Four Lectures on Scalar Curvature

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Unlike manifolds with controlled sectional and Ricci curvatures, those with their *scalar curvatures bounded from below* are not configured in specific rigid forms but display an uncertain variety of flexible shapes similar to what one sees in geometric topology.

Yet, there are definite limits to this flexibility, where determination of such limits crucially depends, at least in the known cases, on two seemingly unrelated analytic means: *index theory of Dirac operators* and the *geometric measure theory,*[1]

The emergent picture of spaces with $Sc.curv \geq 0$, where topology and geometry are intimately intertwined, is reminiscent of *the symplectic geometry*,[2] but the former has not reached yet the maturity of the latter.

The mystery of the scalar curvature remains unsolved.

What follows is an extended account of my lectures, delivered during the Spring 2019 at IHES.

In §1, we give an outline of results, techniques and problems in scalar curvature.

In §2, we spend a few dozen pages on background Riemannian geometry, with another dozen in section 3.3.3 on Clifford algebras and Dirac operators.

In §3, we overview main topics in geometry and topology of manifolds with their scalar curvatures bounded from below, state theorems, explain the ideas of their proofs and formulate a variety of problems and conjectures.

In §§4 and 5, we reformulate, in a more precise and general form, what was stated in the earlier sections and expose technical aspects of the proofs.

In §6, we describe connective links between different facets of the scalar curvature presented in the earlier sections with an emphasis on open problems.

Finally, in §7 we overview metric invariants that are influenced by and/or going along with the scalar curvature.

I have made a maximal effort to lighten the burden on the reader of locating the place where a certain notation or definition was introduced.

Our terminology is displayed in the table of contents.

When returning to the same topic – this happens again and again – we, besides recalling definitions and formulas, explain what is needed for the matter

[1] Spaces of metrics with $Sc \geq \sigma$ on 3-manifolds are amenable to the global study with the *Hamilton's Ricci flow*, which also applies, at the present moment only C^0-locally, in higher dimensions. Also, much topological and geometrical information on 4-manifolds with $Sc \geq \sigma$, for positive as well as negative σ, is obtained, exclusively, with the *Seiberg-Witten equations.*

[2] Geometric invariants associated with the scalar curvature, such as the *K-area*, are linked with the symplectic invariants (see [G(positive) 1996], [Polterovich(rigidity) 1996], [Entov(Hofer metric) 2001], [Savelyev(jumping) 2012[), but this link is still poorly understood.

at hand, rather than referring to earlier sections in the text. Everything needed for understanding a statement on page "x" can be found on a couple of preceding pages.

Contents

1 Preliminaries

1.1 Geometrically Deceptive Definition

The scalar curvature of a C^2-smooth Riemannian manifold $X = (X, g)$, denoted

$$Sc = Sc(X, x) = Sc(X, g) = Sc(g) = Sc_g(x)$$

is a continuous function on X, which is traditionally defined as

the sum of the values of the sectional curvatures at the $n(n-1)$ ordered bivectors of an orthonormal frame in X,

$$Sc(X, x) = Sc(X)(x) = \sum_{i,j} \kappa_{ij}(x), \ \ i \neq j = 1, ..., n,$$

where this sum doesn't depend on the choice of this frame by the Pythagorean theorem.

Algebraically, this formula defines a *second order differential*

$$g \mapsto Sc(g)$$

from the space G_+ of positive definite quadratic differential forms on X to the space S of functions on X, that is characterised uniquely, up to a scalar multiple, by two properties.

⋆ the $g \mapsto Sc(g)$ is *equivariant* under the natural actions of diffeomorphisms of X in the spaces G_+ and S.

⋆ the $g \mapsto Sc(g)$ is *linear in the second derivatives* of g.

To make geometric sense of this, let us summarize basic properties of $Sc(X)$.

$\bullet_1$ *Additivity under Cartesian-Riemannian Products.*

$$Sc(X_1 \times X_2, g_1 + g_2) = Sc(X_1, g_1) + Sc(X_2, g_2).$$

$\bullet_2$ *Quadratic Scaling.*

$$Sc(\lambda \cdot X) = \lambda^{-2} Sc(X), \text{ for all } \lambda > 0,$$

where

$$\lambda \cdot X = \lambda \cdot (X, dist_X) =_{def} (X, dist_{\lambda \cdot X}) \text{ for } dist_{\lambda \cdot X} = \lambda \cdot dist(X)$$

for all metric spaces $X = (X, dist_X)$ and where $dist \mapsto \lambda \cdot dist(X)$ corresponds to $g \mapsto \lambda^2 \cdot g$ for the Riemannian quadratic form g.

Example. The Euclidean spaces are scalar-flat, $Sc(\mathbb{R}^n) = 0$, since $\lambda\cdot\mathbb{R}^n$ is *isometric* to $\mathbb{R}^n$.

$\bullet_3$ *Volume Comparison.* If the scalar curvatures of n-dimensional manifolds X and X' at some points $x \in X$ and $x' \in X'$ are related by the strict inequality

$$Sc(X)(x) < Sc(X')(x'),$$

then the Riemannian volumes of the ε-balls around these points satisfy

$$vol(B_x(X,\varepsilon)) > vol(B_{x'}(X',\varepsilon))$$

for all sufficiently small $\varepsilon > 0$.

Observe that this volume inequality is *additive under Riemannian products*: if

$$vol(B_{x_i}(X,\varepsilon)) > vol(B_{x'_i}(X'_i,\varepsilon)), \text{ for } \varepsilon \le \varepsilon_0,$$

and *for all points* $x_i \in X_i$ and $x'_l \in X'_i$, $i = 1,2$, then

$$vol_n(B_{(x_1,x_2)}(X_1\times X_2,\varepsilon_0)) > vol_n(B_{(x'_1,x'_2)}(X'_1\times X'_2,\varepsilon_0)$$

for all $(x_1,x_2) \in X_i \times X_2$ and $(x'_1,x'_2) \in X'_1\times X'_2$.

This follows from the Pythagorean formula

$$dist_{X_1\times X_2} = \sqrt{dist^2_{X_1} + dist^2_{X_2}}.$$

and the Fubini theorem applied to the "fibrations" of balls over balls:

$$B_{(x_1,x_2)}(X_1\times X_2,\varepsilon_0)) \to B_{x_1}(X_1,\varepsilon_0) \text{ and } B_{(x'_1,x'_2)}(X'_1\times X'_2,\varepsilon_0)) \to B_{x_1}(X'_1,\varepsilon_0),$$

where the fibers are balls of radii $\varepsilon \in [0,\varepsilon_0]$ in X_2 and X'_2.

$\bullet_4$ *Normalisation/Convention for Surfaces with Constant Sectional Curvatures.* The unit spheres $S^2(1)$ have constant scalar curvature 2 and the hyperbolic plane $H^2(-1)$ with the sectional curvature -1 has scalar curvature -2 [3]

It is an elementary exercise to prove the following.

$\star_1$ *The function $Sc(X,g)(x)$ which satisfies $\bullet_1$-$\bullet_4$ exists and unique*;

$\star_2$ *The unit spheres d the hyperbolic spaces with $sect.curv = -1$ satisfy*

$$Sc(S^n(1)) = n(n-1) \textit{ and } Sc(H^n(-1)) = -n(n-1).$$

Thus,

$$Sc(S^n(1)\times H^n(-1)) = 0 = Sc(\mathbb{R}^{2n}),$$

which implies that

the volumes of the small ε-balls in $S^n(1)\times \mathbf{H}^n(-1)$ are "very close" to the volumes of the ε-balls in the Euclidean space $\mathbb{R}^{2n}$.

Also it is elementary to show that the definition of the scalar curvature via volumes of balls agrees with the traditional $Sc = \sum \kappa_{ij}$, where the definition via volumes seem to have an advantage of being geometrically more usable.

[3]The equality $Sc(H^2) = -2$ follows from $Sc(S^2) = 2$ by comparing the volumes of small balls in $S^2\times H^2$ and in $\mathbb{R}^4$.

But this is an illusion:

there is no single known (are there unknown?)
geometric argument, which would make use of this definition.

The immediate reason for this is *the infinitesimal* nature of the volume comparison property: it *doesn't integrate* to the corresponding property of balls of specified, let them be small, radii $r \leq \varepsilon > 0$. [4]

The following *alternative, let it be also only infinitesimal, property* of the scalar curvature seems more promising:

⊛ the inequality $Sc(X,x) < Sc(X',x')$ is equivalent to the following relation between the average mean curvatures of the (very) small ε-spheres $S^{n-1}_x(\varepsilon) \subset X$ and $S^{n-1}_{x'}(\varepsilon) \subset X'$:

$$\frac{\int_{S^{n-1}_x(\varepsilon)} mean.curv(S^{n-1}_x(\varepsilon), s)ds}{vol_{n-1}(S^{n-1}_x(\varepsilon))} > \frac{\int_{S^{n-1}_{x'}(\varepsilon)} mean.curv(S^{n-1}_{x'}(\varepsilon), s')ds'}{vol_{n-1}(S^{n-1}_{x'}(\varepsilon))}.$$

There are also several *non-local inequalities* for the mean curvatures of manifolds B with boundaries S, in terms of the scalar curvatures of B (and sometimes of sizes of B) that we shall see in these lectures, e.g. ⬬ and ■ in section 3.1, but we are still far from the ultimate inequality of this kind.

[*] *Exercise: Spherical Suspension.* Compute the scalar curvature of the *spherical join* of two Riemannian manifolds X_1 and X_2, that is the unit sphere in the product of the Euclidean cones over these manifolds:

$$X_1 * X_2 \subset \mathsf{C}X_1 \times \mathsf{C}X_2,$$

where $\mathsf{C}X = (X \times \mathbb{R}^\times_+, r^2dx^2 + dr^2)$, accordingly

$$\mathsf{C}X_1 \times \mathsf{C}X_2 = (X_1 \times X_2 \times \mathbb{R}^\times_+ \times \mathbb{R}^\times_+, r_1^2dx_1^2 + r_2^2dx_2^2 + dr_1^2 + dr_2^2)$$

and where the hypersurface $X_1 * X_2 \subset \mathsf{C}X_1 \times \mathsf{C}X_2$ is defined by the equation

$$r_1^2 + r_2^2 = 1.$$

(The manifold $X_1 * X_2$ with this metric, which is defined for $r_1, r_2 > 0$, is incomplete; if completed, it becomes singular, unless X_1 and X_2 are isometric to the unit spheres S^{n_1} and S^{n_2}.)

Show, in particular, that if $Sc(X_i) \geq n_i(n_i - 1) = Sc(S^{n_i})$, $n_i = dim(X_i)$, $i = 1, 2$, then

$$Sc(X_1 * X_2) \geq (n_1 + n_2)(n_1 + n_2 - 1).$$

Hint. Use the formula for the curvature of warped products from section 2.4.

1.2 Fundamental Examples of Manifolds with $Sc \geq 0$

Symmetric and homogeneous spaces. Since compact symmetric spaces X have non-negative sectional curvatures κ, they satisfy $Sc(X) \geq 0$, where the equality holds only for flat tori.

[4] An attractive conjecture to the contrary appears in [Guth(volumes of balls-large) 2011], also see [Guth(volumes of balls-width)) 2011].

Since the bi-variant metrics on Lie groups have $\kappa \geq 0$ and since the inequality $\kappa \geq 0$ is preserved under dividing spaces by isometry groups, all compact homogeneous spaces G/H carry such metrics, .[5]

Furthermore,

quotients of compact homogeneous spaces by compact freely acting isometry groups carry metrics with $Sc \geq 0$*,*

where prominent examples of these are

spheres divided by finite free isometry groups.

Thus, in particular,

all homology classes in the classifying spaces $\mathsf{B}(G)$ *of finite cyclic groups* G *are representable by compact manifolds with* $Sc > 0$ *mapped to these spaces.*

But, at the present moment, it is unknown if this remains true for all finite groups G.[6]

On the other extreme, there are no known examples of "$Sc > 0$ *representable*" *non-torsion* homology classes in the classifying spaces of *infinite countable* groups or of (possibly torsion) homology classes in the classifying spaces of groups *without torsion.*

(We shall see in the following sections that majority of known topological obstructions to metrics with $Sc \geq 0$ come from *the rational homology* and *K-theory* of classifying spaces of infinite groups.

Also we shall meet examples – we call these ***S**choen-**Y**au-**S**chick -manifolds* – where non-trivial obstructions to $Sc \geq 0$, which reside in *the integer* homology classes in $\mathsf{B}(\mathbb{Z}^n \times \mathbb{Z}/p\mathbb{Z})$, *vanish for non-zero multiples* of these classes.)

Fibrations. Since the scalar curvature is additive, fibered spaces $X \to Y$ with compact non-flat homogeneous fiberes carry metrics with $Sc > 0$.

(This is seen by scaling metrics in Y by large constants.)

Convex Hypersurfaces. Since convex hypersurfaces in $\mathbb{R}^n$ as well as in general spaces with sectional curvatures $\kappa \geq 0$, their scalar curvatures are also non-negative.

Fano, Uniruled and Calabi-Yau Manifolds. Smooth Fano varieties[7] e.g. complex projective hypersurfaces $X \subset \mathbb{C}P^n$ of degree $\leq n$ admit Kähler metrics g with $Sc > 0$.

In fact, by Yau's solution of the *Calabi conjecture*, Fano varieties carry Kähler metrics with *positive Ricci* curvatures, while hypersurfaces of if degree n+1 carry *Calabi-Yau* Kähler metrics, i.e. with *zero Ricci* curvature.

A distinctive geometric feature of Fano varieties is that they are *uniruled*, i.e. covered by rational curves[8] and it is *conjectured* that, in general,

Uniruled Varieties admit Kähler metrics with $Sc > 0$.[9]

[5] This is also true for non-compact homogeneous spaces the isometry groups of which contain compact semisimple factors.

[6] This was pointed out to me by Bernhard Hanke.

[7] A smooth algebraic variety X is *Fano* if the *anticanonical line bundle* L, that is the top exterior power of the tangent bundle, $L = \wedge^n T(X)$, $n = dimX$, is *ample*, that is the subset Z_x of sections in the space S of all sections of some power $L^{\otimes m}$ that vanish at $x \in X$ has codimension n for all $x \in X$ and the resulting map $x \mapsto Z_x$ from X to the the space of codimension n subspaces in the space S is a *smooth embedding.*

[8] The proof of this relies on Mori's argument of reduction of the general case to that of varieties over *finite fields.*

[9] See [Debarre(lectures] 2003), [Ballmann(lectures) 2006], [Yang-complex(2017)] and references therein.

Conversely, one knows that

($\star$) *compact Kähler manifolds with $Sc > 0$ are uniruled.*

In fact, this is proven in [Heier-Wong(uniruled) 2012] under the weaker assumption of positivity of the *integral* of the scalar curvature, where, observe, this integral depends only on the first Chern class of X and the cohomology class of (the symplectic part ω of) the Kähler metric: $\int_X Sc(X,x)dx = 4\pi/(n-1)!(c_1 \smile [\omega^{n-1}](X))$, for $n = dim_{\mathbb{C}}(X)$.

There is also a non-trivial geometric constraint on $\int_X Sc(X,x)dx$ for general compact Riemannian manifolds X:

this integral can be bounded from above in terms of dimension $dim(X)$, diameter, and a lower bound on the sectional curvature of X, see [Petrunin(upper bound) 2008)].

Yet, it is unclear if there is a *true Riemannian counterpart of ($\star$):*

the literal topological translation of ($\star$) may be deceptive: the connected sum $X = S^2 \times S^2 \# S^2 \times S^2$, which, as we explain below, carries metrics with $Sc > 0$, admits, however, no map of non-zero degree from the total space of any 2-sphere bundle over a surface.

But if one allows

multiparametric families of maps $S^2 \to X$ and/or suitably controlled discontinuities/singularities,

then this X, and apparently all known manifolds X which admits metrics with $Sc > 0$, start looking "topologically unirational".

1.3 Thin Surgery with $Sc > \sigma$

Assumptions. Let an n-dimensional manifolds X bounds a Riemannian manifold X_+, i.e. X_+ is an $(n+1)$-manifold with boundary $\partial X_+ = X$ and let $Z \subset X_+$ be a submanifold, which meets X transversally along its boundary denoted $Y = \partial Z = Z \cap X = \partial X_+$.[10]

If $U \subset X_+$ is a tubular neighbourhood of Z, then the boundary $X' = \partial(X \cup U) \subset X_+$ is a smooth (never mind the corner along $\partial U \cap X$) manifold that implements surgery of X along $Y = \partial Z \subset X$.

Connected Sum Example. If X consists of two connected components, $X = X_1 \sqcup X_2$ and Z is a smooth segments with the ends $y_1 \in X_1$ and $y_2 \in X_2$, then X' is topological connected sum of X_1 and X_2 that is performed by the "tube" $T = \partial U \subset X_+$ joining X_1 and X_2.

Observe that the connected sums and all surgeries performed over X in general can be realized as above by embedding X as a boundary in a larger (non-compact) manifold X_+, where one may assume, if one wishes so, that X_+ *metrically splits* near the boundary $\partial X_+ = X$, i.e. it is isometric to $X \times \mathbb{R}_+$ near X and that $Z \subset X_+$ *agrees with this splitting* by being equal to $Y \times \mathbb{R}_+$ near X.

If Z is compact and $\delta > 0$ is small, then the δ-neighbourhood $U_\delta(Z) \subset X_+$ can be taken for U. It is also clear that if the codimension of Z in X_+ satisfies $k = codim(Y) \geq 3$, e.g. if Y is a curve in a Riemannian 4-manifold, then

[10] Here "boundary" and the equality $Y = \partial Z$ mean that Z *is a manifold-with-boundary*, where this boundary is equal to the intersection of Z with X; this is different from the boundary of Z as a subset in X_+, which is in our examples, where $codim(Z) > 0$, coincides with all of Z.

T_δ with the Riemannian metric induced from X_+ has large positive scalar curvature δ-away from X. Namely, by *Gauss' Theorema Egregium*,

$$Sc(T_\delta) \sim \frac{(k-1)(k-2)}{\delta^2} \text{ for small } \delta \to 0.$$

.

What is more interesting is that the submanifold $X'_\delta = \partial(X \cup U_\delta(Y))$ can be smoothed by slightly perturbing it in the ε-neighbourhood of $Y = X \cap T_\delta$, for $\varepsilon = \varepsilon(\delta) \to 0$ for $\delta \to 0$, such that

the scalar curvature of the resulting submanifold, call it $X'_{\delta,\varepsilon} = \partial(X \cup T'_\delta)$, where T'_δ denotes the smoothed T_δ, becomes almost as positive as that of X.

This is achieved by a local "staircase" construction,[11] that makes U_δ thinner and thinner as you move away from X in the ε-vicinity of Y.

Here is a precise statement.

●–● Proposition: Thin Surgery by Controlled Thickening. Let $\sigma(x_+)$, $x_+ \in X_+$, be a continuous function, such that its restriction to X satisfies $\sigma(x_+) < Sc(X, x_+)$, for all $x_+ \in X$, where the scalar curvature of X is evaluated ted with the Riemannian metric on X induced from X_+.

Let $\delta(z) > 0$ and $\varepsilon(y) > 0$ be continuous positive function on Z and on Y.

Then there exists a family of tubular neighbourhoods $U_{\delta,\varepsilon}(Y), \subset X$ with the following four properties.

$\bullet_1$ the boundary $T_{\delta,\varepsilon} = \partial U_{\delta,\varepsilon}(Y)$ meets $X = \partial X_+$ *tangentially* (rather than transversally) and such that the submanifold

$$X'_{\delta,\varepsilon} = \partial(X \cup U_{\delta,\varepsilon}) \subset X_+,$$

which is, a priori, C^1-smooth, actually is C^∞-smooth.[12]

$\bullet_2$ The scalar curvature of $X'_{\delta,\varepsilon}$ with the metric induced from $X_+ \supset X'_{\delta,\varepsilon}$ satisfies

$$Sc(X'_{\delta,\varepsilon}, x_+) \geq \sigma(x_+) \text{ for all } x_+ \in X'_{\delta,\varepsilon}.$$

Furthermore,

$\bullet_3$ $U_{\delta,\varepsilon}$ is contained in the δ-neighbourhood $U_\delta(Z) \subset X_+$ of $Z \subset X_+$, that is the union of all $\delta(z)$-balls,

$$U_{\delta,\varepsilon} \subset \bigcup_{z\in} B_z(\delta(z)).$$

$\bullet_4$ There exists a positive continuous function $0 < \delta'(z) = \delta'_{\delta,\varepsilon}(z) < \delta(z)$, such that the neighbourhood $U_{\delta,\varepsilon}$ within distance $> \varepsilon$ from Y is equal to the δ'-neighbourhood of Z, that is

$$U_{\delta,\varepsilon} \setminus U_\varepsilon(Y) = U_{\delta'}(Z).$$

[11] See [GL(classification) 1980] and [BaDoSo(sewing Riemannian manifolds) 2018]. This construction also applies to hypersurfaces with *mean.curv* $> \mu$, see [G(mean) 2019] and it extends to families of metrics, see [Ebert-Williams(infinite loop spaces) 2017] and references therein.

Besides there is a *non-local construction* with a similar effect on the scalar curvature that was suggested by Schoen and Yao in [SY(structure) 1979].

[12] Unless stated otherwise, all our manifolds, submanifolds etc. are assumed *smooth* meaning C^∞-smooth.

The domain $U_{\delta,\varepsilon} \subset X_+$ admits a more concrete description if the X_+ and Z metrically split near X, that is $(X_+, Z) = (X, Y) \times \mathbb{R}_+$ near X. Namely, one can take

the δ''-neighbourhood of Z for $U_{\delta,\varepsilon}$, where $\delta''(z) = \delta''_{\delta,\varepsilon}(z)$ is a smooth function on Z.

The most transparent case here is where Y is compact; here one can take $\delta''(z) = \rho(dist(z, X))$ for a suitable function $\rho(d) = \rho_{\delta,\varepsilon}(d)$, where the nature of this ρ is well represented by the following.

Halfspace Example/Exercise. Let $X_+ = \mathbb{R}^n \times \mathbb{R}_+$, where $X = \partial(X_+ \times \mathbb{R}_+) = \mathbb{R}^n \times \{0\}$ and let Z be the half line $\{0\} \times \mathbb{R}_+ \subset \mathbb{R}^n \times \mathbb{R}_+$.

Find the above mentioned function ρ for this pair (X_+, Z) and then derive the general case from this example.

Hint. The scalar curvature of the tube can be calculated either with *Gauss' Theorema Egregium* (section 2.2) or with the Second Main Formula 2.3.A. (A more general statement is formulated in the Cylindrical Extension Exercise in 2.4.)

Why not "$\geq$" instead of "$>$"? One can't replace the strict inequalities $Sc(X) > \sigma$ and $Sc(X''_{\delta,\varepsilon}) > \sigma$ by $Sc(X) \geq \sigma$ and $Sc(X''_{\delta,\varepsilon}) \geq \sigma$, not even in the case of $(X_+, Z) = (\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}_+)$.

In fact the flat metric on $\mathbb{R}^n$ minus a ball $B \subset \mathbb{R}^n$ admits no extension to a complete metric with $Sc \geq 0$ as it follows from the solution of the positive mass conjecture (section 3.11) and/or from non-existence of a complete metric with $Sc \geq 0$ on the punctured torus (sections 4.7, 5.10).

Exercise: Extension of Families of Metrics with $Sc \geq \sigma$. Let X be a smooth manifold, $\sigma(x)$ a continuous function on X and let $\mathcal{S}_{>\sigma} =)S_{>\sigma}(X)$ be the space of Riemannian metrics g on X with $Sc(g, x) > \sigma(x)$.

Given an open subset $U \subset X$, let $\mathcal{S}_{>\sigma}(U)$ denote the space metrics on U with $Sc(g, x) > \sigma(x)$ and, if $Y \subset X$ is a closed subset, let $\mathcal{S}_{>\sigma}(op(Y))$ be the the space of *germs of metrics* with $Sc(g, x) > \sigma(x)$ defined in (arbitrarily small) neighbourhoods $U \supset Y$.

Show that if $codim(Y) \geq 3$, then the natural (restriction) map

$$\mathcal{S}_{>\sigma} \to \mathcal{S}_{>\sigma}(op(Y))$$

is a *Serre fibration.* (Compare with *Chernysh's theorem* as stated in 2.2.3 in [Ebert-Williams(cobordism category) 2019].)

