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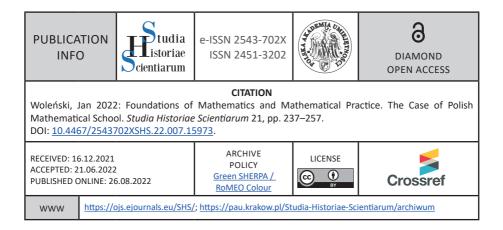# Foundations of Mathematics and Mathematical Practice. The Case of Polish Mathematical School

Abstract

The foundations of mathematics cover mathematical as well as philosophical problems. At the turn of the 20th century logicism, formalism and intuitionism, main foundational schools were developed. A natural problem arose, namely of how much the foundations of mathematics influence the real practice of mathematicians. Although mathematics was and still is declared to be independent of philosophy, various foundational controversies concerned some mathematical axioms, e.g. the axiom of choice, or methods of proof (particularly, nonconstructive ones) and sometimes qualified them as admissible (or not) in mathematical practice, relatively to their philosophical (and foundational) content. Polish Mathematical

School was established in the years 1915–1920. Its research program was outlined by Zygmunt Janiszewski (the Janiszewski program) and suggested that Polish mathematicians should concentrate on special branches of studies, including set theory, topology and mathematical logic. In this way, the foundations of mathematics became a legitimate part of mathematics. In particular, the foundational investigations should be conducted independently of philosophical assumptions and apply all mathematically accepted methods, finitary or not, and the same concerns other branches of mathematics. This scientific ideology contributed essentially to the remarkable development of logic, set theory and topology in Poland.

Podstawy matematyki i praktyka matematyczna. Przypadek Polskiej Szkoły Matematycznej

Abstrakt

Podstawy matematyki obejmują zarówno problemy matematyczne, jak i filozoficzne. Na przełomie XIX i XX w. powstały trzy główne szkoły w podstawach matematyki, tj. logicyzm, formalizm i intuicjonizm. Powstał problem dotyczący tego, w jakim stopniu podstawy matematyki wpływają na matematykę jako taką. Chociaż matematyka była i jest nadal deklaratywnie uważana za niezależną od filozofii, rozmaite kontrowersje wokół niektórych aksjomatów (np. pewnika wyboru) czy metod dowodowych (np. niekonstruktywnych), prowadziły do wniosków na temat ich dopuszczalności (lub nie) w praktyce matematycznej, zależnie od ich treści z punktu widzenia filozofii i podstaw matematyki. Polska Szkoła Matematyczna powstała w latach 1915–1920. Jej program badawczy został nakreślony przez Zygmunta Janiszewskiego (tzw. program Janiszewskiego) i sugerował, że polscy matematycy powinni się koncentrować na określonych dziedzinach badawczych, mianowicie teorii mnogości, topologii i logice matematycznej. W ten sposób, podstawy matematyki stały się w pełni legitymowaną dziedziną

badań matematycznych. W szczególności badania w podstawach matematyki powinny być niezależne od filozoficznych założeń i stosować wszystkie akceptowane metody matematyczne, finitystyczne lub nie, a to samo dotyczy innych gałęzi matematyki. Ta ideologia naukowa istotnie przyczyniła się do rozwoju logiki, teorii mnogości i topologii w Polsce.

Słowa kluczowe: *Polska Szkoła Matematyczna, logika, logicyzm, formalizm, intuicjonizm, teoria mnogości*

1. Introduction

Mutual relations between philosophy of mathematics, the foundations of mathematics (including mathematical logic) and mathematical practice are recently frequently discussed (see, for example, Azzouni 1994, Taylor 1999, Corfield 2003, Mancosu 2008, Ferreirós 2015, Baldwin 2018). Evert Beth (see Beth 1968, Chapter 22) offered a systemization which can serve as a convenient starting point for an analysis of topics involved in the mentioned relation. According to Beth, we might distinguish, the following four interconnected fields of research related to mathematics:

1. Mathematics proper (mathematical activity);
2. Mathematical foundations of mathematics;
3. Philosophy of mathematics;
4. General philosophy.

