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# FORMULAS FOR CALCULATING BESSEL FUNCTIONS OF INTEGRAL ORDER AND COMPLEX ARGUMENT

by

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A B E R D E E N   P R O V I N G   G R O U N D,   M A R Y L A N D

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# FORMULAS FOR CALCULATING BESSEL FUNCTIONS OF INTEGRAL ORDER AND COMPLEX ARGUMENT


## ABSTRACT

Formulas for calculating Bessel functions of integral order and complex argument are derived in this report. Calculations based on these formulas are not subject to the loss of significant figures which occurs in the Taylor and Neumann series when the argument is large and the order is small.

To calculate $J_n(z)$, select an integer $m > n$ and $m >> |z|$ such that $|J_m(z)| << |J_n(z)|$. Calculate $J_m(z)$ and $J_{m+1}(z)$ from the Taylor series, then calculate $J_n(z)$ from the recurrence relation. A similar procedure is used to calculate $I_n(z)$.

To calculate $K_n(z)$, express the quotient $Q_n(z) = K_{n-1}(z)/K_n(z)$ in terms of two Gauss continued fractions. The individual functions $K_n(z)$ and $K_{n-1}(z)$ are obtained from $Q_n(z)$ and the Wronskian relation involving $K_n(z)$, $K_{n-1}(z)$, $I_n(z)$, and $I_{n-1}(z)$. A similar procedure involving Hankel functions is used to calculate $Y_n(z)$ and $Y_{n-1}(z)$.

Cancellation of significant figures is very severe in the Neumann series for $K_n(z)$ when $n$ is small and $z$ is large and real. At the present time, this difficulty is generally overcome by using multiple precision arithmetic, quadrature formulas, or continued fractions with very involved terms. The Gauss continued fractions used in this report are simple in form, rapidly convergent, and are not subject to excessive round-off error.

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## LIST OF SYMBOLS*

| | |
|---|---|
| $z$ | complex variable, $z = x + iy = \rho e^{i\theta}$ |
| $x$ | real part of $z$ |
| $y$ | imaginary part of $z$ |
| $\rho$ | absolute value of $z$ |
| $\theta$ | argument of $z$ |
| $k, \ell, m, n$ | integers |
| $J_n(z)$ | ordinary Bessel function of the first kind |
| $Y_n(z)$ | ordinary Bessel function of the second kind |
| $I_n(z)$ | modified Bessel function of the first kind |
| $K_n(z)$ | modified Bessel function of the second kind |
| $H_n^{(1)}(z)$ | Hankel functions, sometimes called |
| $H_n^{(2)}(z)$ | Bessel functions of the third kind |
| $f(a,b,u)$ | Wall's form of the confluent hypergeometric function |
| $W_{\kappa,m}(z)$ | Whittaker's form of the confluent hypergeometric function |
| $Q_n(z)$ | a quotient. $Q_n(z) = K_{n-1}(z)/K_n(z)$ |
| $P_n^{(1)}(z)$ | quotients of Hankel functions; |
| $P_n^{(2)}(z)$ | see equations (46), (47) |
| $v$ | a complex variable defined in terms of $z$; see equation (21) |
| $F_1(a,b;u)$ | Gauss continued fractions. See equations (30)-(3[illegible]) |
| $G_1(a,b;u)$ | |
| $O$ | of the order of |
| $\Gamma(z)$ | the gamma function |
| $\gamma$ | Euler's constant |

*The notation of Chapter 9, Ref. 5, is used whenever practical.

| | |
|---|---|
| $\alpha$ | normalizing factor in Miller's algorithm |
| $L_n(z)$ | $L_n(z) = \alpha J_n(z) + \beta Y_n(z)$, Eq. (5) |
| $\varepsilon$ | a small quantity |
| Re | the real part of |
| Im | the imaginary part of |
| $\mathbf{R}\{F(z)\}$ | the real part of F(z), Appendix A |
| $\mathbf{I}\{F(z)\}$ | the imaginary part of F(z), Appendix A |
| F(z) | a function of z |
| w | dependent variable in the differential equation $\frac{d^2w}{dz^2} - F(z)w = 0$ |

# I. INTRODUCTION

Linear boundary value problems in cylindrical coordinates may frequently be solved in terms of Bessel functions of integral order. Certain problems in elasticity, viscoelasticity, and fluid flow require Bessel functions of a complex argument in order to satisfy the boundary conditions in a rigorous manner. Taylor and Neumann series are generally used to calculate these functions when the argument is small or moderate in size.[1,2,3*] However, severe cancellation occurs unless all the terms of the series have the same sign. Consequently, multiple precision arithmetic is required in order to obtain final results of the required accuracy.