Exercise+Question. generalize the above to the case where Z is a *piecewise smooth* polyhedral subset of codimension ≥ 3 in X_+.

Then try to generalize this to more general closed subsets Z. (See [G(mean) 2019] for discussion on the corresponding problem for hypersurfaces with $mean.curv > \mu$.)

Discouraging Remark. Despite impressive applications of the above ●–● and its variations to the *topology* of manifolds with $Sc > 0$, e.g.

> the existence of metrics with positive scalar curvatures on *simply connected* manifolds of dimension $n \neq 0, 1, 2, 4 \mod 8$,

and of spaces of metrics with $Sc > 0$, e.g.

> *infiniteness* of the kth homotopy groups of the spaces of such metrics on the spheres S^{4m-k-1} for $m >> k$,

the actual geometry behind "thin construction(s)" is skin-deep: positivity of the scalar curvatures of the n-spheres for $n \geq 2$ and nothing else.

In fact, besides homogeneous spaces, the only known general source of "thickness" with $Sc > 0$ comes from solutions of Monge-Ampere equations on Kähler manifolds.

1.4 Scalar Curvature and Mean Curvature

A simple link between the two notions is provided by the following observation.[13]

Let $X = (X, g)$ be a Riemannian n-manifold with boundary represented by a domain in a slightly larger manifold $X_+ \supset X$ and then embedded to the cylinder $X_+ \times \mathbb{R}$ for

$$X = X_0 = X \times \{0\} \subset X \times \mathbb{R} \subset X_+ \times \mathbb{R}$$

and let $U_\varepsilon = U_\varepsilon(X_0) \subset X_+ \times \mathbb{R}$ be the ε-neighbourhood of $X_0 \subset X_+ \times \mathbb{R}$.

The boundary ∂U_ε consists of two parts: two "ε-copies"

$$X_{\pm\varepsilon} = X \times \{\pm\varepsilon\} \subset \partial U_\varepsilon$$

of X and the complementary semicircular band,

$$\partial X_0 \times S^1_+(\varepsilon) \subset \partial X_0 \times S^1_+(\varepsilon),$$

that is one half of the boundary of the ε-neighbourhood of the boundary $\partial X_0 \subset X_+ \times \mathbb{R}$.

Both parts of the hypersurface ∂U_ε are C^∞-smooth,[14] and ∂U_ε is also C^1-smooth at the common boundary of these parts. But the curvature of the band $\partial X_0 \times S^1_+(\varepsilon) \subset \partial U_\varepsilon$ along the semicircles $\{x\} \times S^1_+(\varepsilon)$, $x \in \partial X_0$, jumps down from ε^{-1} to 0, where this band meats the "flat horizontal" part of ∂U_ε. that is the union of the two "ε-copies" of X,

$$X_{-\varepsilon} \cup X_{-\varepsilon} \subset \partial U_\varepsilon.$$

,

The scalar curvature of this band, computed with the Gauss formula (theorema egregium), interpolates between, roughly, $\varepsilon^{-1} \times mean.curv(\partial X_0)$ at the points not too close to the flat part of ∂U_ε, where it becomes equal to the scalar curvature of X.

if $Sc(g) > \sigma$ and the boundary $\partial X \subset X$ is *strictly mean convex*, i.e. $mean.curv(Y) > 0$,[15] then the boundary ∂U_ε can be C^∞-smoothed by interpolating the curvatures on the two sides of the jump between ε^{-1} and 0, such that

the scalar curvature of the smoothed boundary becomes bounded from below by the scalar curvature of the original metric g on X.

(To see this, look at the $(n-2)$-ball in the n-space, $X_0 = B^{n-2} \subset \mathbb{R}^n$, where the boundary of its ε-neighbourhood can be $O(n-2)$-invariantly smoothed by C^∞-flattening the semicircle $S^1_+(\varepsilon)$ at the ends, while keeping it convex.)

[13] Look at fig 8 in [GL(spin) 1980].

[14] Our Riemannian manifolds are C^∞-smooth unless stated otherwise.

[15] Our coorientation convention is such that convex domains are mean convex according to it.

Since the boundary ∂U_ε is naturally diffeomorphic to the *double* $\mathbb{D}(X)$ obtained by gluing two copies of X along ∂X, this delivers the following

Proposition: Smoothing $\mathbb{D}$-Corner. *There exists an approximation of the natural continuous metric G_0 on the double $\mathbb{D}(X) = X \cup_{\partial X} X$ by smooth metrics G_ε with scalar curvatures bounded from below by $Sc(G_\varepsilon) \geq Sc(X)$.*

Moreover,

strictness of positivity of the mean curvature, can be propagated[16] by a small C^∞-perturbation to such a "strictness" for the scalar curvature all over $\mathbb{D}(X)$, thus making

$Sc(G_\varepsilon)$ *everywhere strictly greater than* $Sc(X)$.

For instance,

the doubles of compact mean convex bounded Euclidean domains carry metrics with positive scalar curvatures,

where the necessary strictness of mean convexity is achieved by small perturbations of the boundaries of these domains.

If you think about this (excessively geometric) construction in intrinsic terms of X, you will realize that the metric G_ε was actually obtained by stretching the original Riemannian metric g of X near the boundary $\partial X \subset X$ *along geodesic segments normal to* ∂X. Then you write down everything in the normal coordinates in a neighbourhood of the boundary $\partial X \subset X$ [17] and arrive at the following proposition.

Miao's Gluing Lemma. Let $X_\circlearrowleft$ be obtained by identifying pairs of points in the boundary of a Riemannian manifold $X = (X, g)$ by an isometric involution $I : \partial(X) \to \partial X$ without fixed points.[18]

If the sums of the mean curvatures at the identified points satisfy

$$mean.curv(\partial X, x) + mean.curv(\partial X, I(x))) > 0 \textit{ for all } x \in \partial X,$$

then the natural continuous Riemannian metric G on $X_\circlearrowleft$ can be approximated by smooth metrics G_ε with their scalar curvatures strictly bounded from below by the scalar curvature of g.[19]

The main step of the poof is stretching g normally to ∂X in a small neighbourhood of ∂X with no decrease of the scalar curvature and without changing the restriction $g|_{\partial X}$, such that the second fundamental form A for the new metric g_{new} on X will match one another at the I-corresponding points, i.e.

$$A_x + A_{I(x)} = 0 \text{ for all } x \in \partial X.$$

We implement such a stretching by extending X with the ε-cylinder $\partial X \times [0, \varepsilon]$ attached to X by the tautological map $\partial X \times \{0\} \to \partial X$ and we endow this cylinder with a family of metrics g_ε defined with the following metrics $h_\varepsilon(t)$ on ∂X.

[16] See section 11.2 in [G(inequalities) 2018].

[17] [Almeida(minimal) 1985], [Miao(corners) 2002], [Bre-Mar-Nev(hemisphere) 2011], [G(billiards) 2014].

[18] This I may be more interesting than interchanging two isometric components of the boundary, such as the involution on the boundary of a centrally symmetric $X \subset \mathbb{R}^n$.

[19] This is similar to preservation of lower bounds on (Alexandrov's) *sectional curvature* under gluing, where the second fundamental form II of the boundary satisfies $\mathrm{II}_x + \mathrm{II}_{I(x)} \geq 0$.

Let A_{old}, A_{new} be quadratic differential forms on ∂X, where A_{old} is equal to the second fundamental form of $\partial_0 = \partial X \times \{0\} = \partial X$ in X and A_{new} is another (desired) quadratic differential form on $\partial X = \partial_\varepsilon = \partial X \times \{\varepsilon\}$.

Let

$$(++) \qquad h_\varepsilon(t) = h + tA_{old} + \frac{t^2}{2\varepsilon}(A_{new} - A_{old}),\ 0 \le t \le \varepsilon.$$

and let

$$g_\varepsilon = h_\varepsilon(t) + dt^2.$$

Then:

(i) *the second fundamental forms of the two boundary parts ∂_0 and ∂_ε for the metric g_ε are equal to A_{old} and A_{new} correspondingly by the Riemann variation formula in 2.1;* [20]

(ii) the scalar curvature of g_ε satisfies,

$$Sc(g_\varepsilon) = \frac{1}{\varepsilon} trace(A_{old} - A_{new}) + O(1)$$

by *Hermann Weyl's tube formula* and Gauss's formula (see 2.3, 2.2).

It is also clear that $(g_\varepsilon)_{|\partial_0} = h$ and $(g_\varepsilon)_{|\partial_\varepsilon} = h + o(\varepsilon)$, which allows a small perturbation of g_ε that makes it equal to h on ∂_ε, while keeping (i) and (ii).[21]

Second Step. Because of the match of the second quadratic forms, the metric G_{new} on $X_\circlearrowleft$ is now C^1-smooth, which allows its painless smoothing, while metric keeping the scalar curvature almost as as positive as that of g, and, due to the strictness condition, even *more positive* than $Sc(g)$.[22]

Besides the above "infinitesimal realtions", there is an amusing similarity between global geometries of n-dimensional *Riemannian manifolds X with positive scalar curvatures* and *mean convex convex hypersurfaces* in the Euclidean space $\mathbb{R}^n$ and in similar spaces.

Although, in many respects mean convex hypersurfaces $Y \subset \mathbb{R}^n$ are better understood then manifolds X with $Sc(X) > 0$, essential geometric properties of Y with $mean.curv \ge \mu$ can be proved at the present moment only in the light of the scalar curvature by means of twisted Dirac operators or minimal hypersurfaces and where the transition from mean curvature to the scalar curvature is most clearly seen in the doubling construction. [23]

Exercises. Let X be a Riemannian n-manifold with a non-empty mean convex boundary. Show the following.

(a) If X has non-negative scalar curvature, then the double of X admits a metric with $Sc > 0$, unless X is Riemannian flat with flat boundary.

For instance, doubles of mean convex domains in $\mathbb{R}^n$ carry metrics with positive scalar curvatures.

(b) If X has non-negative Ricci curvature then either it is diffeomorphic to to a regular neighbourhood of a $(n-2)$-dimensional curve-linear polyhedral subset $P^{n-2} \subset X$, or it is Riemannian flat with flat boundary.

---

[20] These forms are evaluated on the (same unit) vector field $\frac{d}{dt}$.

[21] Details can be found in section 11.5 in [G(inequalities 2018].

[22] This trivially follows from a general "*local h-principle*", see section 11.1 in [G(inequalities 2018] and [Baer-Hanke(local flexibility) 2020].

[23] See [G(mean) 2019], [Lott(boundary) 2020], [Cecchini-Zeidler(scalar&mean) 2021 and section 3.5 for more about it.

For instance, if X is connected orientable of dimension $n = 3$, then it is either diffeomorphic to a handle body, or it is isometric to a flat torus times a segment $[-d, d]$, or to a flat bundle over a flat Klein bottle with the fiber $[-d, d]$.

(c) If X admits an equidimensional isometric immersion to a complete simply connected manifold $\hat{X}^n$ with non-positive sectional curvature, then it is also diffeomorphic to to a regular neighbourhood of an $P^{n-2} \subset X$.

Moreover, if $\hat{X}^n$ is equal to the hyperbolic space $\mathbf{H}^n$ with the sectional curvature -1, then the condition $mean.curv(\partial X) \geq 0$ can be relaxed to $mean.curv(\partial X) \geq -(n-1)$ and if $\hat{X}^n = \mathbb{R}^n$ then one needs only the following integral bound on the negative part M_- of the mean curvature of $Y = \partial X$,

$$\int_Y |M_-(y)|^{n-1} dy \leq (n-1)^{(n-1)} \gamma_{n-1},$$

where

$$M_-(y) = \min(0, mean.curv(Y, y))$$

and γ_{n-1} denotes the volume of the unit sphere S^{n-1}.

Remark/Question. The above integral inequality is sharp, where the equality holds for bands between concentric spheres.

But it is unclear what is the sharp inequality for domains $X \subset \mathbb{R}^n$ with *connected* boundaries Y.

For instance

what is the infimum of $\int_Y |M_-(y)|^2 dy$ for *torical* $Y = \partial X \subset \mathbb{R}^3$, where X is *not diffeomorphic to the solid torus*?

Is there a lower bound on $\int_Y |M_-(y)|^2 dy$ by the topology of X, e.g. by positive $const_n$ times the *simplicial volume* of X? (Compare with the *simplicial volume conjecture* in section 3.13.)

1.5 Topological and Geometric Domination by Compact and non-Compact Manifolds with positive Scalar Curvatures

The global effect of positivity of the curvature of a Riemannian manifold X is a bound on the overall size of X. This, in the case the sectional and Ricci curvatures, can be expressed in terms of simple geometric characteristics of X, e.g. the diameter and the volume, which are defined in purely metric terms with no direct reference to the topology of X.

Positivity of scalar curvature also limits the size of X, geometrically as well as topologically, but here the bounds on geometry in terms of $\inf Sc(X)$ can't be even *properly* formulated without explicit use of the underlying topology of X.

1. *Prelude to Example.* Let g be a Riemannian metric on the Euclidean space $\mathbb{R}^n$ with uniformly positive scalar curvature, i.e. $Sc(g) \geq \sigma > 0$. Then

this g can't be greater than the Euclidean metric in two respects.

(a) *For all $D > 0$, there exist points $y_1, y_2 \in \mathbb{R}^n$ with $dist_{Eucl} \geq D$, such that*

$$dist_g(y_1, y_2) \leq const = const_{n,\sigma}, \text{ to be specific, say, for } const = \frac{2\pi\sqrt{n(n-1)}}{n\sqrt{\sigma}}.$$

(Recall that $n(n-1)$ is the scalar curvature of the unit sphere S^n.)

(b) *For all $\varepsilon > 0$, there exists a smooth surface $S \subset \mathbb{R}^n$, such that*

$$area_g(S) \le \varepsilon \cdot area_{Eucl}(S).$$

On the surface of things, there is nothing particularly topological about these (a) and (b), but the *true comparison relations* between metrics with $Sc > 0$ and the Euclidean ones, which are expressed by means of, in general *non-diffeomorphic, maps* from X to $\mathbb{R}^n$ are inherently topological.

2. Example: *Euclidean non-Domination with $Sc(X) \ge \sigma > 0$.* Let X be an orientable Riemannian manifold of dimension n with uniformly positive scalar curvature, $Sc(X) \ge \sigma > 0$, and let $f : X \to \mathbb{R}^n$ be a smooth proper map [24] with *non-zero* degree.[25] Then,

this f can't be uniformly Lipschitz on the large scale, nor can it be uniformly area non-expanding.

This means the following.

(a*) *For all $D > 0$, there exist points $y_1, y_2 \in \mathbb{R}^n$ with $dist_{Eucl} \ge D$, such that the distance between their pullbacks $f^{-1}(y_1) \subset X$ and $f^{-1}(y_2) \subset X$ is uniformly bounded,*

$$dist_X(f^{-1}(x_1), f^{-1}(x_2)) < const.^{26}$$

(b*) *If X is spin,*[27] *then for all $\varepsilon > 0$, there exist smooth surfaces $S \subset X$, and $\underline{S} \subset \mathbb{R}^n$, such that*

$$area_X(S) \le \varepsilon \cdot area_{\mathbb{R}^n}(\underline{S})$$

and such that the map f sends S diffeomorphically onto $\underline{S}$.

Remarks and Corollaries. (i) The above **1** follows from **2** applied to the identity map $id : (\mathbb{R}^n, g) \to (\mathbb{R}^n, g_{Eucl})$.

(ii) It follows from **2** that

no compact orientable n-manifold X with $Sc(X) > 0$ admits a map f with non-zero degree to the n-torus $\mathbb{T}^n$, [28] *while* **1** *yields this (only) for diffeomorphisms $X \to \mathbb{T}^n$.*

[24] A map is *proper* if "infinity goes to to infinity". Formally: *the pullbacks of compact subsets are compact.*

[25] A sufficient geometric condition for this "*non-zero*" reads: there is a non empty open subset $U \subset \mathbb{R}^n$, such that the pullbacks $f^{-1}(u) \subset X$, $u \in U$, are finite and contain *odd* numbers of points.

[26] If $n = dim(X) \le 9$, a Schoen-Yau kind of argument with minimal hypersurfaces reduces the problem to an auxiliary *spin* manifold with $Sc \ge \sigma$ to which the Dirac theoretic argument applies, see section 5.3 in [G(billiards) 2014]. But if X is *non-spin* of dimension $n \ge 10$, I can't vouch for the proof, since it depends on "desingularization" of minimal varieties from the papers [SY(singularities) 2017] and/or [Lohkamp(smoothing) 2018], which I have not studied in depth.

[27] This a somewhat tricky topological condition, which we shall explain later on. It suffices to say at this point that manifolds homeomorphic to $\mathbb{R}^n$, and more generally, those with vanishing cohomology group $H^2(X;\mathbb{Z}_2)$ are spin.

But, for instance, the connected sum of $\mathbb{R}^4$ with the complex projective plane $\mathbb{C}P^2$ is non-spin.

[28] This was proved in [SY(structure 1979] for $n \le 7$ and in [GL(spin) 1980] for spin manifolds X and all n. Nowadays, (yet unpublished in an academic journal) analysis of singularities of minimal hypersurfaces in dimensions $n \ge 8$ in [Lohkamp(smoothing) 2018] and in [SY(singularities) 2017] yields this result for all, not necessarily spin, compact manifolds X and all n.

Proof. Given a smooth map $f : X \to \mathbb{T}^n$, let $\tilde{f} : \tilde{X} \to \mathbb{R}^n$ be its lift to the $\mathbb{Z}^n$-coverings of both manifolds and apply either **(a*)** or **(b*)** to the ε-scaled map $\varepsilon\tilde{f} : \tilde{X} \to \mathbb{R}^n$ for $\varepsilon \to 0$.

(iii) The proof of **(a*)**, mainly depends the *geometric measure theory*, (see sections 1.6.2, 1.6.5 while **(b*)** relies on an *index theorem for "twisted" Dirac s* (see sections 1.6.1, 1.6.3). At the present day, there is no alternative proof of **(b*)** (not even of **(b)**) and **(b*)** remains unknown for general (non-spin) manifolds $X$of dimension $n \geq 4$.[29]

Motivated by the above example we make the following definition.

Domination by $\mathbf{Sc} > 0$**.** Let $\underline{X}$ be a "nice", say locally contractible topological space, e.g. a cellular or polyhedral one, and let $\underline{h} \in H_n(\underline{X})$ be a homology class. Say that a, possibly open, oriented connected n-manifold X *dominates* $\underline{h}$, if there exists a continuous map $f : X \to \underline{X}$ *locally constant at infinity*, called $\underline{h}$*-dominating map*, which sends the fundamental homology class $[X]$ to $\underline{h}$.

Quasi-Proper Maps. Similarly, if X is a locally compact and countably compact, one defines domination of homology classes with *infinite supports* in $\underline{X}$, where the relevant maps $f : X \to \underline{X}$ are *quasi-proper*, i.e. they extends to continuous maps between the compactified spaces, from $X^{+ends} \supset X$ to $\underline{X}^{+ends} \supset \underline{X}$, obtained by a attaching the sets of ends to these spaces.

In simple words, f is quasi-proper if

for all proper maps $\phi : \mathbb{R}_+ \to X$ (i.e. $\phi(t) \to \infty$ for $t \to \infty$), the composed map $f \circ \phi : \mathbb{R}_+ \to \underline{X}$ is either proper or converges to a point in $\underline{X}$ for $t \to \infty$.

Domination of Manifolds. For instance, if $\underline{X}$ is an oriented n-manifold or a pseudomanifold,[30] and $\underline{h} = [\underline{X}]$, then these "dominations" also called *dominations with degree 1* are just quasi-proper maps $X \to \underline{X}$ of degree 1.

More generally, *domination with degree* $\neq 0$ – we shall meet these many time in these lectures — refers to equividimensional maps of non-zero degrees between orientable manifolds or pseudomanifolds.

Next, if X and $\underline{X}$ are a metric spaces, say that $\underline{h}$ is λ*-Lipschitz dominated* or *distance-wise* λ*-dominated by* X if the map f is λ-Lipschitz, i.e. $dist_{\underline{X}}(f(x_1), f(x_2) \leq \lambda \cdot dist_X(x_1, x_2)$.

Similarly, define *area-wise* λ*-domination*, by the inequality

$$area_{\underline{X}}(f(S)) \leq \lambda \cdot area_X(S),$$

provided that areas of (suitable) surfaces $S \subset X$ and of their images $f(S) \subset \underline{X}$ their are suitably defined in X and $\underline{X}$, e.g. where these are smooth surfaces in Riemannian manifolds.

3. Positive Scalar Curvature Domination Problems. *What are spaces* $\underline{X}$ *and classes* $\underline{h} \in H_n(\underline{X})$*, which can and which can't be dominated by complete Riemannian manifolds* X *with* $Sc(X) > 0$*?*

How much does the answer depend on additional conditions on topology and geometry of a dominating manifold X*?*

When can such a domination be implemented with λ*-Lipschitz or with area* λ*-contracting maps?*

---

[29] All 3-manifolds are spin.

[30] An *n-pseudomanifold* is a triangulated space, where the singular locus, where this space is *not* locally $\mathbb{R}^n$, has codimension (at least) 2.

Notice that **(a*)** says in this regard that

for no $\lambda > 0$, a non-zero multiple of the fundamental homology class $[\mathbb{R}^n]$ can be (large scale) distance-wise λ-dominated by a manifold X with $Sc(X) \geq \lambda > 0$,

Similarly, **(b*)** can be stated as *non-existence of area-wise spin λ-domination.*

4. From Algebraic Topology to Asymptotic Geometry: Topological versus Lipschitz Domination. However trivial, it should be emphasized that the existence of

a positive scalar curvature domination of a compact orientable manifold $\underline{X}$ (or a pseudomanifold) with degree d

implies

positive scalar curvature 1-*Lipschitz* domination of *all covering of $\underline{X}$*, $\underline{X}$, in particular, of the *universal covering $\tilde{\underline{X}}$*, with degrees d.

(Continous maps $f : X \to \underline{X}$ can be approximated by λ-Lipschitz maps; these lift to the coverings and can be made 1-Lipschitz by scaling $X = (X, g) \mapsto \lambda \cdot X = (X, \lambda^2 \cdot g)$.)

Also it must be noted that the Lipschitz domination between open manifolds is more general and versatile relation than the topological domination for compact manifolds.[31]

For instance, only exceptional compact aspherical n-manifolds X dominate the n-torus $\mathbb{T}^n$, but

there is no single example (so far), where the universal covering $\tilde{X}$ of a compact aspherical X wouldn't 1-Lipschitz dominate $\mathbb{R}^n = \tilde{\mathbb{T}}^n$.

In view of this, the topological $Sc{>}0$-domination problem is shifted to a more fruitful geometric one of the (non)existence of

a 1-Lipschitz domination of an open $\underline{X}$ by Riemannian manifolds X with $Sc(X) \geq \sigma > 0$, or – this is most relevant if $\underline{X}$ is complete – by X with $Sc(X) > 0$.

5. Domination Equivalence Conjecture. If a homology class $\underline{h}$ in $\underline{X}$ (here it is an ordinary one, with compact supports) is dominated by a *complete* manifold X with $Sc > 0$, then it also admits a *compact spin domination*, i.e. by a *compact spin* manifold X_o with $Sc(X_o) > 0$.

(It may be safer to assume $n \neq 4$; also, to avoid irrelevant purely topological obstructions to dominability, one should replace "domination of $\underline{h}$" by "domination of a *non-zero multiple* of $\underline{h}$" in some cases.)

Let us stress out that the most essential cases of this conjecture concern homology classes in *aspherical spaces* that are classifying spaces of of discrete groups,

$$\underline{X} = \mathsf{B}(\Pi) = K(\Pi, 1) \text{ for } \Pi = \pi_1(\underline{X}),$$

and that the **main** (topological and naive) $Sc \not> 0$-**conjecture** – the scalar curvature counterpart of *Novikov's higher signatures conjecture* – formulated in the present terms reads:

[Sc $\not>$ 0] *No non-torsion homology class in the classifying space of a countable group can be dominated by a compact manifold with $Sc > 0$.*

[31] This is well demonstrated by aspherical 4- and 5-dimensional manifolds in section 3.10.3.