Mathematics proper, that is, as an academic field, is usually considered as independent of philosophical controversies. Yet it happens that mathematical discoveries are accepted, rejected or evaluated from philosophical point of view. Kronecker rejected set theory as philosophically not acceptable, and Gordan reacted to Hilbert's proof that invariants have a finite basis by words "This is not Mathematics, it is Theology!" – both, Kronecker and Gordan, assumed constructivism (to prove that something exists mathematically means that the object in question is constructed by finite numbers of steps) in philosophy and the foundations of mathematics. However, most mathematicians did not share of their philosophical (that is, related to constructivism) scruples and accepted "non-constructive" results. It also happens that leadings adherents of definite philosophical (foundational) projects go against their views and prove some theorems by methods incoherent

with their "moral" methodological code – Brouwer, the originator of intuitionism, proved the fixed-point theorem in topology by means not intuitionistically acceptable. However, in general, to repeat, working mathematicians see their ordinary scientific practice is autonomous with respect to philosophy.

2. How to understand the name "foundations of mathematics"

I intentionally used the phrase "philosophical (foundational)" in the former section. In fact, borderlines between (2), (3) and (4) are vague. There are various views concerning to which extent the foundations of mathematics are mathematical and to which extent, they involve philosophical issues. Logicism (proposed by Frege at the end of the 19th century and developed by Russell at the beginning of the former century), intuitionism (initiated by Brouwer in the first decade of the 20th century) and formalism (originated with Hilbert also at the beginning of the 20th century), constituted the main traditional (let say about 1920) currents in the foundations of mathematics. They differently accounted the role of logic in mathematics and the question which mathematical theorems and methods, for example, the axiom of choice, transfinite induction, the law of excluded middle, proofs by reductio ad absurdum (indirect demonstrations), are admissible in doing mathematics as such. Yet they explore several general philosophical problems as applied to mathematics, for instance, the existence of mathematical objects or the nature of mathematical knowledge. One can say that the issue concerns not so much ordinary mathematical practice, but rather systematization of mathematics, typically, at least in the case of logicism in a system of a so-called grand logic (like elaborated by Frege or Russell) or, eventually, converting all proofs into formalized in first-order logic (formalism – the first-order thesis: all mathematics can be expressed in elementary logical calculus) or, as in the case of intuitionism, in intuitionistic logic admitting constructive proof-strategies only.

The term "mathematical foundations of mathematics" was invented in order to capture the idea that at least some foundational problems can be effectively treated (perhaps even solved) by genuine mathematical methods. Yet it is very problematic whether such a treatment of the foundations of mathematics can be washed out

of philosophical contents. Take logicism as a foundational program. It says that mathematics is a part of logic. This view requires that all mathematical concepts are defined via logic and all theorems proved by purely logical means. Well, but the logicist must define logic in advance and explain what means the qualification "purely". Omitting details, the logicist should clearly decide whether logic is restricted to first-order or extended to higher-order one. Even if we say that it is a technical issue, its philosophical aspect is clear, similarly as in the case the nature of the axiom of reducibility in the ramified theory of types. An ordinary mathematician might reply something like that "Well, I am not interested in how you identify logic, because I feel entitled to consider all intuitively faithful (correct) deductive means as deserving to be qualified as logical. I see logicism as an interesting attempt to systematize the entire mathematics. I also agree that it looks as valuable intersection of mathematics and philosophy, but this combination has no actual impact on my scientific practice". Similar remarks can be formulated on formalism and intuitionism as foundational positions. All "big three" in the foundations of mathematics, that is, logicism, formalism and intuitionism, were transformed about 1930 into set-theoreticism (set theory is the basis of the entire mathematics and foundational studies should use this theory after all), proof-theoreticism (the foundations of mathematics focus on mathematical proofs analyzed formally, that is, as object of formal theories) and constructivism (constructivity is not only a claim, but also a practice of doing the entire mathematics)[1]. Yet the old perspective remained, because the foundations, even more mathematical as compared with older views of Russell, Hilbert and Brouwer, contain philosophical contents.

