Letters in Mathematical Physics **66:** 65–72, 2003. 65

Formality of Chain Operad of Little Discs

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(Received: 6 December 2002)

Abstract. We prove that the chain operad of little disks is formal in characteristic zero, and discuss briefly the relation with Kontsevich formality in deformation quantization.

Mathematics Subject Classifications (2000). 08C99, 18G55, 55U10.

Key words. braid group, category, formality, little disks operad, nerve, quasi-isomorphism.

1. Introduction

This Letter is a printed version (with minor changes) of [10]. We prove that the $\mathbb{Q}$-chain operad of little disks is formal. This means that it is quasi-isomorphic to its homology operad which is known to be the operad of Gerstenhaber algebras. Maxim Kontsevich [6] has found a proof of a more general result: chain operad of little n-dimensional disks is formal for all n as an operad of coalgebras.

The relation of this formality to Kontsevich's formality theorem is explained in [6]. The key point is that by Deligne's conjecture ([7, 8]) the chain operad of little disks acts on the Hochschild cochain complex of any associative algebra. Therefore, we have a homotopy action of the operad of Gerstenhaber algebras on Hochschild cochain complex of any associative algebra over a field of characteristic zero.

It is explained in [4–6], and [9] how this action implies Kontsevich's formality theorem. To avoid misleading the reader, let us mention that it is only in [6] that the above mentioned homotopy action of the operad of Gerstenhaber algebras is constructed by means of Deligne's conjecture; the other papers use an approach based on the Etingof–Kazhdan dequantization theorem. In the original paper of the author [9], the Etingof–Kazhdan theorem was used in the opposite, i.e. quantization, direction, which resulted in a more complicated proof. The papers [4, 5] use Etingof–Kazhdan theory in a more clever way, thus making the argument more simple and natural.

Being different in establishing the homotopy Gerstenhaber algebra action, the above-mentioned papers use it in the same way to obtain a proof of Kontevich's theorem. Let me briefly outline how it goes. To be specific, consider the case when our associative algebra A is the algebra of polynomials in n variables (the case

★Supported in part by an NSF grant.

$A = C^{\infty}(\mathbb{R}^n)$ is fairly similar). Instead of showing that the differential graded Lie algebra on $C^*(A, A)$, the Hochschild cochains of A, is formal, one proves the formality of the homotopy Gerstenhaber algebra structure on $C^*(A, A)$, which means that there exists a quasi-isomorphism between our homotopy Gerstenhaber algebra structure on $C^*(A, A)$ and the induced Gerstenhaber algebra structure on $HH^*(A, A)$, the Hochschild cohomology of A, the latter being the so-called Nijenjuis–Schouten algebra of polyvector fields. This formality follows by the standard argument from the obstruction theory, which implies even a more general result: let V be a homotopy Gerstenhaber algebra such that the induced Gerstenhaber algebra structure on the cohomology of V is isomorphic to the Nijenhuis–Schouten algebra, then this structure on V is formal.

How does this formality imply Kontsevich's formality? First, whenever we have a (homotopy) Gerstenhaber algebra structure on a complex V, we also have, as its part, a differential graded (homotopy) Lie algebra structure on the shifted complex $V[1]$. A quasi-isomorphism of homotopy Gerstenhaber algebras induces a quasi-isomorphism of these underlying homotopy Lie algebras. Thus, the homotopy Lie algebra structure on $C^*(A, A)[1]$ induced by the homotopy Gerstenhaber structure is quasi-isomorphic to the differential graded Lie algebra structure on $HH^*(A, A)[1]$ determined by the Schouten bracket.

To complete the proof of Kontsevich's formality theorem, it remains to establish a quasi-isomorphism of the homotopy Lie algebra structure on $C^*(A, A)[1]$ induced by the homotopy Gerstenhaber structure on $C^*(A, A)$ with the Lie structure on $C^*(A, A)[1]$ given by the Gerstenhaber bracket (this was not done in the first version of [9]; the author thanks Maxim Kontsevich for pointing at this flaw). It turns out that for each known construction of homotopy Gerstenhaber algebra structure on $C^*(A, A)$, this quasi-isomorphism can be established more or less straightforwardly. This concludes the proof.

