ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 6, Number 4, Fall 1976

FORMS OF SINGULAR ASYMPTOTIC EXPANSIONS IN LAYER-TYPE PROBLEMS

PACO A. LAGERSTROM*

1. **Introduction.** A crucial part of singular perturbation techniques is to find a correct form of the asymptotic expansions used. This form should neither be so general that it precludes an actual construction of a solution nor too restrictive for the problem studied. Other aspects of techniques such as matching are closely tied up with the construction of a correct form. In this paper we shall describe how more sophisticated techniques have developed from what today (1975) seem simple techniques, and also point out some problems which need further study. By analyzing the ideas underlying techniques used in solutions of a class of problem and finding their shortcomings one may extend these techniques so as to make them applicable to more difficult problems. The progressive development of techniques will be illustrated by various examples in this paper. We shall restrict ourselves to layer-type techniques (as defined in Lagerstrom-Casten) and for simplicity we shall mainly deal with comparatively simple second-order differential equations with one small parameter. Other cases will be briefly mentioned, but not discussed, in the last section. In the problems discussed here there will be an inner and an outer expansion. From these it is desirable to construct expansions which are uniformly valid over the entire interval considered. It should be emphasized that most of the equations considered are model examples designed to give simple illustrations of techniques used in solving actual physical problems. The introduction of singular-perturbation techniques have practically always come from applied mathematicians dealing with concrete physical problems.

If ϵ is the small parameter and x is the outer variable, one has to distinguish several functions of ϵ. First the stretching parameter used for the inner layer. It may be ϵ itself so that the inner variable is $x^* = x/\epsilon$. One has, however, also to consider the possibility of coordinate changes $y = f_1(x, \epsilon)$, $y^* = f_2(x, \epsilon)$, where y is of the order of x, and y^* is of the order of x^*. Sometimes the scale of the dependent variable may also be changed. The layer of rapid transition may occur at either endpoint (boundary layer) or in the interior (for which the name shock layer, or shock structure, is used). Thus the inner variable may be of the form, say, $(x - x_0)/\epsilon$ where x_0 is a constant. Secondly one has to find suitable expansion parameters for the inner and outer expansion.

*Research supported in part by NSF Grant GP-32157X2.

These are asymptotic sequences $\delta_j(\epsilon)$ and $\delta_j^*(\epsilon)$. This concept makes clear sense when one deals with Poincaré expansions. However, the a priori assumption that the inner and outer expansions are of the Poincaré form should not be made. One must, however, always have a systematic method for finding the forms of the expansions. The expansions uniformly valid in the entire interval are normally not of the Poincaré form. In many examples, however, the uniformly valid expansion can be simply constructed from the inner and outer expansions. Thirdly there are the gauge functions which form an asymptotic sequence $\zeta_j(\epsilon)$ and which measure to which order an asymptotic expansion is valid in a certain domain, whose endpoints may be functions of ϵ. These concepts are discussed more explicitly in § 2.

§ 3 gives a sequence of examples to illustrate various forms of expansions which may be used. The systematic study of layer-type singular perturbations was started, at least in this country, by Friedrichs and co-workers in the early forties by analyzing the ideas of Prandtl's boundary-layer theory. By a simple model example Friedrichs gave an intuitive justification for Prandtl's boundary-layer theory and also showed how one may obtain higher-order approximations. (Prandtl's proposed technique for obtaining higher-order terms turned out to be incorrect.) Example II analyzes a variant of this model example, essentially along Friedrichs' lines. However, we have added a discussion of the importance of transcendentally small terms. We have also added a negative comment that, due to integrated effects, one cannot find a solution by the inner expansion alone. The important concept of integrated effects is illustrated in Example I by a simple equation due to Kaplun. The same concept will appear in Example III.

Friedrichs' technique was, however, unable to cope with the problem of flow at low Reynolds numbers. His assumption that the expansion parameters and the gauge functions were simply integral powers of the stretching parameters precluded a solution of this problem. By a profound rethinking of the ideas of matching and that of the validity of an expansion, Kaplun, in the mid-fifties, was able to show how the low Reynolds number problem may be solved. This vastly extended the techniques and the understanding of singular perturbations. Examples III and IV are model equations, due to the author, constructed to give simple illustrations of Kaplun's ideas, whose importance have been grasped rather slowly by workers in the field. Example III demonstrates the idea, due to Kaplun, of successively improving the expansion parameters and also gives an example of integrated effects. Example IV shows how the intuitive ideas lead to construction of expansions which are not of the Poincaré form. It also discusses an im-

portant coordinate transformation due to Bush who, by analyzing the original solution, managed to recast it in Poincaré form. The equally important concepts of numbering and grouping of terms of an expansion, which basically are a consequence of Kaplun's ideas, are also discussed. Example V is an example, due to Boa, taken from an actual problem in chemical reactions, which also illustrates the failure of the use of Poincaré expansions.

Example VI is an old nonlinear model equation due to the author, meant to illustrate how in a simple equation one may, depending on the boundary conditions, obtain boundary layers at either endpoint and also interior layers of rapid transitions (shock layers). (Incidentally, this example also illustrates a variety of other phenomena, not discussed here.) The problem of the position of the shock layer is discussed. Computer results are also given for comparison. The inner expansion has been carried to higher order and gives, surprisingly, a uniformly valid solution.

Example VII discusses the use of correction boundary layers. Fraenkel has given two examples to show that Kaplun's intermediate matching method, which is heuristically justified (and whose difficulty of application has often been vastly exaggerated in the literature), is in some cases actually inferior to Friedrichs' formal matching rule, as extended by Van Dyke and Fraenkel. However, it is pointed out here that if one uses the correction boundary layer, i.e., the inner expansion of the exact solution minus the outer expansion, then, granting the assumptions made by Fraenkel, the difficulties disappear.

In order to shorten this paper we have only used examples previously discussed in the literature, to which the reader is referred for analytical details. The main purpose of this paper is to compare systematically the various expansion forms used. However, in many cases the author has introduced new ideas and solutions, not to be found in the literature.

As mentioned above, many of the problems studied here are model equations introduced to illustrate and give credence to solutions of actual problems in fluid mechanics. The reader not interested in this subject may, however, safely skip the references to fluid mechanics and concentrate on the actual model equations which are all relatively simple.

§ 4 briefly mentions some important phenomena not discussed here and also surveys some unsolved difficulties for the examples of this paper.

2. **Basic definitions.** We are given a function $f(x, \epsilon)$ and an asymptotic sequence of gauge functions $\zeta_j(\epsilon)$

$$\zeta_0 = 1, \zeta_{j+1} \ll \zeta_j. \tag{2.1}$$

We may consider transformations

$$y, y^* = \text{functions of } x \text{ and } \epsilon. \tag{2.2}$$

Here y is of the same order as x, and y^* is of different order (larger or smaller). (There may of course be many variables, y, y^*, y^{**}, etc., all of different order). The most general form of approximations would be a sequence

$$F_1(y, y^*, \epsilon), F_2, F_3, \text{etc.}, \tag{2.3a}$$

such that

$$\zeta_j^{-1}(f - F_j) \to 0 \text{ with } \epsilon, \tag{2.3b}$$

uniformly in some interval. The end points of the interval may depend on ϵ. We then say that is an approximation uniformly valid to order ζ_j in the interval considered. If the interval is the entire interval considered for f we say that F_j is uniformly valid everywhere to order ζ_j.

We shall with one exception (Example VII) restrict ourselves to the classical case for which $f(x, \epsilon)$ is defined implicitly by a differential equation (of second order) and two boundary conditions or a periodicity requirement.

