Formation of singularities for a transport equation with nonlocal velocity

By Antonio Córdoba*, Diego Córdoba**, and Marco A. Fontelos***

Abstract

We study a 1D transport equation with nonlocal velocity and show the formation of singularities in finite time for a generic family of initial data. By adding a diffusion term the finite time singularity is prevented and the solutions exist globally in time.

1. Introduction

In this paper we study the nature of the solutions to the following class of equations

$$(1.1) \qquad \theta_t - (H\theta)\,\theta_x = -\nu\Lambda^\alpha\theta, \qquad x \in R$$

where $H\theta$ is the Hilbert transform defined by

$$H\theta \equiv \frac{1}{\pi}PV\int \frac{\theta(y)}{x-y}dy,$$

ν is a real positive number, $0 \leq \alpha \leq 2$ and $\Lambda^\alpha\theta \equiv (-\Delta)^{\frac{\alpha}{2}}\theta$.

This equation represents the simplest case of a transport equation with a nonlocal velocity and with a viscous term involving powers of the laplacian. It is well known that the equivalent equation with a local velocity $v = \theta$, known as Burgers equation, may develop shock-type singularities in finite time when $\nu = 0$ whereas the solutions remain smooth at all times if $\nu > 0$ and $\alpha = 2$. Therefore a natural question to pose is whether the solutions to (1.1) become singular in finite time or not depending on α and ν. In fact this question has been previously considered in the literature motivated by the strong analogy with some important equations appearing in fluid mechanics, such as the 3D Euler incompressible vorticity equation and the Birkhoff-Rott equation modelling the evolution of a vortex sheet, where a crucial mathematical difficulty

lies in the nonlocality of the velocity. Since the fundamental problem concerning both 3D Euler and Birkhoff-Rott equations is the formation of singularities in finite time, the main goal of this paper will be to solve this issue for the model (1.1).

3D Euler equations, in terms of the vorticity vector are

$$(1.2) \qquad \omega_t + v \cdot \nabla \omega = \omega D(\omega)$$

where $D(\omega)$ is a singular integral operator of ω whose one dimensional analogue is the Hilbert transform and the velocity is given by the Biot-Savart formula in terms of ω. In order to construct lower dimensional models containing some of the main features of (1.2), Constantin, Lax and Majda [3] considered the scalar equation

$$(1.3) \qquad \omega_t + v\omega_x = \omega H\omega;$$

with $v = 0$ and showed existence of finite time singularities. The effect of adding a viscous dissipation term has been studied in [13], [16], [17], [15] and [12]. In order to incorporate the advection term $v\omega_x$ into the model, De Gregorio, in [6] and [7], proposed a velocity given by an integral operator of ω. If we take an x derivative of (1.1) and define $\theta_x \equiv \omega$ we obtain a viscous version of the equation (1.3) with $v = -H\theta$ which is similar to the one proposed in [6] and [7].

The analogy of (1.1) with Birkhoff-Rott equations was first established in [1] and [10]. These are integrodifferential equations modelling the evolution of vortex sheets with surface tension. The system can be written in the form

$$(1.4) \qquad \frac{\partial}{\partial t} z^*(\alpha, t) = \frac{1}{2\pi i} PV \int \frac{\tilde{\gamma}(\alpha')d\alpha'}{z(\alpha, t) - z(\alpha', t)}$$

$$(1.5) \qquad \frac{\partial \tilde{\gamma}}{\partial t} = \sigma \kappa_\alpha$$

where $z(\alpha, t) = x(\alpha, t) + iy(\alpha, t)$ represents the two dimensional vortex sheet parametrized with α, and where κ denotes mean curvature. Following [1] we substitute, in order to build up the model, the equation (1.4) by its 1D analog

$$\frac{dx(\alpha, t)}{dt} = -H(\theta)$$

where we have identified $\gamma(\alpha, t)$ with θ. In the limit of $\sigma = 0$ in (1.5) we conclude that γ is constant along trajectories and this fact leads, in the 1D model, to the equation

$$(1.6) \qquad \theta_t - (H\theta)\,\theta_x = 0.$$

There is now overwhelming evidence that vortex sheets form curvature singularities in finite time. This evidence comes back from the classical paper by Moore [9] where he studied the Fourier spectrum of $z(\alpha, t)$ and, in particular, its asymptotic behavior when the wave number k goes to infinity. His

numerical results showed that, up to very high values of k, this asymptotic behavior is compatible with the formation of a curvature singularity in finite time. Although there has been a very intense activity in order to provide a definitive proof of the formation of such a singularity (see discussions and references in [9], [2] and [1]) the existing results are mostly supported in numerics or formal asymptotics and do not constitute a full mathematical proof. The same kind of argument was used in [1] in order to show the existence of singularities for the 1D analog (1.6).

