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FORMS OF THE DISSIPATION FUNCTION FOR A CLASS OF VISCOPLASTIC MODELS

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The minimum properties that allow a dissipation functional to describe a behavior of viscoplastic type are analyzed. It is considered a material model based on an internal variable description of the irreversible processes and characterized by the existence of an elastic region. The dissipation functional derived includes the case of time-independent plasticity in the limit. The complementary dissipation functional and the flow rule are also stated. The model analyzed leads naturally to the fulfillment of the maximum dissipation postulate and thus to associative viscoplasticity. A particular class of models is analyzed, and similarities to and differences from diffused viscoplastic formats are given.

1. Introduction

Strain-rate-dependent behavior is characteristic of many materials at least beyond a certain level of stress, temperature or strain rate. Strain-rate sensitivity and time-temperature superposition effects occur when the time scale of the process is comparable with a characteristic relaxation time of the material. In this paper, we consider the case of a relatively short relaxation time characterizing irreversible phenomena such as plasticity and damage, leading to a generalized viscoplastic model. For such processes, irreversible deformations do not develop instantaneously, and also the apparent yield stress is modified according to the velocity of strain. The same observations apply to other phenomena like damage evolution or hardening, which can be described by additional internal variables in a similar way to the strain (see [Contrafatto and Cuomo 2002] for more details).

Viscoplasticity, introduced systematically by [Rabotnov 1969; Green and Naghdi 1965; Needleman 1988; Krempl 1975; Valanis 1971], to report only some of the earliest contributions to the subject, has received renewed attention in conjunction with the development of advanced models incorporating other phenomena, like hardening-softening behavior, nonassociative flow rules, anisotropy, etc. [Hall 2005; Phillips and Wu 1973; Zienkiewicz et al. 1975]. In most cases, an evolution

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law for the anelastic variables is postulated, which satisfies the second law of thermodynamics, like was done for instance in the original Perzyna [1966] or Duvaut and Lions [1972] proposals, which are largely employed in numerical models. Thermodynamic reformulation of the generalized Perzyna and Duvaut–Lions models were also contributed [Ristinmaa and Ottosen 2000; 1998; Runesson et al. 1999; Perić 1993]. In these papers, the authors, in an attempt to better fit complex material behaviors, proposed an extension of the models using a dynamic yield locus, a function of the internal thermodynamic forces and/or of the viscoplastic strain rates, and a (complementary) dissipation function based on a decomposition of the conjugated forces. They showed that the formulation satisfies the dissipation inequality and that the postulate of maximum dissipation is fulfilled when an associated flow rule is used. Generalization to nonlocal models of plasticity have been proposed [Aifantis et al. 1999; Voyiadjis et al. 2004; Forest 2009], often as a mean for regularizing strain localization in softening materials. Gurtin [2003] proposed a framework for strain gradient small-deformation viscoplasticity. He introduces both polar stresses (third order) and microstresses (second order). However, both vanish when the dependency of the constitutive equations on the strain gradient is disregarded.

In this work, it is shown how viscoplastic constitutive relations can be consistently derived from a properly defined dissipation potential of the irreversible strain rates. The model is implemented within the generalized standard material model, in the definition given by Germain [1962] and Halphen and Nguyen [1975], which derives the constitutive laws from the specification of two potential functionals, the internal energy and the dissipation. Only in the second will an internal time scale be introduced, in order to model a viscoplastic-like behavior. In this way the dissipation inequality will be automatically fulfilled. Conjugated to the dissipation functional, a function of the plastic strain rate, is the complementary dissipation functional, a function of the internal thermodynamic forces. The latter allows one to obtain the flow rules for the plastic rates.

The objective of the paper is to state sufficient conditions for the dissipation functional in order to describe a viscoplastic-type behavior, which in addition admits the existence of an elastic region. Once these conditions will be stated, the complementary dissipation functional will be obtained and from it the flow rule for the internal variables. We will present a case for which the expressions derived can be obtained in a closed form. The answer to a similar question has been given in the case of inviscid plasticity [Eve et al. 1990; Romano et al. 1993]. Therefore, we wish for the dissipation potential for viscoplasticity to include the one for time-independent plasticity as a limit case. The analysis will be carried out in the hypothesis of small deformations so that the kinematic variables will be additively split into reversible and irreversible components. The irreversible component accounts both for time-dependent and time-independent strains.

The main result of the paper is as follows. While a dissipation function for time-independent plasticity has to be a positively homogeneous function (*hodo*) of the plastic strain rate, that is, the gauge function of a set of irreversible strain rates (that will be shown to be polar to the set of the elastic stresses), in order to obtain a time-dependent flow rule, it is necessary to add in the expression of the further dissipation terms, specifically positively homogeneous functions of degree $n > 1$ (*hodn*), which will be recognized to be gauge-like functions. It will be shown that they are powers of the gauge function of a closed set, which again will be identified with the polar set of the elastic domain. Then an expression for the complementary dissipation functional in terms of the gauge function of the elastic domain will be derived. The result will be an overstress model of viscoplasticity.

In the framework of the thermomechanics of dissipative materials [Maugin 1999], the derivation of the constitutive equations from appropriate energy functionals is a standard procedure. However, especially in the case of time-dependent irreversible behavior, usually some specific form of the dissipation potential is postulated, derived from a known rheological model, like in [Houlsby and Puzrin 2002]. The result of this paper relative to a general form of a dissipation functional that gives rise to an overstress viscoplastic model appears new. Although the model is only sufficient for a time-dependent plastic evolution model, there are indications that it may be a general result. For instance, in a recent series of papers, Goddard [2014; Kamrin and Goddard 2014] derived viscoplastic dissipation potential for granular materials starting from Edelen's work [1973] on nonlinear generalization of the classical Rayleigh–Onsager dissipation potentials. In addition to prove a general form of symmetry relations, he presented a form of dissipation potentials for viscoplastic laws that turned out to be a homogeneous function of degree $n > 1$. He also derived a complementary dissipation functional that is analogous to the one obtained in this work in the particular case that the dissipation function is given by only one term.

The theory in this paper is presented for the case of local models of deformation only. Its extension to higher gradient theories like those proposed in [dell'Isola et al. 2015; Placidi 2016; Neff et al. 2014] is possible although there are technical details that need to be carefully analyzed.