The form (2.3) is often too general for the actual construction of approximations. In a layer-type problem one often (exceptions will be noted below) uses Poincaré-type expansions

$$F_n(y, \epsilon) = \sum_{k=0}^{n} \delta_k(\epsilon) A_k(y), \tag{2.4a}$$

$$F_n^*(y^*, \epsilon) = \sum_{k=0}^{n} \delta_k^*(\epsilon) A_k^*(y^*), \tag{2.4b}$$

valid in different intervals and where F_n and F_n^* match in some sense (to be discussed later). The $\delta_k(\epsilon)$ and $\delta_k^*(\epsilon)$ are asymptotic sequences, referred to as expansion parameters. The idea that endpoints of the interval may depend on ϵ was introduced systematically by Kaplun around 1955 as part of his reexamination of the ideas underlying layer-type techniques.

From the expansions (2.4) one may sometimes construct an expansion which is uniformly valid everywhere. The standard construction of the composite expansion yields expansions of the form

$$F_n(y, y^*, \epsilon) = \sum_{j=0}^{n} \delta_j(\epsilon) B_j(y, y^*, \epsilon). \tag{2.5}$$

In Kaplun's method of optimal coordinates (generalized by Van Dyke and Legner) the expansion valid everywhere does not have the form (2.5) but has a more general form. There are also other forms possible. This problem is discussed in Van Dyke [13, pp. 227–228].

3. Examples.

I. An Example of Switchback (Integrated Effect). This example is not a model for a physical problem but will give a simple mathematical illustration of a concept which we shall encounter later. We consider the problem (see Kaplun [7, p. 15])

$$\frac{d}{dx}\left(x^2\frac{df}{dx}\right) = \epsilon, f(1) = 0, f(e^{1/\epsilon}) = e^{-1/\epsilon}. \tag{3.1}$$

The solution is

$$f = x^{-1} - 1 + \epsilon \ln x. \tag{3.2}$$

If we neglect the right-hand side of the equation the solution is

$$\hat{f} = \frac{e^{-1/\epsilon}}{1 - e^{-1/\epsilon}}(1 - x^{-1}). \tag{3.3}$$

Thus the term of order ϵ in (3.1) gives rise to the term $(x^{-1} - 1)$ of order unity in (3.2), and (3.3) has no meaning as an approximation to (3.2). The reason is that ϵ acts over an interval which increases with ϵ, even though its value at the right endpoint decreases fast. Thus the *integrated effect* of a term of formal order ϵ gives rise to a term of order unity in the solution. The terminology as well as the example is due to Kaplun (who in his original unpublished manuscript replaced the second boundary condition by a matching condition). He also used the catchy but less objective term switchback. It indicates that if you assume the solution to be of order ϵ you are forced to backtrack and consider terms of order unity. One may of course make the interval seem less large by introducing the variable $t = \ln x$ and put $g(t, \epsilon) = f(x, \epsilon)$. The problem is then

$$\frac{d^2g}{dt^2} + \frac{dg}{dt} = \epsilon, g(0) = 0, g(1/\epsilon) = e^{-1/\epsilon}, \tag{3.4}$$

and the solution is

$$g = e^{-t} - 1 + \epsilon t. \tag{3.5}$$

Still, the interval increases in length as $1/\epsilon$ which, even if not exponentially large, is large enough to give rise to an integrated effect.

The notion of switchback is of great importance for singular perturbation techniques and has not been studied systematically. Some

discussion is found in Kaplun [7] and in Lagerstrom-Casten [9, p. 97ff].

II. Model Equation for Prandtl's Boundary-Layer Theory. We shall now discuss a simple variant of Friedrichs' model equation referred to in § 1. Several variants have been studied extensively. An especially interesting survey of these, including nonlinear equations, has been given recently by Erdelyi. We shall choose the simple variant discussed in detail in Lagerstrom-Casten [9, p. 76ff]:

$$\epsilon \frac{d^2f}{dx^2} + \frac{df}{dx} - a - 2bx = 0, \tag{3.6a}$$

$$f(0,\epsilon) = 0, f(1,\epsilon) = 1. \tag{3.6b}$$

One may construct outer and inner expansions which, using Fraenkel's notation, are

$$E_k f = \sum_{j=0}^{k} \epsilon^j f_j(x), \tag{3.7a}$$

$$H_k f = \sum_{j=0}^{k} \epsilon^j g_j(x^*), \epsilon x^* = x, \tag{3.7b}$$

where

$$\begin{aligned} f_0 &= (1 - a - b + ax + bx^2), f_1 = 2b - 2bx, f_j = 0, \text{ if } j > 1, \\ g_0 &= C_0(1 - e^{-x^*}), \\ g_1 &= C_1(1 - e^{-x^*}) + ax^*, \\ g_2 &= C_2(1 - e^{-x^*}) + bx^{*2} - 2bx^*, \\ g_j &= 0, \text{ if } j > 2. \end{aligned} \tag{38.b}$$

We note that we have a very simple type of singular perturbation problem. The small parameter ϵ multiplies the highest derivative so that the reduced equation ($\epsilon = 0$ in 3.6a) can satisfy only one boundary condition. By standard methods one finds that the condition at $x = 1$ should be retained for the outer expansion and that a boundary layer of thickness ϵ occurs at $x = 0$. The gauge functions $\zeta_j(\epsilon)$ and the expansion parameters $\delta_j(\epsilon)$ and $\delta_j^*(\epsilon)$ are simply ϵ^j. Furthermore the whole outer expansion may be obtained independently of the inner expansion. By matching, to be discussed below, one finds

$$C_0 = (1 - a - b), C_1 = 2b, C_2 = 0. \tag{3.9}$$

Matching can be done simply by using the formal rule

$$E_k H_k f = H_k E_k f. \tag{3.10}$$

To the author's knowledge this rule was first used systematically by Friedrichs (private communication some thirty years ago). It has been generalized by Van Dyke and discussed thoroughly and corrected by Fraenkel. Later, examples will be given where (3.10) and the generalizations mentioned are inapplicable.

If we form the composite expansion in the standard way

$$C_k f = E_k f + H_k f - E_k H_k f, \tag{3.11}$$

we find

$$E_k C_k f = E_k f, \; H_k C_k f = H_k f, \tag{3.12}$$

which seems to indicate that $C_k f$ is an approximation uniformly valid to order ϵ^k in the entire interval $0 \leqq x \leqq 1$. The reason for subtracting the last term is to avoid duplication due to the fact that $E_k f$ and $H_k f$ contain common terms. This duplication may also be avoided by introducing the perturbation (or correction) boundary-layer. One forms the difference

$$D_k f = f - E_k f \tag{3.13}$$

and obtains the boundary-layer correction to $D_k f$,

$$H_k D_k f = -(C_0 + \epsilon C_1) e^{-x^*}, \; k \geqq 1. \tag{3.14}$$

In this way one eliminates all polynomial terms in $H_k f$ which, one feels, really belong to the outer expansion. The idea of a perturbation (or correction) boundary layer has often been used. However, in most cases, as in the present example, the idea is more of a "nicety." There are only two terms in the inner expansion of $D_k f$, but the computations are hardly shortened. We shall see later, in discussing Example VII, that in certain cases it is of essential importance.

We note that $E_k f$ and $H_k f$ are limit-process expansions, formed in the manner of a Taylor series. Applying the outer limit (x fixed, $\epsilon \to 0$) one obtains f_0. The same limit, applied to $\epsilon^{-1}(f - f_0)$, yields f_1. Similarly, repeated application of the inner limit (x^* fixed, $\epsilon \to 0$) yields g_0, g_1, and g_2.

