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X. FOUNDATIONS WITHOUT SETS

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THE Tarski school has emerged as the dominant school of logic, semantics, and the foundation of mathematics. Three tenets characterize this school: first, set theory constitutes the foundation of mathematics; second, the foundation of mathematics is not strictly speaking part of logic; third, sentences are the primary bearers of truth and of logical validity, two concepts which are properly defined within set theory. From a philosophical point of view, however, these tenets can be shown to be unacceptable. The following alternatives recommend themselves: first, a theory of properties, relations, and propositions (PRPs) constitutes the foundation of mathematics; second, the foundation of mathematics is part of logic proper; third, propositions are the primary bearers of truth and of validity and these two concepts are properly defined within a theory of PRPs.

The argument in outline is this. Intensional logic is undeniably part of logic. The cannonical formulation of intensional logic represents gerundive phrases, infinitive phrases, and "that"-clauses as singular terms which denote PRPs. The logic for these singular terms is thus a theory of PRPs; hence, this PRP theory qualifies as part of logic. Set theory, by contrast, does not qualify as part of logic in any such way. Sets, moreover, play no role in what might be called our naturalistic ontology, nor do sets have any special pragmatic role in pure mathematics or in empirical science. Whatever set theory can accomplish in these areas can be accomplished equally well by PRP theory, which is already a legitimate part of logic. PRP theory should therefore replace set theory as the foundation of mathematics, and given this, the foundation of mathematics should be counted as part of logic after all. Finally, the definitions of truth and of validity are dealt with in terms of two conceptions of PRPs. On one conception, the paradigm PRPs are concepts and thoughts; on the other, the paradigms are the actual qualities, connections, and conditions of things. This dual approach makes it possible to state clearly and precisely the traditional correspondence theory of truth, according to which a thought is true if and only if it corresponds to a condition that obtains. Even Tarski acknowledges the intuitive primacy of this commonsense theory of truth;[1] indeed, the only cogent objection to it has been that a clear and precise statement has been missing. Now once one rejects a semantical treatment of truth in favor of a propositional treatment, uniformity requires one to adopt a propositional treatment of validity as well. The indicated dual approach to PRPs provides such a treatment.

A full exposition of this argument is beyond the scope of a short paper.[2] The purpose of this paper is to present in some detail the part of the argument that is aimed against set theory, for set theory is the cornerstone of the Tarski school. If this part of the argument moves members of the Tarski school, they are unlikely to put up much resistance to the rest of it.

There are, I have indicated, three strategies by means of which a set theorist might attempt to justify his ontology. One strategy is to show that sets are already included in the naturalistic part of our everyday ontology. If they are, then one may assume that whatever justifies this ontology also justifies the ontology of sets. Another strategy is to show that, like the theory of PRPs, set theory is already part of logic. In this case, the ontology of sets would be justified in the same way logic is justified.[3] The third strategy is to show that set theory plays some unique role in mathematics or in empirical science. If it does, then its ontology would be justified pragmatically. None of these strategies is successful, however, as I will now explain.

[1] § 3, Alfred Tarski, "The Semantic Conception of Truth and the Foundation of Semantics", *Philosophy and Phenomenological Research*, vol. 4 (1944), pp. 341–76.

[2] See my "Theories of Properties, Relations, and Propositions", *The Journal of Philosophy*, vol. 76 (1979), pp. 634–48, and *Quality and Concept* (Oxford, 1981), for further discussion of these issues.

[3] The close of the paper presents some remarks on the transcendental justification of the logic for PRPs.

I. The Unnaturalness of Sets

Paul Halmos begins his popular book *Naive Set Theory* (Princeton, 1960) with this observation:

> A pack of wolves, a bunch of grapes, or a flock of pigeons are all examples of sets of things.