In the following section, the results anticipated in Section 1 will be systematically derived. Then a uniaxial example will be presented. Numerical results are not included in this work. The model obtained, also in the case when an explicit form of the flow rule cannot be stated, is amenable to a simple numerical treatment. A detailed examination of the numerical algorithm, which takes advantages of some results established in an earlier work [Contrafatto and Cuomo 2005] will be presented elsewhere.

The paper makes consistent use of convex analysis. Only sporadically are the introduced definitions explicitly stated. The reader can refer to standard texts of convex analysis for the details [Rockafellar 1970]. However, the most important mathematical definitions used in the paper are briefly reviewed in Appendices A and B.

2. Phenomenological constitutive model

The viscoplastic material considered is characterized by the existence of an elastic domain such that no irreversible deformation is associated to stress states belonging to it. Plastic deformations occur otherwise. We consider the case that stresses beyond the elastic limit are allowed (overstress), in which case delayed plastic strains occur. Thus, the description does not cover all the models proposed for viscoplasticity, like power law models, etc.

The standard generalized material model introduced by [Germain 1962; Halphen and Nguyen 1975] is adopted, which can be synthetically described by the following assumptions.

(1) The equilibrium state of the system is described by a set of state variables, which include internal variables in addition to strain. The former account phenomenologically for the modification of the internal structure of the material and rule hardening, damage and other phenomena. In the present paper, the kinematic variables describing the state of the system will be collected in the vector η, which in general includes the macroscopic strain, and other variables, as described in [Contrafatto and Cuomo 2002]. In the present work, no specific constitutive model will be analyzed, so the variable η will be left undefined.

(2) Each kinematic variable is decomposed into a reversible (elastic) and an irreversible part. In the paper, the linearized deformation theory is used so that an additive decomposition into an elastic recoverable part and an inelastic irrecoverable (plastic) strain is considered: $\eta = \eta_e + \eta_p$.

(3) The state of the system is determined by the functionals of the free energy and of the specific dissipation, $e(\eta_e)$ and $d(\dot{\eta}_p)$, the first a function of the reversible part of the internal variables and the second a function of the rate of their irreversible part only.

(4) By standard thermodynamic arguments, the internal driving forces, which in the paper are indicated by τ and which in general include stress and other thermodynamic forces dual to the internal variables, are obtained by differentiating the free energy,

$$\tau = \partial_{\eta_e} e(\eta_e), \tag{1}$$

where the symbol ∂ denotes subdifferentiation, in order to account for the common case of nonsmooth energy functionals. The internal forces are dual to the kinematic variables in the sense of the virtual power

$$P_i = \langle \tau, \dot{\eta} \rangle, \tag{2}$$

where the brackets denote the inner product in the appropriate vector space.

(5) Conjugated potentials are derivable through a Fenchel transformation. The dual potentials are indicated by the index "c":

$$\begin{aligned} e(\eta_e) + e^c(\tau) &= \langle \tau, \eta_e \rangle, \quad \tau \in \partial_{\eta_e} e(\eta_e), \quad \eta_e \in \partial_\tau e^c(\tau), \\ d(\dot{\eta}_p) + d^c(\tau) &= \langle \tau, \dot{\eta}_p \rangle, \quad \tau \in \partial_{\dot{\eta}_p} d(\dot{\eta}_p), \ \dot{\eta}_p \in \partial_\tau d^c(\tau). \end{aligned} \tag{3}$$

Sometimes in the paper, following a consolidated tradition in mathematical papers, instead of the index "c", the conjugated function to $f(x) : X \to \mathbb{R}$ will be indicated by $f^*(x^*) : X^* \to \mathbb{R}$. In the previous expressions, X and X^* are dual vector spaces.

3. The dissipation potential

The main results of the paper are presented in this section. First the case of rate-independent plasticity is examined, recalling classic results concerning the dissipation functional. Then they are generalized to the case of overstress models, in the hypotheses stated in Section 1. Throughout the paper, it will be assumed that the dissipation functional as well as the internal energy potential are convex functions. Nonconvex energy potentials, which have been introduced for several phenomena, are therefore excluded from the present treatment. The minimum properties required for a dissipation functional for reproducing a time-independent plastic behavior were stated in [Romano et al. 1993]; see also [Eve et al. 1990]. The key feature for obtaining time-independent plasticity is that the dissipation function, in addition to being subadditive, be positively homogeneous of degree 1. It has been suggested that a characteristic relaxation time is introduced if the dissipation function is homogeneous of degree 2 in its argument [Maugin 1990]. The aim of this section is to analyze the requisites that give rise to a time-dependent dissipation.

The inviscid case. In order to examine the rate-independent case, the following statements are needed. Their proofs can be found in [Romano et al. 1993]. They follow from the results that a hodo proper convex function $f(x)$ is the support function of the set C°, the polar of the closed convex set $C = \{x : f(x) \le 1\}$, and that, since for a hodo function $f(0) = 0$, the former set coincides with the subdifferential of the function at 0.

Statement. *If the dissipation function is sublinear* (*hodo and subadditive*), *then the thermodynamic forces* τ *given by* (1) *are such that*

$$\tau \in \partial d(\dot{\eta}_p) \subset K = \partial d(0) \tag{4}$$

and

$$d(\dot{\eta}_p) = \sup_{\tau\in k}\langle\tau, \dot{\eta}_p\rangle = \operatorname{supp} K \doteq \psi_K; \tag{5}$$

consequently, *the conjugate dissipation potential is* (K *is convex and closed*)

$$d^c(\tau) = \operatorname{ind} K. \tag{6}$$

In the previous statement, *supp* and *ind* denote the support and the indicator function of a convex set, respectively, and they are defined in Appendix B. In the paper, the support function is also denoted by ψ. A consequence of the above theorem is the following:

Corollary. *For all elastic stress states*, $d^c(\tau) = 0$.

Proof. From Fenchel's equality, one has, assuming $\dot{\eta}_p = 0$,

$$\langle\tau, \dot{\eta}_p\rangle = 0 = d(0) + d^c(\tau),$$

where the thermodynamic force is conjugated to the strain rate, that is, $\tau \in \partial d(0)$. The conclusion follows immediately recalling that $d(0) = 0$. □

It is useful to recall some results of convex analysis that apply to a convex hodo function, as is the case examined in this section.