The system (1.4) and (1.5) with $\sigma = 0$ has the very interesting property of being ill-posed for general initial data. A linear analysis of small perturbations of planar sheets leads to catastrophically growing dispersion relations. Several attempts at regularization were introduced through the incorporation of effects, such as surface tension or viscosity (see [2] for a comprehensive review). In the same spirit we will also study the effects of artificial viscosity terms on the solutions for our model. More precisely we will prove the existence of blow-up in finite time for (1.1) with $\nu = 0$ in Section 2 and, conversely, the global existence of solutions when $\nu > 0$ and $1 < \alpha \leq 2$ in Section 3.

2. Blow-up for $\nu = 0$

The local existence of solutions to (1.1) was established in [1]. In this section we will show the existence of blowing-up solutions to (1.6) for a generic class of initial data.

Let us consider a symmetric, positive, and $C^{1+\varepsilon}(\mathbb{R})$ initial profile $\theta = \theta_0(x)$ such that $\max_x \theta_0 = \theta_0(0) = 1$. We will also assume

$$\mathrm{Supp}(\theta_0(x)) \subset [-L, L] \ .$$

Under these assumptions, it is clear that $\theta(x, t)$ will remain positive (given the transport character of equation (1.1) for $\nu = 0$) and symmetric. Then, $H\theta$ will be antisymmetric and positive for $x \geq L$. This implies the following properties for $\theta(x, t)$:

- $\mathrm{Supp}(\theta(x, t)) \subset [-L, L]$,
- $\max_x \theta = \theta(0, t) = 1$,
- $\|\theta\|_{L^1}(t) \leq \|\theta\|_{L^1}(0)$,
- $\|\theta\|_{L^2}(t) \leq \|\theta\|_{L^2}(0)$.

THEOREM 2.1. *Under the conditions stated above for θ_0, the solutions of (1.1) with $\nu = 0$ will always be such that $\|\theta_x\|_{L^\infty}$ blows up in finite time.*

Proof. Since $\theta_t = -(1 - \theta)_t \equiv -f_t$, $\theta_x = -(1 - \theta)_x \equiv -f_x$ and $H\theta = -H(1 - \theta) \equiv -Hf$, we can write, from (1.6),

$$(2.7) \qquad (1 - \theta)_t = -H(1 - \theta)(1 - \theta)_x \ .$$

We now divide (2.7) by $x^{1+\delta}$ with $0 < \delta < 1$, integrate in $[0, L]$ and obtain the following identity:

$$(2.8) \qquad \frac{d}{dt}\left(\int_0^L \frac{(1-\theta)}{x^{1+\delta}}dx\right) = -\int_0^L \frac{(1-\theta)_x H(1-\theta)}{x^{1+\delta}}dx .$$

Given the fact that θ vanishes outside the interval $[-L, L]$, we can write the right-hand side of (2.8) in the form

$$(2.9) \qquad -\int_0^L \frac{(1-\theta)_x H(1-\theta)}{x^{1+\delta}}dx = -\int_0^\infty \frac{(1-\theta)_x H(1-\theta)}{x^{1+\delta}}dx .$$

In the next lemma we provide an estimate for the right-hand side of (2.9).

LEMMA 2.2. *Let $f \in C_c^\infty(\mathbb{R}^+)$. Then for $0 < \delta < 1$ there exists a constant C_δ such that*

$$(2.10) \qquad -\int_0^\infty \frac{f_x(x)(Hf)(x)}{x^{1+\delta}}dx \geq C_\delta \int_0^\infty \frac{1}{x^{2+\delta}}f^2(x)dx .$$

Proof. First, we recall the following Parseval identity for Mellin transforms:

$$-\int_0^\infty \frac{f_x(x)(Hf)(x)}{x^{1+\delta}}dx = -\frac{1}{2\pi}\int_{-\infty}^\infty \overline{A(\lambda)}B(\lambda)d\lambda \equiv I ,$$

with

$$A(\lambda) = \int_0^\infty x^{i\lambda-\frac{1}{2}-\frac{\delta}{2}}f_x(x)dx ,$$

$$B(\lambda) = \int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}(Hf)(x)dx .$$

Integration by parts in $A(\lambda)$ yields

$$A(\lambda) = -(i\lambda - \frac{1}{2} - \frac{\delta}{2})\int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}f(x)dx .$$