If Halmos is right, then there is as much reason to think sets exist as there is to think packs, bunches, and flocks exist. Now it might be that the idea of a set is somehow "genetically" related to ideas of such naturalistic objects as packs, bunches, and flocks.[4] Nevertheless, it is certain that sets are not the same sort of thing as packs, bunches, flocks, etc. Here are a few of the many reasons. First, packs, bunches, flocks, etc., displace volumes, have mass, and come into and pass out of existence. Sets, by contrast, are non-physical and eternal. Secondly, sets cannot change their members; packs, bunches, flocks, etc., can. If a wolf in[5] a given pack dies (or gives birth), the pack is still the same pack. But the set of wolves-before-the-death (birth) is not the same set as the set of wolves-after-the-death (birth). Thus, a set of wolves and a pack of wolves are different. Thirdly, packs, bunches, flocks, etc. do not exist if nothing is in them; this is not so for sets. If there were no wolves, there would be no packs of wolves. But the set of wolves would exist nonetheless, for it would just be the null set. Indeed, if sets exist, the null set is a set that exists necessarily.[6]

If sets are not the same sort of thing as packs, bunches, flocks, etc., what are they? It is now commonplace to say that sets are collections or classes. What is meant by this? Art collections, social classes, sets of dishes: are these cases of the kind of sets posited in set theory? No, definitely not. They are no more the kind of sets posited in set theory than are packs, bunches, and flocks, etc., and for much the same reasons. First, art collections and sets of dishes can displace volumes, have mass, and come into and pass out of existence. And art collections, social classes, sets of dishes, etc., can change their members.[7] Thirdly, ordinary collections, social classes, and ordinary sets do not exist if nothing is in them. (If China has no aristocrats, it has no aristocracy.)

While these three differences suffice to show that ordinary collections, social classes, and ordinary sets differ from set-theoretical sets, there is perhaps a fourth difference, having special philosophical significance. Ordinary collections, social classes, and ordinary sets do not appear to be sensitive to the distinction between membership ($\in$) and inclusion ($\subseteq$), which is the hallmark of set theory. Consider a billionaire who collects art collections in the style in which Howard Hughes used to collect companies. This man purchases outright entire art collections. Now if his, say, ten art collections contain one Cezanne each, then there are ten Cezannes in his collection of art collections. And in general, if a painting is in an art collection that is itself in a collection of art collections, then the painting is in the collection of art collections. The sets of set theory are not like this. No individual paintings are in the set of art collections; only art collections are. Consider another example. If the individual cup and the individual saucer in a matched cup-and-saucer set are themselves in a full set of dishes, then the matched cup-and-saucer set itself is in the set of dishes. But the set-theoretical set of dishes contains only individual dishes and no cup-and-saucer sets. This sort of difference is especially important philosophically, for without the membership/inclusion distinction we would be unable to construct any of the standard set-theoretical paradoxes.[8]

The moral is that sets show up in none of the above naturalistic ontologies. Those who persist in the attempt to motivate the concept of set along such

[4] At the close of the paper I sketch how the idea of set might be "genetically" related to the idea of a sum (or whole). Incidentally, in the present discussion I do not assume that the naturalistic ontology of packs, bunches, flocks, etc., is justified. The point rather is that, if this ontology is justified, that would confer no justification on the ontology of sets since sets are quite unlike packs, bunches, flocks, etc.

[5] Note that in order not to bias the discussion I will use the natural and neutral locution "is in" (and its cognates) rather than the technical locution $\in$. A moment's reflection will show that in adopting this practice I do not commit any fallacies of equivocation.

[6] For another sort of problem, suppose that by time t a given bunch of grapes has dwindled down to a single grape. Does the bunch still exist? If so, is the bunch = the grape? I am not sure. But notice that in set theory the answers are already prescribed: the singleton of the grape exists, and it is not the same thing as the grape. How bizarre singleton sets and null sets are.

[7] This difference gives rise to another: the time-invariant principle of extensionality, which is supposed to be valid for the sets of set theory, is not valid for ordinary collections, social classes, and ordinary sets. Consider art collections. It is in principle possible that the Tate collection should contain at t exactly those art works that the Guggenheim collection contains at t' ($t \neq t'$) and yet that the Tate collection and the Guggenheim collection should always remain distinct.

