

Formulas and Identities of Inverse Hyperbolic Functions

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Summary. This article describes definitions of inverse hyperbolic functions and their main properties, as well as some addition formulas with hyperbolic functions.

MML identifier: SIN_COS7, version: 7.5.01 4.39.921

The papers [1], [8], [4], [2], [9], [3], [6], [5], and [7] provide the terminology and notation for this paper.

1. PRELIMINARIES

In this paper x, y, t denote real numbers.

Next we state a number of propositions:

- (1) If $x > 0$, then $\frac{1}{x} = x^{-1}$.
- (2) If $x > 1$, then $(\frac{\sqrt{x^2-1}}{x})^2 < 1$.
- (3) $(\frac{x}{\sqrt{x^2+1}})^2 < 1$.
- (4) $\sqrt{x^2+1} > 0$.
- (5) $\sqrt{x^2+1} + x > 0$.

- (6) If $y \geq 0$ and $x \geq 1$, then $\frac{x+1}{y} \geq 0$.
- (7) If $y \geq 0$ and $x \geq 1$, then $\frac{x-1}{y} \geq 0$.
- (8) If $x \geq 1$, then $\sqrt{\frac{x+1}{2}} \geq 1$.
- (9) If $y \geq 0$ and $x \geq 1$, then $\frac{x^2-1}{y} \geq 0$.
- (10) If $x \geq 1$, then $\sqrt{\frac{x+1}{2}} + \sqrt{\frac{x-1}{2}} > 0$.
- (11) If $x^2 < 1$, then $x + 1 > 0$ and $1 - x > 0$.
- (12) If $x \neq 1$, then $(1 - x)^2 > 0$.
- (13) If $x^2 < 1$, then $\frac{x^2+1}{1-x^2} \geq 0$.
- (14) If $x^2 < 1$, then $(\frac{2 \cdot x}{1+x^2})^2 < 1$.
- (15) If $0 < x$ and $x < 1$, then $\frac{1+x}{1-x} > 0$.
- (16) If $0 < x$ and $x < 1$, then $x^2 < 1$.
- (17) If $0 < x$ and $x < 1$, then $\frac{1}{\sqrt{1-x^2}} > 1$.
- (18) If $0 < x$ and $x < 1$, then $\frac{2 \cdot x}{1-x^2} > 0$.
- (19) If $0 < x$ and $x < 1$, then $0 < (1 - x^2)^2$.
- (20) If $0 < x$ and $x < 1$, then $\frac{1+x^2}{1-x^2} > 1$.
- (21) If $1 < x^2$, then $(\frac{1}{x})^2 < 1$.
- (22) If $0 < x$ and $x \leq 1$, then $1 - x^2 \geq 0$.
- (23) If $1 \leq x$, then $0 < x + \sqrt{x^2 - 1}$.
- (24) If $1 \leq x$ and $1 \leq y$, then $0 \leq x \cdot \sqrt{y^2 - 1} + y \cdot \sqrt{x^2 - 1}$.
- (25) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $0 < y - \sqrt{y^2 - 1}$.
- (26) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $0 \leq y \cdot \sqrt{x^2 - 1} - x \cdot \sqrt{y^2 - 1}$.
- (27) If $x^2 < 1$ and $y^2 < 1$, then $x \cdot y \neq -1$.
- (28) If $x^2 < 1$ and $y^2 < 1$, then $x \cdot y \neq 1$.
- (29) If $x \neq 0$, then $\exp x \neq 1$.
- (30) If $0 \neq x$, then $(\exp x)^2 - 1 \neq 0$.
- (31) If $0 < t$, then $\frac{t^2-1}{t^2+1} < 1$.
- (32) If $-1 < t$ and $t < 1$, then $0 < \frac{t+1}{1-t}$.

2. FORMULAS AND IDENTITIES OF INVERSE HYPERBOLIC FUNCTIONS

Let x be a real number. The functor $\sinh' x$ yields a real number and is defined by:

$$\text{(Def. 1) } \sinh' x = \log_e(x + \sqrt{x^2 + 1}).$$

Let x be a real number. The functor $\cosh'_1 x$ yielding a real number is defined by:

(Def. 2) $\cosh'_1 x = \log_e(x + \sqrt{x^2 - 1})$.