With respect to $B(\lambda)$ we can write

$$\begin{aligned}
B(\lambda) &= \int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}\left[\frac{1}{\pi}P.V.\int_{-\infty}^{+\infty}\frac{f(\xi)}{x-\xi}d\xi\right]dx \\
&= \int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}\left[\frac{1}{\pi}P.V.\int_{-\infty}^0\frac{f(\xi)}{x-\xi}d\xi + \frac{1}{\pi}P.V.\int_0^\infty\frac{f(\xi)}{x-\xi}d\xi\right]dx \\
&= \int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}\left[\frac{1}{\pi}\int_0^\infty\frac{f(\xi)}{x+\xi}d\xi + \frac{1}{\pi}P.V.\int_0^\infty\frac{f(\xi)}{x-\xi}d\xi\right]dx \\
&= \int_0^\infty x^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}\left[\frac{-x}{\pi}\int_0^\infty\frac{f(\xi)/\xi}{x+\xi}d\xi + \frac{x}{\pi}P.V.\int_0^\infty\frac{f(\xi)/\xi}{x-\xi}d\xi\right]dx \\
&= \int_0^\infty\left[-\frac{1}{\pi}\int_0^\infty\frac{x^{i\lambda-\frac{1}{2}-\frac{\delta}{2}}}{x+\xi}dx + \frac{1}{\pi}P.V.\int_0^\infty\frac{x^{i\lambda-\frac{1}{2}-\frac{\delta}{2}}}{x-\xi}dx\right]\frac{f(\xi)}{\xi}d\xi
\end{aligned}$$

where we have used Fubini's theorem in order to exchange the order of integration in x and ξ. Using elementary complex variable theory one can write

$$
-\frac{1}{\pi}\int_0^\infty \frac{x^{i\lambda-\frac{1}{2}-\frac{\delta}{2}}}{x+\xi}dx + \frac{1}{\pi}P.V.\int_0^\infty \frac{x^{i\lambda-\frac{1}{2}-\frac{\delta}{2}}}{x-\xi}dx
$$

$$
= \lim_{\substack{R\to\infty \\ \varepsilon\to 0}}\left[-\frac{1}{\pi}\frac{1}{1-e^{2\pi i(i\lambda-\frac{1}{2}-\frac{\delta}{2})}}\int_{\Gamma_1}\frac{z^{i\lambda-\frac{1}{2}-\frac{\delta}{2}}}{z+\xi}dz \right.
$$

$$
\left. +\frac{1}{\pi}\frac{1}{1-e^{2\pi i(i\lambda-\frac{1}{2}-\frac{\delta}{2})}}\int_{\Gamma_2\backslash\{c_1,c_2\}}\frac{z^{i\lambda-\frac{1}{2}-\frac{\delta}{2}}}{z-\xi}dz \right]
$$

$$
\equiv I_1 + I_2
$$

where Γ_1 and Γ_2 are the paths in the complex plane represented in Figures 1 and 2 respectively. Standard pole integration for I_1 and the fact that $\int_{\Gamma_2\backslash\{c_1,c_2\}} = -\int_{\{c_1,c_2\}}$ in I_2 (cf. Lemmas 2.2 and 2.3 in [8] where these integrals had to be computed for a completely different purpose, for instance) yield then

$$
I_1 + I_2 = \left[-\frac{1}{\sin\left((-i\lambda+\frac{1}{2}+\frac{\delta}{2})\pi\right)} + \cot\left((-i\lambda+\frac{1}{2}+\frac{\delta}{2})\pi\right) \right]\xi^{i\lambda-\frac{1}{2}-\frac{\delta}{2}}.
$$

Hence

$$
B(\lambda) = \frac{-1+\cos\left((-i\lambda+\frac{1}{2}+\frac{\delta}{2})\pi\right)}{\sin\left((-i\lambda+\frac{1}{2}+\frac{\delta}{2})\pi\right)}F(\lambda)
$$

with

$$
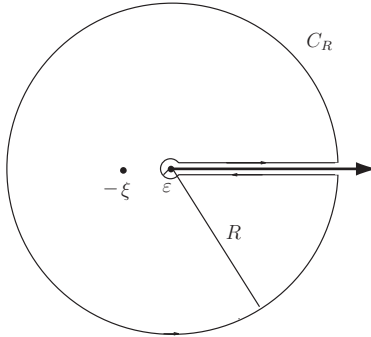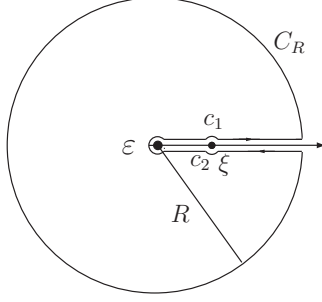F(\lambda) \equiv \int_0^\infty \xi^{i\lambda-\frac{3}{2}-\frac{\delta}{2}}f(\xi)d\xi
$$

Figure 1: Integration contour Γ_1.