Let us now come back to the content of this Letter and present the plan of our proof of formality of the chain operad of little discs.

Step 1. We use a homotopy invariant definition of a little disks operad as in [2]. This definition specifies a class of homotopy equivalent operads in such a way that a conventional little disks operad is in this class as well as any homotopy equivalent operad. The definition of this class can therefore be viewed as a criterion for an operad to be homotopy equivalent to the conventional operad of little disks. This criterion is called *Fiedorowicz's recognition principle*.

Next we pick a representative from the class of little disks operads (which is not the conventional little disks operad). This representative is obtained as a topological realization of a certain simplicial operad which in turn is an operad of nerves of a certain operad of groupoids *PaB*. The latter operad was defined in [1] where it was used to define the Grothendieck–Teichmüller group and Drinfeld's associator.

Step 2. We take a k-linear span of *PaB* and complete it with respect to the augmentation ideal. Drinfeld's associator induces a morphism from this operad to a much simpler one (as in (9)). This induces a quasi-isomorphism of the corresponding operads of simplicial chains. We then explicitly establish the formality of this new operad, which completes the proof. This is done in Section 4 but in a reversed order: we start with the definition of the new operad (which is the target for the map determined by the Drinfeld associator); then we show that the operad of chains of this operad is formal; and finally we construct a map described in the beginning of the paragraph.

2. Little Disks Operad (After[2])

We reproduce the construction of the little disks operad from [2]. First, the symmetric groups in the definition of operad are replaced with the braid groups and we obtain the notion of braided operad. A *topological B_∞-operad X* is defined as a braided operad such that all its spaces $X(n)$ are contractible and the braid group B_n acts freely on $X(n)$. If X and Y are topological B_∞-operads, then so is $X \times Y$ and we have homotopy equivalences

$$p_1\colon X \times Y \to X; \qquad p_2\colon X \times Y \to Y, \tag{1}$$

where p_1, p_2 are the projections.

Let PB_n be the group of pure braids with n strands. Given a topological B_∞-operad X, the corresponding *operad of little disks* is a symmetric operad X' such that $X'(n) = X(n)/PB_n$ with the induced structure maps. The maps (1) guarantee that any two operads of little disks are connected by a chain of homotopy equivalences. It is proved in [2] that the classical operad of May (whose nth space is the configuration space of n disjoint numbered disks inside the unit disk) is an operad of little disks in our sense.

The functor of singular chains $C_*^{\mathrm{sing}}\colon \mathrm{Top} \to \mathrm{Complexes}$ has a natural tensor structure given by the Eilenberg–Zilber map $EZ\colon C_*^{\mathrm{sing}}(X) \otimes C_*^{\mathrm{sing}}(Y) \to C_*^{\mathrm{sing}}(X \times Y)$. Therefore, for a topological operad O, the collection $C_*^{\mathrm{sing}}(O(*))$ has a structure of a dg-operad. The structure map of the ith insertion is

$$C_*^{\mathrm{sing}}(O(n)) \otimes C_*^{\mathrm{sing}}(O(m)) \xrightarrow{EZ} C_*^{\mathrm{sing}}(O(n) \times O(m)) \xrightarrow{o_{i*}^{O}} C_*^{\mathrm{sing}}(O(n+m-1)),$$

where o_i^O is the structure map of the ith insertion in O.

For a small square operad X consider the operad $E_2(X) = C_*^{\mathrm{sing}}(X)$. Any two such operads are quasi-isomorphic, where quasi-isomorphic means connected by a chain of quasi-isomorphisms. In particular, the homology operad of any of $E_2(X)$ is the operad e_2 controlling Gerstenhaber algebras (see Section 4 for the definition of e_2). Our goal is to show that.