6. From $Sc \not> 0$ to $Sc \gneqq 0$. As far as the topology of a complete manifold X is concerned, there is little difference between the conditions $Sc(X) \not> 0$ and $Sc(X) \gneqq 0$,
where, observe, the former corresponds to the bound $\inf Sc \leq 0$ and the latter to $\inf Sc < 0$.

Indeed, according to ***Kazdan's deformation theorem,***

non-existence of a deformation of metric on a complete Riemannian manifold X with $Sc \geq 0$ to a complete metric with $Sc > 0$ implies that X is *Ricci flat,* [Kazdan(complete) 1982].

If $dim(X) = 3$ then then "Ricci flat" implies *Riemannian flat*; and if $n \geq 4$, the ***Cheeger-Gromoll splitting theorem*** shows in most (all?) of our cases that X is *Riemannian flat*, i.e. isometric to the Euclidean space divided by a discrete isometry group.

Thus, as we shall see in several examples later on,

non-existence theorems for $Sc > 0$ yield *rigidity results for* $Sc \geq 0$.

Spin Domination Problem. Non-domination result proven with a use of Dirac operators (these are many, require the dominating manifolds to be *spin*[32]

This could be removed in majority of cases if the following were true.

Unrealistic Conjecture. Compact Riemannian orientable manifolds $\underline{X}$ with positive scalar curvatures can be dominated with *degree* $\neq 0$ by compact Riemannin manifolds X with $Sc \geq 0$ and with *universal coverings spin.*

Exercise. Prove these conjecture for manifolds $\underline{X}$ of dimensions $n \geq 5$ with finite fundamental groups.

Hint. Use Thom's theorem on domination of multiples of homology classes by stably parallelizable manifolds and classification of simply connected manifolds with $Sc > 0$ of dimension ≥ 5 as in ☞of 3.2.

Remark. Proving (maybe disproving?) this conjecture seems possible by the present day tecniqies for manifolds with Abelian fundamental groups.

1.6 Analytic Techniques

The logic of most (all?) arguments concerning the global geometry of manifolds X with *scalar curvatures bounded from below* is, in general terms, as follows.

Firstly, one uses (or proves) the *existence theorems for solutions* Φ *of certain partial differential equations*, where the existence of these Φ and their properties depend on global, topological and/or geometric assumptions $\mathcal{A}$ on X, which are, a priori, unrelated to the scalar curvature.

Secondly, one concocts some *algebraic-differential expressions* $\mathcal{E}(\Phi, Sc(X))$, where the crucial role is played by certain *algebraic formulae* and issuing inequalities satisfied by $\mathcal{E}(\Phi, Sc(X))$ under assumptions $\mathcal{A}$.

Then one arrives at *a contradiction*, by showing that

if $Sc(X) \geq \sigma$*, then the implied properties, e.g. the sign, of* $\mathcal{E}(\Phi, Sc(X))$ *are*

opposite to those satisfied under assumption(s) $\mathcal{A}$.

[32] See section 3.2 for the definition of spin and recall that manifolds with $w_2 = 0$, e.g. *stably parallelizable* ones, are spin.

1.6.1 Spin Manifolds, Dirac Operators $\mathcal{D}$, Atiyah-Singer Index Theorem and S-L-W-(B) Formula

[I] Historically the first Φ in this story were *harmonic spinors* on a Riemannian manifold $X = (X, g)$, that are solutions s of $\mathcal{D}(s) = 0$, where $\mathcal{D} = \mathcal{D}_g$ is the *(Atiyah-Singer)-Dirac* on X.[33]

[I_{yes}]. The existence of *non-zero harmonic* spinors s on certain smooth manifolds X follows from *non-vanishing of the index of* $\mathcal{D}$, where this index, which is *independent of* g, identifies, by the the *Atiyah-Singer theorem of 1963* , with a certain (smooth) topological invariant, denoted $\hat{\alpha}(X)$ (see section 3.2).

Then the relevant formula involving $Sc(X)$ is the following algebraic identity between the squared *Dirac operator* and the (coarse) *Bochner-Laplace operator* $\nabla^*\nabla$ also denoted ∇^2,

[I_{no}]. ***Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) Formula***[34]

$$\mathcal{D}^2 = \nabla^2 + \frac{1}{4}Sc,$$

shows that if $Sc > 0$, then $\mathcal{D}^2 s = 0$ implies that $s = 0$, since

$$0 = \int \langle \mathcal{D}^2 s, s\rangle = \int \langle \nabla^2 s, s\rangle + \frac{Sc}{4}||s||^2 = \int ||\nabla s||^2 + \frac{Sc}{4}||s||^2,$$

where the latter identity follows by integration by parts (Green's formula).

By confronting these *yes* and *no*, André Lichnerowicz[35] showed in 1963 that

$$Sc(g) > 0 \Rightarrow \hat{\alpha}(X) = 0.$$

and proved the following.

Non-Existence Theorem Number One: **Topological Obstruction to $Sc > 0$ for $n = 4k$.** *There exists smooth closed $4k$-dimensional manifolds X, for all $k = 1, 2, \ldots,$ which admit no metrics with $Sc > 0$.*

A decade later, empowered by a general Atiyah-Singer index theorem, Nigel Hitchin extended Lichnerowitz' result to manifolds of dimensions $n = 8k + 1$ and $8k + 2$ and showed, in particular, that

the class of manifolds X with $\hat{\alpha}(X) \neq 0$, that support non-zero g-harmonic spinors all metrics g on X by the Atiyah-Singer theorem, hence no g with $Sc(g) > 0$ by **S-L-W-B** formula, includes certain *homotopy spheres.* [36] [37]

---

[33] All you have to know at this stage about $\mathcal{D}$ is that $\mathcal{D}$ is a certain first order differential on sections of some bundle over X associated with the tangent bundle $T(X)$. Basics on $\mathcal{D}$ are presented in [Min-Oo(K-Area) 2002] and, comprehensively, in [Lawson&Michelsohn(spin geometry) 1989]. Also see sections 3.3.3,4.

[34] All natural selfadjoint geometric second order operators differ from the Bochner Laplacians by *zero order* terms, i.e. (curvature related) endomorphisms of the corresponding vector bundles, but it is remarkable that this in the case of $\mathcal{D}^2$ reduces to *multiplication by a scalar function*, which happens to be equal to $\frac{1}{4}Sc_X(x)$. From a certain perspective, the existence of such an with a wonderful combination of properties is the most amazing aspect of the Atiyah-Singer index theory.

[35] See [Lichnerowitz(spineurs harmoniques) 1963]

[36] See [AS(index) 1971], [Hitchin(spinors)1974].

[37] Prior to 1963, one didn't even know if therere were *simply connected* manifold that would admit *no metric with positive sectional curvature* was known. But Lichnerowicz' theorem, saying, in fact, that

if X is spin, then $Sc(X) > 0 \Rightarrow \hat{A}[X] = 0$

delivered lots of *simply connected manifolds* X that admitted *no metrics with positive scalar*

1.6.2 Inductive Descent with Minimal Hypersurfaces and Conformal Metrics

[II] Another class of solutions Φ of geometric PDE, that are essential for understanding scalar curvature and that are quite different from harmonic spinors, are *solutions to the Plateau problem.*

More specifically, these are *smooth* stable minimal hypersurfaces $Y \subset X$ that represent non-zero integer homology classes from $H_{n-1}(X)$, $n = dim(X)$.

The existence of minimal Y, possibly singular ones, was established by Herbert Federer and Wendell Fleming in 1960, while the smoothness of these Y, that is crucial for our applications, was proven by Federer in 1970 who relied on *regularity* of volume minimizing cones of dimensions ≤ 6 proved by Jim Simons in 1968.

The relevance of these minimal Y of codimension 1 to the scalar curvature problems was discovered by Schoen and Yau who proved in 1979 that

$\bigstar^{codim1}_{\text{mini}}$ if $Sc(X) > 0$ and $Y \subset X$ is a smooth stable minimal hypersurface, then *Y admits a Riemannian metric h with $Sc(h) > 0$.*[38]

In fact, if $dim(Y) = n-1 = 2$, the stability of Y, that is *positivity* of the second variation of the area of Y, implies that (see sections 2.5, 2.4.1)

$$\int_Y (Sc(Y,y) - Sc(X,y))dy \geq 0$$

where the scalar curvature $Sc(Y)$ refers to the metric h_0 in Y induced from the Riemannian metric g of X.

Therefore, positivity of $Sc(X)$ implies positivity of the Euler characteristic of Y, for

$$4\pi\chi(Y) = \int_Y Sc(Y,y)dy \geq \int_Y Sc(X,y)dy > 0.$$

If $m = n-1 \geq 3$, then h is obtained by a *conformal modification* of the metric h_0 on Y,

$$h_0 \mapsto h = (f^2)^{\frac{2}{m-2}} h_0,$$

where, as in the 1975 "conformal paper" by Jerry Kazdan and Frank Warner $f = f(y)$ is the first eigenfunction of the *conformal Laplacian* L on $Y = (Y, h_0)$, that is

$$L_{conf}(f) = -\Delta(f) + \frac{m-2}{4(m-1)} f,$$

where derivation of positivity of the L from positivity of the second variation of $vol_{n-1}(Y)$ relies on the *Gauss formula* suitably rewritten for this purpose by Schoen and Yau and where the issuing positivity of $Sc(f^{\frac{4}{m-2}} h_0)$ follows, as in [Kazdan-Warner(conformal)],[39] by a simple (for those who knows how to do

curvatures, (see section 3.2).

Most of these X have *large Betti numbers*, that, as we know nowadays, is *incompatible* with $sect.curv(X) \geq 0$, but one still doesn't know if there are homotopy spheres not covered by Hitchin's theorem which admit no metrics with positive sectional curvatures.

[38] See [SY(structure) 1979]: *On the structure of manifolds with positive scalar curvature.*

[39] There is more to this paper, than the implication $L_{conf} > 0 \rightsquigarrow \exists g$ with $Sc(g) > 0$ on X. For instance, Kazdan and Warner prove

the existence of metrics g on connected manifolds X, $dim(X) \geq 3$, with prescribed scalar curvatures $Sc(g,x) = \sigma(x)$, for smooth functions $\sigma(x)$, which are negative somewhere on X and

the existence of metrics with $Sc = 0$ on manifolds X, which admits metrics with $Sc \geq 0$.

this kind of things) computation. [40]

Consecutively applied implication $Sc(X,g) > 0 \Rightarrow Sc(Y,h) > 0$ delivers a descending chain of closed oriented submanifolds

$$X \supset Y = Y_1 \supset Y_2 \supset ... \supset Y_i ... \supset Y_{n-2}$$

of dimensions $n-i$ which support Riemannian metrics h_i with $Sc(h_i) > 0$; thus, all connected components of Y_{n-2} must be a spherical.

Thus, Schoen and Yau inductively define a topological class of manifolds ($\mathcal{C}$ in their terms) and prove, in particular, the following.

Non-Existence Theorem Number Two Accompanied by Rigidity Theorem. *Let a compact oriented manifold X of dimension n dominate (a non-zero multiple of the fundamental class of) the n-torus, i.e, X admits a map of non-zero degree to the n-torus $\mathbb{T}^n$,*

$$f : X \to \mathbb{T}^n.$$

If $n \leq 7$,[41] X admits no metric with $Sc > 0$. then X support no metric g with $Sc(g) > 0$.

Moreover, the inequality $Sc(g) \geq 0$ for a metric g on X, implies that g is Riemannian flat and the universal covering of (X,g) is isometric to the Euclidean space $\mathbb{R}^n$.

(The submanifolds Y_i in this case are taken in the homology classes of transversal f-pullbacks of subtori in $\mathbb{T}^n \supset \mathbb{T}^{n-1} \supset ... \supset \mathbb{T}^{n-i} \supset ... \supset \mathbb{T}^2$.)

Remark. The authors of [SY(structure) 1979] say in their paper that it was motivated by problems in general relativity communicated to one of the authors by Stephen Hawking, [42] but I as haven't studied this field I can't judge how much of the current development in geometry of the scalar curvature is rooted in ideas originated in physics.

1.6.3 Twisted Dirac Operators, Large Manifolds and Dirac with Potentials

The index theorem also applies to Dirac operators $\mathcal{D}_{\otimes L}$ that act on spinors with values in Hermitian vector bundles $L \to X$, called *L-twisted spinors*, where *non-vanishing* of the index of $\mathcal{D}_{\otimes L}$ and, thus the existence of non-zero L-twisted harmonic spinors, is *ensured* for bundles L with *sufficiently large top dimensional Chern numbers,* essentially regardless of the topology of the underlying manifold X itself.

On the other hand, the *twisted* S-L-W-(B) formula, which now reads

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \tfrac{1}{4}Sc(X) + \mathcal{R}_{\otimes L},$$

[40] This computation, probably, going back at least hundred years, was brought from the field of infinitesimal geometry to the context of non-linear PDE and *global analysis* by Hidehiko Yamabe in his 1960-paper *On a deformation of Riemannian structures on compact manifolds.*

[41] The dimension restriction was removed in [Lohkamp(smoothing) 2018] and in [SY(singularities) 2017].

[42] It is shown in [Hawking (black holes) 1972], by an argument elaborating on ideas from [Penrose(gravitational collapse) 1965] and resembling those in [SY(structure) 1979], that surface of the event horizon has *spherical topology.* (See [Bengtsson(trapped surfaces) 2011] for more about it.)

shows that such spinors don't exist if the g-*norm* of the curvature of L is *small* compare with the scalar curvature of $X = (X, g)$. Since this norm is inverse proportional to the size of g, large Riemannian manifolds admit *topologically complicated* bundles L with *small* curvatures, which, by the above, shows, as it was observed in [GL(spin) 1980], that, similarly how it is with the sectional and Ricci curvatures,

scalar curvatures of large manifolds must be small.

This delivers confirmation of the main $[Sc \not> 0]$ conjecture from the previous section for certain compact manifolds X, with large fundamental groups, e.g. for X, which support metrics with *non-positive sectional curvatures*:

Spin-non-Domination theorem of $\kappa \leq 0$ by $Sc > 0$. Non-torsion homology classes of complete manifolds $\underline{X}$, with non-positive sectional curvatures can't be dominated by compact (and also by complete) orientable spin manifolds with $Sc > 0$. [43]

In standard terms,

If a compact orientable spin Riemannian manifold X has $Sc > 0$ and $\underline{X}$ is complete with sect.curv$(\underline{X}) \leq 0$, and if

$$f : X \to \underline{X}$$

is a continuous map, then the image of the fundamental class $[X] \in H_n(X)$ is torsion*: some* non-zero *multiple $i \cdot f_*[X] \in H_n(\underline{X})$* vanishes.[44]

For instance,

if $\underline{X}$ is compact of dimension $n = dim(X)$, then all continuous maps $f : X \to \underline{X}$ have *zero* degrees.

Homotopy Invariance of Obstructions to $Sc > 0$ that Issues from $\otimes$ in $\mathcal{D}$. *Non-vanishing of topological invariants* delivered by the twist in $\mathcal{D}_{\otimes L}$ that prevent the existence of metrics with $Sc > 0$ are *stable under toplogical domination* that is, recall, a map $X \to \underline{X}$ of degree ± 1 between orientable manifolds, such that

if such an invariant doesn't vanish for $\underline{X}$, then it doesn't vanish for X either.

(An instance of such an invariant is the $\smile$-product homomorphism $\bigwedge^n H^1(X) \to H^n(X)$, $n = dim(X)$ behind the Schoen-Yau $[Sc > 0]$-non-existence theorem in section 1.6.2 for manifolds mapped to the n-tori)

This is similar to what happens to invariants issuing by the geometric measure theory but very much unlike to those coming from the untwisted index theorem, namely to non-vanishing of $\hat{\alpha}(X)$: the connected sum of two copies of an X with opposite orientations satisfies: $\hat{\alpha}(X\#(-X)) = 0$.

In fact, if X is simply connected of dimension $n \geq 5$, then $\hat{\alpha}(X\#(-X))$ *does admit a metric with $Sc > 0$.* [45]

Dirac with Potentials. The contribution of the connection of L to the Dirac operator can be seen as a vector potential added to $\mathcal{D}$ twisted with a the trivial bundle of rank $= rank(L)$.

Besides, this there are other kinds of – zero order terms – that can significantly influence geometric effects of $\mathcal{D}$.

As far as the scalar curvature is concerned, the first (to the best of my knowledge) potential of this kind (*Cartan connection*) was introduced by Min-Oo in his

[43] See [GL(spin) 1980] and sections 3.2 and 4.7 for more specific statements and proofs.
[44] It's unclear if $f_*[X] \in H_n(\underline{X}$ can be non-zero, yet (odd?) torsion.
[45] I am uncertain about $n = 4$.

proof of the positive mass theorem for hyperbolic spaces, [Min-Oo(hyperbolic) 1989], and, recently, applications of *Callias-type* potentials in the work by Checcini, Zeidler and Zhang have significantly extended the range of the Dirac-theoretic applications to the scalar curvature problems.[46] – –

1.6.4 Stable μ-Bubbles

In general, μ-bubbles $Y \subset X$, are solutions of the "non-homogeneous Plateau equation"

$$mean.curv(Y, y) = \mu(y)$$

for a given function $\mu(x)$ on X.

What we deal with in this paper are *stable μ-bubbles* that are *local minima* of the functional

$$Y \mapsto vol_{n-1}(Y) - \mu(Y_<)$$

where μ is a Borel measure on X and $Y_< \subset X$ is a region in X with boundary $\partial Y_< = Y$ (see section 5).

Often our measure is "continuous", i.e. representable as $\mu(x)dx$, for a continuous function $\mu(x)$ on X,and all basic existence and regularity properties of minimal hypersurfaces automatically extend to μ-bubbles in this case.

And what is especially useful for our purposes, is that the Schoen-Yau form of the the second variation formula neatly extends to μ-bubbles with continuous (and some discontinuous) $\mu \neq 0$.

Example/non-Example. The unit sphere $S^{n-1} \subset \mathbb{R}^n$ (with the mean curvature $n-1$) around the origin is a *stable μ-bubble* for the measure $\mu(x) = (n-1)||x||^{-1}dx$ in $\mathbb{R}^n$ and the same sphere also is the μ-bubble for $\mu(x) = (n-1)dx$; but this μ-bubble is an *unstable* one.

A significant gain achieved with μ-bubbles compared with the "plain" minimal hypersurfaces is due to the *flexibility in the choice of μ*, which can be adapted to the geometry of X, similarly to how one uses *twisted* Dirac operators $\mathcal{D}_{\otimes L}$ on X with " adaptable" unitary bundles $L \to X$.

For example, one obtains this way the following version of Schoen-Yau theorem ★ from section 1.6.2.

$☆^{codim1}_{bbl}$ Let X be a complete Riemannian n-manifold with *uniformly positive* scalar curvature, i.e, $Sc(X) \geq \sigma > 0$. If $n \leq 7$, then

X can be exhausted by compact domains with smooth boundaries,

$$V_1 \subset V_2 \subset ... \subset V_i \subset ...X, \ \bigcup_i V_i = X,$$

where the boundaries ∂V_i, for all $i = 1, 2, ...,$ admit metrics with positive scalar curvatures.

(Here, as in section 1.6.2, this needs additional analytical work to be extended to $n \geq 7$.)

[46] Exposition of Dirac operators with potentials, especially of their recent applications to manifolds with boundaries, are, regretfully, missing from our lectures. The reader has to turn to the original papers by Checcini, Zeidler, Zhang and [Guo-Xie-Yu(quantitative K-theory) 2020]. Also we say very little about the mass/energy theorems for hyperbolic spaces extending that in [Min-Oo(hyperbolic) 1989]; we refer for this subject matter to [Chrusciel-Herzlich [asymptotically hyperbolic) 2003], [Chrusciel-Delay(hyperbolic positive energy) 2019], [Huang-Jang-Martin(hyperbolic mass rigidity) 2019] and [Jang-Miao(hyperbolic mass) 2021] where one can find further references.

1.6.5 Warped FCS-Symmetrization of Stable Minimal Hypersurfaces and μ-Bubbles.

Positivity of the conformal Laplacian $-\Delta + \frac{m-2}{4(m-1)}Sc$ doesn't fully reflect the positivity of the second variation of the volume $vol_{n-1}(Y)$, where the former actually yields positivity of the $-\Delta + \frac{1}{2}Sc$, which is, a priori, smaller then $-\Delta + \frac{m-2}{4(m-1)}Sc$, since $-\Delta \geq 0$ and $\frac{1}{2} > \frac{m-2}{4(m-1)}$.

Remarkably, positivity of the $-\Delta+\frac{1}{2}Sc$ on $Y = (Y, h_0)$ *neatly implies* positivity of the scalar curvature of the (warped product) metric $h^{\rtimes} = h_0(y) + \phi^2(y)dt^2$ for the first eigenfunction ϕ of $-\Delta + \frac{1}{2}Sc$, where this metric is defined on the products of Y with the real line $\mathbb{R}$ and with the unit circle $S^1(1) = \mathbb{T} = \mathbb{R})/\mathbb{Z}$, and where the resulting Riemannian manifolds are denoted

$$\bar{Y}^{\rtimes} = Y \rtimes \mathbb{R} = (Y \times \mathbb{R}, h^{\rtimes}) \text{ and } Y^{\rtimes} = Y \rtimes \mathbb{T} = \bar{Y}^{\rtimes}/\mathbb{Z}.$$

In fact, if $(-\Delta + \frac{1}{2}Sc)(\phi) = \lambda\phi$ with $\lambda \geq 0$, then

$$Sc(h^{\rtimes}(y,t)) = Sc(h_0, y) - \frac{2}{\phi}\Delta\phi(y) = \frac{2}{\phi}\left(-\Delta + \frac{1}{2}Sc(h_0,)\right)(\phi) = \lambda > 0m$$

see sections 5.

The operation

$$Y \rightsquigarrow Y^{\rtimes}$$

is applied in the present case to stable minimal hypersurfaces $Y \subset X$, where the resulting passage $X \rightsquigarrow Y^{\rtimes}$ can be regarded as *symmetrisation* of X (or rather of infinitesimal neighbourhood of $Y \subset X$), because

the metric $h^{\rtimes}$ is invariant under the natural action of $\mathbb{T}$ on $Y^{\rtimes}$ and

$$Y^{\rtimes}/\mathbb{R} = Y \subset X$$

.

This $h^{\rtimes} = h_0(y) + \phi^2(y)dt^2$ defined with the first eigenfunction ϕ of the $-\Delta + \frac{1}{2}Sc$ on Y was introduced by Doris **F**ischer-**C**olbrie and Rick **S**choen[47] who used it for

classification of complete stable minimal surfaces in 3-manifolds X with $Sc(X) \geq 0$, including $X = \mathbb{R}^3$.

Then $h^{\rtimes}$ was used in [GL(complete) 1983], where, with an incorporation of Schoen-Yau's inductive descent, this allowed higher dimensional applications of the following kind.

Given a Riemannian metric g on a product manifold $X = X_0 \times \mathbb{T}^k$, a consecutive symmetrization

$$X = X_0 \rightsquigarrow X_1 = Y_1^{\rtimes}/\mathbb{Z} \rightsquigarrow X_2 = Y_2{}^{\rtimes}/\mathbb{Z} \rightsquigarrow \ldots$$

delivers a $\mathbb{T}^k$-*invariant* metric $\bar{g}$ on $\bar{X}_k = Y_{-k} \times \mathbb{T}^k$, where $Y_{-k} \subset X$ is a submanifold of codimension k which is *homologous to* $X_0 = X_0 \times t_0 \subset X$ and such that the ($\mathbb{T}^k$-invariant) scalar curvature $Sc(\bar{g})$ on $\bar{X}_k$ is *bounded from below* by $Sc(g)$ on $Y_{-k} = \bar{X}_k/\mathbb{T}^k \subset X$.

[47] *The structure of complete stable minimal surfaces Y in 3-manifolds of non-negative scalar curvature.*

Thus, for instance, one obtains a somewhat different proof of the Schoen-Yau theorem for $n \leq 7$:

no metric g on $X = \mathbb{T}^n$ can have $Sc(g) > 0$, because all $\mathbb{T}^n$-*invariant* metrics on $\mathbb{T}^n$ are Riemannian flat.

Non-Compact Case. An apparent bonus of this argument is its applicability to *non-compact complete manifolds.*

Example: Non-domination of $\mathbb{T}^n$ by $Sc > 0$. The n-torus admits no domination by *complete* manifolds X with $Sc(X) > 0$.[48]

For instance, if a closed subset in the torus $Y \subset \mathbb{T}^n$ is contained in a topological ball $B \subset \mathbb{T}^n$, then

the complement $T^n \smallsetminus Y$ *admits no complete metric with* $Sc > 0$.