General philosophy enters mathematics mostly via (2) and (3), but sometimes happens that philosophers claim that a philosophical legitimacy of mathematical results is required; for instance, according to

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[1] Publication of Hilbert-Ackermann 1928, where first-order logic was constructed as a separate calculus and Gödel's discovery of the incompleteness of arithmetic (it showed essential limitations of formalism) were important in this respect. I resign from more extensive remarks on the development after 1930, because new tendencies are not particularly relevant for the topic of the present paper. See Mostowski 1964 for a presentation of set-theoreticism, proof-theoreticism and constructivism, and Lindström, Palmgren, Segerberg, Stoltenberg-Hansen 2009 for the evolution of logicism, formalism and intuitionism.

Roman Ingarden (see Ingarden 1963, p. 335) claimed that the full content of Gödel's theorems can be fully apprehended without after explaining their philosophical assumptions – this way of looking at mathematics is rather abandoned by mathematicians. Another problem, which I entirely neglect, consists in influence of mathematics on general ontology and epistemology – views of Plato, Descartes, Leibniz and Kant provide good examples here. For reasons provided in the last and this paragraph, philosophical and mathematical foundations of mathematics cross each other at many points. Hence, the names "foundations of mathematics", "philosophical foundations of mathematics" (or "philosophy of mathematics") and "mathematical foundations of mathematics" refer to partially the same or similar scope. These terminological facts are well confirmed by titles and contents of related monographs, anthologies, handbooks or textbooks (neglecting older examples, see for instance, Beth 1968, Hatcher 1982, Schirn 1998, George, Velleman 2001, Shapiro 2005, Irvine 2009, Kunen 2011, Roselló 2012, Bedürftig, Murawski 2018, Centrone, Kant, Sarikaya 2019, Hamsik 2020, Linnebo 2020, Cevik 2021). In order to avoid possible misunderstanding, partially due to a very limited way of the above presentation of logicism, formalism and constructivism (see books quoted earlier in this paragraph for a more extensive treatment), the distinction of three schools in the foundations of mathematics does not mean that formalism and constructivism deny the importance of mathematical logic in doing mathematics, logicism and constructivism neglect formal methods or logicism and formalism resign from constructive methods. Thus, all "big three" in the foundations, share some general claims, although they do that in various ways, for instance, as far as the issue concerns the mutual relation of logic and mathematics, logicism considers the first as prior to the second, formalism develops both in parallel (it means that logic can and should use mathematical methods), but constructivism connects logic with mathematical language.

3. Polish Mathematical School: Its origin and development

Polish Mathematical School (PMS, for brevity) provides a very interesting case in the problem of mutual relations between the foundations (philosophical as well as mathematical) and mathematics itself. PMS

was formed in the period 1915–1920 (see Kuzawa 1968; Kuratowski 1980; Duda 2019, Chapter V; I also use fragments of my previously published papers, namely Woleński 1995; 2008). After the beginning of World War I in 1914, Russian troops left Warsaw very soon and the city was occupied by Germans until 1918. The German authorities, in order to gain political sympathies of Poles, agreed to reopen the University of Warsaw in 1916[2]. Four mathematicians, Wacław Sierpiński, Zygmunt Janiszewski, Samuel Dickstein and Stefan Mazurkiewicz, and a philosopher (logician) Jan Łukasiewicz, were appointed as professors or lecturers. This period of the development of PMS finished in 1920, when the first volume of *Fundamenta Mathematicae* appeared. PMS flourished in the interwar period – it also remained active after 1945. It had two branches, one in Warsaw, and second in Lviv (Lwów – in Polish)[3]. For the topic discussed in the present paper, the former is much more important, because the second circle was not so much interested in the foundations of mathematics as the first, although its members, like Stefan Banach or Hugo Steinhaus, shared general foundational views dominant in PMS.