Cancellation is a severe form of round-off error which occurs if the sum of a series is small compared with the largest term. Several significant figures may be lost by subtraction if the calculations are carried out with a fixed number of significant figures. This difficulty is illustrated by the alternating series

$$e^{-x} = 1 - x + \frac{1}{2} x^2 - \frac{1}{3!} x^3 + \ldots$$

since $e^{-x}$ is small when x is large. It is evident that cancellation does not occur when evaluating the series

$$e^{x} = 1 + x + \frac{1}{2} x^2 + \frac{1}{3!} x^3 \ldots$$

Hence, to obtain $e^{-x}$, we first calculate $e^{x}$ and then obtain $e^{-x}$ by division.

Miller[4] has shown that cancellation does not occur if stable recurrence relations are used to calculate Bessel functions of the first kind. An analogous procedure cannot be used for calculating Bessel functions of the second kind, as the corresponding recurrence relations are unstable.**

*References are listed on page 24.

**Miller's Algorithm is discussed on pages 385 and 697, Ref. 5. Stability of various recurrence relations is discussed on page XIII of the Introduction, Ref. 5.

In this paper, the quotient of two modified Bessel functions of the second kind is expressed in terms of two Gauss continued fractions.[6] The original function is then calculated from the Wronskian relation and the functions of the first kind. This procedure eliminates cancellation, so that accurate values of the function can be obtained for a wide range of order and argument. A similar procedure is used to calculate ordinary Bessel functions of the second kind.

If we require a sequence of functions $J_n(z)$, $J_{n+1}(z)$ ... $J_{n+i}(z)$, we calculate $J_{n+i}(z)$ and $J_{n+i+1}(z)$, then calculate the functions of lower order from the recurrence relation. Similarly, if $Y_n(z)$, $Y_{n+1}(z)$ ... $Y_{n+i}(z)$ are required, we calculate $Y_n(z)$ and $Y_{n+1}(z)$, then calculate functions of higher order from the recurrence relations. The routine being developed at the Ballistic Research Laboratories (BRL) will provide the pair of Bessel functions required to start the recurrence process.

## II. ORDINARY BESSEL FUNCTIONS OF THE FIRST KIND

These functions can be calculated accurately from the series*

$$J_n(z) = \frac{z^n}{2^n n!}\left\{1 - \frac{(\frac{1}{2}z)^2}{1!(n+1)} + \frac{(\frac{1}{2}z)^4}{2!(n+1)(n+2)} - \frac{(\frac{1}{2}z)^6}{3!(n+1)(n+2)(n+3)} + \cdots\right\} \quad (1)$$

if $|z|$ is small, or if n is very large compared with $|z|$. However, if z is real and large compared with n, the $n^{th}$ term of the series is large compared with $J_n(z)$; consequently, severe cancellation occurs in summing the series.

We will describe Miller's Algorithm briefly, as it is the basis of most subsequent work in the area. We assume

$$L_{m+1}(z) = 0 \quad (2)$$

$$L_m(z) = 1 \quad (3)$$

*The series and recurrence relations are obtained from references 5 and 7.

where $m >> n$, $m >> |z|$. The functions $L_{m-1}(z)$, $L_{m-2}(z)$ ... $L_n(z)$ are generated from the recursion formula

$$z[L_{k-1}(z) + L_{k+1}(z)] = 2kL_k(z) \tag{4}$$

We express $L_m(z)$ as a linear combination of $J_m(z)$ and $Y_n(z)$:

$$L_m(z) = \alpha J_m(z) + \beta Y_m(z) \tag{5}$$

Then, since $J_m(z)$, $Y_m(z)$, and $L_m(z)$ all satisfy the same linear recurrence relation, we have

$$L_n(z) = \alpha J_n(z) + \beta Y_n(z) \tag{6}$$

When $|z| > k$, $J_k(z)$ increases and $Y_k(z)$ decreases as k decreases; consequently the second term in Eq. (6) is negligible, and we have

$$L_n(z) \sim \alpha J_n(z) \tag{7}$$

The normalizing factor $\alpha$ is obtained from the Neumann series of an elementary function.

$$f(n,z) = \sum_{k=n}^{\infty} a_k J_k(z)$$

or

$$f(n,z) = \alpha \sum_{k=n}^{k=m} a_k L_k(z) \tag{8}$$

It is assumed that $\sum_{m}^{\infty} a_k L_k(z)$ is negligible. This procedure is very effective provided cancellation does not occur in summing the series in Eq. (8). However, it is difficult to find a single function f(n,z) which satisfies this requirement when both n and z vary widely.