[8] To derive the standard paradoxes, one needs both of the following principles:

(a) $Ax \supset x$ is in the collection of *As*.

(b) x is in the collection of *As* $\supset Ax$.

Suppose the collection of *As* is an ordinary collection. In this case, although (a) holds for it, (b) does not, as we have just seen. (b) would hold only if, contrary to fact, there were a membership/inclusion distinction for ordinary collections and "in" meant membership.

naturalistic lines sooner or later find themselves offering the "invisible-plastic-bag" conception. But this, I think, only confirms the point that sets do not fall within our naturalistic ontology.

II. No Basis in Logic

The second strategy for justifying the ontology of sets is to attempt to show that set theory is grounded in logic. The ontology of PRPs can be justified in this way; specifically, we can show that certain syntactic constructions—gerundive phrases, infinitive phrases, and "that"-clauses—are best treated as singular terms denoting PRPs. Now I can think of only one syntactic construction that might in an analogous way serve to justify the ontology of sets. That construction is pluralization, for plurals are devices for talking collectively about things. Let us see how a set theorist might try to show that set theory has a special role to play in the treatment of plurals.

Consider the following sentences:

(1) The peanuts outweigh the pecans.
(2) The counties outnumber the states.

These sentences are not transformed universal conditionals:

(1′) $(\forall x,y)((\text{Peanut}(x) \& \text{Pecan}(y)) \supset \text{Outweigh}(x,y))$
(2′) $(\forall x,y)((\text{County}(x) \& \text{State}(y)) \supset \text{Outnumber}(x,y))$.

For whereas (1′) and (2′) are false, (1) and (2) are true. Provisionally, then, let us represent (1) and (2) as 2-place relational sentences:

(1″) Outweigh (the peanuts, the pecans)
(2″) Outnumber (the counties, the states).

Here the plurals are provisionally treated as (defined or undefined) singular terms. It then becomes appropriate to ask what are the primary semantical correlates of these singular terms. A natural hypothesis is that in (1) the primary semantical correlates of "the peanuts" and "the pecans" are physical sums (specifically, the physical sum of all peanuts and the physical sum of all pecans). On the face of it, this hypothesis seems successful. This gives rise to the presumption that the plurals in (2) should be treated analogously; i.e., this suggests that the primary semantical correlates of "the counties" and "the states" in (2) are also sums. But what kind of sum? Not ordinary physical sums, certainly. Since the ordinary physical sum of the counties is identical to the ordinary physical sum of the states, (2) would be false. Yet on its primary reading (2) is true. Therefore, if one continues to be swayed by the presumption that the primary semantical correlates of "the states" and "the counties" are sums, then a new kind of sum must be hypostasized. These new sums should differ from ordinary sums in at least the following respect: the things in the new sum of *F*s must be exactly those things that satisfy the predicate *F*. But this is precisely what is required of sets according to the abstraction principle of naive set theory. So this gives rise to the further presumption that the new kind of sum that is the primary semantical correlate of the plural "the *F*s" in sentences akin to (2) is in fact a set, specifically, the set of *F*s.

Although the above line of reasoning has a certain appeal, it leads immediately to a fatal dilemma. Consider the following problematical sentences:

The peanuts ***both*** outweigh and outnumber the pecans.

Although the counties occupy exactly the same territory as the states, ***they*** outnumber the states, and, in addition, ***they*** resent federal intervention more than the states do.

The whales once outnumbered the human beings; now, however, ***they*** are nearly extinct.