In order to understand PMS, it is important to take into account some historical facts. Since Poland lost its independence at the end of the 18th century and was partitioned among Russia, Prussia (later Germany) and Austria (later Austro-Hungary), Polish academic life had several difficulties and limitations. Polish scientists undertook several efforts to organize Polish (that is, made by Poles and in Polish language) science. For the origin and development of PMS, at least two facts should be noted. Firstly, the mentioned Dickstein, informally called the Polish ministry of science, popularized many recent achievements in science and mathematics. He published a monograph about the concepts and methods of mathematics (Dickstein 1891) – it informs the foundational works of Bolzano, Cantor, Dedekind, Frege, Grassmann, Hankel, Helmholtz, Kronecker Peano, Riemann, and Weiestrass; it was

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[2] The university was founded in 1816 and closed in 1831 by the Tsarist goverment. It was partially restored as Warsaw Main School in 1862 and closed once again in 1868. In 1870–1915, Warsaw had Imperial (Russian) University.

[3] Note that the name "Polish Mathematical School" is not equivalent to the label "Polish Mathematics", because mathematicians working in Cracow did not belong to PMS.

the first modern book on the foundations of mathematics published in Polish. Dickstein also established a special journal, *Wiadomości Matematyczno-Fizyczne* (Mathematical-Physical News), devoted to recent achievements in mathematics and physics, and a series of books in which Polish translations of works of Riemann, Klein, Helmholtz, Dedekind, Pieri, Enriques, Poincaré and Whitehead were published. Although it is difficult to measure the real influence of these publications, they certainly contributed to the growth of interests in the abstract approach to mathematics in Poland, particularly among young scholars interested in the foundations of mathematics. Secondly, the Lviv University appointed Sierpiński as the professor of mathematics in 1910. He began systematic lectures in set theory, attended by philosophers and mathematicians; he also published one of first textbooks in set theory (Sierpiński 1912) in the world. Janiszewski, defended his Habilitation in Lviv in 1913 – the dissertation concerned topology; Janiszewski's habilitation lecture was devoted to realism and idealism in mathematics – the title indicates his considerable interests in the philosophy of mathematics. In Lviv, Kazimierz Twardowski established, in 1895, a philosophical school (later called the Lviv-Warsaw School; LWS, for brevity) of which many representatives worked in mathematical logic – Twardowski's students included Łukasiewicz and Stanisław Leśniewski, very important persons for the development of PMS (see below).

4. The Janiszewski Program

Janiszewski published two general (and mostly popular – directed to non-specialists in mathematics) papers on logic and the philosophy of mathematics for the 2nd edition of the Guide for Autodicats (see Janiszewski 1915a; 1915b; I also use fragments of my previously published paper, namely Woleński 2008). He presented logic as closely related to mathematics and He saw logic as somehow independent of its applications:

> We note that logistics has no practical profits as its aim, at least directly. Logistic symbolism and the analysis of concepts by their reduction to their primitive elements are not introduced in order to think, argue and write in this way. Similarly, physicists do not create the theory of sounds in order to help musicians in composing or to write notes as

mathematical equations (however, this does not exclude an indirect use of acoustic theories in the development of music). Logistics can contribute to the development of other sciences by discovering new forms of reasoning and by training thinking, but this is only a possibility and would be its by-product. The direct aim of logistics in application to other sciences can only consist in explaining their logical structure (Janiszewski 1915a, p. 454; translation by the present author, cited by Woleński 2008, p. 32).

This is an interesting passage, because it shows that Janiszewski, who studied in France, did not accept skepticism about logic, very characteristic of French mathematicians, and explicitly rejected Poincaré's objections against logic as pointless. As far as the matter concerns the philosophy of mathematics, Janiszewski, limited its scope to (a) the nature of objects and theorems of mathematics (are they *a priori* or not?), and (b) the problem of the existence of mathematical objects and the correctness of some modes of reasoning in mathematics. There is an essential difference between (a) and (b):

> The problems considered in previous sections (that is, concerning (a) – J. W.] are, so to speak, outside of the scope of a mathematician's activity. Independently of any view about these questions or its lack, this fact has no influence, at least no direct one, on work inside mathematics and does not prevent communication between mathematicians. Disregarding what mathematicians think about the essence of natural numbers or mathematical induction, they will use them in the same way. On the other hand, there are controversial problems which have a direct influence on mathematical activity. They concern the validity of some mathematical arguments and the objective side of some mathematical concepts (Janiszewski 1915b, p. 470; translation by the present author, cited by Woleński 2008, p. 32).