In contrast with Miller's algorithm, the author uses Eq. (1) to calculate $J_m(z)$ and $J_{m+1}(z)$, where $m >> |z|$ and $m > n$. The terms of the series are moderate in size, and cancellation is negligible. The function $J_n(z)$ is calculated by repeated use of the recurrence relation

$$z[J_{k-1}(z) + J_{k+1}(z)] = 2kJ_k(z), \tag{9}$$

starting with k = n.

The integer n was obtained from the polynomial approximation

$$J_n(\rho) \sim \frac{\rho^n}{2^n n!} \{1 - \frac{(\frac{1}{2}\rho)^2}{1!(n+1)} + \frac{(\frac{1}{2}\rho)^4}{2!(n+1)(n+2)} + \cdots + \frac{(\frac{1}{2}\rho)^{2\ell}}{\ell!(n+1)\ldots(n+\ell)}\} \tag{10}$$

where $\rho = |z|$ and $\ell$ is the smallest integer for which

$$\frac{(\frac{1}{2}\rho)^{\ell}}{\ell!(n+1)\ldots(n+\ell)} < \varepsilon \tag{11}$$

Then

$$m = n + \ell \tag{12}$$

The small constant $\varepsilon$ was adjusted by trial. The final value was chosen just small enough to guarantee the accuracy of $J_n(z)$ in the range of interest.

## III. MODIFIED BESSEL FUNCTIONS OF THE FIRST KIND

A similar procedure is used to calculate $I_n(z)$ from the series

$$I_n(z) = \frac{z^n}{2^n n!} \{1 + \frac{(\frac{1}{2}z)^2}{1!(n+1)} + \frac{(\frac{1}{2}z)^4}{2!(n+1)(n+2)} + \frac{(\frac{1}{2}z)^6}{3!(n+1)(n+2)(n+3)} + \ldots\} \tag{13}$$

Obviously, no cancellation occurs if z is real and positive, as the terms of the series are all positive. If, however, z is a pure imaginary number, severe cancellation will occur when $|z|$ is large compared with n.

We calculate $I_m(z)$ and $I_{m+1}(z)$ where the integer m is given by Eq. (12). The function $I_n(z)$ is calculated by successive applications of the recurrence relation

$$z[I_{k-1}(z) - I_{k+1}(z)] = 2kI_k(z), \tag{14}$$

starting with $k = m$. This recurrence relation is stable for decreasing index.

A number of functions which have Taylor series expansions and recurrence relations which are stable for decreasing index can be calculated by Miller's algorithm or procedures similar to the one outlined above. If, however, the function has a logarithmic singularity at the origin, no simple method of calculation based on series and recurrence relations appears to be available.

## IV. MODIFIED BESSEL FUNCTIONS OF THE SECOND KIND

These functions can be calculated from the Neumann series

$$K_0(z) = -[\gamma - \log_e 2 + \log_e z]I_0(z)$$

$$+ \frac{(\frac{1}{2}z)^2}{1\cdot 1} + \frac{(\frac{1}{2}z)^4}{1\cdot 2\cdot 1\cdot 2}(1 + \tfrac{1}{2}) + \frac{(\frac{1}{2}z)^6}{1\cdot 2\cdot 3\cdot 1\cdot 2\cdot 3}(1 + \tfrac{1}{2} + \tfrac{1}{3}) + \ldots \tag{15}$$

$$K_n(z) = -[\gamma - \log_e 2 + \log_e z]\, I_n(z) + \frac{1}{2}\sum_{r=0}^{n-1} \frac{(-1)^r (n-r-1)!}{r!} \left(\frac{2}{z}\right)^{n-2r}$$

$$+ \frac{(-1)^n}{2} \sum_{r=0}^{\infty} \frac{(\frac{1}{2}z)^{n-2r}}{r!(n+r)!}\left[1 + \frac{1}{2} + \ldots + \frac{1}{r} + 1 + \frac{1}{2} + \ldots + \frac{1}{n+r}\right] \tag{16}$$

if $|z|$ is small, or if $|z| > n$. However, severe cancellation occurs if $z$ is large, real, and greater than $n$. In this range, we have the approximations

$$K_n(z) \sim \sqrt{\frac{\pi}{2z}}\, e^{-z}, \quad I_n(z) \sim \frac{1}{\sqrt{2\pi z}}\, e^{z}$$

so that $K_n(z)$ is small and $I_n(z)$ is large. It is evident from Eq. (16) that $K_n(z)$ will be calculated from the difference of two large and nearly equal numbers.