In view of our earlier discussion about the nature of set-theoretical sets, if the plurals in these problematical sentences are treated in the same kind of naive surface-syntactical way adopted above in connection with sentence (2), then their primary semantical correlates cannot be sets. (For example, the set of peanuts cannot outweigh the set of pecans since no set weighs anything.) These primary semantical correlates would have to be some further kind of entity (something more akin to packs, bunches, or flocks than to sets). But in this case, uniformity requires us also to identify the primary semantical correlates of the plurals in (2) not with sets but with this further kind of entity. So if the plurals in the above problematical sentences get the naive surface-syntactical treatment that we provisionally gave to (2), then what initially seemed to be a justification for set theory in the logic for (2) evaporates. On the other hand, suppose the plurals in the above problematical sentences are treated in a sophisticated deep-structural way.[9] In this case, we nullify the original presumption

[9] For example, the second problematical sentence cited a moment earlier in the text might be provisionally thought of as follows: $(\exists x,y)(x =$ (being a) county & $y =$ (being a) state & the-x-sum occupies the same territory as the-y-sum & the-x-extension outnumbers the-y-extension & the-typical-x resents federal intervention more than the-typical-y).

that the plurals in (2) ought to be treated on analogy with the plurals in sentence (1) (i.e., the presumption that the plurals in (2) are singular terms whose primary semantical correlates are some sort of sums). This makes sentence (2) fair game for alternative sophisticated treatments; the various treatments of (2) must compete on their own terms. But if the contest is to take place in this stark arena, then, as I will show next, set theory fails to win its motivation from the treatment of plurals in sentences like (2).

To complete the above argument I must show that, if there is no presumption in favor of a set-theoretical treatment of sentences such as (2), then the set-theoretical treatment succumbs to superior competitors. So as not to bias the argument, let us agree provisionally to represent (2) along the following lines:

(2‴) Outnumber($\{x: Cx\}$,$\{x: Sx\}$).

Here $\{x: Cx\}$ and $\{x: Sx\}$ are extensional abstracts; that is, they are (defined or undefined) abstract singular terms for which the following general law holds:

(3) $\{x: Ax\} = \{x: Bx\} \equiv (\forall x)(Ax \equiv Bx)$.

Further, let us allow that for all non-paradox-producing formulas Ax:

(4) $y \in \{x: Ax\} \equiv Ay$

where y is free for x in Ax. And finally, let us allow that (2‴) is true if and only if there is no 1-1 function from $\{x: Sx\}$ onto $\{x: Cx\}$ though there is a 1-1 function from $\{x: Sx\}$ into $\{x: Cx\}$. In this case (2‴) comes out true, as desired. Now consider briefly what seems to me to be the intuitive picture of the semantics for natural language. According to this picture, predicates and formulas do not *refer* to anything; they simply *express.* A formula A, for example, expresses the property, relation, or proposition denoted by a certain associated gerundive phrase, infinitive phrase, or "that"-clause formed from A. For example, Ax expresses the property denoted by the gerundive phrase "being an x such that Ax". For all non-paradox-producing formulas Ax, the following law holds:

(5) being an x such that Ax is predicable of $y \equiv Ay$

where y is free for x in Ax. In symbols: $y \,\Delta\, [Ax]_x \equiv Ay$. In view of this, the extensional abstract $\{x: Ax\}$ may be contextually defined as follows:

(6) $\ldots\{x: Ax\}\ldots$ iff$_{df}$ $(\exists z)$ (z is equivalent to being an x such that Ax, and $\ldots z \ldots$)

where z is free for α in $\ldots\alpha\ldots$[10] In symbols: $\ldots\{x: Ax\}\ldots$ iff$_{df}$ $(\exists z)(y\Delta z \equiv y\Delta[Ax]_x)$ & $\ldots z \ldots)$. And $\in$ may be contextually defined as follows:

(7) $u \in v$ iff$_{df}$ v is predicable of u.

In symbols: $u \in v$ iff$_{df}$ $u\Delta v$. To be convinced of the adequacy of these contextual definitions, notice that, for all non-paradox-producing formulas A and B, the above law (3) follows directly from (5) and (6), and law (4) follows directly from (5), (6), and (7). However, these laws are all that are needed for an adequate treatment of sentences such as (2‴). Thus, extensional abstracts, and sentences such as (2‴), can be adequately treated within the theory of PRPs, a theory already known to be part of logic. Moreover, this is accomplished without having to hypostasize the sets of set theory. So if, as we have agreed, there is no presumption in favor of the set-theoretical treatment of sentences such as (2), then the outlined alternative treatment wins hands down.