Figure 2: Integration contour Γ_2.

and

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - \cos\left((-i\lambda + \frac{1}{2} + \frac{\delta}{2})\pi\right)}{\sin\left((-i\lambda + \frac{1}{2} + \frac{\delta}{2})\pi\right)} (i\lambda + \frac{1}{2} + \frac{\delta}{2}) \, |F(\lambda)|^2 \, d\lambda$$

$$\equiv \frac{1}{2\pi^2} \int_{-\infty}^{\infty} M(\lambda) \, |F(\lambda)|^2 \, d\lambda \, .$$

In order to analyze $M(\lambda)$ we define now

$$z \equiv a + bi \, , \ a \equiv \left(\frac{1}{2} + \frac{\delta}{2}\right)\pi \, , \ b \equiv \lambda\pi$$

which implies, after some straightforward but lengthy computations,

$$(2.11) \quad M(\lambda) = z\frac{1 - \cos\overline{z}}{\sin\overline{z}} = \frac{a\sin a + b\sinh b}{\cosh b + \cos a} + \frac{-a\sinh b + b\sin a}{\cosh b + \cos a}i \, .$$

Since $|F(\lambda)|^2$ is symmetric in λ and the imaginary part of $M(\lambda)$ is antisymmetric,

$$I = \frac{1}{2\pi^2} \int_{-\infty}^{\infty} \text{Re}\,\{M(\lambda)\} \, |F(\lambda)|^2 \, d\lambda \, .$$

Notice now from (2.11) that

$$\frac{1}{C}(1 + |\lambda|) \leq \text{Re}\,\{M(\lambda)\} \leq C(1 + |\lambda|)$$

so that

$$I \geq \frac{1}{2\pi C} \int_{-\infty}^{\infty} |F(\lambda)|^2 \, d\lambda \geq C_\delta \int_0^{\infty} \frac{1}{x^{2+\delta}} f^2(x) dx$$

where we have used the Plancherel identity for Mellin transforms:

$$\int_0^{\infty} \frac{1}{x^{2+\delta}} f^2(x) dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} |F(\lambda)|^2 \, d\lambda \, .$$

This completes the proof of the lemma. $\square$

Remark 2.3. Inequality (2.10) can be extended by density to the restriction to $\mathbb{R}^+$ of any symmetric $f \in C^{1+\varepsilon}(\mathbb{R})$ vanishing at the origin.

In order to complete our blow-up argument, we have, from Cauchy's inequality,

$$\int_0^L \frac{(1-\theta)}{x^{1+\delta}}dx \leq \left(\int_0^L \frac{(1-\theta)^2}{x^{2+\delta}}dx\right)^{\frac{1}{2}} \left(\int_0^L \frac{1}{x^\delta}dx\right)^{\frac{1}{2}}$$

$$\leq \left(\frac{L^{1-\delta}}{1-\delta}\right)^{\frac{1}{2}} \left(\int_0^\infty \frac{(1-\theta)^2}{x^{2+\delta}}dx\right)^{\frac{1}{2}}$$

so that

$$(2.12) \qquad \int_0^\infty \frac{(1-\theta)^2}{x^{2+\delta}}dx \geq C_{L,\delta}\left(\int_0^L \frac{(1-\theta)}{x^{1+\delta}}dx\right)^2.$$

From (2.8), (2.10) and (2.12) we deduce

$$\frac{d}{dt}\int_0^L \frac{(1-\theta)}{x^{1+\delta}}dx \geq C_{L,\delta}\left(\int_0^L \frac{(1-\theta)}{x^{1+\delta}}dx\right)^2$$

which yields a blow-up for

$$J \equiv \int_0^L \frac{(1-\theta)}{x^{1+\delta}}dx$$

at finite time. Since

$$J \leq \int_0^L \frac{(1-\theta)}{x^{1+\delta}}dx \leq \sup_x \frac{1-\theta}{x}\int_0^L \frac{dx}{x^\delta} \leq \frac{L^{1-\delta}}{1-\delta}\sup_x |\theta_x|$$

we conclude that $\|\theta_x\|_{L^\infty}$ must blow up at finite time. This completes the proof of Theorem 2.1. $\square$

Remark 2.4. In fact, numerical simulation by Morlet (see [11]) and additional numerical experiments performed by ourselves (see Figures 3 and 4) indicate that blow-up occurs at the maximum of θ and is such that a cusp develops at this point in finite time.

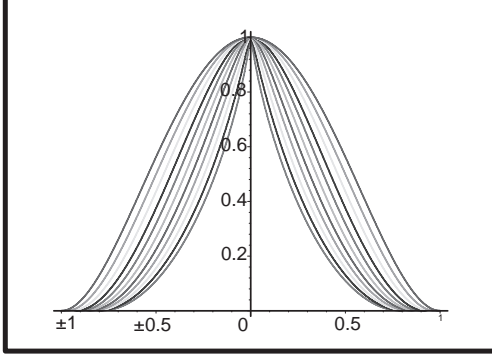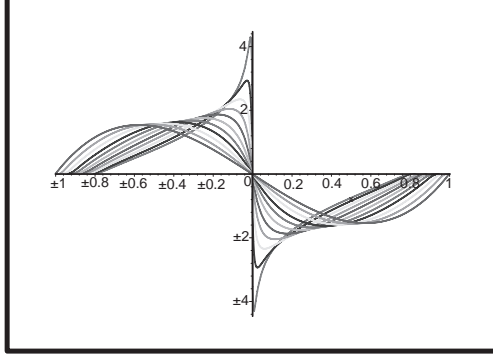The figures below represent the profiles $\theta_x(x,t)$ and $\theta(x,t)$ with initial data

$$\theta_0(x) = \begin{cases} (1-x^2)^2, & if -1 \leq x \leq 1 \\ 0, & otherwise \end{cases}$$

at nine consecutive times.

