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Formation of Gravitationally Bound Primordial Gas Clouds

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The formation of gravitationally bound systems from primordial gas is studied by means of Tolman's solution for dust-like matter. The critical values of density contrast and its growth rate at an initial epoch are derived, which are necessary for an inhomogeneity to condense into a bound system before the appearance of the oldest stars. The matter distributions in an isolated inhomogeneity and an inhomogeneity included in a larger one (which will be a cluster of gas clouds) are followed with time by assuming simple models for inhomogeneities, and it is shown how the bound region spreads outwards. Moreover, the minimum mass of fragments into which the gas clouds may break up is examined.

§ 1. Introduction

The formation of galaxies which are dispersed everywhere at present in the universe has been studied by many authors. Some of them have assumed that small density fluctuations in an early stage of cosmic evolution grow owing to gravitational instability, until they condense into protogalaxies. However, on Lifshitz's assumption[1] that the early stage was very quiet and the fluctuations were statistical, gravitational instability was not effective. This is because the density contrasts of the statistical fluctuations corresponding to galaxies are too small ($\sim 10^{-34}$) to grow to gravitationally bound systems within the cosmic age. The linearized theories were sufficient for the description of such small fluctuations. After Lifshitz, the behavior of the fluctuations has been studied successively in the linear approximation,[2] while Lifshitz's assumption was not always used.

On the other hand, it has been assumed that an early stage of the universe was very turbulent and density fluctuations were not small.[3] At the stage when radiation density was larger than matter density and matter was wholly ionized, matter was strongly kicked by photon particles and hindered from condensing. But, once matter was neutrarized, density fluctuations condensed promptly into bound systems.

However, we do not know at present how quiet or turbulent the early stages were. Accordingly, still we could assume the existence of fluctuations with various amplitudes. In the process of their condensation, gravitation would have played the main role, and, as the density contrasts increased to the order of

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magnitude of unity, the process would have been accelerated by the non-linear effect of gravitation. The non-linear analysis has been performed at first by Bonnor[4] in a simplified model. After that, it has been studied by the author[5] according to perturbation method to the second-order terms. Recently, Kihara[6] and Kondo[7] have investigated the process of condensation (especially in the center) on the basis of fluid dynamic equations in an expanding universe, which have been derived approximately by Nariai and Ueno,[8] and Irvine.[9]

In this paper, we shall study the problem on the growth of a spherically symmetric inhomogeneity from a small, but non-statistical fluctuation to a gravitationally bound system, to be formed before the birth of a galaxy. Our method is due to Tolman's solution[10] for dust-like matter. Accordingly the analysis will be confined to such a stage, that the effect of radiation can be approximately neglected, but that the initial epoch $t=t_i$ of this stage is close to the epoch of the decoupling between matter and radiation. Gas pressure also will be neglected, as long as we consider the inhomogeneity with a size larger than Jeans' wave-length.

In § 2 we review Tolman's solution and the basic information about the background universe. In § 3 we express the spatial curvature difference, time delay, and total energy difference in and around the inhomogeneity from the background, in terms of the initial density contrast and growth rate, and derive the condition of gravitational binding. In § 4 we analyze the behaviors of the inhomogeneity in the central region and examine the critical initial density contrast and growth rate necessary for galaxy formation. In § 5 we assume the simple forms of density contrast, distinguishing whether the inhomogeneity should condense to an isolated gas cloud or a gas cloud in a cluster, and follow with time the change of density distributions in the inhomogeneities. In § 6 we derive the minimum mass of fragments into which the gas cloud may be broken up and examine whether it can evolve to the galaxy. Section 7 is devoted to concluding remarks. In the Appendices A, B and C, the spatial curvature and time delay, the relative density ratio and boundary conditions on the initial density contrast and growth rate, respectively, will be derived.

§ 2. Dynamics of a spherically symmetric dust-like matter

The spherically symmetric adiabatic dust-like matter is described by a line element

$$ds^2=c^2dt^2-\frac{S^2(1+rS'/S)^2}{1-k\alpha(r)r^2}dr^2-S^2r^2(d\theta^2+\sin^2\theta d\varphi^2), \tag{2·1}$$

where r, θ, φ are comoving spatial coordinates, $S=S(t,r)$, and $S'=\partial S/\partial r$. The scalar curvature of three-dimensional space ($t=$const) is expressed as $k\alpha(r)$, in which $k=1, 0, -1$ represent signs of the curvature. Here $\alpha(r)$ (≥ 0) must satisfy an inequality $1-k\alpha(r)r^2>0$.

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From Einstein's field equations,*) we have equations for S and mass density ρ:

$$(\dot{S})^2 - \beta c^2/S = -kc^2\alpha(r), \tag{2·2}$$

$$\kappa\rho = 3\beta S^{-3}(1+rS'/S)^{-1}, \tag{2·3}$$

where $\dot{S}$ denotes $\partial S/\partial t$, $\kappa \equiv 8\pi G/c^2$ (G is Newton's gravitational constant), and β is an integration constant. By the linear transformation of radial coordinate $r \to \tilde{r} = ar$ ($a=$const), β can be given an arbitrary value. Integrating Eq. (2·2), we obtain Tolman's solution:[10),11)]

$$S = \beta^{1/3}(3c/2)^{2/3}(t-T(r))^{2/3}, \qquad (k=0)$$

$$c(t-T(r)) = \frac{\beta}{2k\alpha^{3/2}}(\eta - \sigma(\eta)), \quad S = \frac{\beta}{2k\alpha}(1-\sigma'(\eta)), \quad (k=\pm 1) \tag{2·4}$$

where $\sigma(\eta) = \sin\eta$, $\mathrm{sh}\,\eta$ for $k=1, -1$, respectively, and $\sigma'(\eta) = d\sigma(\eta)/d\eta$.