The main role of the above $\mathbb{T}^k$-symmetrization, however, is *not for the proof of topological non-existence theorems* of metrics with $Sc > 0$ on closed or non-compact complete manifolds, but for the *geometric study of such metrics* on, possibly *non-compact and non-complete,* manifolds X.

In fact, this symmetrization applies to stable minimal hypersurfaces $Y \subset X$ *with prescribed as well as free boundaries*, say with $\partial Y \subset \partial X$ and also to *stable* μ*-bubbles.* [49]

1.6.6 Averaged Curvature of Levels of Harmonic Maps

Recently, Daniel Stern [Stern(harmonic) 2019] found a version of the 3d Schoen-Yau argument for the levels of *non-constant harmonic maps* $f : X \to \mathbb{T}^1$, where, instead of the second variation formula for $area(Y)$, one uses

the Bochner identity, which expresses the Laplace of the norm of the gradient of f in terms of the Hessian of f and the Ricci curvature,

$$\frac{1}{2}\Delta|\nabla f|^2) = |Hess(f)|^2 + Ricci_X(\nabla f, \nabla f).$$

Thus, Stern proved that the average Euler characteristics of these levels $Y_t = f^{-1}(t), t \in \mathbb{T}^1$ satisfies:

Harmonic Map Inequality.

$$4\pi \int_{\mathbb{T}^1} \chi(Y_t)dt \geq \int_{\mathbb{T}^1} dt \int_{Y_t} (|df(y,t)|^{-2}|Hessf(y,t)|^2 + Sc(X,(y,t)))dy.$$

This shows that

$$4\pi \int_{\mathbb{T}^1} \chi(Y_t)dt \geq \int_{\mathbb{T}^1} dt \int_{Y_t} Sc(X,(y,t))dy.$$

[48] Here, as at other similar occasions, singularities of minimal hypersurfaces and of μ-bubbles create complications for $n = dim(X) \geq 8$.

In the present case, if X is spin, this non-domination property follows by a Dirac operator argument from section 6 in [GL(complete) 1983].

If $n = 8$ the perturbation argument from [Smale(generic regularity) 2003] takes care of things.

If $n = 9$ one can still apply Dirac operators to non-spin manifolds, exploiting the fact that singularities of hypersurfaces are at most 1-dimensional, while the obstruction to spin (the second Stiefel-Whitney class) is 2-dimensional, see section 5.3 in [G(billiards) 2014].

If $n \geq 8$ the recent desingularization results presented in [Lohkamp(smoothing) 2018] and in [SY(singularities) 2017] apply to all X.

[49] See section 12 in [GL(complete) 1983], [G(inequalities) 2018] and sections 3.7, **??**).

and implies, among other things, that

if the universal covering of a compact 3-manifolds with positive scalar curvatures is connected at infinity, then the one-dimensional cohomology $H^1(X;\mathbb{Z})$ vanishes.[50]

Indeed, if $H^1(X;\mathbb{Z}) \neq 0$, then X admits a non-constant harmonic map to the circle $\mathbb{T}^1$, where non-singular levels $Y_t \subset \mathbb{X}$ *can't contain spherical components*, because lifts of such a component to the universal covering of X would bound balls on which (the lift of) f would be constant by the maximum principle for harmonic functions. [51]

Vague Questions. Is there an algebraic link between S-L-W-(B) and the above Bochner formula that would connected Dirac operators with harmonic maps?

Do *Dirac harmonic* and/or similar maps bear a relevance to the scalar curvature problem?

1.6.7 Seiberg-Witten Equation

The third kind of Φ are solutions to the 4-dimensional *Seiberg-Witten equation* of 1994, that is the Dirac equation coupled with a certain non-linear equation and where the relevant formula is essentially the same as in [I].

Using these, Claude LeBrun[52] established a non-trivial (as well as sharp)

Fundamental 4D lower bound *on $\int_X Sc(X,x)^2 dx$ for Riemannian manifolds X diffeomorphic to algebraic surfaces of general type.*

1.6.8 Hamilton-Ricci Flow

Hamilton1

The *Hamilton Ricci flow* $\Phi = g(t)$ of Riemannian metrics on a manifold X, that is defined by a *parabolic* system of equations, also delivers a geometric information on the scalar curvature, where the main algebraic identity for $Sc(t) = Sc(g(t))$ reads

$$\frac{dSc(t)}{dt} = \Delta_{g(t)} Sc(t) + 2Ricci(t)^2 \geq \Delta_{g(t)} Sc(t) + \frac{2}{3} Sc(t)^2,$$

which implies by the maximum principle that the minimum of the scalar curvature grows with time as follows:

$$Sc_{\min}(t) \geq \frac{Sc_{\min}(0)}{1 - \frac{2tSc_{min}(0)}{3}}.$$

If $X = (X, \underline{g})$ is a closed 3-manifold of constant sectional curvature -1, then, using the Ricci flow, Grisha Pereleman proved

Sharp 3D Hyperbolic Lower Volume Bound. All Riemannian metrics g on X with $Sc(g) \geq -6 = Sc(\underline{g})$ satisfy

$$Vol(X, g) \geq Vol(X, \underline{g}).$$

[50] It is known that compact 3-dimensional manifolds with $Sc > 0$ are connected sums of space forms and $S^2 \times S^1$, see [GL(complete) 1983] and [Genoux(3d classification) 2013].

[51] In this respect, the surfaces Y_t are radically different from minimal surfaces and μ-bubbles which tend to localize around narrow necks in X, e.g. in "thin" connected sums $\mathbb{T}^3 \# S^3$ described in section 1.3.

[52] [LeBrun(Yamabe) 1999]: *Kodaira Dimension and the Yamabe Problem.*

(See Proposition 93.9 in [Kleiner-Lott(on Perelman's) 2008].)

And, more recently, Richard Balmer, Paula Burkhardt-Guim and Man-Chun Lee, Aaron Naber and Robin Neumayer applied the Ricci flow for regularization of of (limits of) metrics with $Sc \geq \sigma$.[53]

(The logic of the Ricci flow, at least on the surface of things, is quite different from how it goes in the above three cases that rely on *elliptic* equations:

the quantities Φ in the former result from geometric or topological *complexities* of underlying manifolds X, that is necessary for the very existence of these Φ, while the Ricci flow, as a road roller, leaves a uniform terrain behind itself as it crawls along erasing complexity.)

Question. Do 3D-results obtained with the Ricci flow generalize to n-manifolds which have $Sc \geq \sigma$ and which come with *free isometric actions of the tori* $\mathbb{T}^{n-3}$?

For instance, let X^3 be a 3-dimensional Riemannin manifold which admits a hyperbolic metric $\underline{g}$ with sectional curvature -1 and let $X = X^3 \rtimes \mathbb{T}^1$ be a warped product (with $\mathbb{T}^1$-invariant metric), such that $Sc(X) \geq -6$.

Is the volume of $X^3 = X/\mathbb{T}^1$ is bounded from below by that of $(X^3, \underline{g})$?

(It is not even clear if the inequality $Sc(X^3 \rtimes \mathbb{T}^1) \geq -6$ imposes *any lower bound* on the Riemannin metric g of X^3. Namely,

Can such a $g = g_\varepsilon$ satisfy $g \leq \varepsilon \underline{g}$ for a given $\varepsilon > 0$?[54])

1.6.9 Modifications of Riemannian Metrics by a Single Function

Riemannian metrics g on an n-manifold X are given locally by $\frac{n(n-1}{2}$ functions $g_{ij}(x)$, where the scalar curvature $Sc(g)$ is a (messy) non-linear function of these g_{ij} and their first and second derivatives.

There are several constructions of Riemannian metrics on X and of modifications of a given metric g_0 on X by means of a *single* function $\phi(x)$, where the the scalar curvature of the resulting metric $g(\phi) = g(\phi, g_0)$ is expressed by a "nice" non-linear second order differential applied to ϕ.

The simplest and most studied case of this is the conformal transformation $g \mapsto \varphi^2 g$, where for $n \geq 3$ the scalar curvature of this metric is given by the (Yamabe?) equation

$$Sc(\varphi^2 g_0) = -\frac{4(n-1)}{n-2}\varphi^{\frac{n+2}{2}}\Delta\varphi^{\frac{n-2}{2}} + \varphi^2 Sc(g_0),$$

where $\Delta = \Delta_{g_0}$ is the Laplace on functions $\phi = \phi(x)$ on the Riemannian manifold (X, g_0).

We present some properties of this equation, due to Jerry Kazdan and Frank Warner, in section 2.6, which are used in the proof of Schoen-Yau's *non-existence* theorem for metrics with $Sc > 0$ on tori in sections 1.6.2, 2.7.

Also we briefly discuss in 2.6 similar transformations of metrics, where the scaling takes place only in some preferred directions, e.g. in a single direction, where the scalar curvature satisfies a non-linear parabolic (Bartnik-Shi-Tamm) equation, special *solutions* of which used for the proofs of *non-extension* theorems for metrics with $Sc > 0$, see section 3.12.

[53]See [Bamler(Ricci flow proof) 2016], [Burkhart-Guim(regularizing Ricci flow) 2019], [Lee-Naber-Neumayer](convergence) 2019] and section 3.1.3.

[54]An elementary proof of such a bound on g is suggested in [G(foliated) 1991].

Finally, recall Kähler metrics defined with single functions via the $\partial\bar{\partial}$, where, as we mention in section 1.2, Yau's solution of the Calabi conjecture delivers "interestingly thick" metrics with $Sc > 0$ on complex algebraic manifolds.

2 Curvature Formulas for Manifolds and Submanifolds.

We enlist in this section several classical formulas of Riemannian geometry and indicate their (more or less) immediate applications.

2.1 Variation of the Metrics and Volumes in Families of Equidistant Hypersurfaces

(2.1. A) Riemannian Variation Formula. Let h_t, $t \in [0, \varepsilon]$, be a family of Riemannian metric on an $(n-1)$-dimensional manifold Y and let us incorporate h_t to the metric $g = h_t + dt^2$ on $Y \times [0, \varepsilon]$.

Notice that an arbitrary Riemannian metric on an n-manifold X admits such a representation in normal geodesic coordinates in a small (normal) neighbourhood of any given compact hypersurface $Y \subset X$.

The t-derivative of h_t is equal to *twice the second fundamental form* of the hypersurface $Y_t = Y \times \{t\} \subset Y \times [0, \varepsilon]$, denoted and regarded as a quadratic differential form on $Y = Y_t$, denoted

$$A_t^* = A^*(Y_t)$$

and regarded as a quadratic differential form on $Y = Y_t$.

In writing,

$$\partial_\nu h = \frac{dh_t}{dt} = 2A_t^*,$$

or, for brevity,

$$\partial_\nu h = 2A^*,$$

where

ν is the unit normal field to Y defined as $\nu = \frac{d}{dt}$.

In fact, if you wish, you can take this formula for the definition of the second fundamental form of $Y^{n-1} \subset X^n$.

Recall, that the *principal values* $\alpha_i^*(y)$, $i = 1, ..., n-1$, of the quadratic form A_t^* on the tangent space $T_y(Y)$, that are the values of this form on the orthonormal vectors $\tau_i^* \in T_i(Y)$, which *diagonalize* A^*, are called *the principal curvatures* of Y, and that the sum of these is called *the mean curvature* of Y,

$$mean.curv(Y, y) = \sum_i \alpha_i^*(y),$$

where, in fact ,

$$\sum_i \alpha_i^*(y) = trace(A^*) = \sum_i A^*(\tau_i)$$

for *all* orthonormal tangent frames τ_i in $T_y(Y)$ by the Pythagorean theorem.

SIGN CONVENTION. The first derivative of h changes sign under reversion of the t-direction. Accordingly the sign of the quadratic form $A^*(Y)$ of a hypersurface $Y \subset X$ depends on the *coorientation* of Y in X, where our convention is such that

the boundaries of *convex* domains have *positive (semi)definite* second fundamental forms A^*, also denoted II_Y, hence, *positive* mean curvatures, with respect to *the outward* normal vector fields.[55]

(2.1.B) First Variation Formula. This concerns the t-derivatives of the $(n-1)$-volumes of domains $U_t = U \times \{t\} \subset Y_t$, which are computed by tracing the above (I) and which are related to the mean curvatures as follows.

$$[\circ_U] \qquad \partial_\nu vol_{n-1}(U) = \frac{dh_t}{dt} vol_{n-1}(U_t) = \int_{U_t} mean.curv(U_t) dy_t{}^{56}$$

where dy_t is the volume element in $Y_t \supset U_t$.

This can be equivalently expressed with the fields $\psi\nu = \psi \cdot \nu$ for C^1-smooth functions $\psi = \psi(y)$ as follows

$$[\circ_\psi] \qquad \partial_{\psi\nu} vol_{n-1}(Y_t) = \int_{Y_t} \psi(y) mean.curv(Y_t) dy_t{}^{57}$$

Now comes the first formula with the Riemannian curvature in it.

2.2 Gauss' Theorema Egregium

Let $Y \subset X$ be a smooth hypersurface in a Riemannian manifold X. Then the sectional curvatures of Y and X on a tangent 2-plane $\tau \subset T_y(Y) \subset T)y(X)$ $y \in Y$, satisfy

$$\kappa(Y,\tau) = \kappa(X,\tau) + \wedge^2 A^*(\tau),$$

where $\wedge^2 A^*(\tau)$ stands for the product of the two principal values of the second fundamental form form $A^* = A^*(Y) \subset X$ restricted to the plane τ,

$$\wedge^2 A^*(\tau) = \alpha_1^*(\tau) \cdot \alpha_2^*(\tau).$$

This, with the definition the scalar curvature by the formula $Sc = \sum \kappa_{ij}$, implies that

$$Sc(Y,y) = Sc(X,y) + \sum_{i \neq j} \alpha_i^*(y) \alpha_j^*(y) - \sum_i \kappa_{\nu,i},$$

where:

- $\alpha_i^*(y)$, $i = 1, ..., n-1$ are the (principal) values of the second fundamental form on the diagonalising orthonormal frame of vectors τ_i in $T_y(Y)$;
- α^*-sum is taken over all ordered pairs (i,j) with $j \neq i$;

[55] At some point, I found out to my dismay, that this is opposite to the standard convention in the differential geometry. I apologise to the readers who are used to the commonly accepted sign.

[56] This come with the *minus* sign in most (all?) textbooks, see e.g. [White(minimal) 2016], [Cal(minimal(2019].

[57] This remains true for Lipschitz functions but if ψ is (badly) non-differentiable, e.g. it is equal to the characteristic function of a domain $U \subset Y$, then the derivative $\partial_{\psi\nu} vol_{n-1}(Y_t)$ may become (much) larger than this integral.

• $\kappa_{\nu,i}$ are the sectional curvatures of X on the bivectors (ν,τ_i) for ν being a unit (defined up to ±-sign) normal vector to Y;
• the sum of $\kappa_{\nu,i}$ is equal to the value of the Ricci curvature of X at ν,

$$\sum_i \kappa_{\nu,i} = Ricci_X(\nu,\nu).$$

(Actually, Ricci can be defined as this sum.)

Observe that both sums are independent of coorientation of Y and that in the case of $Y = S^{n-1} \subset \mathbb{R}^n = X$ this gives the correct value $Sc(S^{n-1}) = (n-1)(n-2)$.

Also observe that

$$\sum_{i\neq j} \alpha_i\alpha_j = \left(\sum_i \alpha_i\right)^2 - \sum_i \alpha_i^2,$$

which shows that

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - ||A^*(Y)||^2 - Ricci(\nu,\nu).$$

In particular, if $Sc(X) \geq 0$ and Y is *minimal*, that is $mean.curv(Y) = 0$, then

(Sc ≥ −2Ric) $$Sc(Y) \geq -2Ricci(\nu,\nu).$$

Example. The scalar curvature of a hypersurface $Y \subset \mathbb{R}^n$ is expressed in terms of the mean curvature of Y, the (point-wise) L_2-norm of the second fundamental form of Y as follows.

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2$$

for $||A^*(Y)||^2 = \sum_i (\alpha_i^*)^2$, while $Y \subset S^n$ satisfy

$$Sc(Y) = (mean.curv(Y))^2 - ||A^*(Y)||^2 + (n-1)(n-2) \geq (n-1)(n-2) - n\max_i (c_i^*)^2.$$

It follows that *minimal* hypersurfaces Y in $\mathbb{R}^n$, i.e. these with $mean.curv(Y) = 0$, have *negative scalar curvatures*, while hypersurfaces in the n-spheres with all principal values $\leq \sqrt{n-2}$ have $Sc(Y) > 0$.

Let $A = A(Y)$ denote *the shape* that is the symmetric on $T(Y)$ associated with A^* via the Riemannian scalar product g restricted from $T(X)$ to $T(Y)$,

$$A^*(\tau,\tau) = \langle A(\tau),\tau\rangle_g \text{ for all } \tau \in T(Y).$$

2.3 Variation of the Curvature of Equidistant Hypersurfaces and Weyl's Tube Formula

(2.3.A) Second Main Formula of Riemannian Geometry.[58] Let Y_t be a family of hypersurfaces t-equidistant to a given $Y = Y_0 \subset X$. Then the shape operators $A_t = A(Y_t)$ satisfy:

$$\partial_\nu A = \frac{dA_t}{dt} = -A^2(Y_t) - B_t,$$

[58] The first main formula is *Gauss' Theorema Egregium.*

where B_t is the symmetric associated with the quadratic differential form B^* on Y_t, the values of which on the tangent unit vectors $\tau \in T_{y,t}(Y_t)$ are equal to the values of the *sectional curvature* of g at (the 2-planes spanned by) the bivectors $(\tau, \nu = \frac{d}{dt})$.

Remark. Taking this formula for the *definition* of the sectional curvature, or just systematically using it, delivers fast clean proofs of the basic *Riemannian comparison theorems* along with their standard corollaries, by far more efficiently than what is allowed by the cumbersome language of Jacobi fields lingering on the pages of most textbooks on Riemannian geometry. [59]

Tracing this formula yields

(2.3.B) Hermann Weyl's Tube Formula.

$$trace\left(\frac{dA_t}{dt}\right) = -\|A^*\|^2 - Ricci_g\left(\frac{d}{dt}, \frac{d}{dt}\right),$$

or

$$trace(\partial_\nu A) = \partial_\nu trace(A) = -\|A^*\|^2 - Ricci(\nu,\nu),$$

where

$$\|A^*\|^2 = \|A\|^2 = trace(A^2),$$

where, observe,

$$trace(A) = trace(A^*) = mean.curv = \sum_i \alpha_i^*$$

and where *Ricci* is the quadratic form on $T(X)$ the value of which on a unit vector $\nu \in T_x(X)$ is equal to the trace of the above B^*-form (or of the B) on the normal hyperplane $\nu^\perp \subset T_x(X)$ (where $\nu^\perp = T_x(Y)$ in the present case).

Also observe – this follows from the definition of the scalar curvature as $\sum \kappa_{ij}$ – that

$$Sc(X) = trace(Ricci)$$

and that the above formula $Sc(Y,y) = Sc(X,y) + \sum_{i\neq j} \alpha_i^* \alpha_j^* - \sum_i \kappa_{\nu,i}$ can be rewritten as

$$Ricci(\nu,\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - \sum_{i\neq j} \alpha_i^* \cdot \alpha_j^*\right) =$$

$$= \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + \|A^*\|^2\right)$$

where, recall, $\alpha_i^* = \alpha_i^*(y)$, $y \in Y$, $i = 1, ..., n-1$, are the principal curvatures of $Y \subset X$, where $mean.curv(Y) = \sum_i \alpha_i^*$ and where $\|A^*\|^2 = \sum_i (\alpha_i^*)^2$.

[59] Thibault Damur pointed out to me that this formula, along with the rest displayed on the pages in this section, are systematically used by physicists in books and in articles on relativity. For instance, what we present under heading of "Hermann Weyl's Tube Formula", appears in [Darmos(Gravitation einsteinienne) 1927] with the reference to Darboux' textbook of 1897.

2.4 Umbilic Hypersurfaces and Warped Product Metrics

A hypersurface $Y \subset X$ is called *umbilic* if all principal curvatures of Y are mutually equal at all points in Y.

For instance, spheres in the *standard* (i.e. complete simply connected) *spaces with constant curvatures* (spheres $S^n_{\kappa>0}$, Euclidean spaces $\mathbb{R}^n$ and hyperbolic spaces $\mathbf{H}^n_{\kappa<0}$) are umbilic.

In fact these are special case of the following class of spaces .

Warped Products. Let $Y = (Y, h)$ be a smooth Riemannian (n-1)-manifold and $\varphi = \varphi(t) > 0$, $t \in [0, \varepsilon]$ be a smooth positive function. Let $g = h_t + dt^2 = \varphi^2 h + dt^2$ be the corresponding metric on $X = Y \times [0, \varepsilon]$.

Then the hypersurfaces $Y_t = Y \times \{t\} \subset X$ are umbilic with the principal curvatures of Y_t equal to $\alpha_i^*(t) = \frac{\varphi'(t)}{\varphi(t)}$, $i = 1, ..., n-1$ for

$$A_t^* = \frac{\varphi'(t)}{\varphi(t)} h_t \text{ for } \varphi' = \frac{d\varphi(t)}{dt} \text{ and } A_t \text{ being multiplication by } \frac{\varphi'}{\varphi}.$$

The Weyl formula reads in this case as follows.

$$(n-1)\left(\frac{\varphi'}{\varphi}\right)' = -(n-1)^2\left(\frac{\varphi'}{\varphi}\right)^2 - \frac{1}{2}\left(Sc(g) - Sc(h_t) - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2\right).$$

Therefore,

$$Sc(g) = \frac{1}{\varphi^2} Sc(h) - 2(n-1)\left(\frac{\varphi'}{\varphi}\right)' - n(n-1)\left(\frac{\varphi'}{\varphi}\right)^2 =$$

$$(\star) \qquad = \frac{1}{\varphi^2} Sc(h) - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

where, recall, $n = dim(X) = dim(Y) + 1$ and the mean curvature of Y_t is

$$mean.curv(Y_t \subset X) = (n-1)\frac{\varphi'(t)}{\varphi(t)}.$$

Examples. (a) If $Y = (Y, h) = S^{n-1}$ is the unit sphere, then

$$Sc_g = \frac{(n-1)(n-2)}{\varphi^2} - 2(n-1)\frac{\varphi''}{\varphi} - (n-1)(n-2)\left(\frac{\varphi'}{\varphi}\right)^2,$$

which for $\varphi = t^2$ makes the expected $Sc(g) = 0$, since $g = dt^2 + t^2 h$, $t \geq 0$, is the Euclidean metric in the polar coordinates.

If $g = dt^2 + \sin t^2 h$, $-\pi/2 \leq t \leq \pi/2$, then $Sc(g) = n(n-1)$ where this g is the spherical metric on S^n.

(b) If h is the (flat) Euclidean metric on $\mathbb{R}^{n-1}$ and $\varphi = \exp t$, then

$$Sc(g) = -n(n-1) = Sc(\mathbf{H}^n_{-1}).$$

(c) What is slightly less obvious, is that if

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n},$$

then the scalar curvature of the metric $\varphi^2 h + dt^2$, where h is flat, is *constant positive*, namely $Sc(g) = n(n-1) = Sc(S^n)$, by elementary calculation[60]

Cylindrical Extension Exercise. Let Y be a smooth manifold, $X = Y\times\mathbb{R}_+$, let g_0 be a Riemannian metric in a neighbourhood of the boundary $Y = Y\times\{0\} = \partial X$, let h denote the Riemannian metric in Y induced from g_0 and let Y has *constant mean curvature* in X with respect to g_0.

Let X' be a (convex if you wish) ball in the standard (i.e complete simply connected) space with constant sectional curvature and of the same dimension n as X, let $Y' = \partial X'$ be its boundary sphere, let, let $Sc(h) > 0$ and let the mean and the scalar curvatures of Y and Y' are related by the following (comparison) inequality.