Let x be a real number. The functor $\cosh'_2 x$ yields a real number and is defined by:

(Def. 3) $\cosh'_2 x = -\log_e(x + \sqrt{x^2 - 1})$.

Let x be a real number. The functor $\tanh' x$ yields a real number and is defined by:

(Def. 4) $\tanh' x = \frac{1}{2} \cdot \log_e\left(\frac{1+x}{1-x}\right)$.

Let x be a real number. The functor $\coth' x$ yielding a real number is defined as follows:

(Def. 5) $\coth' x = \frac{1}{2} \cdot \log_e\left(\frac{x+1}{x-1}\right)$.

Let x be a real number. The functor $\operatorname{sech}'_1 x$ yields a real number and is defined by:

(Def. 6) $\operatorname{sech}'_1 x = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right)$.

Let x be a real number. The functor $\operatorname{sech}'_2 x$ yielding a real number is defined as follows:

(Def. 7) $\operatorname{sech}'_2 x = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right)$.

Let x be a real number. The functor $\operatorname{csch}' x$ yielding a real number is defined by:

- (Def. 8)(i) $\operatorname{csch}' x = \log_e\left(\frac{1+\sqrt{1+x^2}}{x}\right)$ if $0 < x$,
 (ii) $\operatorname{csch}' x = \log_e\left(\frac{1-\sqrt{1+x^2}}{x}\right)$ if $x < 0$,
 (iii) $x < 0$, otherwise.

The following propositions are true:

- (33) If $0 \leq x$, then $\sinh' x = \cosh'_1 \sqrt{x^2 + 1}$.
 (34) If $x < 0$, then $\sinh' x = \cosh'_2 \sqrt{x^2 + 1}$.
 (35) $\sinh' x = \tanh'\left(\frac{x}{\sqrt{x^2+1}}\right)$.
 (36) If $x \geq 1$, then $\cosh'_1 x = \sinh' \sqrt{x^2 - 1}$.
 (37) If $x > 1$, then $\cosh'_1 x = \tanh'\left(\frac{\sqrt{x^2-1}}{x}\right)$.
 (38) If $x \geq 1$, then $\cosh'_1 x = 2 \cdot \cosh'_1 \sqrt{\frac{x+1}{2}}$.
 (39) If $x \geq 1$, then $\cosh'_2 x = 2 \cdot \cosh'_2 \sqrt{\frac{x+1}{2}}$.
 (40) If $x \geq 1$, then $\cosh'_1 x = 2 \cdot \sinh' \sqrt{\frac{x-1}{2}}$.
 (41) If $x^2 < 1$, then $\tanh' x = \sinh'\left(\frac{x}{\sqrt{1-x^2}}\right)$.
 (42) If $0 < x$ and $x < 1$, then $\tanh' x = \cosh'_1\left(\frac{1}{\sqrt{1-x^2}}\right)$.
 (43) If $x^2 < 1$, then $\tanh' x = \frac{1}{2} \cdot \sinh'\left(\frac{2 \cdot x}{1-x^2}\right)$.
 (44) If $x > 0$ and $x < 1$, then $\tanh' x = \frac{1}{2} \cdot \cosh'_1\left(\frac{1+x^2}{1-x^2}\right)$.
 (45) If $x^2 < 1$, then $\tanh' x = \frac{1}{2} \cdot \tanh'\left(\frac{2 \cdot x}{1+x^2}\right)$.