Janiszewski mentions the character of mathematical definitions (predicative or not) and the admissibility of the axiom of choice as examples of group (b). He rather reports controversies in the philosophy of mathematics without proposing solutions.

The course of the World War I vitalized Polish hopes for independence of the country and inspired a discussion on the future organization of science. In 1916 the Committee of the Mianowski Fund, a special institution supporting Polish science, invited scholars from various fields to formulate remarks and more extensive projects concerning the most effective activities aiming at improving the organization of research. The organizers collected 44 papers, published in the first volume of *Nauka Polska* (Polish Science), a newly established journal devoted to various aspects of scientific research in Poland. Mathematics was represented by the voices of Stanisław Zaremba and Janiszewski. Although the former paper is almost forgotten (see below about some ideas of Zaremba), Janiszewski's contribution (see Janiszewski 1918) gained more fame than any other from the rest of the submitted comments. The main idea of the program in question consisted in promoting various activities for achieving an autonomous position by Polish mathematics. Let me quote the very end of this seminal paper:

> If we do not like to always "lag behind", we must apply radical means and go to the fundamentals of what is wrong. We must create a [mathematical] "workshop" at home! However, we may achieve this by concentrating the majority of our mathematicians in working in one selected branch of mathematics. In fact, this takes place automatically nowadays, but we have to help this process. Doubtless, establishing in Poland a special journal devoted to the only selected branch of mathematics will attract many to research in this field. […]. Yet there is also another advantage of such a journal in building the mentioned "workshop" in ourselves: we would become a technical center for publications in the related field. Others would send manuscripts of new works and have relations with us. […]. If we want to capture the proper position in the world of science, let us come with our own initiative (Janiszewski 1918, p. 18; translation by the present author; cited by Woleński 2008, p. 33).

These words are commonly regarded as the decisive factor of the rise and the subsequent development of modern mathematics in Poland, particularly in the years 1918–1939.

5. The standard interpretation of the Janiszewski Program

According to the standard historical reading, Janiszewski's postulates are interpreted as claiming that Polish mathematicians should concentrate on set theory, topology, foundations of mathematics (including mathematical logic) and their applications in other mathematical fields (I analyzed this issue in Woleński 1995 and 2008, and use the fragments of the article below). However, he did not explicitly point out which field should be chosen as the only branch of mathematics on which Poles could concentrate in the future, although he clearly alluded to something that "takes place automatically"; in particular, the words "set theory", "topology" or "foundations of mathematics" do not occur in Janiszewski 1918. It is possible that Janiszewski's formulations were so careful, because he wanted to avoid a direct clash with Zaremba, a great enemy of new tendencies in the foundations of mathematics, but also a great authority in Polish mathematical community. He was strongly influenced by the French style of doing mathematics, according to which logic was considered as quite marginal in mathematics and regarded as its servant. Although Zaremba's writings contained a lot of logical and foundational topics, he considered them as introductory for mathematics. As I already remarked, Janiszewski, who had also studied in France, represented a quite contrary view about the role of logic. The Janiszewski program as it is commonly understood today was officially announced on the cover of *Fundamenta Mathematicae* by informating that the journal is devoted "set theory, and related issues (direct applications of set theory), Analysis Situs, mathematical logic, axiomatic investigations" (Janiszewski postulated that an international mathematical journal should be published in Poland). Warsaw became the center of the realization of the Janiszewski projects. It was additionally documented by appointing Leśniewski as he professor of philosophy of mathematics in the University of Warsaw. He and Łukasiewicz created the Warsaw Logical School (WLS, for brevity). Both became professors at the Faculty of Mathematics and Natural Science – by the way, it was an (perhaps even, the) phenomenon of a global scale that two philosophers played the role of professors of mathematics. Sierpiński, Mazurkiewicz, Leśniewski and Łukasiewicz belonged to the Editorial Board of *Fundamenta Mathematica* and this fact can be taken as

another sign of how the Janiszewski program generated the position of logic in PMS.