[<] $$\frac{|mean.curv_{g_0}(Y)|^2}{Sc(h,y)} < \frac{|mean.curv(Y')|^2}{Sc(Y')} \text{ for all } y\in Y.$$

Show that

if Y *is compact, there exists a smooth positive function* $\varphi(t)$, $0\le t<\infty$, *which is constant at infinity and such that the the warped product metric* $g = \varphi^2 h + dt^2$ *has*

the same Bartnik data *as* g_0, *i.e.*

$$g|Y = h_0 \text{ and } mean.curv_g(Y) = mean.curv_{g_0}(Y),$$

Then show that

one *can't make* $Sc(g)\ge Sc(X')$ in general, if [<] is relaxed to the corresponding *non-strict* inequality, where an example is provided by the Bartnik data of $Y'\in X'$ itself.[61]

Vague Question. What are "simple natural" Riemannian metrics g on $X = Y\times\mathbb{R}_+$ with given Bartnik data $(Sc(Y), mean, curv(Y))$, where $Y\subset X$ is allowed *variable* mean curvature, and what are possibilities for lower bound on the scalar curvatures of such g granted $|mean.curv(Y,y)|^2/Sc(Y,y) < C$, e..g. for $C = |mean.curv(Y')|^2/Sc(Y')$ for Y' being a sphere in a space of constant curvature.

2.4.1 Higher Warped Products

Let Y and S be Riemannian manifolds with the metrics denoted dy^2 (which now play the role of the above dt^2) and ds^2 (instead of h), let $\varphi>0$ be a smooth function on Y, and let

$$g = \varphi^2(y)ds^2 + dy^2$$

be the corresponding warped metric on $Y\times S$,

Then

(⋆⋆)

$$Sc(g)(y,s) = Sc(Y)(y) + \frac{1}{\varphi(y)^2}Sc(S)(s) - \frac{m(m-1)}{\varphi^2(y)}\|\nabla\varphi(y)\|^2 - \frac{2m}{\varphi(y)}\Delta\varphi(y),$$

[60] See §12 in [GL(complete) 1983].

[61] It follows from [Brendle-Marques(balls in S^n)N 2011] that the the cylinder $S^{n-1}\times\mathbb{R}_+$ admits a complete Riemannian metric g cylindrical at infinity which has $Sc(g) > n(n-1)$, and which has the same Bartnik data as the boundary sphere X'_0 in the hemisphere X' in the unit n-sphere. But the non-deformation result from [Brendle-Marques(balls in S^n) 2011], suggests that this might be impossible for the Bartnik data of *small* balls in the round sphere.

where $m = dim(S)$ and $\Delta = \sum \nabla_{i,i}$ is the Laplace on Y.

To prove this, apply the above c ($\star$) to $l \times S$ for naturally parametrised geodesics $l \subset Y$ passing trough y and then average over the space of these l, that is the unit tangent sphere of Y at y.

The most relevant example here is where S is the real line $\mathbb{R}$ or the circle S^1 also denoted $\mathbb{T}^1$ and where ($\star$) reduces to

$$(\star\star)_1 \qquad Sc(g)(y,s) = Sc(Y)(y) - \frac{2}{\varphi}\Delta\varphi(y).$$ [62]

For instance, if the $L = -\Delta + \frac{1}{2}Sc$ on Y is strictly positive, that is the lowest eigenvalue λ is strictly positive and if φ equals to the corresponding eigenfunction of L, then

$$-\Delta\varphi = \lambda \cdot \varphi - \frac{1}{2}Sc \cdot \varphi$$

and

$$Sc(g) = 2\lambda > 0,$$

The basic feature of the metrics $\varphi^2(y)ds^2 + dy^2$ on $Y \times \mathbb{R}$ is that they are $\mathbb{R}$-*invariant*, where the quotients $(Y \times \mathbb{R})/\mathbb{Z} = Y \times \mathbb{T}^1$ carry the corresponding $\mathbb{T}^1$-*invariant* metrics, while *the* $\mathbb{R}$-*quotients are isometric to* Y.

Besides $\mathbb{R}$-invariance, a characteristic feature of warped product metrics is *integrability* of the tangent hyperplane field normal to the $\mathbb{R}$-orbits, where $Y \times \{0\} \subset Y \times \mathbb{R}$, being normal to these orbits, serves as an integral variety for this field.

Also notice that $Y = Y \times \{0\} \subset Y \times \mathbb{R}$ is totally geodesic with respect to the metric $\varphi^2(y)ds^2 + dy^2$, while the ($\mathbb{R}$-invariant) *curvature* (vector field) *of the* $\mathbb{R}$-*orbits* is equal to the *gradient field* $\nabla\varphi$ extended from Y to $Y \times \mathbb{R}$. coordinates

In what follows, we emphasize $\mathbb{R}$-invariance and interchangeably speak of $\mathbb{R}$-invariant metrics on $Y \times \mathbb{R}$ and metrics warped with factors φ^2 over Y.

Gauss-Bonnet $g^{\rtimes}$-Exercise. Let the above S be the Euclidean space $\mathbb{R}^N$ (make it $\mathbb{T}^n$ if you wish to keep compactness) with coordinates $t_1, ..., t_N$, let

$$\Phi(y) = (\varphi_1(y), ..., \varphi_i(y), ..., \varphi_N(y))$$

be an N-tuple of smooth positive function on a Riemannian mnanifold $Y = (Y, g)$ and define the (iterated t warped product) metric $g^{\rtimes} = g^{\rtimes}_{\Phi}$ on $Y \times S$ as follows:

$$g^{\rtimes} = g(y) + \varphi_1^2(y)dt_1^2 + \varphi_2^2(y)dt_2^2 + ... + \varphi_N^2(y)dt_N^2$$

Show that the scalar curvature of this metric, which, being $\mathbb{R}^N$-invariant, is regarded as a function on Y, satisfies:

$$Sc(g^{\rtimes}, y) = Sc(g) - 2\sum_{i=1}^{N}\Delta_g \log\varphi_i - \sum_{i=1}^{N}(\nabla_g \log\varphi_i)^2 - \left(\sum_{i=1}^{N}\nabla_g \log\varphi_i\right)^2,$$

thus

$$\int_Y Sc(g^{\rtimes}, y)dy \le \int_Y Sc(g, y)dy,$$

and, following [Zhu(rigidity) 2019], obtain the following

[62]The roles of Y and $S = \mathbb{R}$ and notationally reversed here with respect to those in ($\star$)

"Warped" Gauss-Bonnet Inequality for Closed Surfaces Y:

$$\int_Y Sc(g^{\rtimes}, y)dy \le 4\pi\chi(Y)$$

for the (iterated) warped product metrics $g^{\rtimes} = g^{\rtimes}_{\phi}$ for all positive N-tuples of Φ of positive functions on Y. [63]

2.5 Second Variation Formula

The Weyl formula also yields the following formula for the *second derivative* of the $(n-1)$-volume of a cooriented hypersurface $Y \subset X$ under a normal deformation of Y in X, where the scalar curvature of X plays an essential role.

The deformations we have in mind are by vector fields directed by geodesic normal to Y, where in the simplest case the norm of his field equals one.

In this case we have an equidistant motion $Y \mapsto Y_t$ as earlier and the second derivative of $vol_{n-1}(Y_t)$, denoted here $Vol = Vol_t$, is expressed in terms of of the shape $A_t = A(Y_t)$ of Y_t and the Ricci curvature of X, where, recall $trace(A_t) = mean.curv(Y_t)$ and

$$\partial_\nu Vol = \int_Y mean.curv(Y)dy$$

by the first variation formula.

Then, by Leibniz' rule,

$$\partial^2_\nu Vol = \partial_\nu \int_Y trace(A(y))dy = \int_Y trace^2(A(y))dy + \int_Y trace(\partial_\nu A(y))dy,$$

and where, by Weyl's formula,

$$trace(\partial_\nu A) = -trace(A^2) - Ricci(\nu,\nu)$$

for the normal unit field ν.

Thus,

$$\partial^2_\nu Vol = \int_Y (mean.curv)^2 - trace(A^2) - Ricci(\nu,\nu),$$

which, combining this with the above expression

$$Ricci(\nu) = \frac{1}{2}\left(Sc(X) - Sc(Y) - (mean.curv(Y))^2 + \|A^*\|^2\right),$$

shows that

$$\partial^2_\nu Vol = \int \frac{1}{2}\left(Sc(Y) - Sc(X) + mean.curv^2 - \|A^*\|^2\right).$$

In particular, if $Sc(X) \ge 0$ and Y is minimal, then,

$(\int \mathrm{Sc} \ge 2\partial^2 \mathrm{Vol})$
$$\int_Y Sc(Y, y)dy \ge 2\partial^2_\nu Vol$$

(compare with the $(Sc \ge -2Ric)$ in 2.2).

[63]See [Zhu() 2019] and sections 5.9, 7.2 for applications and generalizations.

Warning. Unless Y is minimal and despite the notation ∂_ν^2, this derivative depends on how the normal filed on $Y \subset X$ is extended to a vector filed on (a neighbourhood of Y in) X.

Illuminative Exercise. Check up this formula for concentric spheres of radii t in the spaces with constant sectional curvatures that are S^n, $\mathbb{R}^n$ and $\mathbf{H}^n$.

Now, let us allow a non-constant geodesic field normal to Y, call it $\psi\nu$, where $\psi(y)$ is a smooth function on Y and write down the full second variation formula as follows:

$$\partial^2_{\psi\nu} vol_{n-1}(Y) = \int_Y \|d\psi(y)\|^2 dy + R(y)\psi^2(y)dy$$

for

$$[\circ\circ] \qquad R(y) = \frac{1}{2}\left(Sc(Y,y) - Sc(X,y) + M^2(y) - \|A^*(Y)\|^2\right),$$

where $M(y)$ stands for the mean curvature of Y at $y \in Y$ and $\|A^*(Y)\|^2 = \sum_i (\alpha^*)^2$, $i = 1, ..., n-1$.

Notice, that the "new" term $\int_Y \|d\psi(y)\|^2 dy$ depends only on the normal field itself, while the R-term depends on the extension of $\psi\nu$ to X, unless

Y is minimal, where $[\circ\circ]$ reduces to

$$[**] \qquad \partial^2_{\psi\nu} vol_{n-1}(Y) = \int_Y \|d\psi\|^2 + \frac{1}{2}\left(Sc(Y) - Sc(X) - \|A^*\|^2\right)\psi^2.$$

Furthermore, if Y is volume minimizing in its neighbourhood, then $\partial^2_{\psi\nu} vol_{n-1}(Y) \geq 0$; therefore,

$$[\star\star] \qquad \int_Y (\|d\psi\|^2 + \frac{1}{2}(Sc(Y))\psi^2 \geq \frac{1}{2}\int_Y (Sc(X,y) + \|A^*(Y)\|^2)\psi^2 dy$$

for all non-zero functions $\psi = \psi(y)$.

Then, if we recall that

$$\int_Y \|d\psi\|^2 dy = \int_Y \langle -\Delta\psi, \psi\rangle dy,$$

we will see that $[\star\star]$ says that

the $\psi \mapsto -\Delta\psi + \frac{1}{2}Sc(Y)\psi$ *is greater than*[64] $\psi \mapsto \frac{1}{2}(Sc(X,y) + \|A^*(Y)\|^2)\psi$.

Consequently,

if $Sc(X) > 0$, then the $-\Delta + \frac{1}{2}Sc(Y)$ on Y is positive.

Justification of the $\|d\psi\|^2$ Term. Let $X = Y \times \mathbb{R}$ with the product metric and let $Y = Y_0 = Y \times \{0\}$ and $Y_{\varepsilon\psi} \subset X$ be the graph of the function $\varepsilon\psi$ on Y. Then

$$vol_{n-1}(Y_{\varepsilon\psi}) = \int_Y \sqrt{1 + \varepsilon^2\|d\psi\|^2} dy = vol_{n-1}(Y) + \frac{1}{2}\int_Y \varepsilon^2\|d\psi\|^2 + o(\varepsilon^2)$$

by the Pythagorean theorem

and

$$\frac{d^2 vol_{n-1}(Y_{\varepsilon\psi})}{d^2\varepsilon} = \|d\psi\|^2 + o(1).$$

[64] $A \geq B$ for selfadjoint operators signifies that $A - B$ is positive semidefinite.

by the binomial formula.

This proves $[\circ\circ]$ for product manifolds and the general case follows by *linearity/naturality/functoriality* of the formula $[\circ\circ]$.

Naturality Problem. All "true formulas" in the Riemannian geometry should be derived with minimal, if any, amount of calculation – only on the basis of their "naturality" and/or of their validity in simple examples, where these formulas are obvious.

Unfortunately, this "naturality principle" is absent from the textbooks on differential geometry, but, I guess, it may be found in some algebraic articles (books?).

Exercise. Derive the second main formula 2.3.A by pure thought from its manifestations in the examples in the above *illuminative exercise.*[65]

2.6 Conformal Laplacian and the Scalar Curvature of Conformally and non-Conformally Scaled Riemannian Metrics

Let (X_0, g_0) be a compact Riemannian manifold of dimension $n \geq 3$ and let $\varphi = \varphi(x)$ be a smooth positive function on X.

Then, by a straightforward calculation,[66]

◐ $$Sc(\varphi^2 g_0) = \gamma_n^{-1}\varphi^{-\frac{n+2}{2}}L(\varphi^{\frac{n-2}{2}}),$$

where L is the *conformal Laplace* on (X_0, g_0)

$$L(f(x)) = -\Delta f(x) + \gamma_n Sc(g_0, x) f(x)$$

for the ordinary Laplace (Beltrami) $\Delta f = \Delta_{g_0} f = \sum_i \partial_{ii} f$ and $\gamma_n = \frac{n-2}{4(n-1)}$.

Thus, we conclude to the following.

Kazdan-Warner Conformal Change Theorem. [67] Let $X = (X, g_0)$ be a closed Riemannian manifold, such the the conformal Laplace L is positive.

Then X admits a Riemannian metric g (conformal to g_0) for which $Sc(g) >$ 0.

Proof. Since L is positive, its first eigenfunction, say $f(x)$ is positive[68] and since $L(f) = \lambda f$, $\lambda > 0$,

$$Sc\left(f^{\frac{4}{n-2}} g_0\right) = \gamma_n^{-1} L(f) f^{-\frac{n+2}{n-2}} = \gamma_n^{-1} f^{\frac{2n}{n-2}} > 0.$$

Example: Schwarzschild metric. If (X_0, g_0) is the Euclidean 3-space, and $f = f(x)$ is positive function, then

the sign of $Sc(f^4 g_0)$ is equal to that of $-\Delta f$.

In particular, since the function $\frac{1}{r} = (x_1^2 + x_2^2 + x_3^2)^{-\frac{1}{2}}$, is harmonic, *the Schwarzschild metric $g_{Sw} = \left(1 + \frac{m}{2r}\right)^4 g_0$ has zero scalar curvature.*

[65] I haven't myself solved this exercise.

[66] There must be a better argument.

[67] [Kazdan-Warner(conformal) 1975]: *Scalar curvature and conformal deformation of Riemannian structure.*

[68] We explain this in section 2.9.

If $m > 0$, then this metric is defined for all $r > 0$ and it is invariant under the involution $r \mapsto \frac{m^2}{r}$.

If $m = 0$, this the flat Euclidian metric.

If $m < 0$, then this metric is defined only for $r > m$ with a singularity ar $r = m$.

Non-Conformal Scaling. Let $X = (X, g)$ be a smooth n-manifold, and let $\mathbb{R}^\times_x \subset GL_x(n)$, $x \in X$, be a smooth family of diagnosable (semisimple) 1-parameter subgroups in the linear groups $GL_x(n) = GL_n$ that act in the tangent spaces $T_x(X)$.

Then the the multiplicative group of functions $\phi : X \to \mathbb{R}^\times$ acts on the tangent bundle $T(X)$ by

$$\tau \mapsto= \phi(x)(\tau) \text{ for } \phi(x) \in \mathbb{R}^\times = \mathbb{R}^\times_x \subset GL_x = GL(T_x(X))$$

and, thus on the space of Riemannin metrics g on X.

The main instance of such an action is where the tangent bundle is orthogonally split, $T(X) = T_1 \oplus T_2$, and ϕ acts by scaling on the subbundle T_2.

It is an not hard to write down a formula for the scalar curvature of $g_1 + \phi^2 g_2$, but it is unclear what, in general, would be a workable criterion for solvability of the inequality $Sc(g_\varphi) > 0$ in φ, e.g. in the case where $X = X_1 \times X_2$ and the subbundles T_1 and T_2 are equal to the tangent bundles of submanifolds $X_1 \times x_2 \subset X$, $x_2 \in X_2$, and $x_1 \times X_2 \subset X$, $x_1 \in X_1$.

Yet, in the case of $rank(T_2) = 1$, this equation introduced, I believe, by Robert Bartnik in [Bartnik(prescribed scalar) 1993] was successfully applied to extension of metrics with $Sc > 0$ (see section 3.12)[69]

2.7 Schoen-Yau's Non-Existence Results for $Sc > 0$ on SYS Manifolds via Minimal (Hyper)Surfaces and Quasisymplectic $[Sc \not> 0]$–Theorem

Let X be a three dimensional Riemannian manifold with $Sc(X) > 0$ and $Y \subset X$ be an orientable cooriented surface with minimal area in its integer homology class.

Then the inequality $(\int Sc \geq 2\partial^2 V)$ from section 2.5, which says in the present case that

$$\int_Y Sc(Y, y)dy > 2\partial_\nu^2 area(Y),$$

implies that

Y must be a topological sphere.

In fact, minimality of Y makes $\partial_\nu^2 area(Y) \geq 0$, hence $\int_Y Sc(Y, y)dy > 0$, and the sphericity of Y follows by the Gauss-Bonnet theorem.

And since all integer homology classes in closed orientable Riemannian 3-manifolds admit area minimizing representatives by the geometric measure theory developed by Federer, Fleming and Almgren, we arrive at the following conclusion.

★$_3$ **Schoen-Yau $3d$-Theorem**. *All integer 2D homology classes in closed Riemannian 3-manifolds with $Sc > 0$ are spherical.*

For instance, the 3-torus admits no metric with $Sc > 0$.

[69] Other special cases of this are (implicitly) present in the geometry of Riemannin warped product, in the process of *smoothing corners with* $Sc \geq \sigma$ and in the *transversal blow up* of foliations with $Sc > 0$.

The above argument appears in Schoen-Yau's 15-page paper [SY(incompressible) 1979], most of which is occupied by an independent proof of the existence and regularity of minimal Y.

In fact, the existence of minimal surfaces and their regularity needed for the above argument has been known since late (early?) 60s[70] but, what was, probably, missing prior to the Schoen-Yau paper was the innocuously looking corollary of Gauss' formula in 2.2,

$$Sc(Y) = Sc(X) + (mean.curv(Y))^2 - \|A^*(Y)\|^2 - Ricci(\nu,\nu)$$

and the issuing inequality

$$Sc(Y) > -2Ricci(\nu,\nu)$$

for minimal Y in manifolds X with $Sc(X) > 0$.

For example, Burago and Toponogov, come close to the above argument, where, they bound from below the injectivity radius of Riemannian 3-manifolds X with $sect.curv(X) \le 1$ and $Ricci(X) \ge \rho > 0$ by

$$inj.rad(X) \ge 6e^{-\frac{6}{\rho}},$$

where this is done by carefully analysing minimal surfaces $Y \subset X$ bounded by, a priori very short, closed geodesics in X, and where an essential step in the proof is the lower bound on the first eigenvalue of the Laplace on Y by $\sqrt{Ricci(X)}$.[71]

Area Exercises. Let X be homeomorphic to $Y \times S^1$, where Y is a closed orientable surface with the Euler number χ.

(a) Let $\chi > 0$, $Sc(X) \ge 2$ and show that there exists a surface $Y_o \subset X$ homologous to $Y \times \{s_0\}$, such that $area(Y_o) \le 4\pi$.[72]

(b) Let $\chi < 0$, $Sc(X) \ge -2$ and show that all surfaces $Y_* \in X$ homologous to $Y \times \{s_0\}$ have $area(Y_*) \ge -2\pi\chi$.

(c) Show that (a) remains valid for complete manifolds X homeomorphic to $Y \times \mathbb{R}$.[73]

★^{codim1} **Schoen-Yau Codimension 1 Descent Theorem,** [SY(structure) 1979]. Let X be a compact orientable n-manifold with $Sc > 0$.

If $n \le 7$, then all integer homology classes $h \in H_{n-1}(X)$ are representable by compact oriented $(n-1)$-submanifolds Y in X, which admit metrics with $Sc > 0$.

Proof. Let Y be a volume minimizing hypersurface representing h, the existence and regularity of which is guaranteed by a Federer 1970-theorem[74] and recall that by [★★] in 2.5 the $-\Delta + \frac{1}{2}Sc(Y)$ is positive. Hence, the conformal Laplace $-\Delta + \gamma_n Sc(Y)$ is also positive for $\gamma_n = \frac{n-2}{4n-1} \le \frac{1}{2}$ and the proof follows by Kazdan-Warner conformal change theorem.

[70]Regularity of volume minimizing hypersurfaces in manifolds X of dimension $n \le 7$, as we mentioned earlier, was proved by Herbert Federer in [Fed(singular) 1970], by reducing the general case of the problem to that of minimal cones resolved by Jim Simons in [Simons(minimal) 1968].

[71][BurTop(curvature bounded above)1973],*On 3-dimensional Riemannian spaces with curvature bounded above.*

[72]See [Zhu(rigidity) 2019] for a higher dimensional version of this inequality.

[73]I haven't solved this exercise.

[74][Federer(singular) 1970]: *The singular sets of area minimizing rectifiable currents with codimension one and of area minimizing flat chains modulo two with arbitrary codimension.*

$\bigstar_{\mathbb{T}^n}$ **Mapping to the Torus Corollary.** If a closed orientable n-manifold X admits a map to the torus $\mathbb{T}^n$ with *non-zero degree*, then X admits *no metric with* $Sc > 0$.

Indeed, if a closed submanifold Y^{n-1} is *non-homologous to zero* in this X then it (obviously) admits a map to $\mathbb{T}^{n-1}$ with non-zero degree. Thus, the above allows an inductive reduction of the problem to the case of $n = 2$, where the Gauss-Bonnet theorem applies.

SYS-Manifolds. Schoen and Yau say in [SY(structure) 1979] that their codimension 1 descent theorem delivers a topological obstruction to $Sc > 0$ on a class of manifolds, which is, even in the spin case, [75] is not covered by the twisted Dirac operators methods.

This claim was confirmed by Thomas Schick, who defined, in homotopy theoretic terms, integer homology classes in aspherical spaces, say $h \in H_n(\underline{X})$ and who proved using the codimension one descent theorem that these h for $n \leq 7$ can't be dominated by compact orientable n-manifolds with $Sc > 0$.

In more geometric terms, the n-manifolds X, to which Schick's argument applies, we call them *Schoen-Yau-Schick*, can be described d as follows.

A closed orientable n-manifold is *Schoen-Yau-Schick* if it admits a smooth map $f : X \to \mathbb{T}^{n-2}$, such that the homology class of the pullback of a generic point,

$$h = [f^{-1}(t)] \in H_2(X)$$

is *non-spherical,* i.e. it is not in the image of the *Hurewicz homomorphism* $\pi_2(X) \to H_2(X)$.

Then Schick's corollary to Schoen-Yau's theorem reads.

$\bigstar_{SYS}$ **Non-existence Theorem for SYS Manifolds.** *Schoen-Yau-Schick manifolds of dimensions* $n \leq 7$ *admit no metrics with* $Sc > 0$.

(b) *Exercises.* (b_1) Construct examples of SYS manifolds of dimension $n \geq 4$, where all maps $X \to \mathbb{T}^n$ have zero degrees.

Hint: apply surgery to $\mathbb{T}^n$.

(b_2) Show that if the first homology group $H_1(X)$ of a SYS-manifold has no torsion, then a finite covering of X admits a map with degree one to the torus $\mathbb{T}^n$.

(c) The limitation $n \leq 7$ of the above argument is due a presence of singularities of minimal subvarieties in X for $dim(X) \geq 8$.

If $n = 8$, these singularities were proven to be unstable by Nathan Smale; this improves $n \leq 7$ to $n \leq 8$ in $\bigstar_{SYS}$

More recently, as we mentioned earlier, the dimension restriction was removed for all n by Lohkamp and by Schoen-Yau; the arguments in both papers are difficult and I have not mastered them.[76]

Although the Dirac operator arguments don't apply to SYS-manifolds, they do deliver topological obstructions to $Sc > 0$, which, according to the present

[75] A smooth connected n-manifolds X is *spin* if the frame bundle over X admits a double cover extending the natural double cover of a fiber, where such a fiber is equal to the linear group, (each of the two connected components of) which admits a a unique non-trivial double cover $\tilde{G}L(n) \to GL(n)$.