First the concept of gauge is recalled. A gauge $\gamma(x \mid C)$ of a set C is the function

$$\gamma(x \mid C) = \inf\{\mu \geq 0 : x \in \mu C\}. \tag{7}$$

It can also be thought of as the positively convex hodo function generated by $\operatorname{ind} C(x) + 1$ (Figure 1, left). A gauge function is any function $k(x)$ such that $k(x) = \gamma(x \mid C)$ for some C. The set C for which $k(x)$ is a gauge is exactly $C = \{x : k(x) \leq 1\}$.

Taking the conjugate of gauge functions of convex sets establishes a polarity correspondence between closed convex sets. A set C° is called the polar of C if

$$C^\circ = \{x^* : \operatorname{supp} C(x^*) \leq 1\} = \{x^* : \langle x, x^*\rangle \leq 1 \text{ for all } x \in C\}. \tag{8}$$

It is easy to see that, if C is closed, convex and containing the origin, then the polar set C° is also closed, convex and contains the origin, and the gauge function of C is the support function of C° and vice versa.

In order to extend the polarity correlation to more general convex functions, it is convenient to define the polar of a generic gauge $k(x)$ as

$$\begin{aligned} k^\circ(x^*) &= \inf\{\mu^* \geq 0 : x^* \in \mu^* C^\circ\} = \inf\{\mu^* \geq 0 : \langle x, x^*\rangle \leq \mu^* k(x) \text{ for all } x\} \\ &= \sup_{x\neq 0} \frac{\langle x, x^*\rangle}{k(x)}. \end{aligned} \tag{9}$$

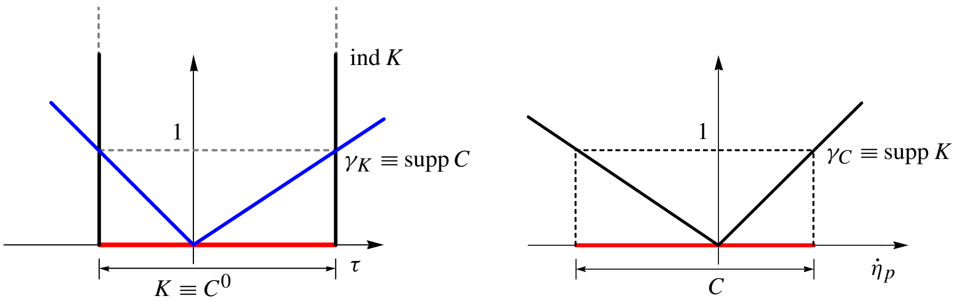

Figure 1. Gauge functions of the polar sets C and C° and the indicator function of the set K. Left: the support function of the set C. Its level set at 1 defines the polar set C°. Right: the support function of the set $K \equiv C^\circ$. Its level set at 1 defines the polar set C.

Then, denoting by C and C° two closed polar sets containing the origin, we have the following polarity correspondences:

spaces	$C \subset X$	$C^\circ \subset X^*$
gauge functions	$k(x)$	$k^\circ(x^*)$
gauges	$\gamma(x \mid C) = \operatorname{supp} C^\circ(x)$	$\gamma(x^* \mid C^\circ) = \operatorname{supp} C(x^*)$

The connection to the plastic potential is readily established by the following:

Statement. *The set* $C = \{\dot\eta_p : d(\dot\eta_p) \le 1\} = \{\dot\eta_p : \langle \tau, \dot\eta_p \rangle \le 1$ *for all* $\tau \in K\}$ *is polar to* $K = \partial d(0)$.

Proof. Let $\tau \in K = \partial d(0)$. Then $\langle \tau, \dot\eta_p \rangle \le d(\dot\eta_p)$ for all $\dot\eta_p$. In particular for $\dot\eta_p \in C$, $d(\dot\eta_p) \le 1$, so $\tau \in C^\circ$. □

The situation is represented in Figure 1 The gauge function of the set K is the support function of the set C whose level set at 1 is the elastic domain, and the gauge function of the set C is the support function of the set K, that is, the dissipation function. The level set at 1 of the dissipation function is the set C. Therefore, for all the plastic strain rates belonging to the boundary of the set C, the rate of dissipation is the same. In the case of associated plasticity with a smooth yield function $g(\tau) - \sigma_0 \le 0$, the plastic strain rate is given by $\lambda \nabla g(\tau)$, and it is easy to see that the plastic strain rates belonging to C are $\dot\eta_p \le \nabla g / \sigma_0$.

The viscoplastic case. The statements above describe a model of inviscid plasticity with a yield function for the generalized stresses. In this section, we shall derive a form of the dissipation function that generalizes the one given in the previous section for inviscid plasticity. The derivation, whose technical details need some care, will be built in several steps. First it will be assumed that it can be assumed for the dissipation function a positively homogeneous function of degree larger than 1.

It will be shown that this kind of function is compatible with the mechanical model of a rate-dependent material, but it doesn't admit the existence of elastic states. A convenient form for this function will also be given. Then it will be shown that taking the dissipation function as the sum of a hodo function plus a function homogeneous of degree $n > 1$ leads to describing the mechanical dissipation of a viscoplastic material with an elastic nucleus. Finally a general form for this class of dissipation functions will be proposed.

As stated above, let's assume that the dissipation function is convex and positively homogeneous of degree $n > 1$ (hodn). Preliminarily, we prove the following:

Statement. *For hodn (closed proper convex) dissipation functions d_n, with $n > 1$, the set $\partial d_n(0)$ contains only the zero element.*

Proof. By definition,

$$\tau \in \partial d_n(0) \quad \Longleftrightarrow \quad \langle \tau, \dot{\eta}_p \rangle \le d_n(\dot{\eta}_p) \quad \text{for all } \dot{\eta}_p. \tag{10}$$

Taking $\dot{\eta}_p = \mu \dot{\eta}_{p0}$, $\mu \ge 0$, one has from (10)

$$\mu \langle \tau, \dot{\eta}_{p0} \rangle \le \mu^n d_n(\dot{\eta}_{p0}) \quad \text{for all } \mu \tag{11}$$

and taking the limit as $\mu \to 0$, it follows that $\tau = 0$. □

The opposite implication is true only for a strictly convex dissipation function:

Statement. *If the dissipation potential, in addition to hodn, is strictly convex, then*

$$0 \in \partial d_n(\dot{\eta}_p) \quad \Longrightarrow \quad \dot{\eta}_p = 0.$$

Proof. If $0 \in \partial d_n(\dot{\eta}_p)$, then $0 \le d_n(\dot{\eta}_{p0}) - d(\dot{\eta}_p)$ for all $\dot{\eta}_{p0}$; that means that $d_n(\dot{\eta}_p)$ is a minimum for d_n, and since $d(\dot{\eta}_p) \ge 0$ for all $\dot{\eta}_p$, the statement follows from the strict convexity of d_n. □

A nonnegative convex hodn function is in general not a gauge, so the results of the previous section related to the inviscid case do not apply. Therefore, more general convex functions conjugate to each other have to be introduced. A real-valued function f is said to be gauge-like if $f(0) = 0$ and the various level sets

$$\{x : f(x) \le \alpha\}, \quad f(0) < \alpha < +\infty,$$

are all proportional, that is, are positive scalar multiples of a single set.

Lemma. *A function f_n, positively homogeneous of degree n, $n > 1$, is a gauge-like function.*

Proof. Since f_n is positively homogeneous of degree n, (a) $f_n(0) = 0 = \inf f_n$; introducing the notations $C_n = \{x : f_n(x) \le 1\}$ and $C_{np} = \{x : f_n(x) \le p\}$,

$$C_{np} = \{x : f_n(x) \le p\} = \{x : p^{-1} f_n(x) \le 1\} = \{x : f_n(p^{-1/n} x) \le 1\}, \tag{12}$$

that is, (b) $C_{np} = p^{1/n} C_n$. Properties (a) and (b) ensure that f_n is a gauge-like function. □

A theorem of convex analysis [Rockafellar 1970, §13] states that a closed convex hodn function f can always be expressed in the form

$$f(x) = \frac{1}{n} k(x)^n, \tag{13}$$

where k is the gauge of the closed set $C = \{x : k(x) \le 1\}$ containing the origin, also known as the Minkowski function (as we have seen it is positively homogeneous, convex and such that $k(0) = 0$).

Before proceeding further, it is convenient to introduce some definitions that allow us to use dimensionless quantities. Let $d_0 = \tau_0 \nu$, with τ_0 a characteristic stress (which may be thought of as an equivalent limit stress), and $\nu = \dot{\eta}_{p0}$ be the inverse of a characteristic time (with $\dot{\eta}_{p0}$ an equivalent strain rate). Setting $\hat{\tau} = \tau/\tau_0$, $\hat{\dot{\eta}}_p = \dot{\eta}_p/\dot{\eta}_{p0}$ and $\hat{d} = d/d_0$, it follows that $C = \{\hat{\dot{\eta}}_p : \hat{d}(\hat{\dot{\eta}}_p) \le 1\} = \{\dot{\eta}_p : d(\dot{\eta}_p) \le d_0\}$. With these notations, the gauge function of C, $k_C(\dot{\eta}_p)$, which as observed on page 223 is equal to the support function of the polar set to C, $C^\circ = \{\hat{\tau} : \langle \hat{\tau}, \hat{\dot{\eta}}_p \rangle \le 1 \text{ for all } \hat{\dot{\eta}}_p \in C\} = \{\tau : \langle \tau, \dot{\eta}_p \rangle \le d_0 \text{ for all } \dot{\eta}_p \in C\}$, which is equal to the elastic domain K (see the statement on page 223), becomes

$$k_C(\dot{\eta}_p) = \operatorname{supp} C^\circ = \sup_{\hat{\tau} \in C^\circ} \langle \hat{\tau}, \hat{\dot{\eta}}_p \rangle = \sup_{\tau \in C^\circ} \frac{\langle \tau, \dot{\eta}_p \rangle}{d_0} = \frac{\operatorname{supp} K}{d_0} \doteq \frac{\psi_K(\dot{\eta}_p)}{d_0}, \tag{14}$$

where the symbol ψ_K denoting the support function of the set K has been introduced for brevity. Similarly, the polar gauge $k^\circ(\tau)$ to C° will be denoted by $j_K(\tau) = \operatorname{supp} C$, which with the notations introduced is equal to

$$j_K(\tau) = \sup_{\dot{\eta}_p \in C} \frac{1}{d_0} \langle \tau, \dot{\eta}_p \rangle. \tag{15}$$

Based on the above lemma, the characterization of the dissipation and of the complementary dissipation functionals is given in the following statement.

Statement. *A positively homogeneous of degree n, $n > 1$, dissipation function (that is, gauge-like), is given by the form*

$$d_n(\dot{\eta}_p) = \frac{1}{n} \frac{[\psi_K(\dot{\eta}_p)]^n}{d_0^{n-1}}, \tag{16}$$

and the conjugated complementary dissipation function is

$$d_n^c(\tau) = \frac{n-1}{n} d_0 [j_K(\tau)]^{n/(n-1)}. \tag{17}$$

Proof. The form (16) follows directly from (13) and (14), the latter giving the gauge function of the set C. In order to prove (17), preliminarily the conjugated function to $(1/n)\psi_K^n$ is evaluated. By definition,

$$\begin{aligned}\Big(\frac{1}{n}\psi_K^n\Big)^c &= \sup_{\dot\eta_p}\Big\{\langle\tau,\dot\eta_p\rangle-\frac{1}{n}[\psi_k(\dot\eta_p)]^n\Big\}\\ &= \sup_{\dot\eta_p}\Big\{\langle\tau,\dot\eta_p\rangle-\frac{1}{n}\Big[\inf_{\mu\ge 0}\mu : d(\dot\eta_p)\le\mu d_0\Big]^n\Big\}\\ &= \sup_{\dot\eta_p}\Big\{\langle\tau,\dot\eta_p\rangle-\frac{1}{n}\Big[\inf_{\mu\ge 0}\mu : \dot\eta_p\in\mu C\Big]^n\Big\} = \sup_{\mu\ge 0}\sup_{\dot\eta_p\in\mu C}\Big\{\langle\tau,\dot\eta_p\rangle-\frac{1}{n}\mu^n\Big\}\\ &= \sup_{\mu\ge 0}\Big\{\mu\Big(\sup_{\dot\eta_p\in C}\langle\tau,\dot\eta_p\rangle\Big)-\frac{1}{n}\mu^n\Big\} = \sup_{\mu\ge 0}\Big\{\mu j_K(\tau)-\frac{1}{n}\mu^n\Big\}\\ &= \frac{n-1}{n}j_K^{n/(n-1)}. \end{aligned}\tag{18}$$

