FOUR CLASSES OF SEPARABLE METRIC INFINITE-DIMENSIONAL MANIFOLDS[1]

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1. **Introduction.** The purpose of this note is to announce some new embedding, homeomorphism, and characterization theorems regarding certain infinite-dimensional manifolds. We list these theorems below along with some of the principal known results in this area. It is expected that these new results will constitute a portion of the author's dissertation and their proofs will appear in a longer paper that is in preparation.

2. **Definitions and notation.** Each infinite-dimensional separable Fréchet space (and therefore each infinite-dimensional separable Banach space) is homeomorphic to s, the countable infinite product of open intervals $(-1, 1)$ (see [**3**]). A *Fréchet manifold* (or *F-manifold*) is a separable metric manifold modeled on s. A *Hilbert cube manifold* (or *Q-manifold*) is a separable metric manifold modeled on the Hilbert cube I^∞, which we represent as the countable infinite product of closed intervals $[-1, 1]$.

Let σ be the set consisting of all points in s having at most finitely many nonzero coordinates and define a *σ-manifold* to be a separable metric manifold modeled on σ. Let Σ be the set consisting of all points in s having at most finitely many coordinates not in $[-\frac{1}{2}, \frac{1}{2}]$ and define a *Σ-manifold* to be a separable metric manifold modeled on Σ.

A subset K of a space X is a *Z-set in* X if K is closed and if for every nonnull homotopically trivial open set U in X, $U \backslash K$ is nonnull and homotopically trivial.

A subset M of a metric space X is said to have the (*finite-dimensional*) *compact absorption property* (or $(f-d)$ cap) *in* X provided that

(1) $M = \bigcup_{n=1}^\infty M_n$, where M_n is a (finite-dimensional) compact Z-set in X, and

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(2) for each (finite-dimensional) compact set K in X, each integer $m>0$, and each $\epsilon>0$, there is an integer $n>0$ and an embedding $h: K \to M_n$ such that $h|K \cap M_m = \text{id}$ and $d(h, \text{id}) < \epsilon$.

An $(f-d)$ cap-*set* for X is a set which has the $(f-d)$ cap in X. We adopt the convention that the symbol $(f-d)$ cap implies two alternative conditions, one for $f-d$ cap and the other for cap. In [2] it is shown that σ is an $f-d$ cap-set for s and Σ is a cap-set for s.

Let X be a space and let $\mathcal{U}$ be an open cover of X. A function $f: X \to X$ is said to be *limited by* $\mathcal{U}$ provided that for each $x \in X$, x and $f(x)$ are both contained in some member of $\mathcal{U}$. By $\mathrm{St}^n(\mathcal{U})$ we will mean the nth star of the cover $\mathcal{U}$.

As in [6], a subset K of a space X is said to be *strongly negligible* provided that given any open cover $\mathcal{U}$ of X, there is a homeomorphism of X onto $X \setminus K$ which is limited by $\mathcal{U}$.

A subset K of I^∞ is said to have infinite deficiency (or *infinite codimension*) if for each of infinitely many different coordinate directions, K projects onto a single interior point of $(-1, 1)$.

3. **Principal results on F, Q, Σ, and σ-manifolds.** The theorems are those of the author unless other authorship is denoted.

I. Characterization of manifolds by homotopy type.

Theorem 1 (Henderson [8]). *If X and Y are F-manifolds of the same homotopy type, then they are homeomorphic.*

Theorem 2. *If X and Y are both σ-manifolds or both Σ-manifolds and are of the same homotopy type, then they are homeomorphic.*

II. Open embedding theorems.

Theorem 3 (Henderson [8]). *If X is an F-manifold, then X can be embedded as an open subset of s.*

Theorem 4. *If X is a σ (or Σ)-manifold, then X can be embedded as an open subset of σ (or Σ).*

III. Product theorems.

Theorem 5 (West [9]). *If K is any countable locally-finite simplicial complex, then $|K| \times s$ is an F-manifold and $|K| \times I^\infty$ is a Q-manifold.*

Theorem 6. *If K is any countable locally-finite simplicial complex, then $|K| \times \sigma$ is a σ-manifold and $|K| \times \Sigma$ is a Σ-manifold.*

Theorem 7. *If X is a σ-manifold and Y is a Σ-manifold of the same homotopy type, then $X \times I^\infty$ and Y are homeomorphic.*

IV. FACTOR THEOREMS.

THEOREM 8 (HENDERSON [8]). *If X is an F-manifold, then there is a countable locally-finite simplicial complex K such that X and $|K| \times s$ are homeomorphic.*

THEOREM 9. *If X is a σ (or Σ)-manifold, then there is a countable locally-finite simplicial complex K such that X is homeomorphic to $|K| \times \sigma$ (or $|K| \times \Sigma$).*

THEOREM 10 (ANDERSON AND SCHORI [5]). *If X is an F-manifold, then X, $X \times s$, and $X \times I^\infty$ are all homeomorphic.*