In the background universe which is homogeneous and isotropic, $S(=\bar{S}(t))$ is given without loss of generality by putting $T(r)=0$ and $\alpha(r)=\mathrm{const}(=\bar{\alpha})$ in Eqs. (2·2), (2·3) and (2·4). In the following, bars on letters will be used to represent background quantities. Hubble's expansion constant and deceleration parameter are defined as usual by $H_0 = (\dot{\bar{S}}/\bar{S})_0$ and $q_0 = -(\ddot{\bar{S}}\bar{S}/\dot{\bar{S}}^2)_0$. Here the suffix 0 stands for the present epoch. In order to fix the above-mentioned constant β, we put $\beta = cH_0^{-1}$. Then, from Eq. (2·2) we obtain

$$\begin{cases} \bar{\alpha} = \bar{k}(2q_0-1)(2q_0)^{-2/3}, \\ \bar{S}_0 = cH_0^{-1}(2q_0)^{-1/3}, \\ \rho_0 = 3H_0^2 q_0/(4\pi G) = 2.06\times 10^{-29} q_0 \quad (\mathrm{g/cm^3}). \end{cases} \tag{2·5}$$

Here we adopt $H_0 = 75$ km/sec/Mpc.[12)] The numerical values of $\bar{\alpha}$, $\bar{S}_0$ are shown in Table I for the values of q_0 consistent with the current astronomical observations.[13)]

Table I. Some values of background quantities.

$\bar{k}$	q_0	$\bar{\alpha}$	$\beta/\bar{S}_0$	$\rho_0\times10^{-29}$	H_0t_0	$(\rho/\rho_r)_i$	$H_0t_i\times10^{-5}$
1	1	0.63	1.26	2.06	0.5708	46	1.805
0	0.5	0	1	1.03	2/3	23	2.108
−1	0.25	0.79	0.79	0.515	0.7536	12	2.383
	0.05	4.18	0.46	0.103	0.8981	2.3	6.667

§ 3. Description of a local inhomogeneity

Now we consider a local inhomogeneity which was born as a small fluctuation from the background at the early stage. Our analysis must be limited to

*) In this paper, the cosmological constant Λ is taken to be zero, only for simplicity.

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the stage after an initial epoch t_i when the scale factor $\bar{S}_i$ is one thousands the present one $\bar{S}_0$. This comes from the applicability of Tolman's solution. Here the epoch t_i has the following properties. (1) At this epoch t_i, the temperature $(T_r)_i$ of cosmic radiation is 2700°K, if its present temperature is assumed to be 2.7°K,[14] and is close to the temperature 4000°K at which matter and radiation decouple. (2) The ratio of the matter density ρ to the radiation density ρ_r at the epoch is larger than unity, as shown in Table I, and the effect of radiation can be approximately neglected after the epoch. Also the gas pressure smaller than the radiation pressure can be neglected as long as the size of the inhomogeneity is large compared to Jeans' wavelength.

For the matter distribution in the inhomogeneity, we assume that the density contrast and its growth rate were small enough at the initial epoch, so that their products were negligible. These quantities are written as

$$\epsilon_1(r) \equiv (\rho/\bar{\rho})_i - 1 = (\delta\rho/\rho)_i \tag{3·1}$$

and

$$\epsilon_2(r) \equiv 1 - (\ln \rho)^{\cdot}_i / (\ln \bar{\rho})^{\cdot}_i \,. \tag{3·2}$$

While $|\epsilon_1(r)|$ and $|\epsilon_2(r)|$ are small compared to unity, the difference of the curvature $\alpha(r)$ in the inhomogeneity to the background curvature $\bar{\alpha}$ is not necessarily small compared to $\bar{\alpha}$. As will be derived in Appendix A, the difference $k\alpha - \bar{k}\bar{\alpha}$ and the time delay $T(r)$ can be expressed as some integrations of $\epsilon_1(r)$ and $\epsilon_2(r)$:

$$k\alpha(r) - \bar{k}\bar{\alpha} = (3\beta/\bar{S}_i)\,(I_1 + 2I_2)\,, \tag{3·3}$$

$$T(r)/t_i = (3/5)\,(I_1 + 3I_2)\,, \tag{3·4}$$

where

$$\begin{cases} I_1 = r^{-3} \displaystyle\int_0^r \epsilon_1(r)\, r^2 dr\,, \\ I_2 = r^{-3} \displaystyle\int_0^r \epsilon_2(r)\, r^2 dr\,. \end{cases} \tag{3·5}$$

These formulas (3·3) and (3·4) hold for $|\epsilon_1|$ and $|\epsilon_2| \ll 1$ (if $\bar{k}=0$) and for $1 \gg |\epsilon_1| + |\epsilon_2| \gg (\bar{\alpha}\bar{S}_i/\beta)^2$ (if $\bar{k} = \pm 1$), which will be satisfied through our analysis.