- (46) If $x^2 > 1$, then $\coth' x = \tanh'(\frac{1}{x})$.
- (47) If $x > 0$ and $x \leq 1$, then $\operatorname{sech}'_1 x = \cosh'_1(\frac{1}{x})$.
- (48) If $x > 0$ and $x \leq 1$, then $\operatorname{sech}'_2 x = \cosh'_2(\frac{1}{x})$.
- (49) If $x > 0$, then $\operatorname{csch}' x = \sinh'(\frac{1}{x})$.
- (50) If $x \cdot y + \sqrt{x^2 + 1} \cdot \sqrt{y^2 + 1} \geq 0$, then $\sinh' x + \sinh' y = \sinh'(x \cdot \sqrt{1 + y^2} + y \cdot \sqrt{1 + x^2})$.
- (51) $\sinh' x - \sinh' y = \sinh'(x \cdot \sqrt{1 + y^2} - y \cdot \sqrt{1 + x^2})$.
- (52) If $1 \leq x$ and $1 \leq y$, then $\cosh'_1 x + \cosh'_1 y = \cosh'_1(x \cdot y + \sqrt{(x^2 - 1) \cdot (y^2 - 1)})$.
- (53) If $1 \leq x$ and $1 \leq y$, then $\cosh'_2 x + \cosh'_2 y = \cosh'_2(x \cdot y + \sqrt{(x^2 - 1) \cdot (y^2 - 1)})$.
- (54) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $\cosh'_1 x - \cosh'_1 y = \cosh'_1(x \cdot y - \sqrt{(x^2 - 1) \cdot (y^2 - 1)})$.
- (55) If $1 \leq x$ and $1 \leq y$ and $|y| \leq |x|$, then $\cosh'_2 x - \cosh'_2 y = \cosh'_2(x \cdot y - \sqrt{(x^2 - 1) \cdot (y^2 - 1)})$.
- (56) If $x^2 < 1$ and $y^2 < 1$, then $\tanh' x + \tanh' y = \tanh'(\frac{x+y}{1+x \cdot y})$.
- (57) If $x^2 < 1$ and $y^2 < 1$, then $\tanh' x - \tanh' y = \tanh'(\frac{x-y}{1-x \cdot y})$.
- (58) If $0 < x$ and $(\frac{x-1}{x+1})^2 < 1$, then $\log_e x = 2 \cdot \tanh'(\frac{x-1}{x+1})$.
- (59) If $0 < x$ and $(\frac{x^2-1}{x^2+1})^2 < 1$, then $\log_e x = \tanh'(\frac{x^2-1}{x^2+1})$.
- (60) If $1 < x$ and $1 \leq \frac{x^2+1}{2 \cdot x}$, then $\log_e x = \cosh'_1(\frac{x^2+1}{2 \cdot x})$.
- (61) If $0 < x$ and $x < 1$ and $1 \leq \frac{x^2+1}{2 \cdot x}$, then $\log_e x = \cosh'_2(\frac{x^2+1}{2 \cdot x})$.
- (62) If $0 < x$, then $\log_e x = \sinh'(\frac{x^2-1}{2 \cdot x})$.
- (63) If $y = \frac{1}{2} \cdot (\exp x - \exp(-x))$, then $x = \log_e(y + \sqrt{y^2 + 1})$.
- (64) If $y = \frac{1}{2} \cdot (\exp x + \exp(-x))$ and $1 \leq y$, then $x = \log_e(y + \sqrt{y^2 - 1})$ or $x = -\log_e(y + \sqrt{y^2 - 1})$.
- (65) If $y = \frac{\exp x - \exp(-x)}{\exp x + \exp(-x)}$, then $x = \frac{1}{2} \cdot \log_e(\frac{1+y}{1-y})$.
- (66) If $y = \frac{\exp x + \exp(-x)}{\exp x - \exp(-x)}$ and $x \neq 0$, then $x = \frac{1}{2} \cdot \log_e(\frac{y+1}{y-1})$.
- (67) If $y = \frac{1}{\frac{\exp x + \exp(-x)}{2}}$, then $x = \log_e(\frac{1+\sqrt{1-y^2}}{y})$ or $x = -\log_e(\frac{1+\sqrt{1-y^2}}{y})$.
- (68) If $y = \frac{1}{\frac{\exp x - \exp(-x)}{2}}$ and $x \neq 0$, then $x = \log_e(\frac{1+\sqrt{1+y^2}}{y})$ or $x = \log_e(\frac{1-\sqrt{1+y^2}}{y})$.
- (69) (The function \cosh)($2 \cdot x$) = $1 + 2 \cdot$ (the function \sinh)(x)².
- (70) (The function \cosh)(x)² = $1 +$ (the function \sinh)(x)².
- (71) (The function \sinh)(x)² = (the function \cosh)(x)² - 1.
- (72) $\sinh(5 \cdot x) = 5 \cdot \sinh x + 20 \cdot (\sinh x)^3 + 16 \cdot (\sinh x)^5$.

$$(73) \quad \cosh(5 \cdot x) = (5 \cdot \cosh x - 20 \cdot (\cosh x)^3) + 16 \cdot (\cosh x)^5.$$

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Received May 24, 2005