The classical formal matching procedure expressed by (3.11) hides the deeper meaning of matching, and in some important cases it is not applicable. In studying flow at low Reynolds numbers (model equations will be discussed below), Kaplun introduced the concept of intermediate matching. We notice that f_0 is an approximation to order

unity in any interval

$$I_1 = [\eta_1(\epsilon) \leqq x \leqq 1], \epsilon \ll \eta_1(\epsilon), \tag{3.15a}$$

and that g_0 is valid in the interval

$$I_2 = [0 \leqq x \leqq \eta_2(\epsilon)], \eta_2 \ll 1. \tag{3.15b}$$

The common domain, or domain of overlap, is then

$$I_3 = \{\eta \mid \epsilon \ll \eta \ll 1\}, \tag{3.15c}$$

that is, if x is of order η with η in I_3, then both expansions are valid.

If the overlap regime is known and contains η, it is easily seen that, if $\eta x_\eta = x$, then

$$\lim_\eta (f_0 - g_0) \equiv \lim_{\epsilon \to 0, x_\eta \text{fixed}} (f_0 - g_0) = 0. \tag{3.16}$$

This determines C_0. In practice it is often superfluous to actually rewrite f_0 and g_0 in terms of x_η. One simply forms $f_0 - g_0$ and notices that some terms cancel identically and others tend to zero with ϵ if the order of x is not too large or too small. However, for pedagogical purposes it is recommended to do the complete version a couple of times, and in difficult cases it may be wise to proceed carefully (see for instance Cole's discussion of the van der Pol relaxation oscillation).

Unlike the older matching principle intermediate matching is not formal but based on intuitive ideas. It also works when, say, the inner expansion is not a limit-process expansion (see Examples IV and V below). The problem is of course to determine the domains of validity of the inner and outer expansions and thus the domain of overlap. Kaplun introduced heuristic principles based on a consideration of the equation for f_j and g_j. This principle is, however, somewhat risky. It is discussed in Lagerstrom-Casten [9]. (After publication of that paper it was found by Casten that the domain of overlap of $E_1 f$ and $H_1 f$ for Example II is somewhat smaller than stated in the paper. The function $\eta = \epsilon \ln \epsilon$ does not lie in the overlap domain to order ϵ even though $\epsilon \ll \eta$. Thus (4.21d) of the paper needs correcting. For the lower limit one may for instance take ϵ^{1-a} where a is an arbitrarily small strictly positive number. This still leaves plenty of overlap. It also furnishes another example of the well-known fact that log functions are treacherous.) One danger is switchback (see I, above), a concept which needs more exploration. Some discussion is given in Kaplun and also in Lagerstrom-Casten.

It may be worthwhile to describe the matching for the present case in very simple terms. The first term of the inner expansion has to pick up $f_0(0)$. In addition it has to satisfy the first inner equation, and

the inner boundary condition, which yields $f_0(0)(1 - e^{-x^*})$. The term e^{-x^*} is irrelevant for the outer expansion. Similarly, ϵg_1 has to pick up the slope of f_0 at $x = 0, f_0'(0) = a$, since $ax = \epsilon a x^*$; in addition it must contain any correction of order ϵ to the outer expansion at $x = 0$ — in this case $2b$ (which comes from $\epsilon f_0''$). This accounts for the terms ax^* and $2b$ in g_1. Finally $\epsilon^2 g_2$ has to pick up the quadratic term of f_0 at the origin $f''(0)(x^2/2) = bx^2 = \epsilon^2 b x^{*2}$, the linear term of $ef_1 = -2\epsilon bx = -2\epsilon^2 bx^*$, and finally the constant of order ϵ^2 for the outer expansion at $x = 0$ (which in this case is zero). If f_0 had had a cubic term $\sim x^3, f_1$ a quadratic term $\sim x^2$ and f_2 a linear term $\sim x$ and f_3 a constant term $f_3(0)$, these would have to be picked up by $\epsilon^3 g_3$. These are terms which need not vanish for x^* large (unlike e^{-x^*}). In the present case, $g_2 \neq 0$ even though $f_k = 0, k > 1$ and $g_k = 0, k > 2$. Similar remarks would apply to the more general case in which $-a - bx$ is replaced by a function $h(x)$ which at $x = 0$ has a longer power series $h(x) = h(0) + xh'(0) + (x^2/2)h''(0) + \cdots$ or a similar function which depends on ϵ. This is the reason for the duplication of terms in the inner and outer expansion. In $D_k f = f - E_k f$ an appropriate number of these terms will disappear. The correction boundary layer in our special case $H_k D_k f$ would terminate with the second term (of order ϵ).

The exact solution of (3.6) is

$$f = (1 - a - b + 2\epsilon b)\frac{1 - e^{-x^*}}{1 - e^{-1/\epsilon}} + bx^2 + ax - 2\epsilon bx, \tag{3.17}$$

and agrees with the composite expansion except for the term $e^{-1/\epsilon}$ which is called transcendentally small because

$$e^{-1/\epsilon} \ll \epsilon^n, \text{ all } n. \tag{3.18}$$

This formula is, however, not uniformly valid in n in the sense that for a fixed $\epsilon < 1$, no matter how small, ϵ^n tends to zero as $n \to \infty$ while $e^{-1/\epsilon}$ is fixed. In fact, $e^{-1/\epsilon}$ may be equal to or larger than ϵ^n for moderate values of n and values of ϵ normally regarded as small. From the power series for e^x one finds that

$$\epsilon^2 > e^{-1/\epsilon}, \epsilon > 0. \tag{3.19a}$$

The smallest exponent for which equality occurs for $\epsilon > 0$ is

$$\begin{gathered} e = 2.718. \\ \epsilon^e \text{ is tangent to } e^{-1/\epsilon} \text{ at } \epsilon = e^{-1} = .3679. \end{gathered} \tag{3.19b}$$

Numerical calculations give the following approximate results

$$\epsilon^3 \leqq e^{-1/\epsilon} \text{ for } .2205 \leqq \epsilon \leqq .5534, \tag{3.19c}$$

$$\epsilon^4 \leqq e^{-1/\epsilon} \text{ for } .1161 \leqq \epsilon \leqq .6995. \tag{3.19d}$$

Denoting the composite expansion by h we find that the error is 0 at $x = 0$, increases with x, and at $x = 1$ it is

$$f - h = (1 - a - b + 2\epsilon b)e^{-1/\epsilon}. \tag{3.20}$$

This is also the relative error since $f(1) = 1$. For $\epsilon = 1/2$ it is approximately $.1353(1 - a)$. These numbers show that the terms called transcendentally small may be numerically important. In (3.1), or a similar equation with $a + 2bx$ replaced by a power series in x, one may truncate the composite expansion for some integer k and find the equation for $f - C_k f$ and the corresponding boundary-conditions for this function and solve this problem taking the transcendentally small terms into account. The possible methods of doing this will not be discussed here. We make two final comments. First, if one writes (3.6) in the inner variable x^*, one obtains, with $g(x^*, \epsilon) = f(x, \epsilon)$ and prime denoting derivative with respect to x^*,

$$g'' + g' - \epsilon a - 2\epsilon^2 b x^{*2} = 0, \tag{3.21a}$$

$$g(0) = 0, g(1/\epsilon) = 1. \tag{3.21b}$$

This does not turn the problem into a regular perturbation problem because terms containing ϵ have an integrated effect as discussed in Example I. One may of course easily solve (3.21) with $\epsilon = 0$ in (3.21a), keeping (3.21b), but the result is wrong. If one replaces ϵ^{-1} by ∞ there is no solution.

As a second comment we observe that the outer *limit* of $f(x)$ at $x = 0$ is 0. Thus the outer limit has a discontinuity at $x = 0$. However, the outer expansion is only valid in some interval which does not contain $x = 0$. Thus the $f_0''(x)$ should not be used at $x = 0$. This is fortunate since otherwise the term $f_0''(x)$ in the equation for f_1 would contain a term proportional to $\delta'(x)$. Thus if limit-processes are used too piously, one obtains equations with the double disadvantage of being complicated and of being wrong.