When we give $\epsilon_1(r)$ and $\epsilon_2(r)$ at the epoch t_i, we can obtain the density distribution at any epoch after t_i, using Eqs. (2·3), (2·4), (3·3) and (3·4). The formulas for $\rho/\bar{\rho}$ will be summarized in Appendix B. In order to get its characteristic value, we shall discuss the condition for the gravitational binding of a spherical surface (r = const). By use of Eqs. (2·3) and (B·2), ρ can be expressed as a function of only r and S. When the surface is bound, i.e. the density in the surface reaches the minimum, we have, therefore, $\dot{S}=0$ or S/β $(=R) = 1/(k\alpha)$ from Eq. (2·2). This condition is satisfied only in the case $k=1$ and η in Eq. (2·4) must be π at that time. The ratio of the density in

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the surface to the background density at the critical epoch is expressed simply in the special case $\bar{k}=0$:

$$(\rho/\bar{\rho})_b = \frac{9\pi^2}{16} / \{4-(\epsilon_1+2\epsilon_2)/(I_1+2I_2)\}. \tag{3·6}$$

It is interesting also to express with ϵ_1 and ϵ_2 the total energy difference in or around the inhomogeneity, since it may be closely related to the type of possible fluctuations. Before we define the difference, we must give the number of constituent particles and the total mass. The number of particles from the center to the surface r is expressed at any time by

$$N(r) = \int_0^r (\rho/m) S^3 (1+rS'/S)(1-k\alpha r^2)^{-1/2} 4\pi r^2 dr$$

$$= (12\pi\beta/\kappa m) \int_0^r (1-k\alpha r^2)^{-1/2} r^2 dr\,,$$

where m is the mass of a particle. The second line is derived by use of Eq. (2·3). The total mass is expressed as

$$M(r) = \int_0^r \rho S^3 (1+rS'/S) 4\pi r^2 dr = (4\pi\beta/\kappa) r^3. \tag{3·7}$$

In fact, if we imagine an virtual empty region $r_1 \leq r \leq r_2$, $M(r_1)$ would be equal to the Schwarzschild mass which should be interpreted as the total mass in the region $r \leq r_1$.[15] The number $N(r)$ and total mass $M(r)$ are independent of t.

On the other hand, the particle number and total mass in the background are similarly given by

$$\begin{aligned} \bar{N}(\bar{r}) &= (12\pi\beta/\kappa m) \int_0^{\bar{r}} (1-\bar{k}\bar{\alpha}\bar{r})^{-1/2} \bar{r}^2 d\bar{r}\,, \\ \bar{M}(\bar{r}) &= (4\pi\beta/\kappa) \bar{r}^3. \end{aligned} \tag{3·8}$$

It is to be noticed, here, that the background radial coordinate $\bar{r}$ should in general be distinguished from r in the universe containing the inhomogeneity. Since the particle numbers in both universes from the centers to the corresponding surfaces r and $\bar{r}$ are equal, it follows that

$$\int_0^r (1-k\alpha r^3)^{-1/2} r^2 dr = \int_0^{\bar{r}} (1-\bar{k}\bar{\alpha}\bar{r}^2)^{-1/2} \bar{r}^2 d\bar{r}\,. \tag{3·9}$$

Now we define the total energy difference (in mass unit from the center to the shell r) by $\Delta M \equiv M(r) - \bar{M}(\bar{r})$, where r and $\bar{r}$ satisfy Eq. (3·9). Then, we obtain from Eqs. (3·7) and (3·8)

$$\Delta M(r)/M(r) = 1-(\bar{r}/r)^3.$$

For the mass range $M < 10^{12} M_\odot$ with which we are concerned, r_b (corresponding to the boundary of the inhomogeneity) is less than 10^{-4}, as seen from Eq.

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(3·7), and Eq. (3·9) can be expanded as the power series of $r_b^2(<10^{-8})$. Substituting Eq. (3·3) for Eq. (3·9) and integrating partially, we obtain

$$\Delta M(r)/M(r) = \frac{9}{4}\frac{\beta}{\bar{S}_i}\frac{r_b^5}{r^3}\int_0^{r/r_b}(\epsilon_1+2\epsilon_2)x^2$$
$$\times\{x^2-(r/r_b)^2\}dx + 0\{(\alpha^2-\bar{\alpha}^2)r^4\}.$$

At the boundary, $\Delta M/M$ is of the order of $(\beta r_b/\bar{S}_i)(\epsilon_1(0)+2\epsilon_2(0)) \ll (\epsilon_1+2\epsilon_2)(0)$ and it can vanish only when $A=B$. Here $A\equiv\int_0^1(\epsilon_1+2\epsilon_2)x^4dx$ and $B\equiv\int_0^1(\epsilon_1+2\epsilon_2)x^2dx$. Outside the boundary, where ϵ_1 and ϵ_2 can be assumed to be zero, we have $\Delta M(r)/M(r)=(9/4)(\beta r_b^2/\bar{S}_i)\{(r_b/r)^3A-(r_b/r)B\}$. In general, $\Delta M(r)/M(r)$ cannot vanish everywhere outside the boundary and it decreases with $1/r$. Only when $B=0$ or $k\alpha=\bar{k}\bar{\alpha}$ (cf. Eq. (3·3)), it decreases with $1/r^3$.

§ 4. Central region of an inhomogeneity

In this section, we deal with the central region of an inhomogeneity. It follows from Eqs. (3·3), (3·4) and (3·5) that

$$k\alpha(0)=\bar{k}\bar{\alpha}+(\beta/\bar{S}_i)(\epsilon_1(0)+2\epsilon_2(0)), \tag{4·1}$$

$$T(0)/t_i=(1/5)(\epsilon_1(0)+3\epsilon_2(0)). \tag{4·2}$$

The numerical values of $\alpha(0)$ are given in Table II for several values of $\epsilon_1(0)+2\epsilon_2(0)$.

Table II. The values of $\alpha(0)$. The case $k=-1$ is denoted by asterisks. In the case $k=0$, $\alpha(0)=0$.