Although the Neumann series is useful when n is large compared with $|z|$, functions of lower order cannot be calculated accurately from the recurrence relation

$$z[K_{n+1}(z) - K_{n-1}(z)] = 2nK_n(z) \tag{17}$$

as the differences of nearly like numbers also occur in the course of these calcula ions. It appears that $K_n(z)$ should be calculated from a formula which does not separate the analytic and logarithmic parts of the function.

This requirement is satisfied by the integral representation*

$$K_n(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} \frac{e^{-z}}{\Gamma(n+\frac{1}{2})} \int_0^\infty e^{-u}\, u^{n-\frac{1}{2}} \left(1 + \frac{u}{2z}\right)^{n-\frac{1}{2}} du \tag{18}$$

which is valid provided z does not lie on the negative half of the real axis. It is related to a form of the confluent hypergeometric function discussed by Wall.**

$$f(a,b;v) = \frac{1}{\Gamma(a)} \int_0^\infty \frac{e^{-u} u^{a-1} du}{(1 + vu)^b} \tag{19}$$

We see that

$$K_n(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z}\, f(a, b;v) \tag{20}$$

*Page 206, Ref. 8

**Page 202, Ref. 8

$$v = \frac{1}{2z} \tag{21}$$

$$a = \frac{1}{2} + n \tag{22}$$

$$b = \frac{1}{2} - n \tag{23}$$

Similarly

$$K_{n-1}(z) = \left(\frac{\pi}{2z}\right)^{\frac{1}{2}} e^{-z} f(a - 1, b + 1; v) \tag{24}$$

We define the quotient function by the equation

$$Q_n(z) = K_{n-1}(z)/K_n(z) \tag{25}$$

Then

$$Q_n(z) = f(a - 1, b + 1; v)/f(a,b;v) \tag{26}$$

We reduce the expression on the right to a form that can be expressed in terms of Gauss continued fractions. The function $f(a,b;v)$ satisfies the recurrence relations

$$f(a, b; v) = f(a + 1, b; v) + bvf(a + 1, b + 1; v) \tag{27}$$

$$f(a, b; v) = f(a, b + 1; v) + avf(a + 1, b + 1; v) \tag{28}$$

Now replace a by a - 1 and b by b + 1 in Eq. (27). We find

$$f(a - 1, b + 1; v) = f(a, b + 1; v) + (b + 1) vf(a, b + 2; v) \tag{29}$$

so that

$$Q_n(z) = \frac{f(a, b + 1; v)}{f(a,b;v)} + \frac{(b + 1) vf(a, b + 2; v)}{f(a,b;v)}$$

On re-arranging this expression, we find

$$Q_n(z) = \frac{f(a, b + 1; v)}{f(a,b;v)} \left[ 1 + \frac{(b + 1) vf(a, b + 2; v)}{f(a, b + 1; v)} \right] \tag{30}$$

or

$$Q_n(z) = F_1(a,b;v)\left[1 + v(b+1)\,G_1(a,b;v)\right]$$

where

$$F_1(a,b;v) = \frac{f(a,\ b+1;\ v)}{f(a,b;v)}$$

and

$$G_1(a,b;v) = \frac{f(a,\ b+2;\ v)}{f(a,\ b+1;\ v)}$$

The functions $F_1(a,b;v)$ and $G_1(a,b;v)$ can be expressed in terms of Gauss continued fractions. We have*

$$\frac{f(a,b;v)}{f(a,\ b-1;\ v)} = \cfrac{1}{1 + \cfrac{av}{1 + \cfrac{bv}{1 + \cfrac{(a+1)v}{1 + \cfrac{(b+1)v}{1 + \cfrac{(a+2)v}{1 + \cfrac{(b+2)v}{1 + \cdots}}}}}}} \qquad (30)$$

Then

$$F_1(a,b;v) = \cfrac{1}{1 + \cfrac{av}{1 + \cfrac{(b+1)v}{1 + \cfrac{(a+1)v}{1 + \cfrac{(b+2)v}{1 + \cdots}}}}} \qquad (31)$$

*Eq. 92.3, page 352, Ref. 6.