III. Model Equation for Incompressible Flow at Low Reynolds Number. The model equation below was introduced by the author to give a simple illustration of Kaplun's solution (and explanation) of the century-old Stokes paradox and the relation of the Stokes and Oseen equations. Kaplun's basic ideas were first published in Lagerstrom-Cole [10] (see "The Method of Kaplun," p. 873ff). The same method was later used by Proudman and Pearson [12] to solve the Whitehead

paradox. As stated earlier, the reader not interested in fluid dynamics need not bother about the technical terms used above. He is, however, advised to keep in mind some intuitive interpretation (for instance as a nonlinear equation for spherically symmetric temperature equilibrium) of the model equation below. In the easily available literature, this equation is most fully discussed in Lagerstrom-Casten [9] and the analytical derivations of various formulas are omitted here.

The equation is

$$\frac{d^2f}{dx^2} + \frac{n-1}{x}\frac{df}{dx} + f\frac{df}{dx} = 0, n = 2, 3, \tag{3.22a}$$

$$f(\epsilon, \epsilon) = 0, f(\infty, \epsilon) = 1. \tag{3.22b}$$

Intuitive reasoning shows that the first term of the outer expansion is $f_0 = 1$. Because of the first boundary condition this cannot be uniformly valid at the "body" (i.e., $x = \epsilon$), and we therefore introduce an inner variable by $\epsilon x^* = x$. The equation for the leading term of the inner expansion is then

$$\frac{d^2g_0}{dx^{*2}} + \frac{n-1}{x^*}\frac{dg_0}{dx^*} = 0, \tag{3.23a}$$

and we must retain the inner boundary condition

$$g_0(x^*, \epsilon) = 0, \text{ at } x^* = 1. \tag{3.23b}$$

We first study the case $n = 2$. The solution of (3.23) is then

$$g_0 = B \ln x^*. \tag{3.24}$$

This does not satisfy the boundary condition at infinity (Stokes paradox), but from the point of view of singular perturbation techniques this is irrelevant. Kaplun's original reasoning, reproduced for the model example in Lagerstrom-Casten [9], makes it intuitively plausible that f_0 and g_0 have a (very small) domain of overlap, and matching gives

$$B = \varphi(\epsilon) + o(\varphi(\epsilon)); \varphi(\epsilon) = -1/\ln \epsilon. \tag{3.25}$$

We notice several significant differences from the previous example. Whether written in x or x^* the highest derivative in (3.22a) is not multiplied by ϵ (or any function thereof which is $o(1)$). As a consequence the outer expansion cannot be determined independently of the inner expansion as in the previous example. The gauge functions $\zeta_k(\epsilon)$ and the expansion parameters are, at least for the beginning of the outer and inner expansions, powers of $\varphi(\epsilon)$ rather than of ϵ, whereas x^* is still determined by dividing x by ϵ . It can also be seen intuitively that

the inner limit of f is zero which of course cannot match with the outer limit which is unity. Thus, from the point of view of matching, the term $g_0(x^*)$ should be considered as the first term of the inner expansion. This shows the idea of correct numbering in (2.4a, b). One should put $\delta_0 = 1$ and $\delta_0^* = B$. The outer expansion $f_0 + \epsilon f_1 + \cdots$, is continued by solving the appropriate equation for f_1 and matching with g_0. This matching also gives the constant b_1 in the second term of the inner expansion, $\varphi^2 b_1 \ln x^*$. Actually all terms of the inner expansion satisfy (3.23), since the last term of (3.22a) remains transcendentally small relative to $\varphi(\epsilon)$. Thus we may always assume $g_j(x^*) = B_j(\epsilon) \ln x^*$. One may then refine the expression for B given by (3.25). If one replaces φ by

$$\bar{\varphi} = -1/(\ln \epsilon + \gamma), \gamma = \text{Euler's constant} = .5772, \tag{3.26}$$

then g_0 still matches with f_0, and the second inner term is zero. (From a numerical point of view this is not an improvement. The "small" parameter φ is ∞ for $\epsilon = 1$, and $\bar{\varphi}$ is ∞ for $\epsilon = e^{-\gamma} = .561$. However, the idea is important and numerically better values for φ may be obtained by considering higher-order terms. This basic idea was discovered by Kaplun in 1955. Van Dyke [13] calls it telescoping and illustrates its importance by a very instructive figure on p. 244. It seems hardly worthwhile to consider transcendentally small terms, due to $f(df/dx)$ until a numerically reasonable value of B has been obtained.) This procedure can be continued, however in order to do so one must carry the outer expansion to a correspondingly high order. In fact the entire inner expansion, if one neglects transcendentally small terms, may in principle be written as $\ln x^*$ times a function of ϵ.

The outer expansion is of less interest in the present context and will not be given here.

We shall now consider the case $n = 3$. The leading term of the inner expansion is now

$$g_0(x^*) = C_0(1 - 1/x^*), C_0 = 1. \tag{3.27}$$

It happens to satisfy the outer boundary condition, but this is a coincidence as will be seen by considering the second term which we assume to be of the form ϵg_1. Its equation is

$$d^2g_1/dx^{*2} + 2/x + dg_1/dx^2 = -(1/x^* - 1)(1/x^*)^2. \tag{3.28}$$

The solution satisfying the inner boundary condition is

$$g_1 = -(\ln x^*)/x^* - \ln x^* + B_1 \ln x^*. \tag{3.29}$$

As will be seen later, g_1 does not satisfy the outer boundary condition (Whitehead's paradox), but again this is irrelevant. Matching with $f_0 + \epsilon f_1$, where

$$f = -E_2(x), \qquad E_n(x) = -\int_x^\infty e^{-t} t^{-n}\, dt, \tag{3.30}$$

gives $B_1 = -\ln \epsilon$. Thus the second term of the inner expansion contains a term of order ϵ and a term of order $\epsilon \ln \epsilon$. From the point of view of matching these should be combined into one term. Thus when the inner expansion has the form (2.4b), it is necessary occasionally to group terms with different coefficients of ϵ into one term.

The occurrence of a term of order ϵ is an example of an integrated effect: A term of formal order ϵ gives rise to a term of order $\epsilon \ln \epsilon$ in the solution. The effect is weaker than in Example I.

Actually in the present example the inner expansion is contained in the outer expansion. This was noticed and explained by Kaplun [7] for the corresponding problem in fluid dynamics and is also discussed in Lagerstrom-Casten [9, pp. 103, 104]. This somewhat freakish phenomenon does not occur in the modified example which we shall discuss below.

For $n = 1$ the problem is regular. In fact the intuitive reasoning that $f_0 = 1$ is no longer valid. See [9, p. 93].

For ϵ large, the expansion in powers of $\eta = 1/\epsilon$ is regular (whereas the corresponding case in fluid dynamics has in general a very complicated expansion). There are many examples which show that if one takes a couple of terms in the expansion, for large and small values of a parameter, the results may agree nicely for some intermediate value of the parameter.