[76] See [Smale(generic regularity) 2003], SY(singularities) 2017], [Lohkamp(smoothing) 2018] and section 3.7.1.

state of knowledge, lie beyond the range of the minimal surface techniques. Here is an instance of this.

$\otimes_{\wedge^k\tilde\omega}$ **Quasisymplectic Non-Existence Theorem.** Let X be a compact $\otimes_{\wedge^k\tilde\omega}$-*manifold* of dimension $n = 2k$, i.e. X is orientable and it carries a *closed 2-form* ω (e.g. a symplectic one), such that $\int_X \omega^k \neq 0$, and such that the lift $\tilde\omega$ of ω to the universal covering $\tilde X$ is *exact*, e.g. $\tilde X$ is contractible.[77]

Then X *admits no metric with* $Sc > 0$.

This applies, for instance, to *even dimensional tori*, to *aspherical* 4-*manifolds with* $H^2(X,\mathbb{R}) \neq 0$ and to *products* of such manifolds[78] but *not* to general SYS-manifolds.

Idea of the Proof. Assume without loss of generality that ω serves as the curvature form of a complex line bundle$L \to X$and let $\tilde L \to \tilde X$ be the lift of L to the universal covering $\tilde X \to X$.

Since the curvature $\tilde\omega$ of $\tilde L$, is exact the bundle $\tilde L$is topologically trivial, hence it can be represented by k-th tensorial power of another line bundle,

$$L = (L^{\frac{1}{k}})^{\otimes k},$$

where the curvature of $L^{\frac{1}{k}}$ is $\frac{1}{k}\tilde\omega$. By Atiyah's L_2-*index theorem*, there are *non-zero harmonic* L_2-*spinors* on $\tilde X$ *twisted with* $L^{\frac{1}{k}}$ for infinitely many k, but the twisted Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) formula applied to large k doesn't allow such spinors for $Sc(\tilde X \geq \sigma > 0$.[79]

Exercise. Show that if X is $\otimes_{\wedge^k\tilde\omega}$, then the classifying map $X \to \mathsf{B}(\Pi)$, where $\mathsf{B}(\Pi) = K(\Pi, 1)$ is the classifying space for the group $\Pi = \pi_1(X)$, sends the fundamental homology class $[X]$ to a *non-torsion* class in $H_n(\mathsf{B}(\Pi))$.

Problem. *Is there a unified approach that would apply to SYS-manifolds and to the above $\otimes_{\wedge^k\tilde\omega}$-manifolds X, e.g. symplectic ones with contractible universal coverings?*

For instance,

do products of SYS and $\otimes_{\wedge^k\tilde\omega}$-manifolds ever carry metrics with positive scalar curvatures?

2.8 Warped $\mathbb{T}^{\rtimes}$-Stabilization and Sc-Normalization

Many geometric properties of Riemannian manifolds $X = (X, g)$ implied by the inequality $Sc(g) \geq \sigma$ follow (possibly in a weaker form) from the same inequality for a larger manifold, say X^*, that, topologically, is the product of X with the a torus, $X^* = X \times \mathbb{T}^N$ for some $N = 1, 2, ...$, where the Riemannian metric g^* on X^* is invariant under the action of $\mathbb{T}^N$ and where $X^*/\mathbb{T}^N$ is isometric to X.

[77] It's enough to have $\tilde X$ spin.

[78] Recently, Chodosh and Li proved that

compact aspherical manifolds of dimensions 4 and 5 admit no metrics with positive scalar curvatures. (See [Chodosh-Li(bubbles) 2020], [G(aspherical) 2020] and section3.10.3)

But this remains problematic for products of pairs of aspherical 4-manifolds.

[79] Atiyah's theorem from [Atiyah(L2) 1976] needs a slight adjustment here, since the action of the fundamental group $\Gamma = \pi_1(X)$ on $\tilde X$ doesn't lift to $L^{\frac{1}{k}}$; yet the fundamental group of the (total space) of the unit circle bundle of L does naturally act on $L^{\frac{1}{k}}$. Also, there is no difficulty in extending Lichnerowicz' vanishing argument to the L_2 case, see §$9\frac{1}{8}$ in [G(positive) 1996].

Surface Examples. Let $X = (X, g)$ be a closed surface and g^* be a $\mathbb{T}^N$-invariant metric on $X \times \mathbb{T}^N$, such that

$$(X \times \mathbb{T}^N, g^*)/\mathbb{T}^N = (X, g).$$

(a) ***Sharp Equivariant Area Inequality.*** If $Sc(g^*) \geq \sigma > 0$, then a special case a theorem by Jintian Zhu,[80] says that

the area of X is bounded the same way as it is for $Sc(g) \geq \sigma$,

$$area(X) \leq \frac{8\pi}{\sigma}.$$

Moreover,

the equality holds only if X^* is the isometric *product $X \times \mathbb{T}^N$.*

(b) (Weakened) $\mathbb{T}^*$***-Stable 2d Bonnet-Myers Diameter Inequality.*** If $Sc(g^*) \geq \sigma$, then

[*BMD*] $$diam(X) \leq 2\pi\sqrt{\frac{N+1}{(N+2)\sigma}} < \frac{2\pi}{\sqrt{\sigma}}.$$

Proof. Given two points $x_1, x_2 \in X$, take two small ε-circles Y_{-1} and Y_{+1} around them, let $X_\varepsilon \subset X$ be the band between them and and apply (the relatively elementary $\mathbb{T}^N$-invariant case of) the $\frac{2\pi}{n}$-*Inequality* from section 3.6.[81]

Non Trivial Torus Bundles. The inequality [BMD] is valid for (all) Riemannian $(N+2)$-manifolds X^* with free isometric $\mathbb{T}^N$-actions:

if $Sc(X^*) \geq \sigma > 0$, then $diam(X^*/\mathbb{T}^N) \leq 2\pi\sqrt{(N+1)/(N+2)\sigma}$.

In fact, the above proof applies, since, *topologically*, the part of X^* that lies over the band $X_\varepsilon \subset X$ *is* the product, $X_\varepsilon \times \mathbb{T}^N$.

It is *unclear*, however, if the areas of $X^*/\mathbb{T}^N$ are bounded in terms of $Sc(X^*)$ for all such X^*.

And, as we shall see later, possible non-triviality of torus bundles create complications for other problems with scalar curvature.

General Question. The above examples suggests that quotients X of manifolds X^* with $Sc(X^*) \geq \sigma$ under free isometric actions of tori have *similar* geometric properties to those of manifolds which have $Sc \geq \sigma$ themselves. But it is unclear how far this similarity goes.

Example. let X be a closed surface and $X^{\rtimes} = X \rtimes \mathbb{T}^1$ be a warped product as described below.

Does the inequality $Sc(X^{\rtimes}) \geq 2$ yield an upper bound on *all of geometry* of X? For instance,

is there a bound on the number of unit discs needed to cover X?

(If $Sc(X) \geq 2$, then X admits a *distance decreasing* homeomorphism from the unit sphere S^2, that can be constructed using the family of boundary curves of concentric discs with center at some point in X.)

Warped Products. As far as geometric applications are concerned, the relevant X^* are (iterated) *warped products*, we denote them $X^{\rtimes}$ and call *warped*

[80] See [Zhu(rigidity) 2019] and 5.9, 7.2 for related inequalities.

[81] Also see §2 in [G(inequalities) 2018] and the proof of theorem 10.2 in [GL(complete) 1983].

$\mathbb{T}^N$*-extensions of* X, that are characterized by the existence of *isometric* sections $X \to X^{\rtimes}$ for $X^{\rtimes} \to X = X^{\rtimes}/\mathbb{T}^N$.

Clearly, metrics $g^{\rtimes}$ on these $X^{\rtimes}$ are

$$g^{\rtimes} = g + \varphi_1^2(x)dt_1^2 + \varphi_2^2(x)dt_2^2 + ... + \varphi_N^2(x)dt_N^2$$

for some positive functions φ_i on X.

Among these we distinguish $O(N)$*-invariant* warped extensions, where the $\mathbb{Z}^N$ covering manifolds $\tilde{X}^{\rtimes} = X \times \mathbb{R}^N$, where

$$\tilde{X}^{\rtimes}/\mathbb{Z}^N = X^{\rtimes},$$

are invariant under the action of the orthogonal group $O(N)$. Thus, $\tilde{X}^{\rtimes}$ are acted upon by the full isometry group of $\mathbb{R}^N$, that is $\mathbb{R}^N \rtimes O(N)$.

Equivalently, the metric in such an $X^{\rtimes}$ is a "simple" warped product: $g^{\rtimes} = g + \varphi^2 d||\bar{t}||^2$ for $\bar{t} = (t_1, t_2, ..., t_N)$, the scalar curvature of which, as we know, 2.4 is

$$Sc(g^{\rtimes})(x, \bar{t}) = Sc(X)(x) - \frac{2N}{\varphi(x)}\Delta_g\varphi(x) - \frac{N(N-1)}{\varphi^2(x)}$$

and which is most simple (and useful) for $N = 1$, where

$$[\rtimes_\varphi] \qquad Sc(g^{\rtimes})(x, \bar{t}) = Sc(X)(x) - \frac{2}{\varphi(x)}\Delta_g\varphi(x).$$

for the Laplace (Beltrami) Δ_g on $X = (X, g)$.

$[\rtimes_\varphi]^N$-Symmetrization Theorem. Let $X = (X, g)$ be a closed oriented Riemannian manifold of dimension $n = m + N$ and let

$$X \supset X_{-1} \supset ... \supset X_{-i} \supset ... \supset X_{-N},$$

be a descending chain of closed oriented submanifolds, where each $X_{-i} \subset X$ is equal to a transversal intersection of $X_{-(i-1)}$ with a smooth closed oriented hypersurface $H_i \subset X$,

$$H_i \cap X_{-(i-1)} = X_{-i}.$$

If $n \leq 7$, then

there exists a closed oriented m-dimensional submanifold $Y \subset X$ homologous to X_{-N} and a warped product $\mathbb{T}^N$-extension $Y^{\rtimes}$ of $Y = (Y, h)$ for the Riemannian metric h on Y induced from g on X, such that the scalar curvature of $Y^{\rtimes}$, that is, being $\mathbb{T}^N$-invariant, is represented by a function on Y, is bounded from below by the Scalar curvature of X on $Y \subset X$,

$$Sc(Y^{\rtimes}, y) \geq Sc(X, y),\ \ y \in Y.$$

Proof. Proceed by induction on codimension $i = 1, 2, , ...N$ and construct submanifolds

$$X \supset Y_1 \supset ... \supset Y_i \supset ... \supset Y_N = Y \subset X$$

as follows.

At the first step, let $Y_1 \subset X$ be a volume minimizing, hence stable, hypersurface homologous to X_{-1} where, the positivity of the second variation implies the positivity of the

$$-\Delta + \frac{1}{2}(Sc(Y_1) - Sc(X)|_{Y_1},$$

for the Laplace $\Delta = \Delta_{h_1}$ on Y_1 with the metric h_1 induced from X and let $\psi_1 > 0$ be the first eigenfunction of this with the positive eigenvalue λ_1, thus

$$-\Delta\psi = \left(\lambda - \frac{1}{2}(Sc(Y,h_1) - Sc(X))\right) \cdot \psi_1.$$

Here, let $h_1^{\rtimes}(y) = h_1(y) + \psi^2 dt^2$ be the warped product metric on $Y_1 \times \mathbb{T}^1$ and observe

$$Sc(h_1^{\rtimes}, y) = Sc(h_1, y) - \frac{2}{\psi}\Delta\psi_1 = Sc(X,y) + 2\lambda_1.$$

Then, at the second step, let $Y_2 \subset Y_1$ be a hypersurface, such that $Y_2 \times \mathbb{T}^1 \subset Y_1 \times \mathbb{T}^1$ is volume minimizing for the metric $h_1^{\rtimes}$, which is equivalent for Y_2 to be volume minimizing in Y_1 with respect to the metric $\psi_1^{l_1} h_1$ for $l_1 = \frac{2}{n-1}$.

Thus we obtain Y_2', where the corresponding metric on $Y_2' \times \mathbb{T}^2$ is

$$h_2' + \psi_1^2 dt_1^2 + \psi_2^2 dt_2^2.$$

Repeating this $N-2$ more times, we arrive at Y_N' and an (iterated) warped product metric

$$h_N' + \sum_{i=1}^{N} \psi_i^2 dt_i^2 \text{ on } Y_N' \times \mathbb{T}^N,$$

which can be symmetrised further to the required $h^{\rtimes}$ by applying the above infinitely many times to hypersurfaces $Y_N' \times T^{N-1} \subset Y_N' \times T^N$ for all subtori $T^{N-1} \subset Y_N' \times T^N$.[82] (The luxury of the extra $O(N)$-symmetry is unneeded for most purposes.)

Exercise. Apply $[\rtimes_\varphi]^N$-symmetrization to n-manifolds with isometric $\mathbb{T}^{n-2}$-actions and prove the above equivariant area inequality by reducing it to the warped product case that was already settled in section 2.4.1.

Symmetrization by Reflections and Convergence Problem. Let Y be a closed minimal co-orientable (i.e. two sided) hypersurface in a Riemannian manifold. If Y is locally volume minimizing, then it admits arbitrarily small neighbourhoods $V_\varepsilon \supset Y$ in X with smooth *strictly mean convex* boundaries. Then by reflecting such a varepsilon in the two boundary components, one obtains manifolds $\hat{V}_\varepsilon$ with isometric actions of $\mathbb{Z} \rtimes \mathbb{Z}_2$.

If these Y are non-singular, e.g. if $dim(X) \leq 7$, then one can take solutions of the isoperimetric problem for these V_ε, where one minimize the volumes of both components of the boundaries of V_ε per given (small) volume contained between them and Y. In this case, $\hat{V}_\varepsilon$, $\varepsilon \to 0$, converge to smooth Riemannian manifolds $V^{\rtimes}$ with isometric actions of $\mathbb{R}$ and with their scalar curvatures bounded from below by $Sc(X)|_Y$.

If Y is singular, the boundaries of these V_ε, even if singular, [83] can be smoothed with positive mean curvatures, but it is unclear if they converge to a reasonable object for $\varepsilon \to 0$: what is *missing for convergence* is a *Harnack type inequality* for the boundary components of $\partial_1, \partial_2 \subset \partial V_\varepsilon$, that is a uniform bound for the ratios of the distances

$$\frac{dist(y, \partial_i)}{dist(y', \partial_i)}, y, y' \in Y,$$

[82] See in, §12[GL(complete)1983], [G(inequalities) 2018] and also the sections 3.7, 5.4 for details of this argument and for generalizations.

[83] If $n = 8$, then, by adapting Nathan Smale's argument, one can show that these V_ε are non-singular for an open dense set of values of ε; but this is problematic for $n \geq 9$.

$i = 1, 2$, and /or of distances $dist(x, x', Y)$, $x, x' \in \partial_i$.

Notice, that "symmetrization by reflections", albeit open to generalizations to singular Y, is not, apparently, applicable, to stable μ-bubbles Y, where the warped product construction does apply. [84]

Symmetrization versus Normalization. $\mathbb{T}^{\rtimes}$-Symmetrization of metrics g typically) makes their scalar curvatures *constant* by paying the price of modification of the topology of the underlying manifolds, $X \rightsquigarrow X \times \mathbb{T}^1$.

As far as sets of "interesting" maps between Riemannian manifolds are concerned a similar effect effect is achieved by keeping the same manifold X but modifying the metric by $g = g(x) \rightsquigarrow g^\circ = g^\circ(x) = Sc(X, x) g(x)$.

In fact, we shall see later in many examples, that

there is a close (but not fully understood) similarity between the sets of λ°-*Lipschitz* maps $(X, g^\circ) \to (Y, h^\circ)$ and of $\mathbb{T}^1$-*equivariant* $\lambda^{\rtimes}$-*Lipschitz* maps $(X \times \mathbb{T}^1, g^{\rtimes}) \to (Y \times \mathbb{T}^1, h^{\rtimes})$ for λ° and $\lambda^{\rtimes}$ related in a certain way.

2.9 Positive Eigenfunctions and the Maximum Principle

Let X be a *compact connected* Riemannian manifold and let

$$\Delta f = \sum_i \nabla_{ii} f = \mathrm{trace\,Hess} f = \mathrm{div\,grad} f$$

denote the Laplace (Beltrami) on X, which, recall, is a *negative* , since

$$\int_X \langle f, \Delta f \rangle dx = -\int_X ||\mathrm{grad} f||^2 dx \le 0$$

by Green's formula.

Non-Vanishing Theorem. Let $s(x)$ be a smooth function, such that the

$$L = L_s : f(x) \mapsto -\Delta f(x) + s(x) f(x)$$

is *non-negative*, that is $\int_X \langle f(x), L f(x) \rangle dx \ge 0$ for all f or, equivalently, if L the lowest eigenvalue $\lambda = \lambda_{min}$ is ≥ 0.[85]

Then

the eigenfunction $f(x)$ associated with λ doesn't vanish anywhere on X.

Start with two lemmas.

1. C^1-*Lemma.* If the minimal eigenvalue of the $f(x) \mapsto L f(x) = -\Delta f(x) + s(x) f(x)$ on a compact Riemannian manifold is *non-negative*, $\lambda = \lambda_{min} \ge 0$, then the *absolute value* $|f(x)|$ of the eigenfunction f associated with λ is C^1-*smooth.*

2. Δ-*Lemma.* Let $f(x)$ be a *non-negative* continuous function on a Riemannian manifold, such that

(i) $f(x)$ *vanishes* at some point in X,

$$f(x_0) = 0,\ x_0 \in X,$$

(ii) $f(x)$ is *not identically zero* in any neighbourhood of the point $x_0 \in X$,

(iii) $f(x)$ is everywhere C^1-*smooth* and it is C^2-*smooth* at the points x where

[84] See §8 in [G(billiards) 2014], §4.3 in [G(inequalities) 2019] and section 5.1 for more about all this.

[85] This is equivalent since our L has *discrete spectrum.*

it doesn't vanish.

Then there exists a sequence of points $x_1, x_2, \ldots \in X$ convergent to x_0, *where* $f(x_i) > 0$ *and such that*

$$\frac{\Delta f(x_i)}{f(x_i)} \to \infty, \text{ for } i \to \infty.$$

Derivation of Non-vanishing Theorem from the Lemmas. Since $|f|$ is C^1 by the first lemma, the Δ-lemma, applied to $|f(x)|$, shows that there exists a point x, where $f(x) \neq 0$ and

$$\frac{\Delta f(x)}{f(x)} = \frac{\Delta |f(x)|}{|f(x)|} > |s(x)|,$$

that is incompatible with $-\Delta f(x) + s(x)f(x) = \lambda f(x) \geq 0$ for $\lambda \geq 0$.

Proof of C^1-Lemma. Recall that the eigenvalues of the $L = L_s = -\Delta + s$ are equal to the critical values of *the energy functional*

$$E(f) = \int_X (\|\mathsf{grad} f(x)\|^2 + s(x))f^2(x)dx$$

on the sphere

$$\|f\|^2 = \int_X f^2(x)dx = 1$$

in the Hilbert space $L_2(X)$ and the critical points of E are represented by eigenfunctions

Indeed,

$$E(f) = \langle f, Lf \rangle = \int_X \langle f(x), Lf(x) \rangle dx$$

by Green's formula and the differential of the quadratic function $f \mapsto \langle f, Lf \rangle$ on the sphere $\|f\|^2 = 1$ is

$$(dE)_f(\tau) = \langle \tau, Lf \rangle \text{ for all for all } \tau \textit{ normal to } f.$$

Thus, vanishing of dE at f on the unit sphere says, in effect, that Lf is a multiple of f, i.e. $Lf = \lambda f$.

All this makes sense in the present case, albeit the space $L_2(X)$ is *infinite dimensional* and L an *unbounded*, because L is an *elliptic* operator, which implies, for compact X, that

the spectrum of L is discrete, bounded from below and all eigenfunctions are smooth.

In particular – this is all we need,

all minimizes of $E(f)$ on the unit sphere, that are, a priori, only Lipschitz continuous, are smooth.[86]

Now, observe that,

taking absolute values of smooth functions $f(x) \mapsto |f(x)|$ doesn't change their energies, as well as their L_2-norms,

$$\||f|\| = \|f\| = \sqrt{\int_X |f|^2(x)dx},$$

[86]Recall that our "smooth" means C^∞ and all our Riemannian manifolds are assumed smooth.

$$E(|f|) = E(f) = \int_X (||\mathsf{grad}|f|(x)||^2 + s(x))|f|^2(x)dx,$$

Indeed, absolute values $|f|(x)$ are Lipschitz for Lipschitz f, hence, they are almost everywhere differentiable functions, such that $\mathsf{grad}|f|(x) = \pm\mathsf{grad}f(x)$ at all differentiability points x of $|f|$.

It follows that the absolute value of the eigenfunction f with the smallest energy $E(f) = \lambda_{min}$ is also a minimizer; hence, *this* $|f|$ *is smooth.* QED.

Poof of Δ*-Lemma.* The common strategy for locating points $x \in X$ with "sufficiently positive" second differential of a function $f(x)$ is by using simple auxiliary functions $e(x)$ with this property and looking for minima points for $f(x) - e(x)$.

The basic example of such a function $e(x)$ in one variable is e^{-Cx}, $x > 0$, for large C, where $\frac{e''}{e} = C^2$, and where observe that the ratio $\frac{e''}{e'} = C$ also becomes large for large C.

It follows that that the Laplacians of the corresponding radial functions in small R-ball $B_y(R)$ in Riemannian manifolds X,

$$e(x) = e_C(x) = e_{y,C}(x) = e^{-C\cdot r_y(x)} \text{ for } r_y(x) = dist(y,x) \leq R$$

satisfy

$$\Delta e(x) \geq C^2 e(x) - C \cdot mean.curv(\partial B_y(r), x) \text{ for } r = r_y(x) = dist(y,x)$$

Now, in order to find a point x close to a given $x_0 \in X$ where $f(x) = 0$, take $y \in X$ very close to x_0, where $f(y) > 0$, let $B_y(R) \subset X$ be the *maximal* ball, such that $f(x) > 0$ in its interior, let

$$e(x) = e_C(x) = e^{-C\cdot r_y(x)} - e^{-C\cdot R}$$

and observe that $e(x)$ vanishes on the boundary of the ball $B_y(R)$ and is strictly positive in the interior. Moreover

$$e(x) \geq \varepsilon\rho,$$

for all x on the geodesic segment between y and x_0 within distance $\geq \rho$ from x_0 for all $\rho_0 \leq R$.

Notice that this $\varepsilon = \varepsilon_C$ albeit *strictly positive*, tends to zero for $C \to \infty$.

Assume without loss of generality that x_0 is the only point in $B_x(R)$ where $f(x)$ vanishes (if not, move y closer to x_0 along the geodesic segment between the two points), let C be *very very large* and see what happens to $f(x)$ and $e(x)$ in the vicinity of $x_0 \in \partial B_y(R)$, say in the intersection

$$U_0 = B_y(R) \cap B_{x_0}(R/3).$$

Observe the following.

• Since $f(x) > 0$ for $x \in B_y(R)$, $x \neq x_0$, and since $e_C(x) \to 0$ for $C \to \infty$ for $r_y(x) = dist(y,x) \geq r_0 > 0$, *the function* $e(x) = e_C(x)$*, for large* C*, is bounded by* $f(x)$ *on the boundary of* U_0,

$$e(x) \leq f(x),\ x \in \partial U_0,$$

where $e(x) < f(x)$ *unless* $x = x_0$.

• Since f is differentiable at x_0 and assumes minimum at this point, the differential df vanishes at x_0, which makes $f(x) = o(\rho)$ for $\rho = dist(x, x_0)$, *there is a part of (the interior of) U_0, where $e(x) > f(x)$.*

Hence, the difference $f(x) - e(x)$ assumes minimum at an interior point $x = x_{y,C} \in U_0$, such that $x = x_{y,C} \to x_0$ for $C \to \infty$ and

$$\frac{\Delta f(x)}{f(x)} \geq \frac{\Delta e(x)}{e(x)} \to \infty.$$

The proof of the Δ-lemma and of the non-vanishing theorem are thus concluded.