and

$$G_1(a,b;v) = \cfrac{1}{1 + \cfrac{av}{1 + \cfrac{(b+2)v}{1 + \cfrac{(a+1)v}{1 + \cfrac{(b+3)v}{1 + \cdots}}}}} \tag{32}$$

To compute these continued fractions, we assume that

$$F_\ell(a,b;v) = \frac{f(a + \ell - 1,\ b + \ell - 1;\ v)}{f(a + \ell - 1,\ b + \ell - 2;\ v)} \tag{33}$$

and

$$G_\ell(a,b;v) = \frac{f(a + \ell - 1,\ b + \ell;\ v)}{f(a + \ell - 1,\ b + \ell - 1;\ v)} \tag{34}$$

Then we can prove that

$$F_\ell(a,b;v) = \frac{1 + (b + \ell)v\ F_{\ell+1}(a,b;v)}{1 + (b + \ell)v\ F_{\ell+1}(a,b;v) + (a + \ell - 1)v} \tag{35}$$

and

$$G_\ell(a,b;v) = \frac{1 + (b + \ell + 1)v\ G_{\ell+1}(a,b;v)}{1 + (b + \ell + 1)v\ G_{\ell+1}(a,b;v) + (a + \ell - 1)v} \tag{36}$$

In order to start the iterative procedures, we assume that

$$F_{\ell+1}(a,b;v) = 0 \tag{37}$$

$$G_{\ell+1}(a,b;v) = 0$$

when

$$\ell = \ell_{max}$$

The integer $\ell_{max}$ is chosen sufficiently large to keep the truncation error within the required bounds.

The individual functions $K_n(z)$ and $K_{n-1}(z)$ are obtained from the Wronskian relation

$$I_n(z)K_{n-1}(z) + I_{n-1}(z)K_n(z) = \frac{1}{z} \tag{38}$$

By definition, we have

$$K_{n-1}(z) = K_n(z)Q_n(z)$$

On eliminating $K_{n-1}(z)$ from these equations, we find

$$K_n(z) = \frac{1}{z[I_{n-1}(z) + I_n(z)Q_n(z)]} \tag{39}$$

Convergence of these continued fractions is rapid if z lies in the right half plane and $|z|$ is large, but becomes much slower as z approaches the origin or a point on the negative half of the real axis. Moreover, the complex zeros of $K_n(z)$, $K_{n-1}(z)$, and $f(a, b + 1; v)$ all lie in the left half plane, and may possibly lead to division by zero for certain values of z. Division by zero cannot occur if $|z| > 2$, lies on the imaginary axis or in the right half plane, and the integer $\ell_{max}$ is sufficiently large.*

Analytic continuation is used in conjunction with the Gauss continued fraction if

$$\text{Re } z < 0$$

and

$$|z| > 2$$

Let $t = -z$; then t lies in the right half plane, so that the functions $K_n(t)$ and $K_{n-1}(t)$ can be calculated as indicated above. $K_n(z)$ and $K_{n-1}(z)$ are then calculated from the following formulas for analytic continuation.

*A detailed proof is given in Appendix A.

$$K_j(z) = K_j(t) - i\pi I_j(t), \quad \text{Im } z \geq 0, \quad j \text{ even} \tag{40}$$

$$K_j(z) = K_j(t) + i\pi I_j(t), \quad \text{Im } z < 0, \quad j \text{ even} \tag{41}$$

$$K_j(z) = -K_j(t) + i\pi I_j(t), \quad \text{Im } z \geq 0, \quad j \text{ odd} \tag{42}$$

$$K_j(z) = -K_j(t) - i\pi I_j(t), \quad \text{Im } z < 0, \quad j \text{ even} \tag{43}$$

These formulas are obtained by comparing the Neumann series for $K_j(t)$ and $K_j(z)$. We note that $K_j(z)$ is discontinuous when we cross the negative real axis.

## V. ORDINARY BESSEL FUNCTIONS OF THE SECOND KIND

These functions are calculated in terms of the Hankel functions $H_n^{(1)}(z)$ and $H_n^{(2)}(z)$, which are linear combinations of the ordinary Bessel functions.