Returning to Janiszewski's views expressed in his paper published in 1915, he distinguished (i) general problems coming from philosophy, like the status of mathematical theorems, the mode of existence of mathematical entities, and (ii) special problems related to the way of defining mathematical properties and entities. The questions from the group (i), for instance, the nature of mathematical entities, cannot be definitely solved by mathematical methods, although mathematic. The situation of problems from the group (ii) appears as fairly different, namely they were considered by Janiszewski as having a mathematical relevance, even if they are subjected to philosophical controversies. Janiszewski pointed out non-predicative constructions, but the axiom of choice provides the most spectacular example. According to Sierpiński:

> Still, apart from our personal inclination to accept the axiom of choice, we must take into consideration [...] its role in the Set Theory and in the Calculus. On the other hand, since the axiom of choice has been questioned by some mathematicians, it is important to know which theorems are proved with its aid and to realize the exact point at which the proof has been based on the axiom of choice; for it has frequently happened that various authors have made use of the axiom of choice in their proofs without being aware of it. And after all, even if no one questioned the axiom of choice, it would not be without interest to investigate which proofs are based on it and which theorems can be proved without its aid – this, as we know, is also done with regard to other axioms (Sierpiński 1965, p. 94; this view was similarly expressed in his writings since 1918; cited by Murawski, Woleński 2008, p. 333).

This is a very instructive formulation of the view that purely mathematical questions should be investigated independently of their philosophical environment. This manner of doing set-theory became typical for PMS.

Let me make a somehow risky historical digression at the point. Twardowski urged that philosophy should be scientific. Disregarding that this claim, as show various attempts of transforming philosophy into one

of the so-called special sciences, is controversial and differently justified, the founder of the Lvov-Warsaw School saw the future of scientific philosophy in using a clear language and justifying philosophical statement by proper methods. In particular, scientific philosophy must abandon traditional speculative problems. Twardowski used the label "metaphysicism" for the way of doing philosophy by considering so-called "great" problems as, for instance, the existence of God or immortal soul. Philosophy should concentrate on clearly formulated problems, because only such can be scientifically decided. Twardowski did not deny that metaphysics has an importance for worldviews and is fully legitimate in this respect, but he sees no chance to solve such question in a way used in science. Doubtless, Sierpiński had to discuss the foundational problems of mathematics with Janiszewski, and the mentioned papers of the latter give an indirect evidence that basic points of the Janiszewski program were (or at least, could be) formulated in discussions in the Lviv mathematical circle. I have no evidence that Twardowski participated in these debates or other concerning the future of mathematics in Poland. However, I am inclined to think that he, as a person strongly interested in organization of science, could exchange some general views with Sierpiński – both were colleagues at the same faculty. It is notable that Twardowski wanted to see Polish philosophy as comparable with the world philosophy, similarly as Janiszewski's aim consisted in making Polish mathematics as capturing "the proper position in the world of science". Anyway, we can eventually say that the group (i) in the Janiszewski program exemplifies a kind of metaphysicism and thereby should be considered as belonging not to mathematics proper, but to a philosophical worldview of particular mathematicians. A very important consequence of this perspective consists in making investigations on the problems from the group (ii) independent of "big" philosophical controversies.

6. The Warsaw Logical School

Leśniewski and Łukasiewicz, two leaders of WLS, were philosophers by their studies (and PhDs obtained in philosophy proper, that is, strict philosophy as it was called at the time) with an additional mathematical training (I analyzed this issue in Woleński 2008, and use the fragments of the article below). They attracted several young mathematicians and philosophers to work in the domain of mathematical logic, in

particular (I preserve alphabetical order) Stanisław Jaśkowski, Adolf Lindenbaum, Andrzej Mostowski, Moses Presburger, Jerzy Słupecki, Bolesław Sobociński and Alfred Tarski; the last became the third central figure of WLS. Thus, this group had double roots, mathematical and philosophical – the latter going back to the LWS. According to Tarski

> Almost all researchers, who pursue the philosophy of exact sciences in Poland, are indirectly or directly the disciples of Twardowski (Tarski 1992, p. 20; this letter was written in 1936; cited by Woleński 2008, p. 41).