It might be objected that no economy follows from adopting this contextual treatment of extensional abstracts since sets have already entered the picture through an independent pathway, namely, through extensional semantics. According to Frege's semantical theory, all meaningful expressions have two kinds of meaning: sense and reference. Frege identified the references of predicates (and open sentences) with what he called functions. But since at least the time of Tarski's work in extensional semantics, it has been common instead to view the reference of a predicate (open sentence) as a set, namely, the set of

The expressions "the-x-sum," "the-x-extension" and "the-typical'-x" can then be treated as contextually defined singular terms. For example, if "the-x-extension" is represented by "$\{z: x$ is predicable of $z\}$," it can be contextually defined in the way suggested in the text a bit later. Incidentally, it might be crucial that x ranges over properties rather than sets, given the extensionality of sets. Is it not true that the typical policemen $\neq$ the typical short-order cook (or at least that the ideal policeman $\neq$ the ideal short-order cook) and that this would be so even if, because of widespread moonlighting practices, all and only policemen coincidentally turned out to be short-order cooks?

[10] This is just the first-order analogue of Russell's higher-order "no-class" definition of extensional abstracts (A. N. Whitehead and B. Russell, *Principia Mathematica*, Vol. I, Cambridge, 1910, pp. 71–81). Incidentally, even Quine acknowledges, "Classes may be thought of as properties in abstraction from any differences which are not reflected in differences of instances." (*Mathematical Logic*, revised ed., Cambridge, 1951, pp. 120–1).

things that satisfy the predicate (open sentence). That is, on this view the reference of the predicate *F* is the set of *F*s. Since extensional semantics already makes use of sets here, no economy of theory is gained (so someone might argue) by giving extensional abstracts such as $\{x: Fx\}$ the above property-theoretic treatment. In fact, for those persuaded by this set-theoretic semantical theory, it is only natural to identify the primary semantical correlate of the extensional abstract $\{x: Fx\}$—and thus also the plural "the *F*s"—with the reference of the predicate *F*, i.e., with the set of *F*s.

This objection, it seems to me, has gotten the proper order of the argument turned around. What good reason is there for accepting the extensional semantical theory? After all, the natural, intuitive picture of the semantics for predicates and formulas is Russell's, not Frege's. According to this picture, predicates and formulas do not refer to anything; they simply express. The primary semantical correlates of predicates and formulas are just the properties, relations, and propositions expressed by them. What point, then, is there in having a Fregean two-kinds-of-meaning semantics for predicates and formulas rather than the simpler, more natural Russellian one-kind-of-meaning semantics? Nothing is gained theoretically since a Fregean semantics for predicates and formulas can be derived from a Russellian semantics when the above property-theoretic treatment of extensional abstracts is adopted in the metalanguage. In this sense, a Fregean semantics for predicates and formulas provides no more information than its simpler Russellian counterpart. So one can hardly justify a set-theoretical treatment of extensional abstracts and plurals by appealing to the set-theoretical content in an unnatural and informationally superfluous semantical theory.

The conclusion then is this. Neither a naive surface syntactical approach to plurals nor a sophisticated deep-structural approach justifies the ontology of sets.

III. The Dispensability of Sets

On the basis of the foregoing it appears that the ontology of sets does not fall within our naturalistic ontology and also that set theory is not part of logic. There remains one more strategy by which one might try to justify the ontology of sets: perhaps set theory is uniquely useful in pure mathematics or in empirical science. This strategy, however, also comes to naught.