$\bar{k}$	q_0	$\epsilon_1(0)+2\epsilon_2(0)$					
		10^{-4}	10^{-3}	3×10^{-3}	6×10^{-3}	10^{-2}	3×10^{-2}
1	1	0.756	1.890	4.410	8.190	13.23	37.80
0	0.5	0.1	1.0	3.0	6.0	10.0	30.0
−1	0.25	0.715*	0	1.588	3.970	7.146	23.82
	0.05	4.131*	3.713*	2.785*	1.393*	0.463	9.740

We find from Table II that, in spite of smallness of $\epsilon_1(0)$ and $\epsilon_2(0)$, the curvature in the inhomogeneity can take large values. This situation was already explained by Peebles[11] in an analogue of "a slightly dried-out apple." Here, $1/\bar{\alpha}$ and $1/\alpha$ correspond to the radius of the apple and the radius of curvature of the wrinkled skin, respectively.

Now we consider the epoch $t_b(0)$ when the central region is gravitationally bound. Since the central density $\rho_0(0)$ is derived from Eqs. (2·3) and (2·4)

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by putting $\eta=\pi$ and $k=1$, the ratio of $\rho_b(0)$ to the background density $\bar{\rho}$ can be expressed as

$$(\rho(0)/\bar{\rho})_b=\{\alpha(0)\zeta\}^3, \tag{4·3}$$

where

$$\zeta\equiv\{1-\bar{\sigma}'(\pi)\}/(2\bar{k}\bar{\alpha}) \qquad \text{for } \bar{k}=\pm1\,.$$

In the case $\bar{k}=0$, we obtain the central value of Eq. (3·6), i.e.

$$(\rho(0)/\bar{\rho})_b=(9/16)\pi^2\fallingdotseq5.55\,,$$

which is independent of $\epsilon_1(0)$ and $\epsilon_2(0)$. Here $T(0)/t_b(0)\,(\ll T(0)/t_i\ll1)$ was neglected. The above value of $(\rho(0)/\bar{\rho})_b$ is consistent with the one derived by Kihara[6] in a different approach. The values in the cases $\bar{k}=\pm1$ are evaluated and shown in Table III. In the case $\bar{k}=1$, $(\rho(0)/\bar{\rho})_b$ becomes smaller (and

Table III. $(\rho(0)/\bar{\rho})_b$.

$\bar{k}$	q_0	$\epsilon_1(0)+2\epsilon_2(0)$					
		10^{-4}	10^{-3}	3×10^{-3}	6×10^{-3}	10^{-2}	3×10^{-2}
1	1	1.55	3.72	4.73	5.10	5.27	5.46
0	0.5	5.55	5.55	5.55	5.55	5.55	5.55
-1	0.25	/	/	8.79	6.78	6.22	5.75
	0.05	/	/	/	/	121	8.29

$t_b(0)/t_0$ larger) with the decrease of $\epsilon_1(0)+2\epsilon_2(0)$. If $\epsilon_1(0)=\epsilon_2(0)=0$, $(\rho(0)/\bar{\rho})_b$ is of course unity and $t_b(0)$ is the epoch of the maximum expansion of the background universe itself. In the case $\bar{k}=-1$, $(\rho(0)/\bar{\rho})_b$ becomes larger with the decrease of $\epsilon_1(0)+2\epsilon_2(0)$, and at last reaches infinity for some finite value of $\epsilon_1(0)+2\epsilon_2(0)$. This means that the condensation of density fluctuation is more difficult in the open model, since the expansion rate is larger.

The ratio of $t_b(0)$ to the present age t_0 is also evaluated and tabulated in Table IV. The subsequent epoch $t_{\mathrm{inf}}(0)$ at which the central region collapses

Table IV. $t_b(0)/t_0$. Daggers denote the case $t_b(0)/t_0>1$. Oblique lines denote the case in which there exists no gravitationally bound state.

$\bar{k}$	q_0	$\epsilon_1(0)+2\epsilon_2(0)$					
		10^{-4}	10^{-3}	3×10^{-3}	6×10^{-3}	10^{-2}	3×10^{-2}
1	1	4.19†	1.06†	0.297	0.117	0.0572	0.0118
0	0.5	74.5†	2.36†	0.454	0.160	0.0745	0.0143
-1	0.25	/	/	1.04†	0.264	0.109	0.0179
	0.05	/	/	/	/	5.55†	0.0575

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and reaches the state of infinite density in the limit is given by putting $\eta=2\pi$ and its values are equal to 2 times $t_b(0)$.

The present age of the background universe is $(7\sim12)\times10^9$ y, if $H_0^{-1}=13\times10^9$ y (cf. Table I). But the ages of some old stars are estimated to be $\sim10^{10}$ y,[16] according to the theory of stellar evolution. Hence, the protogalaxies in which the stars were born must have been formed at the epoch whose age is less than t_0 by a factor of about 10 at least. Moreover, the primordial gas clouds would have condensed before the formation of protogalaxies. It follows, therefore, that $t_{\rm inf}(0)/t_0=2t_b(0)/t_0<1/10$. In comparison with Table IV, we find that, in the cases $\bar{k}=0$, $1(q_0=1)$ and $-1(q_0=0.25)$, $\epsilon_1(0)+2\epsilon_2(0)>10^{-2}$, and, in the case $\bar{k}=-1(q_0=0.05)$, $\epsilon_1(0)+2\epsilon_2(0)>3\times10^{-2}$.

§ 5. Simple models of inhomogeneities

Some bound gas clouds may have been born as members of a cluster. They would have been formed from inhomogeneities included in a larger inhomogeneity isolated in the expanding universe. But there may be an another type of bound gas clouds, which themselves are isolated in the universe. They would have condensed from a single inhomogeneity. In this section, we follow with time the change of the density distributions in the inhomogeneities of these two types separately.