Next, observing that $(\lambda f(x))^c=\lambda f^c(x^*/\lambda)$ and the fact that ψ_K is a positively homogeneous function, we finally obtain

$$\begin{aligned} d_n^c = \Big(\frac{1}{n}\frac{\psi_K^n}{d_0^{n-1}}\Big)^c &= \frac{1}{d_0^{n-1}}\frac{n-1}{n}[j_K(\tau d_0^{n-1})^{n/(n-1)}]\\ &= \frac{n-1}{n}d_0[j_K(\tau)]^{n/(n-1)}. \end{aligned}$$

□

So if the dissipation potential d_n is positively homogeneous of degree n, its conjugate is positively homogeneous of degree $m=n/(n-1)$, with $1/n+1/m=1$.

Summarizing, it has been found that, if the dissipation function is a positively hodn, the only stress state conjugated to zero dissipation is zero, that is, the elastic domain reduces to the zero element alone. For any other stress state, the rate of plastic deformation is given by

$$\dot\eta_p\in\partial_\tau\frac{n-1}{n}[k^\circ(\tau)]^{n/(n-1)}=[k^\circ(\tau)]^{1/(n-1)}\partial_\tau k^\circ(\tau),\tag{19}$$

that is, the rate of plastic deformation is proportional to a gauge; therefore, an overstress effect is found. The larger n is, the smaller the plastic rate is. From Fenchel's identity, it can also be obtained that, if τ and $\dot\eta_p$ are a conjugated pair, the strain rate associated to an internal force $p\tau$ is $p^{1/(n-1)}\dot\eta_p$. Only the case $n=2$ yields proportional strain rates; in this case, both the dissipation function and the conjugated dissipation d_n^c are positively homogeneous of degree 2.

In order to obtain a viscoplastic model with a threshold value for the stress and that reduces to inviscid plasticity as the relaxation time vanishes, the dissipation functional may then be taken as a sum of closed convex proper hodn functions for increasing values of $n\ge 1$. In this way, the dissipation function and, according to (17), also the complementary dissipation function are expressed as series expansions. In consideration of the finding that, if d_n is positively homogeneous of degree n, d_n^c is positively homogeneous of degree $n/(n-1)$, two series expansions are considered for the dissipation functional, which can both be expressed in the form

$$d(\dot{\eta}_p) = \psi_K(\dot{\eta}_p) + \sum_{n=2}^{N} \frac{1}{n} \frac{1}{(\tau_0 \nu)^{q-1}} [\psi_K(\dot{\eta}_p)]^q \tag{20}$$

with

$$q = n \quad \text{or} \quad q = \frac{n}{n-1}. \tag{21}$$

In the following development, the choice will be left unspecified, and only in the final example will the two cases be differentiated. In the expression of the dissipation appears the sum of a hodo sublinear functional, and of other hodn terms, with $n > 1$.

Remark. For the dissipation function defined by (20), $\partial d(0) = \partial \psi_K(0) = K$.

Proof. The observation follows from the fact that the functional (20) is the sum of proper convex functions, the relative interior of the domain of which have common points. In these hypotheses, one has $\partial d(0) = \partial \psi_K(0) \cup \partial \psi_K^2(0) \cup c \cup \cdots$, but $\partial \psi_K^n(0) = \{0\}$, $n > 1$. □

It is now possible to proceed to evaluate the conjugate dissipation function. Since the relative interiors of the domains of the addends of the dissipation function have obviously common points, the subgradient of the function (20) is given by the infimal convolution of the addends. Recalling that $\psi_K^c = \operatorname{ind} K$, using (17),

$$d^c(\tau) = \inf \left\{ \operatorname{ind} K(\tau_1) + \sum_{n=2}^{N} \tau_0 \nu \frac{q-1}{q} [j_K(\tau_n)]^{q/(q-1)} : \sum_{n=1}^{N} \tau_n = \tau \right\}. \tag{22}$$

Some particular cases are examined. If $N = 2$ and $q = n$, (22) becomes

$$d^c(\tau) = \inf\{\operatorname{ind} K(\tau_1) + \tfrac{1}{2}\tau_0 \nu j_K(\tau_2)^2 : \tau_1 + \tau_2 = \tau\} = \tfrac{1}{2}\tau_0 \nu \inf\{j_K(\tau - \tau_1)^2 : \tau_1 \in K\}. \tag{23}$$

The infimum in (23) is the square of the minimum distance between the vector of the internal forces and the admissible domain in the norm induced by j_K.

In the case when $N = p$ and $q = n$ and all terms but the first and the p-th are null, one has

$$\begin{aligned} d^c(\tau) &= \inf \left\{ \operatorname{ind} K(\tau_1) + \tau_0 \nu \frac{p-1}{p} j_K(\tau_2)^{p/(p-1)} : \tau_1 + \tau_2 = \tau \right\} \\ &= \tau_0 \nu \frac{p-1}{p} \inf\{j_K(\tau - \tau_1)^{p/(p-1)} : \tau_1 \in K\}. \end{aligned} \tag{24}$$

In the general case, (22) can be rewritten as

$$d^c(\tau) = \inf\left\{\sum_{n=2}^{N} \tau_0 \nu \frac{q-1}{q} j_K(\tau_n)^{q/(q-1)} : \sum_{n=2}^{N} \tau_n = \bar{\tau} = \tau - \tau_1\ \tau_1 \in K\right\}, \quad (25)$$

where $\bar{\tau}$ is the overstress.

The flow rule. The next step is to obtain an explicit form for the flow rule, which is done through the evaluation of the subgradient of the complementary dissipation function, which gives the set of the irreversible strain rates compatible with the constitutive equation.