Discussion. The non-vanishing theorem, which, probably, goes back to Rayleigh, is often used without being even explicitly stated as, for instance, by Kazdan and Warner in their "conformal change" paper. But I couldn't find an explicit reference on the web, except for the paper by Doris Fischer-Colbrie and Rick Schoen, where they prove such a non-vanishing for non-compact manifolds needed for their

non-existence theorem for non-planar stable minimal surfaces in $\mathbb{R}^3$.

Their argument relies on the *"strong maximum principle"* for the L, for which they refer to pp. 33-34 of the canonical Gilbarg-Trudinger textbook, where the relevant case of this principle is stated (on p. 35 in the 1998 edition which is available on line) after the proof of theorem 3.5 as follows.

"Also, if $u = 0$ at an interior maximum (minimum), then it follows from the proof of the theorem that $u = 0$, irrespective of the sign of c."

(The assumptions of the theorem specifically rule out c with variable signs, where this $c = c(x)$ is the coefficient at the lowest term in the equation $Lu = a^{ij}(x)D_{ij}u + b^i D_i u + c(x)u = 0$ introduced on p. 30.)

What is actually proven in this book on about twenty lines on p. 34, is a version of "Δ-lemma" for L.

In our proof, we reproduce what is written on these lines, except for "*direct calculation gives*" that is replaced by an explicit evaluation of $\Delta e(x)$ [87]

The following (obvious) corollary to the non-vanishing theorem will be used for construction of stable symmetric μ-bubbles in sections 5.2, 5.4.

Uniqueness/Symmetry Corollary. *If X is compact connected, then the lowest eigenfunction f of the L is unique up to scaling. Consequently, if L is invariant under an action of an isometry group on X, then, even if X is disconnected, there exists a positive f invariant under this action.*

[87] In truth, the only non-evident aspect of the argument resides with the identities $(e^{-Cx})' = -Ce^{-Cx}$ and $(e^{-Cx})'' = (-Ce^{-Cx})' = C^2e^{-Cx}$ with the issuing inequalities $(e^{-Cx})'' >> e^{-Cx}$ and $(e^{-Cx})'' >> |(e^{-Cx})'|$, which *can't be done by just staring* at the exponential function. (The appearance of e^x, that is an isomorphism between the *additive* $\mathbb{R}$ and *multiplicative* $\mathbb{R}^\times_+$ with all its counterintuitive properties, is amazing here – there is nothing visibly multiplicative in Δ; besides, the geometric proof of the existence of e^x via the *conformal* infinite cyclic covering map $\mathbb{C} \to \mathbb{C} \setminus \{0\}$ and analytic continuation is non-trivial.)

The rest of the proof is geometrically effortless: you *just look* at the graph Γ_e of the function $e(x) = \exp -C \cdot dist(y, x)$ in a small R-ball $B \subset X$ outside zero set of f with the center of your choice, such that B touches this set at x_0, and let $C = C_i \to \infty$. Then you *see* a tiny region in this ball close to x_0, where Γ_e mounts above Γ_f, and you take the point in X just under the *top* of this mountain, i.e. where the distance measured vertically between the two graphs is maximal, for you $x = x_i$.

Exercises. (a) **Multi-Dimensional Morse Lemma.** Show that two *non-coinciding* volume minimizing hypersurfaces in the same indivisible homology integer homology class of an orientable manifold X have *empty intersection and that, consequently, volume minimizing hypersurfaces must be invariant under symmetries of* X.[88]

(b) Generalize this to μ-bubbles, that are boundaries of domains V in a Riemannian manifold X that *minimize* the functional

$$V \to vol_{n-1}(\partial V) - \int_V \mu(x)dx$$

for a smooth function $\mu(x)$. (Unit spheres $S^{n-1}\mathbb{R}^n$ are *not minimizing* μ-bubbles for $\mu = (n-1)dx$.)

(b) **Courant's Nodal Theorem.** Show that the that is the number of connected components of the complement to the "k-th nodal set", i.e. the zero set of the k-th eigenfunction of $L = L_s = \Delta + s$ on a compact connected manifold, can't have more than k connected components.

Question. Is there a counterpart to this for non-quadratic functionals in spaces of functions, or, even better, spaces of hypersurfaces?

3 Topics, Results, Problems

3.1 Scale Persistent Criteria for $Sc \geq \sigma$ for Smooth and non-Smooth Metrics

Scale persistence of a geometric property P applicable to compact n-dimensional Riemannian manifolds V with boundaries, means the following:

if such a P is satisfied by small neighbourhoods of all points in a Riemannian n-manifold X then it is satisfied for all domains V in X.

Two classical examples of these are the following characteristic properties of surfaces with *non-negative sectional curvatures* κ and of n-dimensional manifolds with *non-negative Ricci curvatures.*

In the case of the sectional curvature we formulate such a property as a *comparison inequality* for *geodesic quadrilaterals* as follows.

$\square_{\kappa\geq 0}$ *All convex, e.g. geodesic, quadrilaterals in surfaces with non-negative curvatures, call them* $\square \subset X$, *have the greatest of their four angles at least* $90°$:

$$\max_{i=1,\dots,4} \angle_i(\square) \geq \frac{\pi}{2}.$$

In fact, the sum of the four angles of such a $\square$ must be $\geq 2\pi$ by the *Gauss-Bonnet inequality* for compact surfaces V with (quadrilateral in the present case) boundaries Θ:

if $\kappa(V) \geq 0$, *then*

$$\int_\Theta curv(\Theta, \theta)d\theta \leq 2\pi.$$

[88] This was used by Marston Morse to show that

if the $(n-1)$-dimensional homology group of some covering of a compact Riemannian n-manifold, doesn't vanish then the universal covering $\tilde{X}$ of X contains an infinite minimal hypersurface the image of which under the covering map $\tilde{X} \to X$ is compact.

Morse was concerned in his paper "*Recurrent Geodesics on a Surface of Negative Curvature*" with the case of $n = 2$ but his argument, transplanted to the environment of the geometric measure theory, applies to manifolds of all dimensions n.

(The curvature of $\Theta = \partial V$ at a vertex of V with the angle α is the point-measure with the weight $\pi - \alpha$.)

It is also clear that $\square_{\kappa\geq 0}$ is *sufficient*, as well as necessary, for $\kappa \geq 0$:

if $\kappa(X,x) < 0$, then there exist (small) geodesic quadrilaterals in X around x with all angles $< \frac{\pi}{2}$.

Thus,

local validity of $\square_{\kappa\geq 0}$ implies the global one.

(Also notice that if all four angles of a convex $\square$ with $\kappa(\square) \geq 0$ are $\leq \frac{\pi}{2}$, then this $\square$ is isometric to a plane Euclidean rectangular.)

Next, turning to Ricci, observe that the inequality $\bigcirc_{Ricci\geq 0}$ stated below says, in effect, that the mean curvatures of the boundaries of compact manifolds with $Ricci \geq 0$ can't be greater than these of Euclidean balls of comparable size.

$\mathsf{P}_{Ricci\geq 0}$. If $Ricci(V) \geq 0$, then the minimum of the *mean curvature* of the boundary of V is related to the *inradius* of V by the inequality

$$\inf_{v\in\partial V} mean.curv(\partial V, v) \leq \frac{n-1}{inrad(V)},\ n = dim(V),$$

where

$$inrad(V) = \sup_{v\in V} dist(v, \partial V),$$

where our sign convention for the mean curvature is such that convex domains have $mean.curv \geq 0$ and where

the (opposite) inequality

$$mean.curv(\partial V, v) \geq \frac{1}{inrad(V)},$$

implies that V is isometric to the Euclidean ball of radius $R = inrad(V)$.

All this follows from Hermann Weyl's tube formula applied to concentric spheres in V around the point $v \in V$ farthest from the boundary.

Let us now state two inequalities that characterize n-manifolds X with non-negative scalar curvatures, where

the first one says that *cubical domains $V \subset X$ can't be "more mean convex", than rectangular solids in the Euclidean space*,

and the second one, that applies to domains $V \subset X$ with smooth boundaries, claims that

these boundaries *can't be simultaneously greater in size and "more mean convex" than convex hypersurfaces in the Euclidean space.*

I. $\blacksquare$-Inequality. Let V be a Riemannian n-manifold diffeomorphic to the cube $[0,1]^n$.

If $Sc(V) \geq 0$ and if all $(n-1)$ faces $\partial_i \subset \partial V$, $i = 1, ..., 2n$, have $mean.curv(\partial_i) \geq 0$, then the supremum of the dihedral angles between the (tangent spaces of) $(n-1)$-faces at the points in the $(n-2)$ faces satisfy

$$\blacksquare \qquad \sup_{i,j} \angle_{ij}(V) \geq \frac{\pi}{2}.$$

This may serve as a criterion for $Sc(X) \geq 0$, since

(□) *the inequality* $Sc(X,x) < 0$ *implies the existence of a small (topologically) cubical mean convex neighbourhood* $V \subset X$ *of* x *which violates* ■:

all dihedral angles of V *are everywhere* $> \frac{\pi}{2}$.

II. ⬬ **-Inequality.** Let V be a Riemannian manifold diffeomorphic to the n-ball the boundary of V of which has positive mean curvature,

$$mean.curv(\partial V) > 0,$$

let $\underline{V} \subset \mathbb{R}^n$ be a convex domain with smooth boundary and let

$$f : \partial V \to \partial \underline{V}$$

be a diffeomorphism.[89]

If $Sc(V) \geq 0$, *then the differential of* f *can't be everywhere strictly smaller than the ratio of the mean curvatures of the two boundaries: there exists a point* $v \in \partial V$, *such that*

⬬$_{\geq}$
$$||df(v)|| \geq \frac{mean.curv(\partial V, f(v))}{mean.curv(\partial \underline{V}, v)}.$$

(⬭$_{<}$) Conversely, *the inequality* $Sc(X,x) < 0$ *implies the existence of* $V \subset X$, $\underline{V} \subset \mathbb{R}^n$ *and of a diffeomorphism* $f : \partial V \to \partial \underline{V}$, *such that*

$$||df(x)|| < \frac{mean.curv(\partial V, f(v))}{mean.curv(\partial \underline{V}, v)} \text{ for all } v \in \partial V.$$

We indicate the proofs of ■ and ⬬ in the next section, and refer to section 3.4 for a generalization of ⬬ to topologically non-trivial manifolds V; below, we turn to manifolds with $Sc \geq \sigma \neq 0$.

Corollaries of **I** and **II** for *manifolds* X *with* $Sc(X) \geq \sigma$ *for* $\sigma \gtrless 0$. The inequalities ■ and ⬬, when applied to manifolds X multiplied by surfaces S with scalar curvatures $-\sigma$, yield

geometric criteria for $Sc(X) \geq \sigma$ *for all* σ.

The geometric meaning of this, if any, is obscure; possibly, it can be expressed in terms of 2-parametric families of domains V_s, $s \in S$. But the following generalizations of ⬬ to $\sigma > 0$ and of ■ to $\sigma < 0$ are geometrically transparent.

⬬**-Comparison Theorem for Sc > 0.** Let V and $\underline{V}$ be compact Riemannian n-manifolds with smooth boundaries, where $\underline{V}$ has constant sectional curvature $+1$ and the boundary $\partial \underline{V}$ is convex and and let $f : V \to \underline{V}$ be a diffeomorphism.

Then either

there exists a point $v \in V$, *where the norm of the exterior square of the differentials of* f *is bounded from below by*

$$|| \wedge^2 df(v)|| \geq \frac{Sc(V,v)}{n(n-1)}$$

or, as earlier,

[89] It is enough to assume that f is a smooth map with *positive degree* as it will become clear later on.

there exists a point $v' \in \partial V$, such that

⬬$'_{>0}$ $$\|df(v')\| \geq \frac{mean.curv(\partial V, f(v'))}{mean.curv(\partial \underline{V}, v')}.$$

■-Comparison Theorem for $\mathbf{Sc(V)} > \sigma < \mathbf{0}$. Let

$$(\mathbf{H}^n, g_{hyp}) = \mathbb{R}^1 \rtimes \mathbb{R}^{n-1} = (\mathbb{R}^1 \times \mathbb{R}^{n-1}, dt^2 + e^{2t}dx^2)$$

be the hyperbolic space with sectional curvature -1 represented as the warped product in the normal horospherical coordinates, let

$$\underline{V} = [0,1] \times [0,1]^{n-1} \subset \mathbb{R}^1 \rtimes \mathbb{R}^{n-1} = \mathbf{H}^n$$

and observe that all dihedral angles in $\underline{V}$ are $\frac{\pi}{2}$, all "side faces" are geodesic flat, while the "bottom" $\{0\} \times [0,1]^{n-1} \subset \underline{V}$ and the "top" $\{1\} \times [0,1]^{n-1} \subset \underline{V}$, have mean curvatures $-(n-1)$ and $n-1$ respectively.

The corresponding comparison inequality for cubical Riemannian manifolds V diffeomorphic to $[0,1] \times [0,1]^{n-1}$ reads.

Let all dihedral angles of V be $\leq \frac{\pi}{2}$, let all ("side") faces $\partial_i \subset V$, except for $\partial_0 = \{0\} \times [0,1]^{n-1}$ and $\partial_1 = \{1\} \times [0,1]^{n-1}$, have non-negative mean curvatures and let

$$mean.curv(\partial_0) \geq -(n-1) \text{ and } mean.curv(\partial_1) \geq n-1.$$

Then the scalar curvature of V can't be everywhere greater than that of $\mathbf{H}^n$,

■$'_{<0}$ $$\inf_{v \in V} Sc(V, v) \leq -n(n-1).$$

Remarks. (a)The *proofs* of these are indicated in the sections 3.1.1 below.

(b) Probbaly – figuring this out this way or another can't be too difficult – these ⬬$'_{>0}$ and ■$'_{<0}$ *characterizes* $Sc \geq \pm 1$.

(c) Granted (b), either of ■$'_{<0}$ or ⬬$'_{<0}$ can be used for characterization of $Sc \geq \sigma$ for all σ by passing to products of X with S^2 or $\mathbb{H}^2$ as we did earlier.

(c) The proof of ■$'_{<0}$ for $n \geq 9$, which relies on stable μ-bubbles, needs (a slight generalization of) the desingularization theorem from [SY(singularities) 2017] or of such a result from [Lohkamp(smoothing) 2018].

(d) Probably, a combination of ideas from [Min-Oo(hyperbolic) 1989] and from recent papers by Cecchini, Zeidler, Lott and Guo-Xie-Yu on index theorems for manifolds with boundaries[90] may provide an alternative proof of ■$'_{<0}$ for all n.

■ *And* ⬬ *for Continuous Riemannian Metrics.* One can define mean convexity and, more generally, lower bound of the mean curvatures from below for boundaries ∂V of domains V in a metric space X, whenever one has a notion of the volume/measure for ∂V as follows.

$mean.curv(\partial V, v) > m$, $v \in \partial V$, if there exists a sequence of subdomains $V_i \subset V$ with the following properties.

(i) The difference between V and V_i contains a neighbourhood v in V for all i and it converges to v for $i \to \infty$, i.e. $V \setminus V_i$ is contained in the δ_i-ball around v for $\delta_i \to 0$.

[90]See [Cecchini-Zeidler(Scalar&mean) 2021], [Lott(boundary) 2020], [Guo-Xie-Yu(quantitative K-theory) 2020.

(ii) the volume of ∂V_i is bounded in terms of the volume of the part of ∂V outside V_i and the Hausdorff distance between the boundaries of V and ∂V_i as follows:

$$vol(\partial V_i) < vol(\partial V) - m \cdot vol(\partial V \smallsetminus V_i) \cdot dist_{Hau}(\partial V, \partial V_i).$$

With this "mean curvature", the definitions of ■ and ⬬ as well as $■_{<0}$ and $⬬_{>0}$ automatically extend to continuous, even only Borel, Riemannian metrics.

Question. Do ■ and ⬬ define the same concept of $Sc(g) \geq 0$ for *continuous* Riemannian metrics g?

3.1.1 Reflection Orbihedra and Trapped Minimal Hypersurfaces

$(1_{\geq 0})$ Idea of the Proof of ■. Reflect $V = (V, g)$ as a cube in $\mathbb{R}^n$ in the $(n-1)$-faces, let $\hat{V}$ be the resulting *universal orbi-covering* manifold with an action of the relection group Γ that is isomorphic of a finite extension the group $\mathbb{Z}^n$ for cubical V, and let $\hat{g}$ be the (singular path) metric on $\hat{V}$ that Γ-invariantly extends g from V naturally embedded to $\hat{V}$.

If the mean curvatures of all codimension one faces are ≥ 0, all dihedral angles of V are $\leq \frac{\pi}{2}$ and the dihedral angle at a point v on some codimension two face of V satisfies the strict inequality $\angle_{ij}(v), \frac{\pi}{2}$, then one can *approximate* $\hat{g}$ *by smooth* Γ*-invariant metrics* $\tilde{g}$ for, such that $Sc(\tilde{g}) > \sigma$ for $\sigma = \inf_{v \in V}(Sc(V, v))$.[91]

Thus, if we assumed that $Sc(V) \geq 0$, this $\tilde{g}$ would descend to a metric on the torus $\hat{V}/\mathbb{Z}^n$ with $Sc > 0$ and the proof of ■ follows by contradiction: *there is no metrics with positive scalar curvatures on the torus.*

$(2_{\leq 0}.)$ *About* $■_{<0}$. Here we reflect V around the $2(n-1)$ "side faces" and, thus, after smoothing, reduce $■_{<0}$ to the comparison inequality for hyperbolic cusps $\mathbb{R}^1 \rtimes \mathbb{T}^{n-1} = \mathbf{H}^n/\mathbb{Z}^{n-1}$. [92]

The available proofs of this inequality, apply only for $n = dim(X) \geq 3$, while the case $n = 2$ follows by a simple argument that we suggest as an elementary exercise to the reader.

Remark. The proof of the comparison inequality for hyperbolic cusps relies on stable μ-bubbles $\hat{Y}$ between pairs of $(n-1)$ tori in the cusps $\mathbb{R}^1 \rtimes \mathbb{T}^{n-1}$ where $mean.curv(\hat{Y}) \geq n-1$.

This suggests a similar proof directly in V with relevant μ-bubbles $Y \subset V$ having free boundaries in the *side boundary* of V that is in the union of the "side" faces i.e. all, except for the two corresponding to the bottom $\{0\} \times [0,1]^{n-1}$ and the top of the cube $\{1\} \times [0,1]^{n-1}$.

But since the mean curvatures of the side faces are only assumed *non-negative*, such a bubble with mean curvature $\geq n-1$ may meet a side face in the interior and render the argument invalid.

It is unclear how to formulate the existence of needed $Y \subset V$ without direct appeal to reflections in the "side mirrors".

(3) Other Reflection Groups. The above construction applies to all *Riemannian reflection orbifolds*, that are manifolds (V, g_0) with corners that serve

[91] Working this out in detail requires some patience, see [G(billiards) 2014]) and [Nuchi(cube) 2018].

[92] See §$5\frac{5}{6}$ in [G(positive) 1996], §9 in [G(inequalities) 2018].

as fundamental domains Δ of reflection groups Γ, that act on the corresponding orbi-covering manifolds $\hat{V}$ as follows.

Let g be a Riemannian metric on such a V, which satisfies the following $2\frac{1}{2}$ conditions.

$\bullet_1$ the codimension 1 faces ∂_i of V are g-mean convex:

$$mean.curv_g(\partial_i V) \geq 0;$$

$\bullet_4$ the g-dihedral angles $\angle_{ij}$ of V at the codimension 2 faces of V are bounded by the corresponding g_0-angles of V,

$$\angle_{ij}(V,g) \leq \angle_{ij}(V,g_0);^{93}$$

$\bullet_2^<$ there is a point v on some codimension 2 face of V, where the above inequality is strict,

$$\angle_{ij}(V,g)(v) < \angle_{ij}(V).$$

Then, as earlier for the cubical V, one can show, that

the natural singular metric $\hat{g}$ on $\hat{V}$ can be approximated by smooth $\tilde{g}$ with $Sc(\tilde{g}) > \inf_v Sc(V,g)(v)$.

About Examples. There are few $V \subset \mathbb{R}^n$ and Euclidean reflection groups, to which the above applies. In fact, all such V are the products of segments and triangles with the angles $(60°, 60°, 60°)$, $(60°, 30°, 90°)$ and $(45°, 45°, 90°)$.

But there are lots of non-Euclidean orbifold V, e.g. with right-angled corners, (see [Davis(orbifolds) 2008]), the universal orbi-coverings $\hat{V}$ of which are *hyper-Euclidean* and, hence, admit no Γ-invariant metrics with $Sc > 0$ (see sections 1.6.3, 3.3). Therefore,

the conditions $\bullet_1$,$\bullet_2$ and $\bullet_2^<$ imply that $\inf_{v\in V} Sc(g,v) \leq 0$ for these V.

But, in general, the following problem, solutions of special cases of which are spread throughout this paper, remains widely open,

⬠-Problem. Let V be a compact manifold with corners, e.g. a closed manifold, or, at the other end of the spectrum, diffeomorphic to a convex polyhedron in the Euclidean space. Find necessary (ideally, necessary and sufficient) conditions on V for the existence of a Riemannian metric g on V, such that:

(i) the scalar curvature is bounded from below by a given real number, $Sc \geq \sigma$,

(ii) the mean curvatures of the codimension 1 faces, call them V_i, are are similarly bounded from below, $mean.curv_g(V_i) \geq \mu_i$

(iii) the dihedral angles of all codimension 2 faces are bounded by given numbers, say, $\angle_{ij} \leq \alpha_{ij}$.

The above ■-comparison theorem provides an instance of such a condition with $\sigma \leq 0$ (this, moreover, characterizes metrics with $Sc \geq \sigma$), but the inequality, ⬬$'_{>0}$ for $\sigma > 0$, unlike ■ involve the distance geometry of V.

It $n = 2$, then it is not hard to show (an exercise to the reader) that

if $\sigma \geq 2$ then no k-gonal Riemannian (surface) V may have the faces (edges) with curvatures $\geq \mu \geq -\varepsilon$ and the angles $\leq \alpha \leq \varepsilon$ for a sufficiently small $\varepsilon = \varepsilon_k > 0$.

[93] All dihedral g_0-angles are $\frac{\pi}{k}$, $k = 3, 4, ...$, where k are half-orders of the stabilizer subgroup of the corresponding faces ∂_{ij}. Thus, all dihedral angles of (V,g) must be $\leq \pi/2$.

But it is unclear if this condition is $\mathbb{T}^{\rtimes}$-stable, i.e. extends to $\mathbb{T}^{n-2}$-invariant (warped product) metrics $g^{\rtimes}$ on $V \times \mathbb{T}^{n-2}$, and thus, would allow the reduction of higher dimensions n to $n = 2$ by the (warped FCS) $T^{\rtimes}$-symmetrization.

(The full solution of the ⬠-problem remains unsettled even for $n = 2$.)

(6) Minimal Hypersurfaces in Cubical V. At the core of the proof of ■ lies non-existence of metrics with $Sc > 0$ on the torus $X = \hat{V}/\mathbb{Z}^n$, which in turn, can be proved in two different ways: by the Schoen-Yau's descent with minimal hypersurfaces or with a use of twisted Dirac operators on X

To pursue the latter, one has to describe/construct/analyse twisted harmonic spinors on $\hat{V}$ in terms of V with no use to the orbi-covering condition $V \rightsquigarrow \hat{V}$, such that it would be applicable to (more) general manifolds V with corners.[94]

The picture of minimal hypersurfaces in X is more transparent in this respect, where those homologous to the coordinate subtori in X may originate from V, namely from

minimal hypersurfaces $Y \subset V$, which separate pairs of opposite $(n-1)$-faces in V.[95]

In general, such Y do not exist, since they may escape the interior of Y in the course of volume minimization, but if $mean.curv(\partial_i V) > 0$ and $\angle_{ij}(V) < \frac{\pi}{2}$, then the "boundary walls" ∂_i "trap" Y inside V.

Indeed, the first inequality shows that, in the course of minimisation, the interior of Y can't touch ∂V by the maximum principle and the second one keeps the boundary of Y away from faces Y is suppose to separate.[96]

What is non-obvious here is the nature of singularities at the boundary of Y which may create complications in consecutive inductive steps of descent method, even for $n \leq 7$, where there is no singularities away from the $n-2$-faces of V.

