1})}{1-(1-u)^p}\prod_{k=1}^n\frac{u}{1-(1-u)^{x_k}}\right]_n,$$

and we obtain the first formula

$$\operatorname{Tr}\left(\prod_{k=1}^n\frac{1}{1-\zeta^{x_k}}\right)=\left[\frac{p\langle u\rangle_{p-1}}{\langle u\rangle_p}\prod_{k=1}^n\frac{u}{\langle u\rangle_{x_k}}\right]_n.$$

Next

$$\frac{p(1-(1-u)^{p-1})}{1-(1-u)^p}=p\frac{(1-u)-(1-u)^p}{(1-u)(1-(1-u)^p)}$$

$$=\frac{p}{1-u}-\frac{pu}{(1-u)(1-(1-u)^p)}\simeq-\frac{pu}{1-(1-u)^p}(1+u+u^2+\dots),$$

and we obtain the second formula

$$\operatorname{Tr}\left(\prod_{k=1}^n\frac{1}{1-\zeta^{x_k}}\right)\simeq\sum_{m=0}^n\left[\frac{pu}{1-(1-u)^p}\prod_{k=1}^n\frac{u}{1-(1-u)^{x_k}}\right]_m.$$

The lemma is therefore proven. □

In §3 the functions of the fixed points $\gamma_p(x_1,\dots,x_n)$ having values in the ring $\Omega_U\otimes Q$,

$$\gamma_p(x_1,\dots,x_n)=\left[\frac{u}{\Psi^p(u)}\prod_{k=1}^n\frac{u}{x_k\Psi^{x_k}(u)}\right]_n$$

were constructed. By considering the composition of the function γ_p with the Todd genus $T\colon\Omega_U\to Z$, we obtain a function (which we continue to denote by $\gamma_p(x_1,\dots,x_n)$) which associates to a set of weights the rational number mod $p\mathbf{Z}_p$

$$\gamma_p(x_1,\dots,x_n)=\left[\frac{pu}{1-(1-u)^p}\prod_{k=1}^n\frac{u}{1-(1-u)^{x_k}}\right]_n=\left[\frac{pu}{\langle u\rangle_p}\prod_{k=1}^n\frac{u}{\langle u\rangle_{x_k}}\right]_n$$

which is such that under the conditions of Theorem 2 the number

$$\sum_{j=1}^n\gamma_p(x_1^{(j)},\dots,x_n^{(j)})$$

is a p-adic integer and coincides modulo p with the Todd genus.

We now recall the Conner–Floyd equation introduced in [13]. If the group Z_p acts complexly on the manifold M^n with fixed points $\mathcal{P}_1, \dots, \mathcal{P}_q$, at which it has the set of weights $(x_1^{(j)}, \cdots x_n^{(j)})$, $j = 1, \dots, q$, then the Conner–Floyd equation

$$\sum_{j=1}^{q} u \prod_{k=1}^{n} \frac{u}{x_k^{(j)} \Psi^{x_k^{(j)}}(u)} = 0$$

is satisfied, where u is the formal variable which generates the ring $\Omega_U[[u]]$ under the relations $p\Psi^p(u) = 0$ and $u^n = 0$. Consequently there is an element $\phi \in \Omega_U[[u]]$ such that the equation

$$\sum_{j=1}^{q} \frac{u}{\Psi^p(u)} \left(\prod_{k=1}^{n} \frac{u}{x_k^{(j)} \Psi^{x_k^{(j)}}(u)} \right) = p\phi$$

is valid in the ring $\Omega_U[[u]] \otimes Q$. Thus, if $(x_1^{(j)}, \dots, x_n^{(j)})$ are the sets of weights of the action of the group Z_p on the manifold M^n, then they are related by the Conner-Floyd equation

$$\sum_{j=1}^{q} \frac{u}{\Psi^p(u)} \left[\prod_{k=1}^{n} \frac{u}{x_k^{(j)} \Psi^{x_k^{(j)}}(u)} \right]_m \simeq 0, \quad m = 0, \dots, n-1.$$

By considering the Todd genus $T\colon \Omega_U \to Z$, we obtain the Conner–Floyd equation

$$\sum_{j=1}^{q} \left[\left(\frac{pu}{1-(1-u)^p} \prod_{k=1}^{n} \frac{u}{1-(1-u)^{x_k^{(j)}}} \right) \right]_m \simeq 0, \quad m = 0, \dots, n-1,$$

corresponding to the Todd genus.