Now, one could expect that Warsaw logicians, due to their philosophical pedigree, defended a definite standpoint in the philosophy of mathematics. However, this supposition is not correct, except of Leśniewski, who constructed a kind of a grand logic, that is, a system covering the entire mathematics. Sobociński contrasted him and Łukasiewicz in the following manner.

> There is an interesting contrast between [...] Łukasiewicz and Leśniewski. The latter was also a philosopher by training; he too moved away from philosophy and avoided even philosophical "asides" in his npublished work. But, unlike Łukasiewicz, he held that one could find a "true" system in logic and in mathematics. His systematization of the foundations of mathematics was not meant to be merely postulational; he wished to give, in deductive form, the most general laws according to which reality is built. For this reason, he had little use for any mathematical or logical theory which, even though consistent, he did not consider to be in accord with fundamental structural laws of reality. [...]. Łukasiewicz had no such preoccupations. He did not try to construct a definite system of the foundations of the deductive systems. His aims were, on the other hand, to provide exact and elegant structures for many domains of our thinking where such had either been wanting or insufficient (Sobociński 1957, pp. 42–43; cited by Woleński 2008, p. 42).

Łukasiewicz's position as described by Sobociński was commonly shared in WSL and PMS. In fact, members of WLS had very different

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views concerning the existence of mathematical objects, for instance. Łukasiewicz was close to Platonism, but Tarski – to nominalism. Yet some important Polish logical discoveries were motivated by philosophy, for example, Łukasiewicz's many-valued logic by the problem of determinism or Tarski's theory of truth – by the classical truth-definition going back to Aristotle.

7. The general foundational standpoint of PMS

Due to its general standpoint, PMS was neutral with respect to logicism, formalism and intutionism (I analyzed this issue in Woleński 1995 and Murawski, Woleński 2008, and use the fragments of the articles below). It concerned not only purely philosophical issues, but also means employed in metamathematics, that is, mathematical treatment of mathematical theories – the word "metamathematics" can be considered as equivalent to the label "mathematical foundations of mathematics". Tarski expressed the standpoint of PMS in the following words:

(1) As an essential contribution of the Polish school to the development of metamathematics one can regard the fact that from the very beginning it admitted into metamathematical research all fruitful methods, whether finitary or not (Tarski 1954, p. 713; cited by Woleński 1995, p. 386).

(2) No particular philosophical standpoint regarding the foundations of mathematics is presupposed in the present work (Tarski 1930, p. 62; page-reference to English tr.).

(3) We would of course fully dispose of all the problems involved [that is, concerning inaccessible cardinals – J. W.] if we decided to enrich the axiom system of set theory by including (so to speak, on a permanent basis) a statement which precludes the existence of 'very large' cardinals, e. g., by a statement to the effect that every cardinal $> \omega$ is strongly incompact. Such a decision, however, would be contrary to what is regarded by many as one of the main aims of research in the foundations of set theory, namely, the axiomatization of increasingly large segments of 'Cantor's absolute'. Those who share this attitude are always ready to accept new 'construction principles', new axioms securing the existence of new classes of 'large' cardinals (provided that they appear to be consistent with old axioms), but are not prepared to accept

any axioms precluding the existence of such cardinals – unless this is done on a strictly temporary basis, for the restricted purpose of facilitating the metamathematical discussion of some axiomatic systems of set theory (Tarski 1962, p. 124; cited by Woleński 1995, pp. 385–386).

These methodological explanations fully concur with position illustrated by the earlier quoted Sierpiński's account of the axiom of choice.

Consequently, PSM did not investigate mathematical problems as dependent of philosophical, even foundational assumptions (see Murawski 2014 for a general account). This attitude resulted in a rapid development of mathematical foundations of mathematics in Warsaw. In particular, every foundational problem could be investigated from many general standpoints, for instance, Boolean algebra, as a logical system, formal calculus and as something related to intuitionistic logic. Another outcome consisted in application, mostly by Tarski, Lindenbaum and Presburger, of foundational results to specific mathematical theories, like topology, arithmetic of natural and real numbers or geometry (see Mc Farland A., McFarland J., Smith J. 2014). It was a novelty, because the standard view (Leopold Löwenheim and Thoralf Skolem was an exception in this respect) at the time restricted the role of the foundations to providing the mathematical method and being a scheme of systematization of mathematics. Thus, although formalists and intuitionists declared that mathematics should respect finitism or constructivism in applying mathematical methods, their practice did not frequently obey this declaration. PMS had not such dilemmas. To sum up, PMS accepted two principles:

(P1) all commonly accepted mathematical methods should be applied in metamathematical investigations;

(P2) metamathematical research cannot be limited by any *a priori* accepted philosophical standpoint (this principle was already anticipated by Dickstein) (cited by Murawski, Woleński 2008, p. 324).