In order to be specific, we examine the particular case that the function d^c is given by (23) or (24). Since the function inside the infimum operation is positive, it is possible to interchange the power with the infimum operation so that, applying the chain rule of subdifferentiation,

$$\dot{\eta}_p \in \partial d^c(\tau) = \tau_0 \nu \inf\{j_K(\tau - \tau_1)^{1/(p-1)} : \tau_1 \in K\}\, \partial\xi(\tau), \quad (26)$$

having indicated with ξ the infimum of the gauge function

$$\xi(\tau) = \inf\{j_K(\tau - \tau_1) : \tau_1 \in K\}. \quad (27)$$

For evaluating its subdifferential, it is first observed that, if $\tau \in K$, $\xi(\tau) = 0$; hence, $\partial\xi(\tau) = 0$. If $\tau \notin K$, then one has $j_k(\tau) = \mu_\tau \geq 1$. Set $\tau_{10} = \mu_\tau^{-1}\tau$ so that $j_K(\tau_{10}) = 1$, $\tau_{10} \in \partial K$, the boundary of K. The infimum operation in (27) can then be rewritten as

$$\inf\{j_K(\mu_\tau \tau_{10} - \tau_{10} - \bar{\tau}_1) : \tau_{10} + \bar{\tau}_1 \in K\}, \quad (28)$$

where the vector $\bar{\tau}_1$ must be such that

$$\langle \bar{\tau}_1, \dot{\eta}_p \rangle \leq 0 \quad \text{for all } \dot{\eta}_p \in N_K(\mu_\tau^{-1}\tau) = N_{\mu_\tau K}(\tau),$$

N_K being the tangent cone to K at the point τ_{10}.

From the convexity of K, it follows that the infimum in (28) is attained for $\bar{\tau}_1 = 0$ so that $\inf\{j_K(\tau - \tau_1 : \tau_1 \in K\} = j_K(\mu_\tau \tau_{10} - \tau_{10}) = \mu_\tau - 1$.

Statement. *The subdifferential of the function $\xi(\tau)$ is given by*

$$\partial\xi(\tau) = \frac{1}{\tau_0}\gamma, \quad \gamma \in N_K(\mu_\tau^{-1}\tau), \quad (29)$$

where $N_K(\mu_\tau^{-1}\tau)$ is the normal cone to K at the point τ/μ_τ.

Proof. The normal cone to K at $\mu_\tau^{-1}\tau$ is

$$N_K(\mu_\tau^{-1}\tau) = \left\{\dot{\eta}_p : \left\langle \dot{\eta}_p, \bar{\tau} - \frac{\tau}{\mu_\tau}\right\rangle \leq 0 \text{ for all } \bar{\tau} \in K\right\}. \quad (30)$$

By definition, the subdifferential of ξ is

$$\partial\xi(\tau)=\left\{\frac{1}{\tau_0}\hat{\dot{\eta}}_p:\langle\hat{\dot{\eta}}_p,\bar{\tau}-\hat{\tau}\rangle\leq\xi(\bar{\tau})-\xi(\tau)=\mu_{\bar{\tau}}-\mu_\tau\right\}. \tag{31}$$

Dividing by μ_τ, one has

$$\mu_\tau^{-1}\langle\hat{\dot{\eta}}_p,\bar{\tau}-\tau\rangle\leq\mu_{\bar{\tau}}/\mu_\tau-1$$

and the last difference is smaller than 0 if $\bar{\tau}/\mu_\tau\in K$. □

Then from (26), the rate of plastic deformation can be represented as

$$\dot{\eta}_p=\nu(\mu_\tau-1)^{1/(p-1)}\gamma,\quad \gamma\in N_K(\mu_\tau^{-1}\tau). \tag{32}$$

In this way, the flow rule has been characterized.

Corollary. *The plastic strain rates in the case when the function d^c is given by (23) or (24) are elements of the normal cone to K at the point τ/μ_τ.*

A similar conclusion holds for the more general expressions of the dissipation potential as a power expansion, similar to what has been suggested by Goddard [2014]. The general case will be examined in a forthcoming paper.

Remark. According to (26) and (31), the viscoplastic strain rate is normal to the static yield surface at the closest point projection of the current stress state, where the definition of the closest point projection is in the sense of the Minkowski norm. This model, thus, does not include the generalization of the Duvaut–Lions model proposed by Simo [Simo and Govindjee 1991; Simo et al. 1988], which uses as the norm the complementary elastic energy $e^c(\tau-\tau_1)$, $\tau_1\in K$.

Remark. The model obtained is associative, in the sense of the above corollary. Furthermore, it can be immediately applied to the case of hardening plasticity coupled with damage, once a generalized yield domain is defined, as proposed in [Contrafatto and Cuomo 2002]. The choice of the dissipation potential is completely independent of that of the internal energy. Notice that in the present model the same viscosity constant applies to the plastic strain rate and to the rate of the hardening variable. In order to model different time scales for the two phenomena, it would be necessary to introduce two different dissipation functions, both of the type (20): one for the plastic strain rate and the other for the rate of the plastic hardening. Investigating this case is however beyond the limits of this work.

Remark. In the case when the elastic domain K has corner points, they are reflected in the flow rule as indicated by the inclusion of (31). The case of inviscid plasticity is naturally recovered when the relaxation time vanishes.

In order to obtain an explicit expression for the complementary dissipation potential and for the flow rule to be more convenient for algorithmic developments, we introduce the classical yield function. In the continuum mechanics practice, the elastic domain K, rather than being defined as the polar set to $C = \{\hat{\dot{\eta}}_p : \hat{d}(\hat{\dot{\eta}}_p) \le 1\}$, is directly introduced as the level set of a function $g(\tau)$, which possesses the properties

(1) $\inf g = g(0)$ and

(2) the level sets $\{\tau : g(\tau) \le c\}$, $g(0) \le c \le +\infty$, are all proportional.