THEOREM 2.1. *Any operad $E_2(X)$ is quasi-isomorphic to its homology operad e_2.*

It suffices to prove this theorem only for one operad X of little disks.

3. Realization of E_2

3.1. OPERAD OF CATEGORIES PaB_n (AFTER [1])

First, let us reproduce from [1] the definition of the category PaB_n. Let B_n (PB_n) be the group of braids (pure braids) with n strands, let S_n be the symmetric group. Let p: $B_n \to S_n$ be the canonical projection with the kernel PB_n. We assume that the strands of any braid are numbered in the order determined by their origins.

The objects of the category PB_n are parenthesized permutations of elements $1, 2, \ldots, n$ (that is pairs (σ, p), where $\sigma \in S_n$ and π is a parenthesizing of the non-associative product of n elements). The morphisms between (σ_1, π_1) and (σ_2, π_2) are such braids from B_n that any strand joins an element of σ_1 with the same element of σ_2, in other words, $\mathrm{Mor}((\sigma_1, \pi_1), (\sigma_2, \pi_2)) = p^{-1}(\sigma_2^{-1}\sigma_1)$. The composition law is induced from the one on B_n. The symmetric group S_n acts on PaB_n via renumbering the objects $T_\sigma(\sigma_1, \pi_1) = (\sigma\sigma_1, \pi_1)$ and it acts identically on morphisms.

The collection of categories PaB_n form an operad. Indeed, the collection $\mathrm{Ob}PaB_*$ forms a free operad in the category of sets generated by one binary noncommutative operation. Let us describe the structure map o_k of the insertion into the kth position on the level of morphisms. Suppose we insert y: $(\sigma_1, \pi_1) \to (\sigma_2, \pi_2)$ into x: $(\sigma_3, \pi_3) \to (\sigma_4, \pi_4)$. We replace the strand number $\sigma_3^{-1}(k)$ of the braid x by the braid y made very narrow.

3.2. OPERAD OF CLASSIFYING SPACES

We have the functor of taking the nerve N: **Cat** $\to \Delta^o$**Sets** and the functor of topological realization $|\ |$: Δ^o**Sets** $\to$ **Cellular spaces**. These functors behave well with respect to the symmetric monoidal structures, therefore the collection of cellular complexes $X_n = |NPaB_n|$ forms a cellular operad. One checks that this operad is a little disks operad. Indeed, let PaB'_n be the category whose objects are pairs (x, y), where x belongs to the braid group B_n and y is a parenthesizing of the non-associative product of n elements, and there is a unique morphism between any two objects. We have a free left action of B_n on PaB'_n: $(x, y) \to (gx, y))$ and a braided operad structure on PaB'_* (the structure maps are defined similarly to PaB_n). One checks that the corresponding operad of classifying spaces is a topological B_∞-operad and that the corresponding little disks operad is isomorphic to X_*.

Consider the corresponding chain operad. Let $C_*(NPaB_n)$ be the chain complex over $\mathbb{Q}$ of $NPaB_n$ as a simplicial set. The collection $C_*(NPaB_n)$ forms a dg-operad (via the Eilenberg–Zilber map). Since $C_*(NPaB_n)$ is just a bar complex of the category PaB_n, this operad will be denoted by $C_*(PaB_*)$. We have a canonical quasi-isomorphism of operads $C_*(PaB_*) \to C_*^{\mathrm{sing}}|NPaB_*|$. Therefore, it suffices to construct a quasi-isomorphism of $C_*(PaB_*)$ and e_2.

4. Operad of Algebras A_n^{pb} and Construction of Quasi-isomorphism

By definition [3], A_n^{pb} is the algebra over $\mathbb{Q}$ of power series in the noncommutative variables

$$t_{ij}, 1 \leqslant i, \qquad j \leqslant n; \qquad i \neq j; \qquad t_{ij} = t_{ji} \tag{2}$$

with relations

$$[t_{ij} + t_{ik}, t_{jk}] = 0. \tag{3}$$

Let I_n be the double-sided ideal generated by all t_{ij}. We have a canonical projection

$$\chi\colon A_n^{pb} \to A_n^{pb}/I_n \cong \mathbb{Q}. \tag{4}$$

The symmetric group S_n acts naturally on A_n^{pb} so that $T_\sigma t_{ij} = t_{\sigma(i)\sigma(j)}$. The collection A_n^{pb} forms an operad in the category of algebras in a well-known way. The map of the insertion into the ith position $o_i\colon A_n^{pb} \otimes A_m^{pb} \to A_{n+m-1}^{pb}$ looks as follows.