THEOREM 11 (ANDERSON AND SCHORI [5]). *If X is a Q-manifold, then X and $X \times I^\infty$ are homeomorphic.*

THEOREM 12. *If X is a σ-manifold and I^n is any n-cell, then X, $X \times \sigma$, and $X \times I^n$ are all homeomorphic.*

THEOREM 13. *If X is any Σ-manifold, then X, $X \times \Sigma$, and $X \times I^\infty$ are all homeomorphic.*

V. RELATIONSHIPS BETWEEN F, Q, Σ, AND σ-MANIFOLDS.

THEOREM 14. *If X is a σ (or Σ-manifold), then X can be embedded as an $f-d$* cap *(or* cap*)-set for an F-manifold and also for a Q-manifold.*

THEOREM 15 (ANDERSON [2]). *If X is I^∞ or s and M, N are $(f-d)$* cap*-sets for X, then there is a homeomorphism of X onto itself taking M onto N.*

THEOREM 16. *If X is an F or Q-manifold and M is an $f-d$* cap *(or* cap*)-set for X, then M is a σ (or Σ)-manifold.*

THEOREM 17. *If X is an F or Q-manifold, M and N are $(f-d)$* cap*-sets for X, and $\mathfrak{U}$ is an open cover of X, then there is a homeomorphism of X onto itself which takes M onto N and is limited by $\mathfrak{U}$.*

THEOREM 18 (ANDERSON [2]). *If M is an $(f-d)$* cap*-set for I^∞, then $I^\infty \backslash M$ is homeomorphic to s.*

THEOREM 19. *If X is any Q-manifold and M is an $(f-d)$* cap*-set for X, then $X \backslash M$ is an F-manifold which is of the same homotopy type as X.*

VI. SUBSETS AND SUPERSETS OF F, Q, Σ, AND σ-MANIFOLDS.

THEOREM 20 (ANDERSON, HENDERSON AND WEST [6]). *A necessary and sufficient condition that a closed subset K of an F-manifold be a Z-set is that K be strongly negligible.*

THEOREM 21. *A necessary and sufficient condition that a closed subset K of a σ or Σ-manifold be a Z-set is that K be strongly negligible.*

THEOREM 22. *Let M be an $(f-d)$* cap-*set for an F or Q-manifold X and let K be a Z-set in X. Then $M\backslash K$ is an $(f-d)$* cap-*set for X.*

THEOREM 23. *Let M be an $(f-d)$* cap-*set for an F or Q-manifold X and let K be a countable union of (finite-dimensional) compact Z-sets in X. Then $M\cup K$ is an $(f-d)$* cap-*set for X.*

VII. HOMEOMORPHISM EXTENSION THEOREMS.

THEOREM 24 (ANDERSON AND MCCHAREN [4]). *Let X be an F-manifold, let K_1, K_2 be Z-sets in X, let $\mathfrak{U}$ be an open cover of X, and let h be a homeomorphism of K_1 onto K_2 such that there is a homotopy $H:K_1\times I\to X$ for which $H_0=\mathrm{id}$, $H_1=h$, and $H(\{x\}\times I)$ is contained in some member of $\mathfrak{U}$, for each $x\in K_1$. Then h can be extended to a manifold homeomorphism which is limited by* $\mathrm{St}^4(\mathfrak{U})$.

THEOREM 25. *Let X be a σ or Σ-manifold, let K_1, K_2 be Z-sets in X, let $\mathfrak{U}$ be an open cover of X, and let h be a homeomorphism of K_1 onto K_2 such that there is a homotopy $H:K_1\times I\to X$ for which $H_0=\mathrm{id}$, $H_1=h$, and $H(\{x\}\times I)$ is contained in some member of $\mathfrak{U}$, for each $x\in K_1$. Then h can be extended to a manifold homeomorphism which is limited by* $\mathrm{St}^{28}(\mathfrak{U})$.

VIII. COMPLETE EXTENSIONS OF Σ AND σ-MANIFOLDS.

THEOREM 26. *Let X be a σ (or Σ)-manifold and let Y be a complete separable metric space containing X. Then there is an F-manifold Z such that $X\subset Z\subset Y$ and X is an $f-d$* cap *(or* cap*)-set for Z.*

IX. INFINITE DEFICIENCY.

THEOREM 27 (ANDERSON [1]). *Let X be I^∞ or s and let K be a closed subset of X. A necessary and sufficient condition that K be a Z-set in X is that there exists a homeomorphism of X onto itself taking K onto a set having infinite deficiency.*

THEOREM 28 (CHAPMAN [7]). *Let X be an F-manifold and let K be a closed subset of X. A necessary and sufficient condition that K be a Z-set in X is that there exists a homeomorphism h of X onto $X\times s$ such that $\pi_s\circ h(K)$ has infinite deficiency.*

THEOREM 29. *Let X be a Q-manifold and let K be a closed subset of X. A necessary and sufficient condition that K be a Z-set in X is that there exists a homeomorphism h of X onto $X\times I^\infty$ such that $\pi_{I^\infty}\circ h(K)$ has infinite deficiency.*

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