The Polish view, as we can call it, has a particular consequence consisting in a freedom of accepting philosophical opinions sometimes being at odds with applied methods. Perhaps Tarski was an extreme example of this

practice. He used all admissible mathematical methods in his logical works, in particular, infinitary ones, usually associated with Platonism in the philosophy of mathematics, but he contributed to all mentioned grand projects by the idea of logical concepts as invariants (related to logicism), the theory of consequence operations (a component of formalism) and the topological semantics for intuitionistic logic. Tarski stressed that this methodological attitude, sometime labelled as "methodological Platonism", became a characteristic feature of almost the entire Polish school. On the other hand, he had explicit sympathies to empiricism, nominalism, reism and finitism (he even called himself "a tortured nominalist"). Due to a strict departure of mathematics and philosophy as well as locating them on different levels, no contradiction occurs in Tarski's position, although we certainly encounter here an example of a cognitive dissonance to some extent. Probably Tarski saw the situation in such a way and perhaps it explains why he usually abstained from a wider elaboration of his philosophical views, at least in his writings. Tarski was more involved in philosophy in oral debates concerning general philosophical issues as well those related to mathematics. It is well documented by the records of discussions between Tarski, Carnap, Quine and Russell at Harvard in 1940–1941 (see Frost-Arnold 2013) (cited by Murawski, Woleński 2008, pp. 324–325).

Perhaps Mostowski's characterization of the cognitive situation in the foundations of mathematics is particularly instructive. He said:

> These [general philosophical] problems are of a philosophical nature and we can hardly expect to solve them within the limits of mathematics alone and by applying only mathematical methods (Mostowski 1955, p. 3).

and, similarly:

> We see that the issue between Platonists, formalists and intuitionists is as undecided to-day as it was fifty years ago (Mostowski 1964, p. 14).

Yet he stressed that these general questions led to more specific and subjected to a formal treatment, namely (I) the role and limits of axiomatic method; (II) the constructive tendencies in mathematics; (III) the axiomatization of logic; (IV) the decision problem. Observe that these considerations are very similar to that of Janiszewski (see above). Mostowski's general standpoint was such:

> Thus, as we see, the investigations of the foundations of mathematics are not without importance although they do not stand for a full investigation on the foundations of mathematics. Their results are to use for mathematics as well as for philosophy. In this sense they fulfil the tasks assigned to them (Mostowski 1955, p. 42; cited by Murawski, Woleński 2008, pp. 328–329).

He believed that the future belongs to constructivism, but it was rather a faith than a mathematically justified view.

8. Final remarks

Thus, we see a continuity of views from Janiszewski to Mostowski. Although members of PMS, particularly logicians, had definite philosophical views, their scientific practice was subordinated to what is in mathematics as such. In other words, mathematical practice and its technical needs should be always prior to philosophical problems. Perhaps it was one of roots of the success of this school in two directions: (a) replacing logicism by set-theoreticism; (b) developing mathematical foundations of mathematics. Appreciating the role of Hilbert and other people in the rise and development of metamathematics, Poland was the country in which mathematical foundations of mathematics were practiced in an especially fruitful way, also by "hard" mathematicians, like Sierpiński (works on inaccessible cardinals, also joint papers with Tarski) and Kazimierz Kuratowski (investigations of transfinite reasoning and, with Tarski, on logical operations and projective sets). However, it should be explicitly noted that Janiszewski's postulate "let us come with our own initiative" was the most successfully realized in mathematics based on set theory. Yet the works of this school in the foundations of mathematics essentially contributed to the climate in which other mathematical investigations successfully developed.

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