3. The effect of viscosity.

Below we study the effect of viscosity ($\nu > 0$) on the solutions of (1.1) with positive initial datum. First

Figure 3: $\theta(x,t)$ Figure 4: $\theta_x(x,t)$

LEMMA 3.1. *Let θ be a C^1 solution of* (1.1) *in $0 \leq t \leq T$, with a nonnegative initial datum $\theta_0 \in H^2(R)$. Then,*

$$(3.13) \quad 1) \qquad 0 \leq \theta(x,t) \leq \|\theta_0\|_{L^\infty} ,$$

$$(3.14) \quad 2) \qquad \|\theta\|_{L^1}(t) \leq \|\theta_0\|_{L^1} ,$$

$$(3.15) \quad 3) \qquad \|\theta\|_{L^2}(t) \leq \|\theta_0\|_{L^2} \qquad and \qquad \int_0^T \|\Lambda^{\frac{\alpha}{2}}\theta\|_{L^2}^2 dt \leq \frac{\|\theta_0\|_{L^2}^2}{2\nu} .$$

Proof. Since

$$\frac{d}{dt}\int |\Delta\theta|^2 dx = \int \Delta\theta\Delta(H(\theta)\theta_x)dx \leq C\|\Delta\theta\|_{L^2}^3$$

we have local solvability up to a time $T = T(\|\theta_0\|_{H^2(R)}) > 0$ (without any restriction upon the sign of θ_0). Let us also observe that the same result is true for the periodic version of (1.1): $-\pi \leq x \leq \pi$,

$$Hf(x) = P.V.\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f(x-y)}{\tan\frac{y}{2}}dy.$$

We shall prove (3.13) first in the periodic case: Let us define $M(t) \equiv \max_x\theta(x,t)$, $m(t) \equiv \min_x\theta(x,t)$. It follows from the H. Rademacher theorem that the continuous Lipschitz functions $M(t)$, $m(t)$, admit ordinary derivatives at almost every point t. Then we may argue as in references [4] and [5] to conclude that, at each point of differentiability, $M'(t) \leq 0$ and $m'(t) \geq 0$, implying (3.13).

Let $\phi \in C_0^\infty(R)$ be such that $\phi \geq 0$, $\phi(x) \equiv 1$ in $|x| \leq 1$ and $\phi(x) \equiv 0$ when $|x| \geq 2$. With $R > 0$ let us consider $\theta_0^R(x) = \phi(\frac{x}{R})\theta_0(x)$ and let $\theta^R(x,t)$ be the solution of the periodic problem (1.1) with initial data θ_0^R in $-\pi R \leq x \leq \pi R$, $0 \leq t \leq T = T(\theta_0)$.

We have that $0 \leq \theta^R(x,t) \leq \|\theta_0\|_{L^\infty}$ with uniform estimates for $\nabla_x\theta^R$, $\frac{\partial}{\partial t}\theta^R$. By compactness, we obtain a sequence θ^{R_j}, $R_j \to \infty$, converging uni-

formly on compact sets to θ, the solution of (1.1) with initial data θ_0. Then estimate (3.13) follows.

To obtain inequality (3.14) we proceed as follows:

$$\frac{d}{dt}\int \theta dx = \int H\theta\theta_x dx = -\int \theta\Lambda\theta dx = -\|\Lambda^{\frac{1}{2}}\theta\|_{L^2}^2,$$

because $\int \Lambda^\alpha \theta dx = 0$.

Next, observe that

$$\frac{1}{2}\frac{d}{dt}\int \theta^2 dx = \int \theta\theta_x H\theta dx - \nu\int \theta\Lambda^\alpha\theta dx$$

$$= -\frac{1}{2}\int \theta^2\Lambda\theta dx - \nu\int |\Lambda^{\frac{\alpha}{2}}\theta|^2 dx.$$

On the other hand

$$\int \theta^2\Lambda\theta dx = \int\int \frac{[\theta(x)+\theta(y)]}{2}\frac{(\theta(x)-\theta(y))^2}{(x-y)^2}dxdy \geq 0$$

and the proof of the third part of the lemma follows.