As for the background universe, we adopt only the flat model ($\bar{k}=0$) in this section.*) For the functional forms of $\epsilon_1(r)$ and $\epsilon_2(r)$, it should be noticed that ϵ_2 plays effectively the same role as $\epsilon_1/2$ in the system of equations for ρ.**) Accordingly, we shall put $\epsilon_2(r)=0$ without loss of generality and assume for $\epsilon_1(r)$ the simple form which has the single peak in the center and satisfies the boundary conditions given in Appendix C.

(a) *An isolated inhomogeneity*

We take the functional form $\epsilon_1(r)=\epsilon_1(0)\times f(x)$, where $f(x)=(1-x)^2(1+2x-(27/5)x^2)$ for $x\leq1$ and 0 for $x>1$ with $x\equiv r/r_b$. This form was chosen as example in such a way that $\epsilon_1(r)$ may be a smooth function expressed as a polynomial of r of the lowest order for mathematical simplicity and $A=\int_0^1\epsilon_1x^4dx$ (cf. § 3) vanish. The corresponding $I_1(r)$ is given by $(1/3)\epsilon_1(0)\times g(x)$, where $g(x)=1-(126/25)x^2+(32/5)x^3-(81/35)x^4$ for $x\leq1$ and $(8/175)x^{-3}$ for $x>1$. The behaviors of $f(x)$ and $g(x)$ are shown in Fig. 1.

If we adopt the value 10^{-2} for $\epsilon_1(0)$ so as to make a galaxy formation possible in a reasonable way, then the ratio of the density in the inhomogeneity

*) The density distribution in the case of the other models will have forms similar to the one in the flat case, while the growth rate is somewhat different, as the results in the previous section show.

**) As can be seen from the formulas in Appendix B, ϵ_1 and ϵ_2 appear in the form of either $\epsilon_1+2\epsilon_2$ or $\epsilon_1+3\epsilon_2$, but the terms containing the latter do not contribute to the value of $\rho/\bar{\rho}$, because they are much smaller.

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to that of the background is calculated (cf. Appendix B) and found to change as shown in Fig. 2. The central region is bound first at $\tau(\equiv H_0 t)=0.05$ and thereafter the bound region expands gradually to the outside direction. Till $\tau=0.2$, a nucleus with high density is formed and the mass in the range $r/r_b \leq 0.6$ is bound. This mass contracts further and the mass in the range $r/r_b \leq 0.7$ is bound at present.

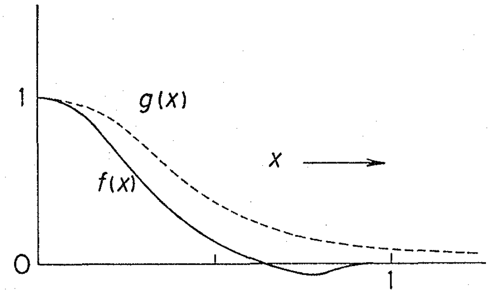

Fig. 1. The initial density contrast ($\propto f(x)$) and the curvature difference ($\propto g(x)$) in an isolated inhomogeneity.

The density ratio in the trough decreases first with time, but tends after $\tau=0.2$ to increase from internal region. Even in the range $r \geq r_b$, we find that the density ratio increases in a monotonic way. These are because a positive curvature is given to the spaces in such regions (or in the Newtonian language a negative energy is given to the matter) owing

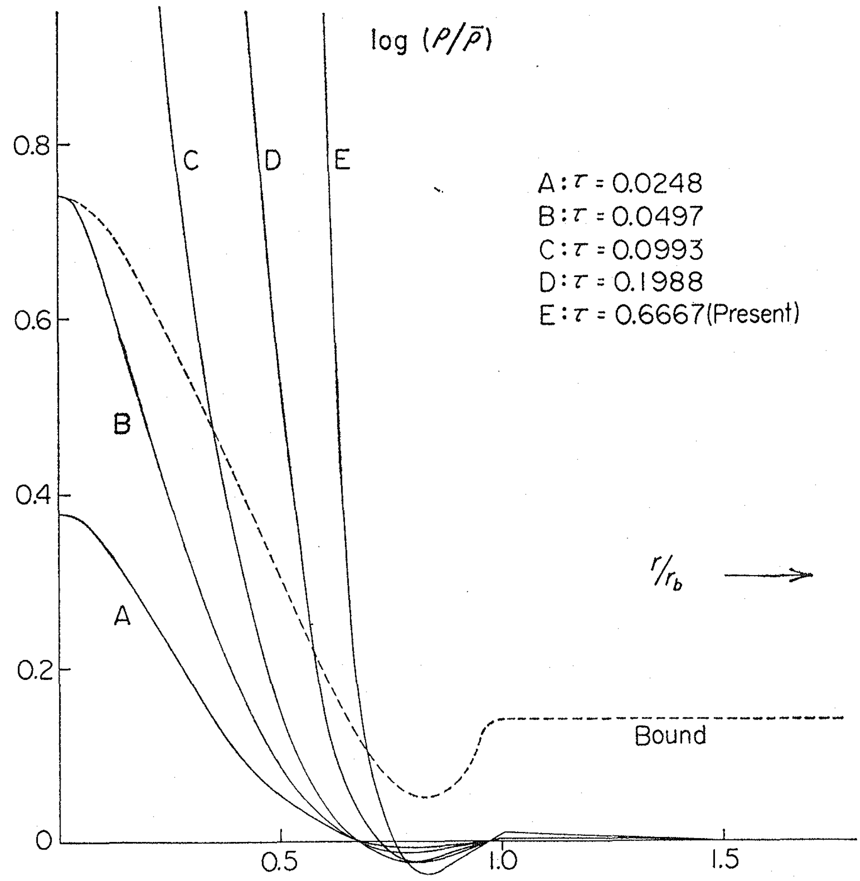

Fig. 2. The ratios of the matter density in an isolated inhomogeneity to that of the background universe ($\bar{k}=0$), at times between $\tau_i=H_0 t_i=2\times10^{-5}$ and $\tau_0=H_0 t_0=2/3$. The broken line represents the ratio in each surface at the epoch when it is gravitationally bound.