Let

$$\phi(k) = \begin{cases} k, & k \leqslant i; \\ k+m-1, & k > i. \end{cases}$$

Then

$$o_i(t_{pq} \otimes 1) = \begin{cases} t_{\phi(p)\phi(q)}, & p, q \neq i; \\ \sum_{r=i}^{i+m-1} t_{r\phi(q)}, & p = i; \end{cases}$$

$$o_i(1 \otimes t_{pq}) = t_{i+p-1, i+q-1}.$$

Any algebra with unit over $\mathbb{Q}$ gives rise to a $\mathbb{Q}$-additive category C_A with one object. Denote by $\mathbb{Q}$Cat the category of small $\mathbb{Q}$-additive categories, and by $\mathbb{Q}\mathrm{Cat}' = \mathbb{Q}\mathrm{Cat}/C_{\mathbb{Q}}$ the over-category of $\mathbb{Q}$Cat over $C_{\mathbb{Q}}$. Its objects are the elements of $\mathrm{Mor}_{\mathbb{Q}\mathrm{Cat}}(x, C_{\mathbb{Q}})$, where $x \in \mathbb{Q}\mathrm{Cat}$. A morphism between ϕ and ψ, where $\phi\colon x \to C_{\mathbb{Q}}$; $\psi\colon y \to C_{\mathbb{Q}}$, is a morphism $\sigma\colon x \to y$ in $\mathbb{Q}$Cat such that $\sigma\psi = \phi$. This category has a clear symmetric monoidal structure. We have the functor of nerve $N^{\mathbb{Q}}\colon \mathbb{Q}\mathrm{Cat}' \to \Delta^o\mathrm{Vect}$, which is the straight analogue of the nerve of an arbitrary category, and the functor $C_*\colon \Delta^o\mathrm{Vect} \to$ Complexes. Both of these functors have tensor structure (on the latter functor it is defined via the Eilenberg–Zilber map), therefore we have a through functor $\mathbb{Q}\mathrm{Cat}' \to$ Complexes and the induced functor

$$\mathbb{Q}\mathrm{Cat}'\text{-Operads} \to \mathrm{d}g\text{-Operads},$$

which will be denoted by $C_*^{\mathbb{Q}}$.

The map (4) produces a morphism $\chi_*\colon C_{A_n^{pb}} \to C_{\mathbb{Q}}$ and defines an object $O_A(n) \in \mathbb{Q}\mathrm{Cat}'$. The operad structure on A_n^{pb} defines an operad structure on the collection $O_A(n)$. The complex $C_*^{\mathbb{Q}}(O_A(n))$ looks as follows: $C_n^{\mathbb{Q}}(O_A(k)) \cong A_k^{pb\otimes n}$;

$$\begin{aligned} \mathrm{d}a_1 \otimes \cdots \otimes a_n = \chi(a_1)a_2 \otimes \cdots \otimes a_n - a_1a_2 \otimes \cdots \otimes a_n + \cdots \\ + (-1)^{n-1} a_1 \otimes \cdots \otimes a_{n-1}\chi(a_n). \end{aligned}$$

This is the bar complex for $\mathrm{Tor}^{A_n^{pb}}(A_n^{pb}/I_n, A_n^{pb}/I_n)$.

Let e_2 be the operad of graded vector spaces governing the Gerstenhaber algebras. It is generated by two binary operations: the commutative associative multiplication of degree zero, which is denoted by $\cdot$, and the commutative bracket of degree -1 denoted by $\{,\}$.