3.1. *Global existence with $\alpha > 1$.*

THEOREM 3.2. *Let $0 \leq \theta_0 \in H^2(R)$, $\nu > 0$ and $\alpha > 1$. Then there exists a constant C, depending only on θ_0 and ν, such that for $t \geq 0$:*

$$(3.16) \qquad\qquad 1) \qquad \|\Lambda^{\frac{1}{2}}\theta\|_{L^2}(t) \leq C\ ,$$

$$(3.17) \qquad\qquad 2) \qquad \|\Lambda\theta\|_{L^2}(t) \leq C(1+t)\ ,$$

$$(3.18) \qquad\qquad 3) \qquad \|\Delta\theta\|_{L^2}(t) \leq Ce^{Ct^3}\ .$$

Proof. Integration by parts and the formula for the Hilbert transform

$$2H(fH(f)) = (H(f))^2 - f^2$$

yield

(3.19)

$$\frac{1}{2}\frac{d}{dt}\int |\Lambda^{\frac{1}{2}}\theta|^2 dx = \int \Lambda\theta\theta_x H\theta dx - \nu\int |\Lambda^{\frac{1}{2}+\frac{\alpha}{2}}\theta|^2 dx$$

$$= -\int \theta H(\theta_x H\theta_x)dx - \nu\int |\Lambda^{\frac{1}{2}+\frac{\alpha}{2}}\theta|^2 dx$$

$$= -\frac{1}{2}\int \theta(H\theta_x)^2 dx + \frac{1}{2}\int \theta(\theta_x)^2 dx - \nu\int |\Lambda^{\frac{1}{2}+\frac{\alpha}{2}}\theta|^2 dx$$

$$\leq \|\theta_0\|_{L^\infty}\|\Lambda\theta\|_{L^2}^2 - \nu\int |\Lambda^{\frac{1}{2}+\frac{\alpha}{2}}\theta|^2 dx.$$

Since

$$\|\Lambda\theta\|_{L^2}^2 \leq R^{2-\alpha}\|\Lambda^{\frac{\alpha}{2}}\theta\|_{L^2}^2 + R^{1-\alpha}\|\Lambda^{\frac{1}{2}+\frac{\alpha}{2}}\theta\|_{L^2}^2,$$

by taking R sufficiently large and applying inequality (3.15), we obtain the desired inequality (3.16).

Applying Λ operator to both sides of equation (1.1), multiplying by $\Lambda\theta$ and integrating in x, we obtain

$$(3.20) \qquad \frac{1}{2}\frac{d}{dt}\int |\Lambda\theta|^2 dx = \int \Lambda\theta\Lambda(\theta_x H\theta)dx - \nu\|\Lambda^{1+\frac{\alpha}{2}}\theta\|_{L^2}^2$$

$$= -\frac{1}{2}\int (\theta_x)^2\Lambda\theta dx - \nu\|\Lambda^{1+\frac{\alpha}{2}}\theta\|_{L^2}^2$$

$$\leq C\int |\Lambda\theta|^3 dx - \nu\|\Lambda^{1+\frac{\alpha}{2}}\theta\|_{L^2}^2 \ ,$$

where we have used the isometry of Hilbert transform in L^2, integration by parts and finally Cauchy's inequality together with the boundedness of Hilbert transform in L^3.

In order to estimate $\|\Lambda\theta\|_{L^3}$ we make use of Hausdorff-Young's inequality

$$(3.21) \qquad \|\Lambda\theta\|_{L^3} \leq \|\widehat{\Lambda\theta}\|_{L^{\frac{3}{2}}} = \left(\int |\xi|^{\frac{3}{2}}|\widehat{\theta}(\xi)|^{\frac{3}{2}}d\xi\right)^{\frac{2}{3}}.$$

Picking now $\bar{\alpha}\in(1,\alpha)$ and using Cauchy's inequality we obtain

$$\left(\int |\xi|^{\frac{3}{2}}|\widehat{\theta}(\xi)|^{\frac{3}{2}}d\xi\right)^{\frac{2}{3}} \leq \left(\int |\xi|^{2+\bar{\alpha}}|\widehat{\theta}(\xi)|^2 d\xi\right)^{\frac{1}{3}}\left(\int |\xi|^{1-\bar{\alpha}}|\widehat{\theta}(\xi)|d\xi\right)^{\frac{1}{3}}$$

$$\equiv I_1^{\frac{1}{3}}\cdot I_2^{\frac{1}{3}}.$$

For I_1 we get

$$(3.22) \qquad I_1 = \int_{|\xi|\leq R} |\xi|^{2+\bar{\alpha}}|\widehat{\theta}(\xi)|^2 d\xi + \int_{|\xi|\geq R} |\xi|^{2+\bar{\alpha}}|\widehat{\theta}(\xi)|^2 d\xi$$

$$\leq R^{2+\bar{\alpha}}\|\theta\|_{L^2}^2 + \frac{1}{R^{\alpha-\bar{\alpha}}}\int_{|\xi|\geq R} |\xi|^{2+\alpha}|\widehat{\theta}(\xi)|^2 d\xi$$

$$\leq R^{2+\bar{\alpha}}\|\theta\|_{L^2}^2 + \frac{1}{R^{\alpha-\bar{\alpha}}}\|\Lambda^{1+\frac{\alpha}{2}}\theta\|_{L^2}^2.$$