Properties (1) and (2), using the result of the lemma on page 224, ensure that the yield function is a gauge-like function, so it has to be of the form $h(k(\tau))$, with $k(\tau)$ the gauge function. In particular it can be obtained as the composition of the gauge function of K and of a nondecreasing, nonnegative, convex, lower semicontinuous function h [Rockafellar 1970, Theorem 15.3]. However, it is convenient to take the function g to be positively homogeneous, that is,

$$K = \{\tau : j_K(\tau) \le 1\} = \{\tau : g(\tau) \le \tau_0\} \quad \Longrightarrow \quad g(\tau) = \tau_0 j_K(\tau), \tag{33}$$

where τ_0 is the level of g corresponding to the boundary of the set K. Then

$$\xi(\tau) = \inf\{g(\tau - \tau_1)/\tau_0 : \tau_1 \in K\} = (g(\tau)/\tau_0 - 1)_+ \tag{34}$$

with $(x)_+ = (x + |x|)/2$. The subgradient $\partial\xi(\tau)$ can then be evaluated as $\partial\xi(\tau) = \partial g(\tau)/\tau_0$ so that, if the function g is differentiable at τ, then the subgradient is composed by a unique vector, representing the outward normal to $\mu_\tau K$ in τ, coinciding with the normal to K in τ/μ_τ. More generally, if $g(\tau) = \sup_i g_i(\tau)$, with each g_i supposed differentiable, then if τ/μ_τ is a corner point of K, the subgradient is the convex combination of the normals ∂g_i to $\mu_\tau K$ in τ.

The flow rule, in the case when the dissipation function is given by (24), is then expressed as

$$\dot{\eta}_p = \partial d_n^c(\tau) = \nu \left[\frac{g(\tau) - \tau_0}{\tau_0} \right]_+^{1/(p-1)} \partial g(\tau). \tag{35}$$

Remark. Equation (35) in the case $p = 2$ coincides with the formulation of Perzyna, the contents of the brackets being the overstress function.

Remark. From (35) it is observed that, using for the dissipation a power function greater than 2 of the support function of K, the complementary functional is a power function less than 2 of the Minkowski distance from the admissible domain. This can be interpreted as a stress dependency for the relaxation time (viscosity parameter).

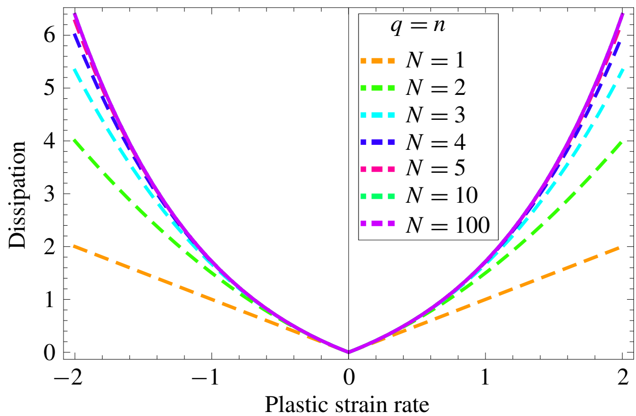

Figure 2. Dissipation function (36) for subsequent truncations of the series expansion.

4. Uniaxial exemplification

The main results of the previous section are now summarized and graphically illustrated in reference to the uniaxial case.

A slightly different expression for the dissipation function with respect to (20) is considered:

$$d(\dot{\eta}_p) = \psi_K(\dot{\eta}_p) + \sum_{n=2}^{N} \frac{1}{n!} \frac{1}{(\tau_0 \nu)^{n-1}} [\psi_K(\dot{\eta}_p)]^n. \tag{36}$$

The motivation for introducing the factorial of n lies in the fact that the form (36) is the series expansion of $\exp[\psi_K] - 1$.

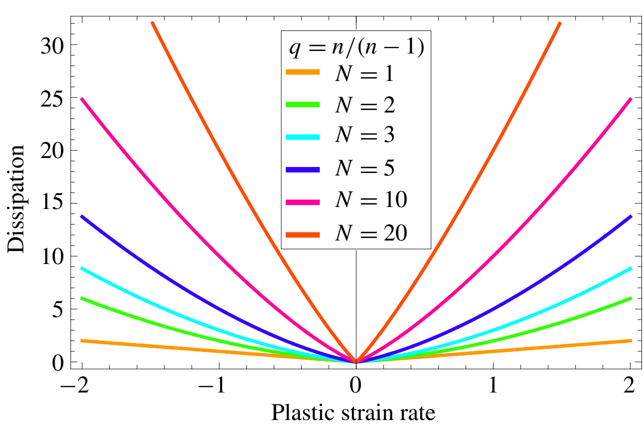

Figure 3. Dissipation function (20), $q = n/(n-1)$, for subsequent truncations of the series expansion.

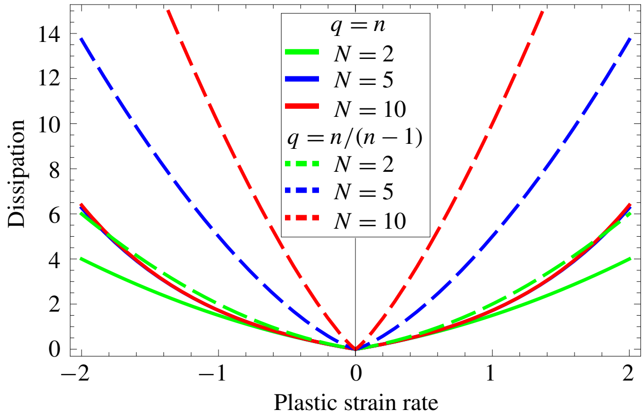

Figure 4. Comparison of the dissipation functions, (36) (solid lines) and (20) for $q = n/(n-1)$ (dashed lines).

Figure 2 shows the dissipation in an uniaxial case with $\tau_0 = 1$. It can be observed that, adding terms to the series for larger N, the dissipation function tends to converge to a limit value. The case given by (20) with $q = n/(n-1)$ is represented in Figure 3. In this case, adding terms to the series, the dissipation increases and tends toward a linear form, that is, the viscosity parameter tends to diverge, and a sort of inviscid plasticity is recovered for a wider elastic domain.

A comparison of the two forms is shown in Figure 4. The "exponential expansion" appears to yield lower values for the dissipation for the same N.