These operations satisfy the Leibniz identity

$$\{ab, c\} = a\{b, c\} + (-1)^{b(c+1)}\{a, c\}b$$

and the Jacoby identity

$$(-1)^{|a|}\{a, \{b, c\}\} + (-1)^{|a||b|+|b|}\{b, \{a, c\}\} + (-1)^{|a||c|+|b||c|+|c|}\{c, \{a, b\}\} = 0.$$

We have a morphism of operads

$$k\colon e_2 \to C_*^{\mathbb{Q}}(O_A), \tag{5}$$

which is defined on $e_2(2)$ as follows:

$$k(\cdot) = 1 \in C_0^{\mathbb{Q}}(O_A); \qquad k(\{,\}) = t_{12} \in C_1^{\mathbb{Q}}(O_A).$$

Direct check shows that this map respects the relations in e_2.

PROPOSITION 4.1. *The map k is a quasi-isomorphism of operads.*

Proof. Let $\mathfrak{g}_n$ be the graded Lie algebra generated by the elements (2) and relations (3), and the grading is defined by setting $|t_{ij}| = 1$. Then the universal enveloping algebra $U\mathfrak{g}_n$ is a graded associative algebra, and A_n^{pb} is the completion of $U\mathfrak{g}_n$ with respect to the grading. The algebras $U\mathfrak{g}_*$ form an operad with the same structure maps as in A_*^{pb}. The inclusion

$$U\mathfrak{g}_n \to A_n^{pb} \tag{6}$$

is a morphism of operads. We have a canonical projection $\chi\colon U\mathfrak{g}_n \to \mathbb{Q}$, therefore the collection $C_{U\mathfrak{g}_*}$ forms an operad in $\mathbb{Q}\mathrm{Cat}'$ and we have a dg-operad $C_*^{\mathbb{Q}} C_{U\mathfrak{g}_*}$ which will be denoted by $C_*^{\mathbb{Q}} U\mathfrak{g}_*$. The injection (6) induces a morphism of operads

$$i_*\colon C_*^{\mathbb{Q}}(U\mathfrak{g}_*) \to C_*^{\mathbb{Q}}(O_A). \tag{7}$$

It is clear that $\mathrm{Tor}_*^{A_n^{pb}}(A_n^{pb}/I_n, A_n^{pb}/I_n)$ is the same as the completion of

$$H_*(\mathfrak{g}_n) \cong \mathrm{Tor}_*^{U\mathfrak{g}_n}(\mathbb{Q}, \mathbb{Q}) \cong H_*(C_*^{\mathbb{Q}}(U\mathfrak{g}_n))$$

with respect to the grading induced from $\mathfrak{g}_n$.

Let us study $H_*(\mathfrak{g}_n)$. We have a natural injection $\mathfrak{g}_{n-1} \to \mathfrak{g}_n$. One sees that the Lie sub-algebra $\imath_n \subset \mathfrak{g}_n$ generated by t_{nk}, $k = 1, \ldots, n-1$ is free and is an ideal in $\mathfrak{g}_n$. Also, we have $\mathfrak{g}_n = \imath_n \oplus \mathfrak{g}_{n-1}$ in the category of vector spaces. The Serre–Hochschild spectral sequence $E_{*,*}^2 = H_*(\mathfrak{g}_{n-1}, H_*\imath_n) \Rightarrow H_*(\mathfrak{g}_n)$ collapses at E^2 and shows that

$$H_*(\mathfrak{g}_n) \cong H_*(\mathfrak{g}_{n-1}) \oplus \left(\bigoplus_{k=1}^{n-1} H_*(\mathfrak{g}_{n-1})\right)[-1], \tag{8}$$

where the first summand is the image of $H_*(\mathfrak{g}_{n-1})$ under the injection $\mathfrak{g}_{n-1} \to \mathfrak{g}_n$. This implies by induction that the homology of $\mathfrak{g}_n$ is finite-dimensional, therefore the map (7) is a quasi-isomorphism (because the homology of the right-hand side is the completion of the homology of the left-hand side).