With respect to I_2 one can estimate

$$(3.23) \quad I_2 = \int_{|\xi|\leq 1} |\xi|^{1-\bar{\alpha}}|\widehat{\theta}(\xi)|d\xi + \int_{|\xi|\geq 1} |\xi|^{1-\bar{\alpha}}|\widehat{\theta}(\xi)|d\xi$$

$$\leq \int_{|\xi|\leq 1} |\widehat{\theta}(\xi)|d\xi + \int_{|\xi|\geq 1} |\xi|^{\frac{1}{2}}|\widehat{\theta}(\xi)||\xi|^{\frac{1}{2}-\bar{\alpha}}d\xi$$

$$\leq \int_{|\xi|\leq 1} \|\theta\|_{L^1}d\xi + \left(\int_{|\xi|\geq 1} |\xi|^{1-2\bar{\alpha}}d\xi\right)^{\frac{1}{2}}\left(\int_{|\xi|\geq 1} |\xi||\widehat{\theta}(\xi)|^2 d\xi\right)^{\frac{1}{2}}$$

$$\leq \|\theta\|_{L^1} + c_\alpha\|\Lambda^{\frac{1}{2}}\theta\|_{L^2} \ .$$

From (3.23), (3.14) and (3.16) it follows that

(3.24) $I_2 \leq C.$

Hence by (3.21), (3.22) and (3.24) we get

(3.25) $\|\Lambda\theta\|_{L^3} \leq C^{\frac{1}{3}}(R^{\frac{2+\bar{\alpha}}{3}}\|\theta\|_{L^2}^{\frac{2}{3}} + \frac{1}{R^{\frac{\alpha-\bar{\alpha}}{3}}}\|\Lambda^{1+\frac{\alpha}{2}}\theta\|_{L^2}^{\frac{2}{3}})\,.$

To finish let us take R sufficiently large together with (3.20), (3.25) and (3.15) to conclude that

$$\frac{1}{2}\frac{d}{dt}\int|\Lambda\theta|^2 dx \leq C\|\theta_0\|_{L^2}^2$$

from which (3.17) follows.

Finally let us consider

(3.26) $\frac{1}{2}\frac{d}{dt}\|\Delta\theta\|_{L^2}^2 = \int \Delta\theta\Delta(\theta_x H\theta)dx - \nu\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2$

$$\leq C\|\Delta\theta\|_{L^\infty}\|\Delta\theta\|_{L^2}\|\Lambda\theta\|_{L^2} - \nu\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2$$

and let us observe that

(3.27) $\|\Delta\theta\|_{L^\infty}^2 \leq C(\|\Lambda^{2+\frac{\alpha}{2}}\theta\|_{L^2}^2 + \|\theta\|_{L^2}^2).$

Therefore, by Holder's inequality,

$$\|\Delta\theta\|_{L^\infty}\|\Delta\theta\|_{L^2}\|\Lambda\theta\|_{L^2} \leq \frac{\delta}{2}\|\Delta\theta\|_{L^\infty}^2 + \frac{1}{2\delta}\|\Delta\theta\|_{L^2}^2\|\Lambda\theta\|_{L^2}^2,$$

and inequality (3.27) we estimate the first term at the right-hand side of (3.26), and conclude that choosing δ small enough,

$$\frac{d}{dt}\|\Delta\theta\|_{L^2}^2 \leq C(\|\Lambda\theta\|_{L^2}^2\|\Delta\theta\|_{L^2}^2 + \|\theta\|_{L^2}^2)$$

which implies the estimate

$$\|\Delta\theta\|_{L^2}^2 \leq \|\Delta\theta_0\|_{L^2}^2 e^{C\int_0^t \|\Lambda\theta\|_{L^2}^2 ds} + C\int_0^t \|\theta\|_{L^2}^2 e^{\int_s^t \|\Lambda\theta\|_{L^2}^2 d\sigma}ds.$$

By (3.15) and (3.17), (3.18) then follows for some large enough C.

3.2. *Small data results for* $\alpha = 1$. In the critical case $\alpha = 1$ we have the following global existence result for small data.

THEOREM 3.3. *Let* $\nu > 0$, $\alpha = 1$, $0 \leq \theta_0 \in H^1$ *and assume that the initial data satisfy* $\|\theta_0\|_{L^\infty} < \nu$. *Then there exists a unique solution to* (1.1) *which belongs to* H^1 *for all time* $t > 0$.