IV. Model Equation for Compressible Flow at Low Reynolds Numbers. This example was introduced by the author to show that, contrary to what used to be stated in the literature, the Stokes equation is linear for incompressible flow only by accident. It should not be derived by linearization but by a limit process applied to the full equation. Again forgetting about fluid mechanics we consider the problem

$$\frac{d^2 f}{dx^2} + \frac{1}{x}\frac{df}{dx} + f\frac{df}{dx} + \left(\frac{df}{dx}\right)^2 = 0, \tag{3.31a}$$

$$f(\epsilon, \epsilon) = 0, f(\infty, \epsilon) = 1. \tag{3.31b}$$

It differs from (3.22) for $n = 2$ by the addition of $(df/dx)^2$ to the equation. Again we find that the first term of the outer expansion is $f_0 = 1$ and that it should match with a function g_0 satisfying the equation corresponding to (3.23).

$$\frac{d^2g_0}{dx^{*2}} + \frac{1}{x^*}\frac{dg_0}{dx} + \left(\frac{dg_0}{dx^*}\right)^2 = 0, \tag{3.32a}$$

$$g_0(x^*, \epsilon) = 0 \text{ for } x^* = \epsilon^{-1}x = 1. \tag{3.32b}$$

The general solution of (3.31) is

$$g_0 = \ln(1 + B \ln x^*). \tag{3.33}$$

Note that (3.32) is nonlinear, and hence B no longer enters as a multiplicative constant. Matching gives

$$B = \varphi(\epsilon)(e - 1), \varphi(\epsilon) = -1/\ln \epsilon. \tag{3.34}$$

Again, as in the previous case for $n = 2$, one may cancel some subsequent terms of the inner expansion by successively refining the choice of B. This necessitates finding corresponding terms of the outer expansion which will not be discussed here. (For details see [9] and references given there). The reason is that if $B = (dg_0/dx^*)_{x^*=1}$ (which in a physical problem often is a physically significant quantity) is known, one has two boundary conditions for (3.31a) at $x^* = 1$ and can obtain a complete solution. Furthermore the term $f(df/dx)$ is transcendentally small compared to $\varphi(\epsilon)$. The same two statements are valid for the previous example for $n = 2$. (In example II ($n = 2$) all $g_j(x^*)$ obey the same equation. In the present case this is not true. However, successive terms $g_j(x^*)$, using $B = (e - 1)\varphi(\epsilon)$, may be obtained from $g_0(x^*, B)$, using a better value of B and developing in $\varphi(\epsilon)$.) The parenthetical comment following (3.26) still applies to the problem of considering transcendentally small terms.

The essential feature of the present example is that the inner expansion is not an inner-limit expansion (it is not of the Poincaré form (2.4b)) even if the terms are numbered and grouped properly. In fact, g_0 would contribute to every term in the inner-limit expansion. The leading term in the inner-limit expansion, assuming B small, is

$$\tilde{g}_0(x^*) = B \ln x^*. \tag{3.35}$$

Now $\tilde{g}_0(x^*)$ can be easily matched to $f_0 = 1$ by various rules. The only trouble is that the value thus obtained for B is wrong, it is in fact the value obtained in Example III (for $n = 2$). The reason is, of course, that $\tilde{g}_0(x^*)$ does not overlap with $f_0 = 1$. One needs the full expression (3.33) for matching. Kaplun's heuristic ideas for domains of validity

(together with his extension theorem) show that one must have a solution of the full equation (3.31) in order to obtain overlap with the domain of validity. Incidentally, for the present equation Cohen and Lagerstrom (unpublished) have given a rigorous proof that the statements made above are correct. This may be gratifying. However, to the authors a proof of a result obtained by intuitive reasoning is uninteresting unless the proof itself introduces some new mathematical ideas. A proof that such a result is wrong would have been much more interesting since it would require a serious rethinking of the underlying ideas.

We have thus found an example for which the inner expansion is not of Poincaré form. The leading term is a function of x^* and ϵ which is not of the form $\delta_0^*(\epsilon)A_1^*(x^*)$. There is still, however, a well-defined method for constructing it.

Using the above solution as a starting point, Bush has shown that by a change of variables it may be brought into Poincaré form. If one defines

$$y = -\ln x, y^* = \varphi(\epsilon)y, \varphi = -1/\ln \epsilon, \tag{3.36}$$

the inner expansion is then of the form

$$Y_0(y^*) + \varphi(\epsilon)Y_1(y^*) + \cdots, \tag{3.37a}$$

where

$$Y_0(y^*) = \ln[1 + (e - 1)(1 - y^*)]. \tag{3.37b}$$

This may have certain advantages but it does not invalidate the rule that one should not always look for a Poincaré-type expansion.

V. An Example from Theory of Chemical Reactions. Boa [1] has studied the following equation

$$d^2u/dt^2 + (A^2/u^2 + 2u - \nu)du/dt + A^2(1 - 1/u) = 0, \tag{3.38}$$

for $\nu \gg 1$. There then exists a limit cycle which is a relaxation oscillation. If one uses singular perturbation techniques for finding the relaxation oscillation, see for instance Cole [3, p. 38ff], one finds that the equation for the very rapid part of the motion is

$$d^2u/dt^2 + (A^2/u^2 + 2u - \nu)du/dt = 0. \tag{3.39}$$

In order to get reasonable results one needs to keep both the term A^2/u^2 and the term $2u$. Although for ν large we use layer-type techniques the solution will obviously not be of the Poincaré form.

VI. A NONLINEAR MODEL EQUATION FOR SHOCK LAYERS, ETC. The equation is

$$\epsilon\, d^2y/dx^2 + y\, dy/dx - y = 0, \tag{3.40a}$$

with the boundary conditions

$$y(x_0) = Y_0, y(x_1) = Y_1. \tag{3.40b}$$

This example was introduced by the author in the fifties for a Caltech discussion group on singular perturbation. Its purpose was to show how a shock layer can occur and how its position may be determined. This layer may in fact start as a boundary layer at $x = 0$ and then, depending on the values of Y_0 and Y_1, wander across the interval (in the interior it is a shocklayer) and finally becomes a boundary layer at $x = 1$. An unexpected bonus was the discovery by Kaplun of the possibility of a corner layer. Full details of the first approximation are given by Cole [3, p. 29ff] who also discusses another unexpected phenomenon, that of the transition layer. We refer to Cole [3, p. 29ff.] for a discussion of the first approximation and shall discuss here some aspects of higher order approximations and give some values obtained by computing. For future references we shall give various forms of (3.40). Putting

$$\begin{gathered} \sqrt{\epsilon}\, t = x, \sqrt{\epsilon}\, f(t, \epsilon) = y(x, \epsilon), \\ v = df/dt = dy/dx, \end{gathered} \tag{3.41a}$$

we obtain

$$d^2f/dt^2 + f\, df/dt - f = 0. \tag{3.41b}$$

Putting

$$\epsilon u = x, g(u, \epsilon) = y(x, \epsilon), w = dg/du = v/\epsilon, \tag{3.42a}$$

we obtain

$$d^2g/du^2 + g\, dg/du - \epsilon g = 0. \tag{3.42b}$$

The equations are invariant under an obvious group, translation of the independent variable. We may first assume that $f(0) = 0$, and hence $f''(0) = 0$. This yields solutions odd in t. The general discussion will refer to odd solutions. For specific boundary-value problems we shall have to introduce a shift of the origin. The phase-plane equations for (3.41) are

$$df/dt = v, dv/dt = f(1 - v). \tag{3.43}$$

The phase portrait is given in [3, p. 35]. We shall only consider the

case for which $v > 1$ (although the general discussion will also be applicable to other cases). For $\epsilon \ll 1$ there is then a layer of rapid transition given by a hyperbolic tangent. We shall first assume it to be centered around the origin. By a shift of the origin this layer may become a boundary layer or a shock layer in a given region.

Integrating (3.43) we find, assuming $v > 1$,

$$f^2 + v + \ln(v - 1) = C, \tag{3.44a}$$

and, for $f(t)$ odd,

$$f^2/2 + (v - v(0)) + \ln \frac{v - 1}{v(0) - 1} = 0. \tag{3.44b}$$

We note that as $|f| \to \infty$, $v \to 1$, so that for t large $f \sim t$. We also see that v has a maximum $v(0)$ at $f = 0$ (for $v > 1$).