$$H_n^{(1)}(z) = J_n(z) + iY_n(z) \tag{44}$$

$$H_n^{(2)}(z) = J_n(z) - iY_n(z) \tag{45}$$

The quotient functions

$$P_n^{(1)}(z) = H_{n-1}^{(1)}(z)/H_n^{(1)}(z) \tag{46}$$

$$P_n^{(2)}(z) = H_{n-1}^{(2)}(z)/H_n^{(2)}(z) \tag{47}$$

are derived from the integral representations

$$H_n^{(1)}(z) = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{e^{i(z - \frac{1}{2}n\pi - \frac{1}{4}\pi)}}{\Gamma(n + \frac{1}{2})} \int_0^\infty e^{-u} u^{n-\frac{1}{2}} \left(1 + \frac{iu}{2z}\right)^{n-\frac{1}{2}} du \tag{48}$$

$$-\frac{1}{2}\pi < \text{avg } z < \frac{3}{2}\pi$$

$$H_n^{(2)} = \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} \frac{e^{-i(z - \frac{1}{2}n\pi - \frac{1}{4}\pi)}}{\Gamma(n + \frac{1}{2})} \int_0^\infty e^{-u} u^{-\frac{1}{2}} \left(1 - \frac{iu}{z}\right)^{n-\frac{1}{2}} du \tag{49}$$

$$-\frac{3}{2} < \text{avg } z < \frac{1}{2}\pi$$

On referring to Eqs. (19), (21), (22), and (23), we find

$$v = i/2z, \quad P_n^{(1)}(z) = i\, Q_n(v) \tag{50}$$

if $\quad \text{Im } z \geq 0$

$$v = -i/2z, \quad P_n^{(2)}(z) = -iQ_n(v) \tag{51}$$

if $\quad \text{Im } z < 0$

The restrictions on Im z insure that

$$\text{Re } v \geq 0$$

in both cases. As Eqs. (50) and (51) taken together account for the entire complex z plane, additional formulas for analytic continuation are not required.

We now derive Wronskian formulas which involve only one type of Hankel function. We have

$$J_{n-1}(z)\, Y_n(z) - J_n(z)\, Y_{n-1}(z) = -2\pi/z \tag{52}$$

On referring to Eqs. (44) and (45), we find

$$Y_n(z) = i[J_n(z) - H_n^{(1)}(z)] \tag{53}$$

$$Y_n(z) = i[J_n(z) - H_n^{(2)}(z)] \tag{54}$$

We use these relations to eliminate $Y_n(z)$ and $Y_{n-1}(z)$ from Eq. (52).

$$J_{n-1}(z)\, H_n^{(1)}(z) - J_n(z)\, H_{n-1}^{(1)}(z) = 2\pi i/z \tag{54}$$

$$J_{n-1}(z)\, H_n^{(2)}(z) - J_n(z)\, H_{n-1}^{(2)}(z) = -\, 2\pi i/z \tag{56}$$

On referring to Eqs. (50) and (51), we find

$$H_n^{(1)}(z) = \frac{2\pi i}{z[J_{n-1}(z) - P_n^{(1)}(z)\, J_n(z)]} \tag{57}$$

$$H_n^{(1)}(z) = P_n^{(1)}(z)\, H_n^{(1)}(z) \tag{58}$$

$$H_n^{(2)}(z) = -\frac{2\pi i}{z[J_n(z) - P_n^{(2)}(z)\, J_n(z)]} \tag{59}$$

$$H_{n-1}^{(2)}(z) = P_n^{(2)}(z)\, H_n^{(2)}(z) \tag{60}$$

We calculate $Y_n(t)$ and $Y_{n-1}(t)$ from Eqs. (50), (53), (57), and (59), if Im $z \leq 0$. If Im $z \geq 0$, we use Eqs. (51), (54), (58), and (60). This involved sequence of calculations has been checked by extensive calculations on the BRLESC.

## VI. RESULTS AND CONCLUSIONS

We have derived an accurate and efficient method for calculating Bessel functions of the second kind for integral order and complex argument. The formulas given here have been programmed in both the FORAST and FORTRAN IV programming languages.

In general, $J_n(x)$ and $Y_n(x)$ can be calculated to 14 decimal places if x is positive and less than n. The calculations are accurate to 14 significant figures if $n < x$. The functions $I_n(x)$ and $K_n(x)$ can also be calculated to 14 significant figures. It is difficult to check the accuracy of the calculations when the argument is complex, as sufficiently accurate tables are not available. These remarks apply to the range.

$$2 < x < 25$$

$$0 \leq n \leq 25$$

Existing tables were used whenever possible. In addition, quadrature formulas based on Gaussian quadrature along paths of steepest descent in the complex plane were also derived. The calculations were time-consuming but highly accurate. Calculations based on these quadrature formulas were used as a basis of comparison whenever tables were not available.

In the routines under development at BRL, Taylor and Neumann series are used when

$$|z| < 2$$

No cancellation occurs for these small values of the argument. The Hankel asymptotic expansions will be used when $|z|$ is large compared with n. The routines are being written in both the FORAST and FORTRAN IV programming languages.