We are now going to make the isomorphism (8) more specific. First, note that $\mathfrak{g}_2$ is one-dimensional, therefore $H_0(\mathfrak{g}_2) = \mathbb{Q}$; $H_1(\mathfrak{g}_2) = \mathbb{Q}[-1]$; $H_i(\mathfrak{g}_2) = 0$ for $i > 1$. The operadic maps of insertion into the kth position o_k: $U\mathfrak{g}_{n-1} \otimes U\mathfrak{g}_2 \to U\mathfrak{g}_n$, where $k = 1, 2, \ldots, n-1$ induce maps $o_k^*: H_*(\mathfrak{g}_{n-1}) \otimes H_*(\mathfrak{g}_2) \to H_*(\mathfrak{g}_n)$, and the $(k+1)$-th summand in (8) is equal to $o_k^*(H_*(\mathfrak{g}_{n-1}) \otimes H_1(\mathfrak{g}_2))$. The induction argument shows that

(1) The homology operad $n \mapsto H_*(\mathfrak{g}_n) \cong H_*(C_*^{\mathbb{Q}}(U\mathfrak{g}_n))$ is generated by $H_*(\mathfrak{g}_2)$, therefore the homology operad of $C_*^{\mathbb{Q}}(O_A(n))$ is generated by the homology of $C_*^{\mathbb{Q}}(O_A(2))$.
(2) The total dimension of $H_*(\mathfrak{g}_n)$ and of the homology of $C_*^{\mathbb{Q}}(O_A(n))$ is $n!$.

The first statement implies that the map (5) is surjective on the homology level, and the second statement means that the map (5) is bijective since $\dim e_2(n) = n!$.

Let $\mathbb{Q}(PaB_n)$ be the $\mathbb{Q}$-additive category generated by PaB_n. We have a map $\mathbb{Q}(PaB_n) \to C_{\mathbb{Q}}$ sending all morphisms from PaB_n to Id. Thus, $\mathbb{Q}(PaB_n) \in \mathbb{Q}\mathrm{Cat}'$. The operadic structure on PaB_* induces the one on $\mathbb{Q}PaB_*$.

Any associator $\Phi \in A_3^{pb}$ over $\mathbb{Q}$ produces a map of operads

$$\phi\colon \mathbb{Q}(PaB_*) \to O_A(*). \tag{9}$$

Indeed, define ϕ on Ob PaB_n by sending any object to the only object of $O_A(n)$. There are only two objects in PaB_2, let us denote them x_1x_2 and x_2x_1. The morphisms between these two objects correspond to the non-pure braids. Let $x \in B_2$ be the generator. We define $\phi(x) = e^{t_{12}/2}$. Take the two objects $(x_1x_2)x_3$ and $x_1(x_2x_3)$ of PaB_3 corresponding to the identical permutation $e \in S_3$, and the morphism i between them, corresponding to the identical braid $e_b \in B_3$. Define $\phi(i) = \Phi$. Since the operad PaB_* is generated by x and i, these conditions define ϕ uniquely. The definition of the associator is equivalent to the fact that ϕ is well-defined. This construction is very similar to the one from [1].

The map ϕ produces a map of operads $C_*^{\mathbb{Q}}PaB_* \cong C_*^{\mathbb{Q}}(\mathbb{Q}(PaB_*)) \to C_*(O_A)$. It is well known that the homology operad of C_*PaB_* is e_2. It is easy to check that ϕ is a quasi-isomorphism for $* = 2$ and hence it is a quasi-isomorphism of operads (since e_2 is generated by $e_2(2)$). By Proposition 4.1, k is a quasi-isomorphism. Thus, the chain operad C_*PaB_* is quasi-isomorphic to e_2. □

Acknowledgements

I would like to thank Boris Tsygan, Paul Bressler, and Maxim Kontsevich for their help. I am especially grateful to Daniel Sternheimer, the initiator of this special issue.

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