We shall now study the perturbation problem, with (3.40a) as the outer equation and (3.42b) as the inner equation. Assuming

$$y = y_0 + \epsilon y_1 + \epsilon^2 y_2 \cdots, \tag{3.45}$$

we find (disregarding the solution $y_0 = 0$ since we study the case $v > 1$) that the odd solution is

$$y_0 = x, y_j = 0, \text{for } j > 0. \tag{3.46}$$

Shifts in the x-origin will be discussed in connection with specific boundary-value problems. We notice that (3.46) is consistent with the fact that the linear term of f at infinity is t. Since (3.40a) is equivalent to (3.41b), it is clear that the expansion of y in powers of ϵ needs to be complemented by an inner expansion.

For the solution of the inner equation we assume

$$g \cong g_0 + \epsilon g_1 + \cdots. \tag{3.47}$$

The equation for g_0 is

$$d^2 g_0/du^2 + g_0\, dg_0/du = 0, \tag{3.48}$$

which has the odd solution, depending on a parameter β which we may assume $\geqq 0$,

$$g_0 = \beta \tanh(\beta \mu/2). \tag{3.49}$$

(3.48) has the obvious group

$$g_0 \to a g_0, u \to a^{-1} u, w_0 \to a^2 w_0 \tag{3.50}$$

which corresponds to the integral

$$g_0{}^2/2 + w_0 = w_0(0). \tag{3.51}$$

From this derive

$$g_0^2(\infty)/2 = w_0(0) \tag{3.52a}$$

which also follows from the exact solution,

$$g_0(\infty) = \beta,\ w_0(0) = \beta^2/2. \tag{3.52b}$$

We notice that the group (3.50) takes one phase curve into another and could be derived directly from the integral (3.49). There is a corresponding, but considerably more complicated, group derivable from the integral (3.44). The integral may also be used to relate the behavior of f at infinity and at zero. However, we shall not pursue this further in the present paper.

The equations for the higher order approximations are

$$g_1'' + (g_0 g_1)' = g_0, \tag{3.53a}$$

$$g_2'' + (g_0 g_2)' = g_1(1 - g_1'), \text{ etc.} \tag{3.53b}$$

The left hand side of the equations are the same, for a given g_0. An odd solution of the homogeneous equation is found by differentiating g_0 with respect to β,

$$\psi_1 = \frac{\partial g_0}{\partial \beta} = \tanh(\beta u/2) + \frac{\beta u}{2\cosh^2(\beta u/2)}. \tag{3.54a}$$

An even solution is found by differentiating with respect to u

$$\psi_2 = (2/\beta^2)\, g_0' = \cosh^{-2}(\beta u/2). \tag{3.54b}$$

A particular solution for g_1, $\psi_p^{(1)} = \psi_p$, may be found by quadrature. Since, for large values of u, $g_0 \sim \beta$ we expect $\psi_p \sim u$, u large. This is verified by detailed computation. Defining

$$G_0 = \int_0^u g_0(s)\, ds = 2 \ln \cosh(\beta u/2), \tag{3.55a}$$

$$H_0 = e^{G_0} = \cosh^2(\beta u/2) \frac{\cosh \beta u + 1}{2} = \frac{\beta^2}{2g_0'} \tag{3.55b}$$

$$\begin{aligned} K_0 &= \int_0^u H_0(s)\, ds = \frac{\sinh \beta u}{2\beta} + u/2 \\ &= (1/\beta) \sinh(\beta u/2) \cosh(\beta u/2) + u/2. \end{aligned} \tag{3.55c}$$

we find a particular solution from

$$g_1' + g_0 g_1 = G_0 \tag{3.56}$$

which gives

$$\psi_p = H_0^{-1}(u) \int_0^u H_0(s)G_0(s)\, ds. \tag{3.57a}$$

To better show its behavior for large u we transform it by integration by parts, using (3.55c),

$$\begin{aligned}\psi_p = {} & \ln\cosh(\beta u/2)((2/\beta)\tanh(\beta u/2) + uH_0^{-1}) \\ & - (1/\beta)\tanh(\beta u/2) + H_0^{-1}(u/2 - (2/\beta)\int_0^{\beta u/2} s\tanh s\, ds).\end{aligned} \tag{3.57b}$$

For large values of u we find, within terms exponentially small in u,

$$\psi_1 \sim 1, \psi_2 \sim 0, \psi_p \sim u - 1/\beta - (2/\beta)\ln 2. \tag{3.58}$$

Thus for u large $g_1(1 - g_1')$ is exponentially small and the equation for g_2 has a particular solution which is exponentially small for u large. The homogeneous equation for g_2, and for any g_k, $k \geqq 1$, has the same general solution as that for g_1.

We note that if g_1 obeys (3.56), then $g_1'(0) = 0$. This is then true individually for ψ_p and for ψ_2 which may be obtained from (3.57a) by shifting the lower limit of integration. Thus, using (3.54a)

$$\psi_p'(0) = \psi_2'(0) = 0, \psi_1'(0) = \beta. \tag{3.59}$$

We shall now discuss boundary-value problems for the region $x_0 = 0, x_1 = 1$. The various types of phenomena occurring depend on the values of $Y_0 = y(0)$ and $Y_1 = y(1)$. This, as well as the role of the equation for y as an outer equation and the equation for g as an inner equation, are discussed by Cole [3] and will not be repeated here. We shall assume that Y_j is of order unity, (which includes $Y_0 = 0$), in other words independent of ϵ.

First consider the case of a boundary-layer at $x = 0$. The first outer solution is

$$y_0 = x + Y_1 - 1, \tag{3.60a}$$

and the first inner solution is

$$g_0(u) = \beta\tanh\left[\frac{\beta}{2}(u + k)\right], \tag{3.60b}$$

where

$$\beta = Y_1 - 1 \tag{3.60c}$$

and k is determined from

$$Y_0 = \beta\tanh(\beta k/2). \tag{3.60d}$$

If we now consider $g_0 + \epsilon g_1$ we see that at large distances in u its leading term is $\epsilon u = x$ which matches with (3.46). Since g_0 already matches the constant term of the outer solution we must put

$$g_1 = \psi_p + C\psi_1 \tag{3.61a}$$

(ψ_2 is negligible for u large) and select C so that for u large the constant term of g_1 is zero.

From (3.58) we find

$$C = 1/\beta + (2/\beta) \ln 2, \tag{3.62a}$$

which gives, from (3.59)

$$w_0'(0) + \epsilon w_1'(0) = \beta^2/2 + \epsilon(1 + 2 \ln 2).$$

However $g_0 + \epsilon g_1$ has now an error of order ϵ at $x = 0$. A simple way of handling this is a correction of k. Writing $k = k_0 + \epsilon k_1 + \epsilon^2 k_2 \cdots$ we satisfy the boundary condition to higher order by demanding

$$g_0(k) + \epsilon g_1(k) + \cdots = Y_0. \tag{3.63}$$

Note that a shift in w does not affect the matching conditions. The first term k_0 is identical with the k determined by (3.60d). Since the constant of the outer solution has already been matched we must write each g_ℓ, $\ell > 0$, as

$$g_\ell = \psi_p^{(\ell)} + D_\ell \psi_2, \tag{3.64}$$

where for $\ell > 1$, $\psi_p^{(\ell)}$ is a particular solution which is exponentially small for u large. The D_ℓ are then in principle determined by (3.64). Thus it seems that in principle our perturbation method may be used to satisfy the boundary condition at $x = 0$. However, at $x = 1$ there will be an "exponentially small" error which, as we have seen in Example II, may be significant. We shall not discuss this difficulty here. Instead we shall turn to another problem which shows the difficulty even more clearly.