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to the appearance of the inhomogeneity.

(b) *A large-scale inhomogeneity*

Let us consider an inhomogeneity in another large-scale inhomogeneity which was uniform in the neighbourhood of the smaller one. The density contrast ϵ_1 in the larger one is estimated to be $\sim 3\times 10^{-3}$, because it must be bound and contract only by the present epoch. So, we shall take $\epsilon_1(r)=3\times 10^{-3}+10^{-2}f(x)$ as the functional form in the neighbourhood of the smaller inhomogeneity. The

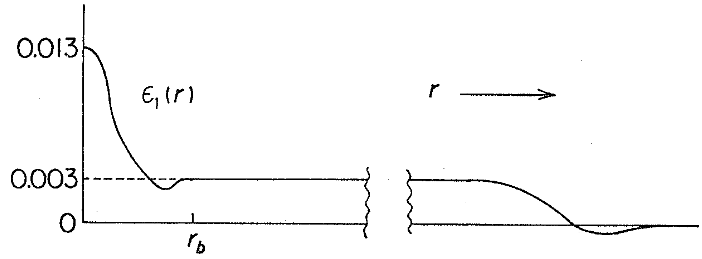

Fig. 3. The initial density contrast in an inhomogeneity included in a larger inhomogeneity.

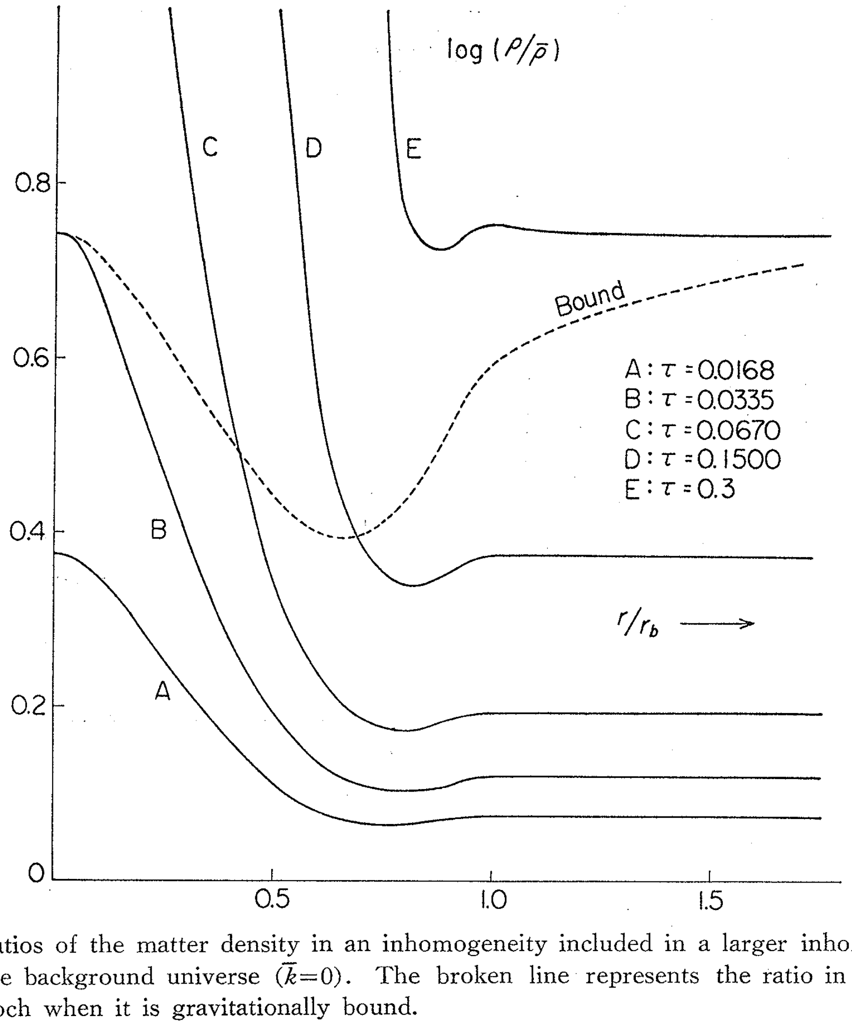

Fig. 4. The ratios of the matter density in an inhomogeneity included in a larger inhomogeneity to that of the background universe ($\bar{k}=0$). The broken line represents the ratio in each surface at the epoch when it is gravitationally bound.

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behavior of $\epsilon_1(r)$ is shown in Fig. 3. The density distribution in the inhomogeneity is calculated at several epochs and is shown in Fig. 4.

It is seen from Fig. 4 that at time $\tau=0.034$ the central region is bound. Till $\tau=0.2$, a nucleus is formed and the mass in the range $r/r_b \leq 0.7$ is bound. The process of condensation is rapid in comparison with that of an isolated inhomogeneity. At time $\tau=0.3$, the region $r/r_b \leq 1$ also is bound and subsequently the gas density outside the bound region continues to increase with time. However, if a large-scale angular momentum is contained in the larger inhomogeneity, the rotational motion of condensed inhomogeneities becomes rapidly large, so that a cluster of inhomogeneities and the gas among them can avoid the collapse by centrifugal forces.