The procedure outlined in this report can be extended to Bessel functions of fractional order, and also to Whittaker's function $W_{k,m}(z)$. The Gauss continued fraction is most effective precisely where the series development is plagued with cancellation.

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## APPENDIX A

We must show that division by zero will not occur in calculating the Gauss continued fractions and the quotient function $Q_n(z)$ provided

$$|z| > 2$$

and

$$\text{Re } z \geq 0.$$

The functions $K_n(z)$ and $K_{n-1}(z)$ have no zeros in this region.* It follows from Eqs. (24) and (25) that f(a, b; v) and f(a - 1, b + 1; v) are also free of zeros. It is shown below that f(a, b + 1; v) is also free of zeros in this region; consequently, division by zero cannot occur in Eqs. (26) and (30).

The proof is based on an oscillation theorem due to Einar Hille:**

"Let F(z) be real and positive when z is real and greater than $x_1$, analytic throughout a region D including the real axis for $\mathbf{R}(z) > x_1$, and such that either

$$\mathbf{R}\{F(z)\} > 0 \text{ or } \mathbf{I}\{F(z)\} \neq 0$$

in D; let W(z) be a solution of

$$\frac{d^2w}{dz^2} - F(z)w = 0$$

and such that $W(z) \to 0$ as $z \to \infty$ in D along a parallel to the real axis, then under very general assumptions, W(z) has no zero nor extremum in D."

The function f(a, b+; v) can be expressed in terms of Whittaker's function $W_{k,m}(z)$, which satisfies a differential equation of the required form.***

*Page 511, Ref. 8.
**Example 4, page 527, Ref. 9; pages 360 and 361, Ref. 10.
***Chapter XVI, pages 337-354, Ref. 11.

$$\frac{d^2W_{k,m}(z)}{dz^2} + \left[ -\frac{1}{4} + \frac{k}{z} + \frac{\frac{1}{4} - m^2}{z^2} \right] W_{k,m}(z) = 0 \qquad \text{(A-1)}$$

$$W_{k,m}(z) = \frac{e^{-\frac{1}{2}z} z^k}{\Gamma(\frac{1}{2} - k + m)} \int_0^\infty u^{-k - \frac{1}{2} + m} (1 + \frac{u}{z})^{k - \frac{1}{2} + m} e^{-u} du \qquad \text{(A-2)}$$

Now

$$f(a, b + 1; v) = \frac{1}{\Gamma(n + \frac{1}{2})} \int_0^\infty u^{n - \frac{1}{2}} (1 + vu)^{n - \frac{3}{2}} e^{-u} du \qquad \text{(A-3)}$$

A comparison of Eqs. (A-2) and (A-3) shows that

$$v = \frac{1}{z}, \quad m = n - \frac{1}{2}, \quad k = -\frac{1}{2} \qquad \text{(A-4)}$$

Eq. (61) now becomes

$$\frac{d^2W_{-\frac{1}{2}, n - \frac{1}{2}}(z)}{dz^2} - \left[ \frac{1}{4} + \frac{1}{2z} + \frac{n^2 - n}{z^2} \right] W_{-\frac{1}{2}, n - \frac{1}{2}}(z) = 0 \qquad \text{(A-5)}$$

The function F(z) of Hille's theorem is

$$F(z) = \frac{1}{4} + \frac{1}{2z} + \frac{n^2 - n}{z^2} \qquad \text{(A-6)}$$

Two cases arise. If $n = 0$ or $n = 1$,

$$\text{Re}\{F(z)\} = \frac{1}{4} + \frac{x}{x^2 + y^2}$$

The function Re{F(z)} is positive outside the circle

$$(x + 2)^2 + y^2 = 4$$

and negative within it. The region D consists of the entire complex z plane outside the semi-infinite strip bounded by the semicircle

$$y = \pm \sqrt{4 - (x + 2)^2}, \quad -2 \leq x < 0$$

and the lines

$$y = \pm 2, \quad x < -2.$$

When $n \geq 2$, we must consider both the real and imaginary parts of F(z) in delineating the region D. We assume

$$\mathrm{Re}\{F(z)\} > 0 \text{ in } D_1$$

$$\mathrm{Im}\{F(z)\} > 0 \text{ in } D_2$$

and

$$\mathrm{Im}\{F(z)\} < 0 \text{ in } D_3$$

$$\mathrm{Re}\{F(z)\} = \frac{1}{4} + \frac{x}{2(x^2 + y^2)} + \frac{(n^2 - n)(x^2 - y^2)}{(x^2 + y^2)^2} \qquad \text{(A-7)}$$

$$\mathrm{Im}\{F(z)\} = -\frac{y}{2(x^2 + y^2)} - \frac{2(n^2 - n)xy}{(x^2 + y^2)^2} \qquad \text{(A-8)}$$