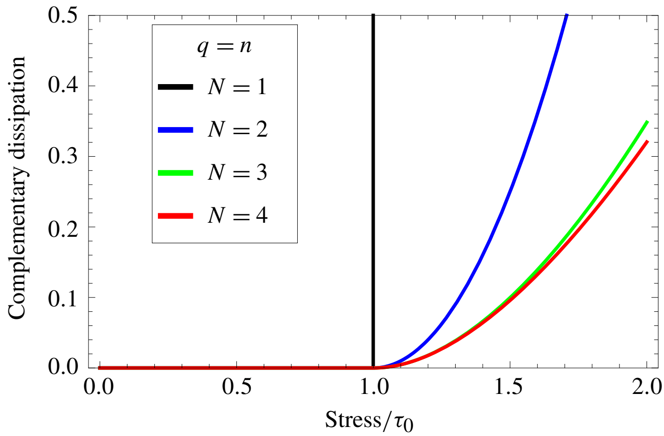

Figure 5. Complementary dissipation function (22) for the case $q = n/(n-1)$ for increasing number of functions in the series expansion.

The complementary dissipation for the case (36) is given by

$$d^c(\tau)=\inf\left\{\text{ind } K(\tau_1)+\sum_{n=2}^{N}\tau_0\nu\frac{n-1}{n}[(n-1)!]^{n/(n-1)}[j_K(\tau_n)]^{n/(n-1)}:\sum_{n=1}^{N}\tau_n=\tau\right\}. \tag{37}$$

It is represented in the uniaxial case in Figure 5, where also the case of inviscid plasticity has been represented. Larger values of N appear to act as mollifying parameters for the indicator function of the elastic domain.

5. Conclusions

The main results of the paper can be summarized as follows.

(1) We have given a formulation for the dissipation functional of a time-dependent dissipating material within the framework of the standard generalized material model; it has been shown that for the model to include an elastic domain the dissipation functional must be at least the sum of a positively homogeneous functional plus other hodn terms, with $n > 1$. A form of the dissipation potential has been proposed, based on a sum of powers of the support function of the elastic domain, which degenerates into the dissipation function of time-independent models when a viscosity parameter tends to 0. This form is not unique, but it seems to be the simplest one compatible with the standard generalized material model that guarantees fulfillment of the dissipation inequality and that preserves all the essential properties of time-independent plasticity.

(2) The complementary dissipation functional, useful for the numerical implementation of the model, has been derived in a general form as the infimal convolution of gauge functions of the elastic domain. For the case that only two terms appear in the dissipation functional, and particularly for the commonly employed case that the second one is homogeneous of degree 2, the infimal convolution has been solved explicitly. Similarly, the relevant expressions for the flow rule have been derived.

(3) The general case of the dissipation function obtained as a power expansion of the support function of the elastic domain will be treated in a future paper. However, from a uniaxial exemplification, it seems that the series eventually converges to a limit form of the function.

Appendix A: Homogeneous functions

A function $f(x)$ is called positively homogeneous (of degree 1) (*hodo*) if

$$f(\alpha x)=\alpha f(x), \quad \alpha\geq 0. \tag{38}$$

A function is called positively homogeneous of degree $n > 1$ (*hodn*) if

$$f(\alpha x) = \alpha^n f(x), \quad \alpha \geq 0. \tag{39}$$

In the text, the term "positively" will often be omitted for brevity.

Appendix B: Review of some results of convex analysis

Let $\{x \in X\}$ be a linear vector space. A function $f(x) : X \to \overline{\mathbb{R}}$ is called convex if

$$f((1-\lambda)x_1 + \lambda x_2) \leq (1-\lambda) f(x_1) + \lambda f(x_2), \quad \lambda \in (0, 1). \tag{40}$$

If the inequality in (40) is fulfilled strictly, the function is said to be strictly convex. The domain of f is

$$\operatorname{dom} f = \{x \in X : f(x) < +\infty\}. \tag{41}$$

The function f is said to be proper if $\operatorname{dom} f \neq \varnothing$ and $f(x) > -\infty$ for all $x \in X$.

Let X be a topological real reflexive Banach space. The topological dual space to X, X^*, is the space of the linear functionals defined on X. The value of a functional $x^* \in X^*$ at x is denoted by $\langle x^*, x\rangle$. If X is a Hilbert space, then $\langle x^*, x\rangle$ is a scalar product and $X^{**} = X$.

Let $f_1, f_2, \ldots, f_n$ be proper functions on a linear space X. The function

$$f(x) := \inf\{f_1(x_1) + \cdots + f_n(x_n) : x_1 + \cdots + x_n = x,\ x_i \in X,\ i = 1, \ldots, n\} \tag{42}$$

is called the infimal convolution, and it is convex.

The support function of a convex set $K \subset X$ is the element of X^*

$$\operatorname{supp} K = \sup_{y \in K}\{\langle y, x\rangle\}. \tag{43}$$

The indicator function of a set A is

$$\operatorname{ind} A = \begin{cases} 0 & \text{if } x \in A, \\ +\infty & \text{if } x \notin A. \end{cases} \tag{44}$$

Given a set C, a gauge of the set C is defined as

$$\gamma_C(x) = \gamma(x \mid C) = \inf\{\mu : x \in \mu C,\ \mu \geq 0\}, \tag{45}$$

also called the Minkowski gauge functional.

A functional $f : X \to \overline{\mathbb{R}}$ is said to be sublinear if

(i) $f(\alpha x) = \alpha f(x)$ when $\alpha \geq 0$ (positive homogeneity) and

(ii) $f(x + y) \leq f(x) + f(y)$ (subadditivity).

A sublinear functional is a generalization of a norm on a linear space.

A function $f(x)$ is said to be lower semicontinuous at x_0 if there exists a neighborhood $U_0(x)$ such that

$$\text{there exists } \varepsilon > 0 \quad \text{such that} \quad f(x) - f(x_0) > \varepsilon \quad \text{for all } x \in U_0(x). \tag{46}$$

Subdifferential and conjugacy. Let $f : X \to \overline{\mathbb{R}}$ be a proper convex function. The subdifferential of f at x is the set

$$\partial f(x) = \{x^* \in X^* : f(y) - f(x) \geq \langle x^*, y - x \rangle \text{ for all } y \in X\}. \tag{47}$$

The function $f^*(x^*) : X^* \to \overline{\mathbb{R}}$ is called conjugate to $f(x) : X \to \overline{\mathbb{R}}$ if

$$f^*(x^*) = \sup_{x \in X} \{\langle x, x^* \rangle - f(x)\}. \tag{48}$$

From the definition, it follows that

$$f(x) + f^*(x^*) \geq \langle x, x^* \rangle. \tag{49}$$

The equality sign in (49) holds only if $x^* \in \partial f(x)$.

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