Definition 4. The Conner–Floyd functions $CF(x_1, \dots, x_n)_m$, $m = 0, \dots, n-1$, of fixed points are the functions which associate with each set of weights $(x_1, \dots, x_n)$ the rational numbers

$$CF(x_1, \dots, x_n)_m = \left[\frac{pu}{\langle u \rangle_p} \prod_{k=1}^{n} \frac{u}{\langle u \rangle_{x_k}} \right]_m, \quad m = 0, \dots, n-1.$$

Summarizing, we obtain the following theorem.

Theorem 5. *The Atiyah–Bott and Conner–Floyd functions of fixed points and the functions $\gamma_p(x_1, \dots, x_n)$ are related by the equation*

$$AB(x_1, \dots, x_n) - \gamma_p(x_1, \dots, x_n) \simeq \sum_{m=0}^{n-1} CF(x_1, \dots, x_n)_m.$$

We can now answer the question about the relation of the formulas for fixed points taken from the Atiyah–Bott theory and cobordism theory.

Let $f\colon M^n \to M^n$ be a holomorphic transverse mapping of period p of the compact complex manifold M^n, and let $\mathcal{P}_1, \dots, \mathcal{P}_q$ be its fixed points. Let the mapping $df|\mathcal{P}_j$ in the tangent space at the fixed point $\mathcal{P}_j$ have eigenvalues $\lambda_k^{(j)} = \exp(2\pi i x_k^{(j)}/p)$, $k = 1, \dots, n$. Then the formula which expresses the Todd genus in terms of the weights $(x_1^{(j)}, \dots, x_n^{(j)})$, taken from the Atiyah–Bott theorem, has the form

$$T(M^n) \simeq \sum_{j=1}^{q} \sum_{m=0}^{n} \left[\frac{pu}{1-(1-u)^p} \prod_{k=1}^{n} \frac{u}{1-(1-u)^{x_k^{(j)}}} \right]_m \tag{6}$$

(see Theorem 2 and Lemma 3). A similar formula, from cobordism theory, has the form

$$T(M^n) \simeq \sum_{j=1}^{q} \left[\frac{pu}{1-(1-u)^p} \prod_{k=1}^{n} \frac{u}{1-(1-u)^{x_k^{(j)}}} \right]_n, \tag{7}$$

and the difference between the first and second formula is exactly the sum in the Conner–Floyd equation, expressed for the Todd genus $T\colon \Omega_U \to Z$:

$$\sum_{m=0}^{n-1} \left(\sum_{j=1}^{q} \left[\frac{pu}{1-(1-u)^p} \prod_{k=1}^{n} \frac{u}{1-(1-u)^{x_k^{(j)}}} \right]_m \right) \simeq 0,$$

(see §IV of [13]).

In conclusion the authors wish to point out that out of the fundamental results of this paper the two different proofs of the theorem concerning the relation of the cohomology operations to the Hirzebruch series were obtained independently (and in the text of §1 both proofs are presented).

The basic concepts, the general assertions about formal power systems and the principal examples given of them, particularly the "square modulus" systems of type 2, to a large measure are due to Novikov, while the investigation of the logarithm;

of these systems by means of the Chern–Dold character, the precise definition and investigation of the ring of coefficients of the "two-valued formal groups" and their connection with Sp-cobordism are for the most part due to Buhštaber.

The remaining results were obtained in collaboration, while the important lemma of §3 and also Lemma 3 of the Appendix, were proved at our request by D. K. Faddeev, to whom the authors express their deep gratitude, We also thank Ju. I. Manin and I. R. Šafarevič for discussions and valuable advice concerning the theory of formal groups and algebraic number theory.

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