§ 6. The minimum mass of fragments in the primordial gas cloud

The bound gas cloud formed from the primordial gas may break up into small-scale fragments owing to gravitational instability, if density fluctuations arise in the gas cloud. The minimum size of fragments is given by Jeans' wavelength $\lambda_J=\{2\pi kT_m/(G\rho m_p)\}^{1/2}$, where T_m is the gas temperature, m_p the proton mass and k the Boltzmann constant. The minimum mass of fragments is of the order of $M_J=(4/3)\pi\rho\lambda_J^3$. Since $T_m \propto \rho^{2/3}$ for adiabatic process, $\lambda_J \propto \rho^{-1/6}$ and hence $M_J \propto \rho^{1/2}$.

Now let us evaluate M_J at the epoch t_b of gravitational binding ($\dot{\rho}=0$), when M_J reaches the minimum ($(M_J)_b$). If we assume $T_m=T_r$ ($=4000°$K) at the decoupling time, we obtain by use of Eqs. (2·3), (2·4) and (2·5)

$$(M_J)_b=2.6\alpha^{3/2}(0)\,q_0^{-1}M_\odot$$

at the center. Since $\alpha(0)$ must be ~ 10 (cf. § 4), it follows that

$$(M_J)_b=(8.1\times10^2,\ 1.6\times10^3,\ 3.2\times10^3 \text{ and } 1.6\times10^4)\,M_\odot$$

at the center for $q_0=1,\ 0.5,\ 0.25$ and 0.05, respectively. In the envelope, the density ρ_b at time t_b is smaller than the central one $\rho_b(0)$ by a factor of $<10^2$, so that $(M_J)_b$ ($\propto \rho_b^{1/2}$) is smaller than the above values by a factor of 10 at most. We conclude, therefore, that, as long as the cloud is not much cooled at the contracting stage, ordinary stars with mass $M \sim M_\odot$ cannot be formed in the primordial gas cloud and it will evolve to a supermassive star[17] or a system of massive stars with mass $M \gtrsim 10^2 M_\odot$. In either cases, the primordial gas cloud will not become directly the ordinary galaxy we observe at present, but will explode after the rapid evolution.

§ 7. Concluding remarks

In this paper, the cosmological constant Λ has been neglected for simplicity. However, if we assume the background model which has a longer time scale

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and is consistent with the present observations,[18] we could make somewhat smaller our critical values of the initial density contrast and growth rate.

The primordial gas clouds will condense to supermassive stars or systems of massive stars and evolve rapidly to release hot gas to the surroundings through explosive phenomena. Thereafter, galaxies may be formed in the turbulent, hot gas, as was proposed by Doroshkevich et al. and Takeda et al.[17] At this stage, condensation due to thermal instability[19] will play some role. Moreover, the remnants of exploded supermassive or massive stars may be the nuclei of newly born galaxies. The exploded gas will be able to accrete again to the remnants through gravitational attraction, and ordinary stars will be formed around them, as the gas will be cooled by the heavy elements in the exploded gas.

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Appendix A

$\alpha(r)$ and $T(r)$

Equation (2·3) can be rewritten as

$$(S^3r^3)'=9\beta r^2/\kappa\rho\,.$$

If we integrate it with respect to r at time $t=t_i$, we obtain

$$S_i^3=(9\beta/\kappa r^3)\int_0^r \rho_i^{-1}r^2dr$$

$$=(9\beta/\kappa\bar{\rho}_i r^3)\,\{r^3/3-\int_0^r \epsilon_1(r)\,r^2dr+r^3 0(\epsilon_1^2)\}$$

by use of Eq. (3·1). Since $\bar{S}_i^3$ is given by $3\beta/\kappa\bar{\rho}_i$, it follows that

$$\delta S_i/\bar{S}_i=S_i/\bar{S}_i-1=-I_1(r)+0(\epsilon_1^2), \tag{A·1}$$

where $I_1(r)$ is defined by Eq. (3·5) in the text.

Next, if we differentiate Eq. (2·3) with respect to t and consider the values at time $t=t_i$, we have

$$(\dot{\rho}/\rho)_i=-3(\dot{S}/S)_i-r(S'/S)_i^{\cdot}+0(\epsilon_1^2), \tag{A·2}$$

where we should notice that $(S'/S)_i=\{(\delta S)'/S\}_i\sim\epsilon_1$. Eliminating $\dot{S}$ and $\dot{S}'$ in Eq. (A·2) by use of Eq. (2·2), we obtain from Eq. (3·2)

$$\epsilon_2=-\tfrac{3}{2}(\delta S/\bar{S})_i-\tfrac{1}{2}k\alpha S_i/\beta+\tfrac{1}{2}\bar{k}\bar{\alpha}\bar{S}_i/\beta-\tfrac{1}{6}r(\bar{S}_i/\beta)\,(3\beta S_i'/S_i^2+k\alpha')$$
$$+0\ (\text{Second order terms with respect to } \epsilon_1,\ \alpha\bar{S}_i/\beta \text{ and } \bar{\alpha}\bar{S}_i/\beta). \tag{A·3}$$

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Here, $(\bar{S}_i/\beta)^2$ is about $10^{-5}\sim10^{-6}$, since $\bar{S}_i/\bar{S}_0=10^{-3}$ and $\bar{S}_0/\beta=1\sim5$ (cf. Table I). If we insert Eq. (A·1) into $(\delta S)_i$ in Eq. (A·3), we obtain

$$k\left(\alpha+\frac{r}{3}\alpha'\right)-\bar{k}\bar{\alpha}=(\beta/\bar{S}_i)\,(\epsilon_1+2\epsilon_2)$$
$$+\,(\beta/\bar{S}_i)\,\{0(\epsilon_1^2)+0(\bar{\alpha}\bar{S}_i/\beta)^2+0(\alpha\bar{S}_i/\beta)^2\}.$$

This is integrated as

$$k\alpha-\bar{k}\bar{\alpha}=(3\beta/\bar{S}_i)\,(I_1+2I_2)+(\beta/\bar{S}_i)\,\{0(\epsilon_1^2)+0(\bar{\alpha}\bar{S}_i/\beta)^2+0(\alpha\bar{S}_i/\beta)^2\},\qquad (\text{A}\cdot4)$$

where $I_2(r)$ is defined by Eq. (3·5).