Proof. From the previous inequality (3.19) we have for $\alpha = 1$

$$(3.28) \qquad \frac{1}{2}\frac{d}{dt}\int|\Lambda^{\frac{1}{2}}\theta|^2 dx \leq \|\theta_0\|_{L^\infty}\|\Lambda\theta\|_{L^2}^2 - \nu\int|\Lambda\theta|^2 dx$$
$$= (\|\theta_0\|_{L^\infty} - \nu)\|\Lambda\theta\|_{L^2}^2,$$

which implies that if $\|\theta_0\|_{L^\infty} < \nu$, then

$$(3.29) \quad \|\Lambda^{\frac{1}{2}}\theta\|_{L^2}(t) \leq \|\Lambda^{\frac{1}{2}}\theta_0\|_{L^2} \quad and \quad \int_0^t \|\Lambda\theta\|_{L^2}^2 ds \leq C\|\Lambda^{\frac{1}{2}}\theta_0\|_{L^2}^2.$$

From (3.20) we get

$$(3.30) \qquad \frac{1}{2}\frac{d}{dt}\int|\Lambda\theta|^2 dx \leq \frac{1}{2}\int|\Lambda\theta|^3 dx - \nu\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}^2.$$

Since

$$\|\Lambda\theta\|_{L^3}^3 \leq \|\Lambda\theta\|_{L^2}^2 \cdot \|\Lambda\theta\|_{\text{BMO}}$$

and

$$\|\Lambda\theta\|_{\text{BMO}} \leq C\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}$$

(we refer to [14] for the corresponding definitions and properties of the functions of bounded mean oscillation (BMO)), we obtain

$$\frac{1}{2}\frac{d}{dt}\int|\Lambda\theta|^2 dx \leq C\|\Lambda\theta\|_{L^2}^2\|\Lambda^{\frac{3}{2}}\theta\|_{L^2} - \nu\|\Lambda^{\frac{3}{2}}\theta\|_{L^2}^2$$
$$\leq \frac{C^2}{4\nu}\|\Lambda\theta\|_{L^2}^4.$$

Together with inequalities (3.29) this allows us to complete the proof of the theorem.

Universidad Autónoma de Madrid, Madrid, Spain
E-mail address: antonio.cordoba@uam.es

Consejo Superior de Investigaciones Científicas, Madrid, Spain
E-mail address: dcg@imaff.cfmac.csic.es

Universidad Rey Juan Carlos, Madrid, Spain
E-mail address: mafontel@escet.urjc.es

References

[1]   G. R. Baker, X. Li, and A. C. Morlet, Analytic structure of 1D-transport equations with nonlocal fluxes, *Physica D* **91** (1996), 349–375.

[2]   A. L. Bertozzi and A. J. Majda, *Vorticity and the Mathematical Theory of Incompresible Fluid Flow*, Cambridge University Press, Cambridge, 2002.

[3]   P. Constantin, P. Lax, and A. Majda, A simple one-dimensional model for the three dimensional vorticity, *Comm. Pure Appl. Math.* **38** (1985), 715–724.

[4]   A. CÓRDOBA and D. CÓRDOBA, A maximum principle applied to Quasi-geostrophic equations, *Comm. Math. Phys.* **249** (2004), 511–528.

[5]   D. CHAE, A. CÓRDOBA, D. CÓRDOBA, and M. A. FONTELOS, Finite time singularities in a 1D model of the quasi-geostrophic equation. *Adv. Math.* **194** (2005), 203–223.

[6]   S. DE GREGORIO, On a one-dimensional model for the three-dimensional vorticity equation, *J. Statist. Phys.* **59** (1990), 1251–1263.

[7]   ———, A partial differential equation arising in a 1D model for the 3D vorticity equation, *Math. Methods Appl. Sci.* **19** (1996), 1233–1255.

[8]   M. A. FONTELOS and J. J. L. VELÁZQUEZ, A free boundary problem for the Stokes system with contact lines, *Comm. Partial Differential Equations* **23** (1998), 1209–1303.

[9]   D. W. MOORE, The spontaneous appearance of a singularity in the shape of an evolving vortex sheet, *Proc. R. Soc. London* A **365** (1979), 105-119.

[10] A. MORLET, Further properties of a continuum of model equations with globally defined flux. *J. Math. Anal. Appl.* **221** (1998), 132-160.

[11] ———, Some further results for a one-dimensional transport equation with nonlocal flux, *Comm. Appl. Anal.* **1** (1997), 315–336.

[12] T. SAKAJO, On global solutions for the Constantin-Lax-Majda equation with a generalized viscosity term, *Nonlinearity* **16** (2003), 1319–1328.

[13] S. SCHOCHET, Explicit solutions of the viscous model vorticty eqution, *Comm. Pure Appl. Math.* **41** (1986), 531–537.

[14] E. STEIN, *Harmonic Analysis*: *Real Variable Methods*, *Orthogonality and Oscillatory Integrals*, *Princeton Math. Series* **43**, Princeton Univ. Press, Princeton, NJ, 1993.

[15] Y. YANG, Behavior of solutions of model equations for incompressible fluid flow, *J. Differential Equations* **125** (1996), 133–153.

[16] M. VASUDEVA, The Constantin-Lax-Majda model vorticity equation revisited, *J. Indian Inst. Sci.* **78** (1998), 109–117.

[17] M. VASUDEVA and E. WEGERT, Blow-up in a modified Constantin-Lax-Majda model for the vorticity equation, *Z. Anal. Anwend.* **18** (1999), 183–191.