Consider pure mathematics first. Here set theory is used in an entirely abstract way to aid in and to unify the study of such matters as cardinality, order, mapping, etc. Let x be an arbitrary non-empty set. I will say that x_0 is an ultimate element of x if and only if $x_0 \in x_1 \in x_2 \in \ldots \in x$ and nothing is in x_0 itself. Now, it is a matter of complete indifference what the ultimate elements are of any set that might be contemplated in pure mathematics. Hence, as far as pure mathematics is concerned, the study of sets can be limited to those sets whose only ultimate element is the null set. The theory of such sets is called ***pure set theory***. We may conclude, therefore, that if set theory should turn out to have a unique role to play in pure mathematics, that role can be filled by pure set theory.

It turns out, however, that the axioms of pure set theory can be interpreted as being, not about sets at all, but instead about properties of an appropriate kind.[11] To see one way in which this can be done, consider the usual motivation given for Zermelo's axioms for pure set theory, namely, the motivation provided by the iterative conception of set.[12] Beginning with the null set ϕ, one constructs in stages a hierarchy of new sets by means of a power operation:

Stages	1	2	...	α	...
Sets	$\{\phi\}$	$\{\phi,\{\phi\}\}$	...	$\{y$: y is a set and every element of y belongs to a set constructed prior to $\alpha\}$	...

The elements of these constructed sets are all pure sets. And if $\in$ is interpreted as expressing the set-membership relation and if, for every stage in the hierarchy, there is a later stage that immediately follows no stage, then the union of these constructed sets is a model for Zermelo's axioms for pure set theory. However, on analogy with the iterative conception of set, there are also iterative conceptions of properties. The easiest to describe is the iterative conception of what may be called pure *L*-determinate Carnapian properties. Carnapian properties are identical if and only if they are necessarily equivalent.[13] A property x is *L*-determinate if and only if, necessarily, x's instances are necessarily x's instances,

[11] Since this proposal works for first-order pure set theories, which countenance sets of sets, sets of sets of sets, etc., it goes well beyond Russell's original "no-class" construction, which works only for discourse about sets of non-sets.

[12] See, e.g., George Boolos, "The Iterative Conception of Set," *The Journal of Philosophy*, vol. 67 (1971), pp. 215–31.

[13] In "Theories of Properties, Relations, and Propositions" (*ibid.*) and *Quality and Concept* (*ibid.*), Carnapian properties are called type 1 properties.

i.e., $\Box(\forall y)(y\Delta x \equiv \Box y\Delta x)$. On the iterative conception, one begins with the necessarily null Carnapian property $\Lambda =_{df} [x \neq x]_x$, and one then constructs[14] in stages a hierarchy of new properties by means of a power operation:

Stages	1	2	. . .	α	. . .
Properties	$[y = \Lambda]_y$	$[y = \Lambda \vee y = [x = \Lambda]_x]_y$	. . .	[y is an L-determinate Carnapian property predicable only of things of which a property constructed prior to α is predicable]	. . .

The elements of these constructed properties are all pure L-determinate Carnapian properties. If $\in$ is interpreted as expressing the predication relation and if, as before, for every stage there is a later stage that immediately follows no stage, then the union of these constructed properties is a model for Zermelo's axioms for pure set theory.[15] Consider the axiom of extensionality, for example: since Carnapian properties are identical if necessarily equivalent, pure L-determinate Carnapian properties will be identical if they have the same pure L-determinate Carnapian instances; but this is just what the axiom of extensionality says when pure set theory is interpreted as a theory of pure L-determinate Carnapian properties. Now this property-theoretic interpretation of the axioms of pure set theory is just as well motivated and natural as the usual set-theoretic interpretation. Yet when these axioms are so interpreted, the resulting theory has all the mathematical utility as it does on the usual set-theoretic interpretation. It follows, therefore, that pure mathematics provides no pragmatic justification for set theory. A theory of properties is just as satisfactory an instrument for doing pure mathematics.