We consider the case of an interior shock layer. The corresponding range of Y_0 and Y_1 is given in [3]. The outer solution is discontinuous. It has a left part and a right part

$$y_{0L} = Y_0 + x, \; y_{0R} = Y_1 - 1 + x. \tag{3.65a}$$

The discontinuous jump occurs where $y_{0L} + y_{0R} = 0$, that is, at

$$x = x_d = (1 - Y_0 - Y_1)/2. \tag{3.65b}$$

The inner solution (shock layer) is to leading order

$$g_0 = \beta \tanh(\beta/2)(u - x_d), \; 2\beta = Y_1 - Y_0 - 1. \tag{3.66}$$

For the next approximation we may choose

$$g_1 = \psi_p(u - x_d) + C_1\psi_1(u - x_d), \tag{3.67}$$

where as before, C_1 is chosen so that the constant term at infinity vanishes, i.e., by (3.62a). But, unlike the previous example, the inner solution satisfies no boundary conditions, only matching conditions. Thus our scheme gives us no guidance for what to do with the ψ_2 part of g_1 or, more generally, of $g_\ell, \ell > 0$. We know only that if $\psi_p^{(\ell)}$, $\ell > 1$ is exponentially small for $u - x_d$ large, then the coefficient of ψ_1 in g_ℓ must be zero.

There is a further difficulty. Let us define x_c as the crossing point of the exact solution, i.e.,

$$y(x - x_c) = 0. \tag{3.68}$$

Let us take a concrete example

$$\begin{aligned} Y_0 &= -1/2, Y_1 = 1, \\ x_d &= 1/4, \beta = 1/4. \end{aligned} \tag{3.69}$$

As ϵ tends to infinity the solution tends to the straight line

$$y = 3x/2 - 1/2. \tag{3.70}$$

We expect x_c to vary with ϵ. (3.70) gives us $x_c = 1/3$ at $\epsilon = \infty$, and, as ϵ decreases to zero, we expect x_c to decrease to $x_d = 1/4$. Numerical computations (the author is indebted to Elliot Fischer, graduate student at Caltech, for doing all the numerical work), using the shooting method, give the following table

ϵ	x_c	$(dy/dx)_{xc}$	$(dy/dx)_{x=0}$	$(dy/dx)_{x=1}$
.5	(.3211, .3212)	1.587486	1.499577	1.290778
.1	(.2836, .2837)	1.960218	1.455651	1.016623
.05	(.2635, .2636)	2.364419	1.318689	1.000242
.025	(.2523, .2524)	3.084508	1.101985	1.000000
.01	(.2500, .2501)	5.1407	1.000970	1.000000

Table 1. $y(0) = -1/2, y(1) = 1, y(x - x_c) = 0$

For ϵ small the shooting method becomes very sensitive to the assumed value of the slope at $x = 0$, and for $\epsilon = .01$ about a dozen trials were needed. We note from Table 1 that dy/dx approaches its asymptotic value $(=1)$ better at $x = 1$ than at $x = 0$ which of course is due to the fact that $1 - x_c$ is considerably larger than x_c.

The results agree with the qualitative ideas. They also show that for practical purposes, ϵ starts to get "small" only below .05. This fact is further confirmed if one performs a perturbation method treating ϵ as large. The problem then becomes a regular perturbation problem and y may be developed in powers of $\eta = 1/\epsilon$ whose coefficients are polynomials in x the computation of which is straight-forward but tedious. For $\epsilon = 1/2$, $\eta = 2$ four terms give very good results. This can no longer be expected when η is in the neighborhood of .01. As anticipated x_c moves from 1/3 to $x_d = 1/4$ as ϵ decreases. However, our present method does not indicate how this is to be described analytically. An examination of the exponentially small errors in the boundary conditions seems to be necessary. (The value of $g_0 + \epsilon g_1$, as computed above, with $\epsilon = .01$ and $x_c = x_d = -1/4$, is $y = 1.000$ at $x = 1$ and $y = -.5039$ at $x = 0$. Thus the boundary condition at $x = 1$ is better satisfied than that at $x = 0$. As explained above this is to be expected since the function has more time to reach its asymptotic value as x varies from 1/4 to 1 than when x varies on an interval only a third as long.)

The integral (3.44b) determines v as a function of f with $v(0)$ as parameter. Thus we have, since $y = \sqrt{\epsilon} f$,

$$\epsilon^{-1/2} = \int_{Y_0\epsilon^{-1/2}}^{Y_1\epsilon^{-1/2}} df/v = I(v(0), \epsilon). \tag{3.71}$$

For given values of Y_0 and Y_1 we then find $v(0)$ as a function of ϵ and, in principle, the boundary-value problem can be solved. This cannot be done analytically but may be adaptable to perturbation methods. In general, any improvement of the perturbation solution found above will have to involve transcendentally small terms.

The deadline for the present paper prevents the author from examining the present example further. It is hoped that the discussion will be continued in a subsequent paper.

VII. Examples by Fraenkel. In a study of the formal matching principle (3.10) and its generalizations, Fraenkel [5] gives examples in order to show that it may be much superior to matching based on overlap. A different interpretation of Fraenkel's result will be given here. Fraenkel ([5], part I, pp. 216, 217) first studies a given function and then (part II, p. 245ff) a function defined by a two-point boundary-value problem. We shall only discuss the first example and then present some abstract arguments about the general situation. We present a nonessential simplification of Fraenkel's function. Let

$$0 < \epsilon \ll 1, \epsilon \leqq x \leqq 1, \epsilon x^* = x, U(x^*) = \sum_{j=0}^{N} (x^*)^{-j},$$

(3.72)

$$V(x) = \sum_{j=0}^{N} x^j, f(x, \epsilon) = U(x^*)V(x).$$

We shall assume that N is a finite number large enough to make the example interesting. One finds

$$E_p f = \left(\sum_{j=0}^{p} (x^*)^{-j} \right) V(x), H_p f = U(x^*) \sum_{j=0}^{p} x^j, \tag{3.73a}$$

$$H_p E_p f = \left(\sum_{j=0}^{p} (x^*)^{-j} \right) \left(\sum_{j=0}^{p} x^j \right) = E_p H_p, \tag{3.73b}$$

$$f - C_p f = o(\epsilon^p). \tag{3.73c}$$

($C_p f$ is defined as in (3.11)). Now

$$f - E_p f \ll \epsilon^q \text{ implies } x^* \gg \epsilon^{-q/(p+1)}, \tag{3.74a}$$

$$f - H_q f \ll \epsilon^q \text{ implies } x^* \ll \epsilon^{-1/q+1)}. \tag{3.74b}$$

If we want overlap to order ϵ^q we must then have $p \geqq q^2 + q$. Thus assume that $V(x)$ is defined above, but that $U(x^*)$ has some unknown coefficients to be determined by matching, or more generally that f is defined by a differential equation. To determine the coefficients in $H_2 f$ by overlap we need to find $E_6 f$, whereas the much simpler matching principle $H_2 E_2 f = E_2 H_2 f$ suffices. To study this in more generality we shall make the following assumptions which are certainly fulfilled in the above example and also in the example given by Fraenkel ([5], part II, p. 245ff). We assume that $f(x, \epsilon)$ has an outer Poincaré expansion $E_p f$ and an inner expansion $H_q f$. The inner expansion is used near $x = 0$ or near $x = \epsilon$. In these expansions one should carefully group and number the terms as discussed above and also by Fraenkel. (His reasoning differs from ours. It is not certain that the results always agree.) We furthermore assume

$$H_p E_p f = E_p H_p f, \tag{3.75a}$$

and that $f - C_p f = o(\zeta_p(\epsilon))$ in the entire interval where the ζ_p are suitable gauge functions, say ϵ^p. Instead of f we now consider the difference function defined by

$$D_k f = f - E_k f. \tag{3.75b}$$

We now expand $D_k f$ and compare the expansion with $D_k f$. One finds

$$D_k f - H_k D_k f = f - E_k f - H_k f + H_k E_k f. \tag{3.76}$$

By assumption this difference is $o(\zeta_k)$ in the entire interval. Similarly

$$D_k f - E_k D_k f = f - E_k f \tag{3.77}$$

which is $o(\zeta_k)$ in any interval $[a, 1]$, $a > 0$ (and hence by Kaplun's extension theorem in a slightly larger interval $\eta(\epsilon) \ll x \leqq 1$, $\eta \ll 1$). Thus the inner and outer expansions of $D_k f$ overlap. Actually, the inner approximation, the perturbation boundary-layer $H_k D_k f$, is a uniformly valid approximation to $D_k f$ in the entire interval. If one then adds $E_k f$ to $H_k D_k f$ one obtains a composite expansion of f valid to order ζ_k.