Moreover, if we substract from $T(r)$ in Eq. (2·4) the counterpart for the background universe and eliminate η, we obtain

$$cT(r)=\{\sqrt{S(\beta-k\alpha S)}/(k\alpha)-\sqrt{\bar{S}(\beta-\bar{k}\bar{\alpha}\bar{S})}/(\bar{k}\bar{\alpha})\}_i$$
$$-\beta\{\sigma^{-1}(\sqrt{\alpha S/\beta})/(\alpha^{3/2}k)-\bar{\sigma}^{-1}(\sqrt{\bar{\alpha}\bar{S}/\beta})/(\bar{\alpha}^{3/2}\bar{k})\}_i\,,\qquad (\text{A}\cdot5)$$

where the case $\bar{k}=0$ or $k=0$ is included in the above equation as the limit $\bar{k}\to0$ or $k\to0$. Expanding the right-hand side of Eq. (A·5) as a power series of $(\delta S)_i$, $\alpha\bar{S}_i$ and $\bar{\alpha}\bar{S}_i$ and eliminate δS_i and α by Eqs. (A·1) and (A·4), we have

$$T(r)/t_i=\frac{3}{5}(I_1+3I_2)+0(\epsilon_1^2)+0(\bar{\alpha}\bar{S}_i/\beta)^2+0(\alpha\bar{S}_i/\beta)^2.\qquad (\text{A}\cdot6)$$

Appendix B

The density distribution at any epoch is described by the formulas summarized below, which are derived by eliminating S', $\alpha(r)$ and $T(r)$ from Eq. (2·3). Instead of S, $\bar{S}$ and t, we employ $R\equiv S/\beta$, $\bar{R}\equiv\bar{S}/\beta$ and $\tau\equiv ct/\beta=H_0t$ for convenience. Then we have

$$\rho/\bar{\rho}=(\bar{R}/R)^3/(1+\Phi),\qquad (\text{B}\cdot1)$$

where

$$\Phi=-\sqrt{R(1-k\alpha R)}/R^2\times\Big[\frac{3}{5}\tau_i\{\epsilon_1+3\epsilon_2-3(I_1+3I_2)\}$$
$$+\frac{3}{2}\frac{1}{R_i\alpha}\{R(3-k\alpha R)/\sqrt{R(1-k\alpha R)}-3\sigma^{-1}(\sqrt{\alpha R})/\sqrt{\alpha}\}$$
$$\times\{\epsilon_1+2\epsilon_2-3(I_1+2I_2)\}\Big],\qquad (\text{B}\cdot2)$$

$$\tau=\frac{3}{5}\tau_i(I_1+3I_2)+\{-\sqrt{R(1-k\alpha R)}+\sigma^{-1}(\sqrt{\alpha R})/\sqrt{\alpha}\}/(k\alpha)$$
$$=\{-\sqrt{\bar{R}(1-\bar{k}\bar{\alpha}\bar{R})}+\bar{\sigma}^{-1}(\sqrt{\bar{\alpha}\bar{R}})/\sqrt{\bar{\alpha}}\}/(\bar{k}\bar{\alpha}).\qquad (\text{B}\cdot3)$$

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If ϵ_1 and ϵ_2 are given, we can calculate R and $\bar{R}$ at time τ by use of Eq. (B·3), and, inserting them into Eqs. (B·1) and (B·2), we can obtain $\rho/\bar{\rho}$.

Appendix C

Boundary conditions for the differences ϵ_1 and ϵ_2

The functional forms of $\epsilon_1(r)$ and $\epsilon_2(r)$ cannot be arbitrary, because the metric (2·1) should be smoothly adjusted in any interface. This condition has been derived as the boundary condition by O'Brien and Synge.[20]

Between the quantities interior and exterior to the interface, we have three equations from the boundary condition:

$$(S)_{\rm in} = (S)_{\rm ex}\,, \tag{C·1a}$$

$$(S')_{\rm in} = (S')_{\rm ex}\,, \tag{C·1b}$$

$$(\alpha)_{\rm in} = (\alpha)_{\rm ex}\,. \tag{C·1c}$$

On the other hand, the curvature $\alpha(r)$ in Eq. (3·3) is continuous, because it is given as an integration of ϵ_1 and ϵ_2, and the condition (C·1c) is automatically satisfied. Now, let us examine conditions (C·1a) and (C·1b) at time t_i. The difference $\delta S = S - \bar{S}$ is also continuous, because it is expressed as an integration of ϵ_1, as shown in Appendix A. However, it is necessary that

$$(\epsilon_1)_{\rm in} = (\epsilon_1)_{\rm ex} \tag{C·2a}$$

in order that $S'(=\delta S')$ is continuous. At an arbitrary epoch $t(>t_i)$, S is still continuous, because, in Eq. (2·4) determining S, $T(r)$ is continuous as well as $\alpha(r)$. However, by differentiating Eq. (2·4) with respect to r, we find that $\epsilon_2(r)$ must be continuous in order that S', and therefore I_2, are continuous:

$$(\epsilon_2)_{\rm in} = (\epsilon_1)_{\rm ex}\,. \tag{C·2b}$$

Equations (C·2a) and (C·2b) are the necessary and sufficient conditions for the adjustment of metric.

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