Consider next the role of sets in the empirical sciences. True, there are occasions when it is useful to talk collectively of individuals with which a given empirical science deals. Usually, however, this job is done by such objects as sums, packs, bunches, flocks, tribes, species, constellations, arrays, ordinary collections, social classes, ordinary sets, etc. But let us suppose there are occasions when these naturalistic objects do not suffice and when it appears more useful to talk of set-theoretical sets. Let us call the sets postulated for these purposes empirical sets. Notice, however, that all talk of empirical sets can easily be reinterpreted as talk about empirical properties: the relation of belonging-to-an-empirical-set would be understood as the relation of having-an-empirical-property and the relation of identity-among-empirical-sets would be understood as the relation of equivalence-among-empirical-properties.[16] For example, when the principle of extensionality for empirical sets is interpreted this way, it becomes a trivial tautology. Now since talk of empirical sets can easily be interpreted in terms of empirical properties, it follows that empirical properties possess whatever utility empirical sets might appear to have in the sciences.

Summing up, we have seen that both pure set theory and applied set theory can be interpreted as theories of properties. Therefore, in view of the conclusion that the ontology of sets does not fall within our naturalistic ontology and the conclusion that set theory is not part of logic, there is simply no justification for positing an ontology of sets. Sets have no place in a rational view of reality.

One wonders, then, what is the origin of set-theoretical thought. In my quite speculative closing remarks I will sketch an idealized "genetic" account of the concept of set. The ontology of PRPs plays a central role in the explanation of the activities of our rational faculty, including in particular such activities as ontological construction, explanation, and justification. In this way, the ontology of PRPs has a transcendental justification. Now given self-consciousness, the ontology of PRPs is at hand from the

[14] Since platonists object to the idea that properties are really "constructed," one might prefer to think of this hierarchy as a stage-by-stage certification of sub-portions of the extension of the predication relation over the field of properties.

[15] The following is another iterative hierarchy of properties that does the same job:

Stages	1	. . .	α	. . .
Properties	$[y = \Lambda]_y$	. . .	$[(\exists u)(u$ is a sum of things of which a property constructed prior to α is predicable & $y = [v$ is in $u]_v)]_y$	. . .

Here we use the notion of sum from a part/whole calculus that permits summation of abstract, as well as concrete, objects (as in Nelson Goodman's *The Structure of Appearance*, Cambridge, 1951). An advantage of this iterative hierarchy is that we need not build in the requirement that properties be Carnapian; the union of the properties in this hierarchy constitutes a model for pure set theory with or without that requirement.

[16] This approach to applied set theory resembles Russell's original "no-class" treatment in *Principia Mathematica* (*ibid.*).

start of, or at least very early in, our conceptual development. We may assume that our early conceptual equipment includes, in addition, the part/whole relation. Used on its own, this relation permits us to make unlimited finite additions to our ontology in accordance with the following principle: for any finite number of given items, there exists a unique whole, or sum, of those items. However, once the part/whole relation is used in combination with the ontology of PRPs, we put ourselves in a position to make unlimited infinite extensions to our ontology in accordance with the following principle: for every non-null property, there exists a unique whole, or sum, containing all things of which the property is predicable. This major ontological extension is a significant step in the direction of an ontology of sets. For, like these newly posited sums, sets are characteristically thought of as being generated by properties. And of course both sums and sets satisfy a principle of extensionality: sums containing the same things are identical, and sets containing the same things are identical. However, the ontology of sums is formally unlike the ontology of sets in two important respects. First, there is no sum generated by null properties. Secondly, even though everything of which a given non-null property is predicable is in the sum generated by it, typically the property is not predicable of everything in the sum; typically, non-equivalent properties can generate the same sum. To arrive at the full ontology of sets, one therefore does two things. First, one fills in the "gap" left by null properties; i.e., one fabricates a null sum. Secondly, one refashions the concept of a sum so that the generating properties are predicable of all and only the things in the sum—or equivalently, so that all and only equivalent properties generate the same sum. Sets are thus a new, artificial kind of sum: they are generated by any sort of property whatsoever, including null properties, and the things in them reflect as directly as possible the identity of the properties that do the generating.

While this new kind of sum is formally constructible, it has absolutely no place in nature or in logic, and there is no call to introduce it into mathematics or empirical science. The ontology of ordinary sums and PRPs serves perfectly well already.

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