We find

$$\mathrm{Re}\{F(z)\} > 0 \text{ if } x^2 > y^2$$

of if

$$\sqrt{x^2 + y^2} > 4(n^2 - n)$$

If we set

$$\mathrm{Im}F(z) = 0$$

we find

$$y = 0$$

or

$$[x + 2(n^2 - n)]^2 + y^2 = 4(n^2 - n)$$

We see that $\mathrm{Im}\{F(z)\} \neq 0$ provided z lies outside this circle and does not lie on the x axis. The regions $D_1$, $D_2$, and $D_3$ are shown below.

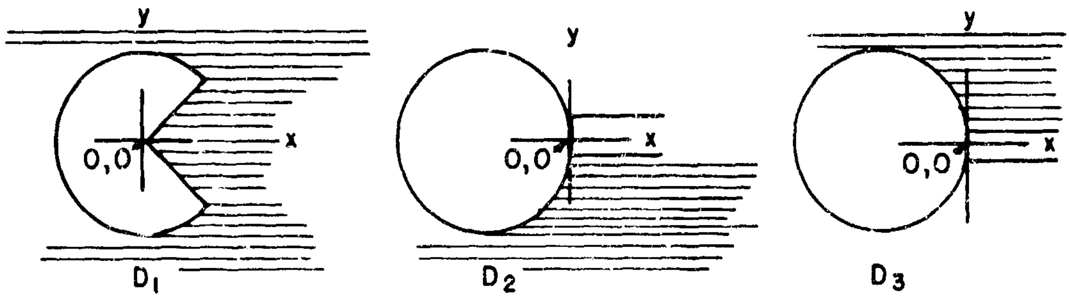


Figure A-1. Zero-Free Regions of $W_{-\frac{1}{2}, n - \frac{1}{2}}(z)$

The region D contains these three regions. The entire right-half plane and the imaginary axis, exclusive of the origin, lies in D.

If arg z < π, Whittaker's function has the asymptotic expansion

$$W_{k,m}(z) \sim e^{-z} z^k \quad 1 + \frac{m^2 - (k - \frac{1}{2})^2}{z} + O(\frac{1}{z^2}) \tag{A-9}$$

and hence tends to zero as $z \to \infty$ along a path parallel to the x axis. Consequently, $W_{-\frac{1}{2}, n - \frac{1}{2}}(z)$ has no zeros in D, and division by zero will not occur in Eq. (30). This conclusion has been verified by extensive calculations on the BRLESC. The functions occurring in Eq. (33) and Eq. (34) may be analyzed in the same way. The parameter k of Whittaker's function must be negative or zero if we are to be certain that the right half plane is free of zeros. This condition is satisfied for the cases under consideration.

These results show that the continued fractions for $F_1(a, b; v)$ and $G_1(a, b; v)$, Eqs. (31) and (32), have no poles in the right half plane or the imaginary axis. Hence, these continued fractions converge uniformly in this region.*

*Theorem 92.2, page 351, Ref. 9.

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13. ABSTRACT


Formulas for calculating Bessel functions of integral order and complex argument are derived in this report. Calculations based on these formulas are not subject to the loss of significant figures which occurs in the Taylor and Neumann series when the argument is large and the order is small.

To calculate $J_n(z)$, select an integer $m > n$ and $m >> |z|$ such that $|J_m(z)| << |J_n(z)|$. Calculate $J_m(z)$ and $J_{m+1}(z)$ from the Taylor series, then calculate $J_n(z)$ from the recurrence relation. A similar procedure is used to calculate $I_n(z)$.

To calculate $K_n(z)$, express the quotient $Q_n(z) = K_{n-1}(z)/K_n(z)$ in terms of two Gauss continued fractions. The individual functions $K_n(z)$ and $K_{n-1}(z)$ are obtained from $Q_n(z)$ and the Wronskian relation involving $K_n(z)$, $K_{n-1}(z)$, $I_n(z)$, and $I_{n-1}(z)$. A similar procedure involving Hankel functions is used to calculate $Y_n(z)$ and $Y_{n-1}(z)$.


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