Thus, to the author, Fraenkel's examples show, not that matching based on overlap is unwieldy, but that it may be of great advantage to work with the correction boundary layer, i.e., one should consider the expansions of $D_k f$ rather than those of f. As mentioned, in the Example II the use of $D_k f$ is a nonessential alternative method, but in Fraenkel's examples it is of essential importance.

4. Conclusions. We have seen through a succession of examples how the techniques of layer-type singular perturbations has been successively refined. The classical Example II was, at the time it was introduced by Friedrichs, very illuminating. However, because of its simple nature it has also been misleading. The singular nature of the problem was obvious from the fact that the small parameter multiplied the highest order derivative (this, unfortunately was for some time considered the criterion for a layer-type problem). Each term may be constructed by limit process, and the inner and outer expansions are of Poincaré type with ϵ^j as expansion parameters, and the term multiplied by ϵ^j is the $(j + 1)$th term in each expansion so there is no problem of numbering or grouping of terms. There are no integrated effects and the composite expansion is formed in a very simple way. However, even in this example the transcendentally small terms should be considered (which happens to be simple for the problem in question). A major break with or rather extension of the, by present-day standards, simple methods which work for Example II was made by Kaplun. His method leads to a solution of Examples III and IV. In this case the expansion parameters are not integral powers of ϵ and there are integrated effects (to the authors knowledge the first example of an integrated effect was found by Proudman-Pearson, essentially using Kaplun's methods). Proper attention must be paid to the correct ordering and grouping of terms. Since we still deal with Poincaré type expansions they may be technically obtained (a posteriori!) as

limit process expansions, but this hides their true nature. Examples IV and V show that one should not a priori assume that the expansions are of Poincaré type.

There are many remaining problems. The question of transcendentally small terms must be further examined. The use of coordinate transformations (as exemplified by Bush's treatment of Example IV) should be investigated systematically. The same is true for the construction of composite expansions. Various methods are discussed by Van Dyke [13, pp. 227–228]. However, difficulties arise when the inner solution does not have exponential decay (as in Example IV) and when there is an interior shock layer. The variation of the position of the shock layer with ϵ may pose problems (as in Example VI). Also, while the author believes that his discussion of Example VII is correct, an intuitive justification is needed. More generally, Kaplun's heuristic ideas why equations have solutions with domain of overlap must be further developed, especially in view of integrated effects. There is also the problem why matching outside the domain of overlap can be made. For instance, in Example III one may find the unknown constant in $g_0(x^*)$ by applying the outer limit to the difference $f_0 - g_0$. Kaplun told the author (in connection with the real physical problem) that this was obvious but gave no explanation why it was obvious.

The author wishes to emphasize that all examples, except Example VII ultimately come from "honest" physical problems. One may of course invent odd-ball problems which may show a bewildering variety of the forms of the expansions of their solutions. Some of these may be useful for getting new ideas. A main conclusion of this paper, and of the study of other equations, is the platitudinous advice that if existing techniques do not work for a specific problem, try to extend these techniques to suit the problem, but try to understand *why* they work.

We may finally ask the question: Why, in this age of highspeed computing, use perturbation techniques (regular or singular)? To the author the importance of an expansion is that it shows the qualitative form of the solution and the dependence on the parameter. For Example VI the solution $g_0 + \epsilon g_1$ gives us a good idea of the qualitative nature of the solution, even if it is not numerically reliable except for very small values of ϵ. On the other hand the expansion of the same problem for $\eta = 1/\epsilon$ small works numerically well even for $\epsilon = 1/2$, but the terms in the expansion are dumb polynomials which do not give a quick insight into the nature of the solution and its dependence on parameters.

The author has the pious hope that perturbation solutions may serve as a guidance for computer solutions, even though this idea seems at present to be going out of fashion. However, the author knows of several instances for which extensive (and expensive) computer results have disagreed with the qualitative ideas of what the solution should look like, and the computer results have subsequently been shown to be erroneous. Conversely, it must be admitted that computer experiments have lead to new qualitative results, subsequently explained analytically. Computation is also useful in determining the crucial question of the range for which ϵ may be regarded as small.

Partial differential equations have not been considered here. A bewildering variety of forms are needed. Many examples are given in [3]. The phenomenon of sublayers is often of importance. Only one example will be mentioned here, which shows how, in dealing with actual physical problems, one is forced to introduce some very sophisticated ideas which are far removed from the by now standard ideas of singular perturbation methods. This is a new technique for dealing with sublayers successfully applied by Stewartson, Messiter and Sychev. References and a short summary of this technique is given in Lagerstrom [8, pp. 208–209, 211–212].

Another topic omitted here is the problem of several small parameters. The order of one parameter relative to the other parameter as both tend to zero is crucial in such cases.

References

1. James A. Boa, *A model biochemical reaction*, Doctoral Thesis, Caltech, 1974. The example used here will appear in J. of Math. Analysis and Applications.

2. Wm. B. Bush, *On the Lagerstrom mathematical model for viscous flow at low Reynolds number*, SIAM J. Appl. Math. **20** (1971), 279–287.

3. J. D. Cole, *Perturbation Methods in Applied Mathematics*, Blaisdell, Waltham, Mass., 1968.

4. A. Erdelyi, *A Case History in Singular Perturbations*. Lecture given at the International Conference on Differential Equations, University of Southern California, September 3–7, 1974. Proceedings published by Academic Press, 1975.

5. L. E. Fraenkel, *On the method of matched asymptotic expansions*, Pro. Cambridge Philos. Soc., Part I, 65, pp. 209–231, Part II, 65, pp. 233–261; Part III, 65, pp. 263–283, (1969).

6. K. O. Friedrichs, *Theory of viscous fluids*. Fluid Dynamics, Chapter 4. Brown Univ., 1942.

7. Saul Kaplun, *Fluid Mechanics and Singular Perturbations*, P. A. Lagerstrom, L. N. Howard and C. S. Liu, eds., Academic Press, New York, 1967.

8. P. A. Lagerstrom, *Solutions of the Navier-Stokes Equation at Large Reynolds Number*, SIAM J. Appl. Math., **28** (1975), 202–214.

9. P. A. Lagerstrom and R. G. Casten, *Basic Concepts Underlying Singular Perturbation Techniques*, SIAM Review, **14** (1972), 63–120.

10. P. A. Lagerstrom and J. D. Cole, *Examples illustrating expansion pro-*

cedures for the Navier-Stokes equations, J. Rational Mech. Anal., **4** (1955), 817–882.

11. Hartmut H. Legner, *On optional coordinate and boundary-layer theory*, Ph.D. Thesis, Stanford, July 1971.

12. I. Proudman and J. R. A. Pearson, *Expansions at small Reynolds numbers for the flow past a sphere and a circular cylinder*, J. Fluid Mech., **2** (1957), 237–262.

13. M. Van Dyke, *Perturbation Methods in Fluid Mechanics*, Academic Press, New York, 1964. Annotated edition, Parabolic Press, P.O. Box 3032, Stanford, 1975.

DEPARTMENT OF APPLIED MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CALIFORNIA 91125