Recently, however, Chao Li [Li(rigidity) 2019] established necessary regularity property of minimal $Y \subset V$ at the corners of V and thus gave a direct proof of ■ for $n \leq 7$ by Schoen-Yau's inductive decent method with minimal hypersurfaces separating pairs of opposite $(n-1)$-faces in V.

An advantage of the direct approach is applicability to a class of non-cubical manifolds V with corners, which are *not amenable to reflections*, namely to products $V = [0,1]^{n-2} \times \bigcirc \subset \mathbb{R}^n$, where $\bigcirc \subset \mathbb{R}^2$ is a convex polygon.

no metric g on such a V, for which the codimension 1 faces are mean convex and all dihedral angles are bounded by the Euclidean ones of V, can have $Sc(g) > 0$.(See section 3.16 for more about it.)

However, reflections reveal a fuller picture of the geometry of V, not limited to minimal hypersurfaces between opposite faces, but also including those reflected in various $(n-1)$-faces which correspond to the minimal Γ-invariant hypersurfaces in the universal orbi-covering $\hat{V}$ of V.

Plateau Billiard Problem. Given a Riemannin manifold V with (smooth or cornered) boundary, study minimal subvarieties in V with the *reflection boundary condition.*

[94]Relevant harmonic spinors on $\hat{V}$ may be not Γ-invariant but interesting classes of such spinors are.

[95]Complete minimal subvarieties in $\hat{Y} \subset \hat{V}$ correspond to non-compact singular $Y \in V$ that reflect in the codimension 1 faces alike to billiard trajectories in the case $dim(\hat{Y}) = 1$.

[96]In the case of non-strict inequalities, the minimal Y may touch these two faces, even coincide with one of them but the interior of Y can't touch the boundary of V by the maximum principle.

3.1.2 MC-Normalization of Hypersurfaces with Positive Mean Curvatures and *Sc*-Normalized Convex Area Extremality Theorem

(5) Reduction of ■ to ⬬ and the Proof of ⬬. Such a reduction, which provides an alternative proof of ■, is achieved by smoothing the corners of V. Then ⬬ is proved by doubling V and applying the following.

[⬭] **Convex Area Extremality Theorem.** Let $\underline{X} \subset \mathbb{R}^{n+1}$ be a compact smooth convex hypersurface, let $\underline{g}$ be the Riemannian metric on $\underline{X}$ induced from the ambient Euclidean space $\mathbb{R}^{n+1} \supset \underline{X}$ and let g be another Riemannian metric on $\underline{X}$ with non-negative scalar curvature, $Sc(g) \geq 0$.

Denote by $\underline{g}^\circ$ and g° *normalizations* of these metrics by their respective scalar curvatures,

$$\underline{g}^\circ(\underline{x}) = Sc(\underline{g}, \underline{x}) \cdot \underline{g}(\underline{x}) \text{ and } g^\circ(\underline{x}) = Sc(g, \underline{x}) \cdot g(\underline{x}).$$

(These metrics vanish, where the scalar curvatures vanish.)

If n is even,[97] *then there exists smooth surface $S \subset \underline{X}$, on which both functions $Sc(\underline{g}, \underline{x})$ and $Sc(g, \underline{x})$ are strictly positive and such that*

$$area_{g^\circ}(S) \leq area_{\underline{g}^\circ}(S).$$

In words,

The Sc-normalization of *no Riemannian metric* with non-negative scalar curvature on a *convex* Euclidean hypersurface can't be *area wise greater* than the *Sc*-normalized *original metric* on this hypersurface that is induced from the Euclidean space.

This is a special case of *Spin-Area Convex Extremality Theorem* (see [X_{spin}→⬭] in sections 3.4, 3.4.1 that is derived from curvature estimates for the twisted Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) formula due to Uwe Goette and Sebastian Semmelmann. Earlier, these estimates and the issuing extremality were established by Marcelo Llarull for the spheres S^n for all n, while the idea of *Sc*-normalisation, which is crucial for geometric applications, was suggested by Mario Listings.[98]

(6) Problem. Find an elementary (whatever this means) proof of ■ in the case where (V, g) admits an isometric embedding to $\mathbb{R}^n$.

Exercise. Give a direct proof of ■ for *convex* $V \subset \mathbb{R}^n$.

Hint. Show that if all $\frac{n(n-1)}{2}$ dihedral angles at a vertex $v \in V$ are $< \frac{\pi}{2}$, then the spherical measure of the set of the supporting hyperplanes of V at v is $> \frac{1}{2^n} vol(S^{n-1})$.

It is known (private communication by Karim Adiprasito) that

no convex polyhedron admits an infinitesimal deformation, which decreases its dihedral angles but it is unknown if a polyhedron P' combinatorially equivalent to P may have all its dihedral angles $\angle'_{ij} < \angle_{ij}$.

[97] Conjecturally, this parity assumption is unneeded.

[98] See [Goette-Semmelmann(symmetric)2002], [Llarull(sharp estimates)1998], [Listing(symmetric spaces) 2010] and compare with §5$\frac{4}{9}(D)$ in [G(positive) 1996]. Also note that recently, John Lott [Lott(boundary 2020] suggested a direct proof of a non-Sc an non-MC-normalized version of ⬬ by establishing *index and vanishing theorems for Dirac operators on manifolds with boundaries.* Probably, a minor adjustment of his argument will deliver the full normalized ⬬.

Conjecturally, there is no such P' even among *curve-linear* polyhedra with *mean convex faces.*

At the present time, this is confirmed for for special polyhedra P, e.g. those with all dihedral angles $\leq \frac{\pi}{2}$. (see section 3.18).

(7) On the Proof of □ and ⬭. Construction of a small strictly mean convex $V \subset X$ with rectangular corners needed for □ proceeds by induction on n, where the resulting V looks like a solid $[0, l_1] \times) \times [0, l_2 \times ... \times [0, l_n]$ with $l_1 >> l_2 >> ...' >> l_n$. (See [G(billiards) 2014] or do it yourself.)

Then the proof of ⬭ follows by smoothing these corners (another exercise for the reader).

Small domains $V \subset X$, especially for ⬭, obtained this way are fairly artificial. It would be nicer to have exp-images of ellipsoids from $T_x(X)$ at a point where $Sc(X, x) < 0$, or small perturbation of these in X.

But, probably, such a B can't be a ball, unless X has constant sectional curvature.

(8) Normalization of Metrics by Mean Curvatures . The relations ⬬$_\leq$ and ⬬$_=$ for $f : \partial V \to \partial W$ becomes more transparent if the Riemannian metrics in the hypersurfaces $\partial V \subset V$ and in $\partial W \subset \mathbb{R}^n$ induced from the ambient spaces, call them g on ∂V and h on ∂W, are rescaled by (the squares of) their mean curvatures, denoted here $m(v) = mean.curv(\partial V, v)$, $v \in \partial V$ and $m(w) = mean.curv(\partial W, w)$, $w \in \partial W$,

$$g = g(v) \mapsto g^\natural = m(v)^2 \cdot g(v), v \in \partial V, \text{ and } h = h(w) \mapsto h^\natural = m(w)^2 \cdot h(w), w \in \partial W.$$

Now, the inequality ⬬$_\leq$ says that the map f is *distance non-increasing* with respect to the *MC-scaled* metrics $g^\natural$ and and $h^\natural$, while ⬬$_=$ expresses the isometry between these metrics established by f.

Exercise. Let $V \subset \mathbb{R}^n$ be a convex polyhedron and $V_i \supset V$ be a decreasing sequence of larger convex subsets in $\mathbb{R}^n$ with smooth boundaries, which converge to V, i.e.

$$V_1 \supset V_2 \supset ... \supset V_i \supset ... \supset V \text{ and } \bigcap_i V_i = V.$$

Describe the limit of the metrics spaces $(\partial V_i, m_i^2 \cdot g_i)$, where $m_i = m_i(v)$, $v \in \partial V_i$, denote the mean curvatures of the boundaries ∂V_i and g_i are the Riemannian metrics on these $\partial_i V$ induced from $\mathbb{R}^n$.

(To make it more specific, let ∂V_i be (very closely) pinched between the boundaries of ε_i- and $(\varepsilon_i + \varepsilon_i^i)$-neighbourhoods of V, i.e.

$$U_{\varepsilon_i}(V) \subset V \subset U_{\varepsilon_i + \varepsilon_i^i}(V),$$

where $\varepsilon_i \to 0$ for $i \to \infty$.)

Do the same for the (induced Riemannian) metrics g_i on ∂V_i *normalized by their scalar curvatures*, $g_i \leadsto Sc(g_i) \cdot g_i$, and then for the other *symmetric functions* s_k, $k = 1, 2, ...n - 1$, *of the principal curvatures* $\alpha_1, \alpha_2, ...\alpha_{n-1}$ of ∂V_i, i.e. $g_i \leadsto s_k^{\frac{2}{k}} \cdot g_i$.

(10) *Problem.* Is there a theory of singular spaces with $Sq \geq \sigma$, that is built on the basis of ■, ⬬ or more powerful inequalities?

Example. Let $U \subset \mathbb{R}^{n+1}$ be a convex subset, the principal curvatures α_i of the boundary $X = \partial U$ of which satisfy

$$\sum_{i>j} \alpha_i \alpha_j \geq \sigma \geq 0.$$

at all regular points of X.

Albeit in general singular, X can be neatly approximated by the $C^{1,1}$-regular boundary X_ε of the ε neighbourhood of U, where the induced Lipschitz continuous Riemannian metric g_ε on X_ε g_ε be C^0 can be approximated– – this is obvious in the present case – by C^∞-metrics with $Sc \geq \sigma$.

Thus, (the intrinsic) path metric on X *must share all its geometric properties with smooth manifolds which have* $Sc \geq \sigma$.

Exercise. Show that X satisfies the inequalities ■, ⬬.[99]

(11) *Category Theoretic Perspective on normalized Riemannian metrics.* The above suggests that the geometry of Riemannian manifolds $X = (X, g)$, where $Sc(g) > 0$ without bounary is well depicted by the Sc-*normalised metric* $Sc(X) \cdot g$ and that maps, which are *1-Lipschitz with respect to the* Sc-*normalised metrics* can be taken for morphisms in the category of manifolds with $Sc > 0$.

Now, If X does have a boundary and this boundary is mean convex, the normalization of X by $Sc(X)$ and of ∂X by $mean.curv(\partial X)$ do not agree on ∂X.

Alternatively, one may use positivity of the mean curvature of ∂X for blowing up the metric of X near ∂X keeping $Sc > 0$, as it is done in fill-ins3.

(11) Polyhedral Localization Conjecture. Let $\underline{V} \subset \mathbb{R}^n$ be a convex polyhedron an let

V be a compact oriented Riemannian n-manifold with corners, of "combinatorial class" of $\underline{V}$, which means there exists a *proper corner continous map* $f : V \to \underline{V}$ of degree one, where "proper corner" means "face respecting": the $(n-1)$ faces $V_i \subset \partial V$ ae equal the pullbacks of the corresponding faces $\underline{V}_i \subset \partial \underline{V}$.

Let the boundary of V be *more mean convex* than that of $\underline{V}$, i.e. all $(n-1)$-faces $V_i \subset \partial V$ have non-negative mean curvatures. and the dihedral angles of V along the $(n-2)$-faces are strictly bounded by the corresponding angles in $\underline{V}$,

$$\angle(V_i, V_j) \leq \angle(\underline{V}_i, \underline{V}_j).$$ [100]

Then, conjecturally,

[⊡] V contains domains $U_\varepsilon \subset V$ with corners, which have *arbitrarily small diameters*,

$$diam(V_\varepsilon) \leq \varepsilon > 0,$$

which are also in the combinatorial class of $\underline{V}$ and such that their boundaries are *more mean convex* than that of $\underline{V}$.

"Cubical" Remark/Theorem. The ■-inequality together with its contraposition (□), that is the existence of cubical mean convex domains with acute dihedral angles in manifolds with $Sc < 0$, (see section3.1) imply the validity of [⊡] for cubical, $\underline{V} = [0,1]^n$ and V *homeomorphic* to $\underline{V}$.

[99] I haven't done this exercise.

[100] This strict "$<$", rather than more natural "$\leq$", is used here to avoid possible technical complications with the rigidity problem (see sections 3.18 and **??**)

Now, a close look at the proofs of the ■ in section 3.1.1 the ■ (also see 3.18 and 4.4) apply to more general V:

The proofs with the Dirac operator works for *all spin* manifolds V, while the calculus of variation methods needs no spin, but it become cumbersome for $n \geq 8$ and especially so for $n \geq 9$, due to possible singularities of minimal hypersurfaces of dimension≥ 7 (see section 3.7.1 for more about it.)

However, these proofs, especially the Dirac-operator theoretic one, do not directly pinpoint the small cube with acute angles in V, as they also need the local property (□) of $Sc < 0$.[101]

In fact, one can construct U_ε, at least for $n \leq 8$, to get such a cube U_ε arguing with minimal hypersurfaces, roughly as follows.

Given an *admissible* $U \subset V$, i.e. a mean convex cubical domain with acute dihedral angles, let us push the faces U_i in one after another little by little keeping $mean.curv.0$ and $\angle_{ij} \leq \frac{\pi}{2}$.

This process stops when we arrive at some $U = U_{min}$, where all faces a (locally) volume minimizing with free boundaries on the unions of th remaining faces. Now one can slightly move each face, say U_1 in, $U_1 \rightsquigarrow U_{1,\delta}$ keeping the dihedral angles equal $\frac{\pi}{2}$, but now such that $U_{1,\delta}$ is everywhere *mean concave* rather than mean convex. Thus, we obtain a smaller admissible domain, say $U' \subset U = U_{min}$ namely the band between $U_1 \rightsquigarrow U_{1,\delta}$ in U.

If $n \leq 8$ one can indeed arrange this process to arrive at an ε-small "cube" $U = U_\varepsilon$, but, in general, this "cube" may be only as small as the singularities of the minimal hypersurfaces are: the "cube minimization process" can, a priori, converge to an $(n-8)$-dimensional closed subset $U_\bullet \subset V$.

About Dimension $n = 9$. *If $n = 9$, the domains U_ε, are spin.*

Indeed, U_ε are localized near a subset $U_\bullet$ of dimension ≤ 1, while the obstruction to spin, that is the second Stiefel-Whitney number $w_2 \in H^2(V; \mathbb{Z}_2)$, resides in dimension 2.

At this point, the Dirac theoretic argument applies to U_ε and, together with (□), yields [⊡] for cubical $\underline{V}$ with $dim(\underline{V}) = 9$. (see [G(billiards) 2014]).

3.1.3 C^0-Limits of Metrics g with $Sc(g) \gtrless \sigma$

Let X be a smooth Riemannian manifold, let $G = G(X)$ the space of C^2-smooth Riemannian metrics g on X and let $G_{Sc \geq \sigma} \subset G$ and $G_{Sc \leq \sigma} \subset G$, $-\infty < \sigma < \infty$, be the subsets of metrics g with $Sc(g) \geq \sigma$ and with $Sc(g) \leq \sigma$ respectively.

Then:

A: **C^0-Closure Theorem.** The subset $G_{Sc \geq \sigma} \subset G$ is closed in G with respect to the C^0-topology:

uniform limits $g = \lim g_i$ of metric g_i with $Sc(g_i) \geq \sigma$ have $Sc \geq \sigma$, provided these g are C^2-smooth in order to have their scalar curvature defined.

B: **C^0-Density Theorem.** The subset $G_{Sc \leq \sigma} \subset G$ is dense in G with respect to the C^0-topology.

Moreover, *all $g \in G$ admit fine (which is stronger than uniform for non-compact X) approximations by metrics with scalar curvatures $\leq \sigma$.*

Short Proof of A. Let us show that violation of ⬬ for a smooth metric g on a manifold X, that is (⬭) from the previous section, implies this for g_ε for

[101] Our conjecture doesn't mention any curvature and we want the proof to be also like that.

sufficiently small $\varepsilon = ||g - g_\varepsilon||_{C^0}$.

Indeed let the boundary ∂V of a compact strictly mean convex domain $V \subset X = (X, g)$ admits a smooth map f of degree one to the boundary of a convex $W \subset \mathbb{R}^n$, the norm of the differential of which satisfies:

$$||df(v)|| < \frac{mean.curv(\partial W, f(v))}{mean.curv(\partial V, v)}.$$

If g_ε is close to g, then there exists a smooth $V_\varepsilon \subset X$, the boundary of which is δ-*close* to ∂V and its g_ε-mean curvature is δ-*close* to the g-mean curvature of ∂V, where $\delta \to 0$ for $\varepsilon \to 0$, and where "δ-close" means the following. diffeomorphisms exists a smooth $(1+\delta)$-Lipshitz map[102] $\nu : \partial V_\varepsilon \to \partial V$, i.e. $||d\nu|| \le 1 + \delta$, such that

$$dist_g(x, \nu(x)) \le \delta \text{ for all } x \in \partial V_\varepsilon$$

as well as

$$|mean.curv_{g_\varepsilon}(\partial V_\varepsilon, x) - mean.curv_g(\partial V, \nu(x))| \le \delta.$$

In fact, one can take the g-normal projection of the δ-neighbourhood of ∂V to ∂V restricted to ∂V_ε for this ν, where, observe, this projection $\partial V_\varepsilon \to \partial V$, albeit *not necessarily a diffeomorphism* for small $\varepsilon \to 0$, can be C^0-*approximated by diffeomorphisms.* [103]

About Alternative Proofs. Instead of ⬬ one can use ■ but the available argument in [G(billiards) 2014]) is unpleasantly convoluted.

A streamlined proof based on *Hamilton-Ricci flow* was suggested by Richard Bamler and further developed by Paula Burkhardt-Guim who has shown, in particular, that

($\star$) *if a continuous metric g on a smooth manifold X admits a C^0-approximation by metrics g_ε with $Sc(g_\varepsilon) \ge \varepsilon \to 0$, then X admits a smooth metric with $Sc \ge 0$.*

Moreover,

g can be C^0 approximated by metrics with $Sc \ge 0$.

Thus,

continuous metrics which are C^0-limits of of smooth metrics metrics g_i with $\lim Sc(g_i) \ge -\varepsilon \to 0$ *have the same kind of geometries as metrics with $Sc \ge 0$.*

Question. Do *Lipschitz metrics*[104] are similar to continuous ones in this respect for suitable limits $g_i \to g$?

About B. This is a special case of *the curvature h-principle* discovered by Joachim Lohkamp,[105] whose proof in depends on a (more or less) direct, yet, elaborate, geometric construction, which also shows that

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[102] A map between metric spaces, $f : X \to Y$, is λ-*Lipschitz* if $dist_Y(f(x_1), f(x_2)) \le \lambda dist_X(x_1, x_2)$); a λ-Lipschitz map is λ-*bi-Lipschitz* if it is one-to-one and the inverse map is also λ-Lipschitz.

[103] The existence of such a V_ε and its properties must be a common knowledge among experts on the geometric measure theory but I couldn't find a reference and written down a proof in section 10.2 of [G(Hilbert) 2011].

[104] A measurable Riemannian metric g on a smooth n-manifold X is *Lipschitz* if it is locally *bi-Lipschitz* equivalent to the Euclidean metric on (a domain in) $\mathbb{R}^n$, see [Ivanov(Lipschitz) 2008]. Notice that the natural domains X for such g are *Lipschitz*, rather than smooth, manifolds that are defined by *bi-Lipschitz atlases* on X, see [NSLipschitz) 2007].

[105] [Lohkamp(negative Ricci curvature) 1994].

($\star$) *the metics with Ricci < 0 are C^0-dense in he space of all Riemannian metrics on X.*

(If, contrary to A, the space of metrics with $Sc \geq 0$ were dense, there would be no hope for a non-trivial geometry of such metrics.)

(The C^0-closure theorem for the scalar curvature looks similar to

Eliashberg's C^0-Closure Theorem, which claims that

C^0-limits of symplectomorphisms, are again symplectomorphisms, provided they

are C^1-smooth and C^1-invertible.

But, unlike $Sc \geq 0$, non-smooth such limits are significantly *more flexible* and geometrically *less constrained* than smooth symplectomorphisms[106]

Weak convergence of metrics and convergence of manifolds. Besides uniform convergence, there are other metric conditions on sequences of metrics that preserve positivity of the scalar curvature in the limit, where the simplest unkown case is the following.

Let smooth Riemannian metrics g_i *converge in measure* to an also smooth g, i.e. the measure of the subset, where the $|g(x) - g_i(x)| \geq \varepsilon$ tends to 0 for $i \to \infty$.

Do the inequalities $Sc(g_i) \geq 0$, imply that also $Sc(g) \geq 0$?

This is most likely to hold true if the *Lipschitz distance*[107] between g and g_i remains bounded by a constant independent of $i \to \infty$. [108]

In geometry, however, natural limits are not these of metrics but those of Riemannian manifolds, with no fixed topological background,

$$X_i = (X_i, g_i) \to X = (X, g),$$

where, relevantly for us, such limits, even when drastically changing topology, may preserve positivity of the scalar curvature; yet, some natural geometric limits, e.g. the *Hausdorff* and *intrinsic flat* ones may uncontrollably change scalar curvature.[109]

Conjecture: *Quantification of C^0-Convergence.* Let g_0 and g_ε be smooth Riemannian metrics on the ball B^n, such that the C^0-norm of the difference $g_\varepsilon - g_0$ is bounded by ε, or (almost) equivalently the identity map $(B^n, g_0) \to (B^n, g_\varepsilon)$ is $\left(1 + \frac{\varepsilon}{2}\right)$-bi-Lipschitz.

Then there exist positive constants (large) $c_0 > 0$ and (small) $\varepsilon_0 > 0$, which depend only on g_0, such that if $\varepsilon \leq \varepsilon_0$, then the scalar curvature of g_ε at the center of the ball satisfies,

$$Sc(g_0(0)) \geq \inf_{x \in B^n} (Sc(g_\varepsilon(x))) - c_0 \varepsilon.$$

[106] [Buhovsky-Opshtein(C^0-symplectic)2014], [Bu-Hu-Sey(C^0 counterexample) 2016]; yet, some symplectic geometry, if properly understood, passes the C^0-barrier [Bu-Hu-Sey(C^0-symplectic) 2020].

[107] This is the maximum of the Lipschitz constants of the identity map $V \to V$ with respect the pairs of metrics, $(V, g) \to (V, g_i)$ and $(V, g_i) \to (V, g)$.

[108] Beware of examples implied by theorem 1.4 in [Brun-Han(large and small) 2009]).

[109] See sections 3.1.3, 6.8 for examples (and counter examples), of various kind of behaviour of the scalar curvature under such convergence.

Motivating Example. If $g_\varepsilon = (1\pm\varepsilon)^2 g_0$, then $||g_\varepsilon - g_0|| = 2\varepsilon + o(\varepsilon)$ and $|Sc(g_\varepsilon) - Sc(g_0)| = O(\varepsilon)$.

Exercise. Prove this conjecture for $n = 2$.

Remark. Probably, a close look at the proof of A will yield the conjecture for radial (i.e. $O(n)$-invariant) metrics g_0 (compare with approximation corollary in §$5\frac{5}{6}$ from [G(positive)1996] as well as the inequality $Sc(g_0, 0) \geq \inf_{x\in B^n}(g_\varepsilon) - c_0\varepsilon^{\frac{1}{2}}$.

3.2 Spin Structure, Dirac Operator, Index Theorem, $\hat{A}$-Genus, $\hat{\alpha}$-Invariant and Simply Connected Manifolds with and without $Sc > 0$

Let $L \to X$ be a real orientable vector bundle of rank r and $F \to X$ be the oriented frame bundle of L. If $r \geq 2$ the fundamental group of the fiber $F_x = SL(k)$ is infinite cyclic and if $k \geq 3$ this group is cyclic of order 2. In both cases, F comes with a canonical double cover $\tilde{F}_x \to F_x$.

The bundle L is called *spin*, if $\tilde{F}_x \to F_x$ extends to a double cover $\tilde{F} \to F$, and smooth orientable manifold X is *spin* if its tangent T(X) bundle is spin.

Extension of the covering $\tilde{F}_x \to F_x$, if it exists, is, in general, non-unique. In the case of of $L = T(X)$ such an extension is called a *spin structure* on X.

When you speak of spin, it is common in geometry and for a good reason, to reduce the structure group of L from $SL(r)$ to $SO(r) \subset SL(r)$ and to deal with the orthonormal frame bundle $OF \to X$ instead of F, where the double cover group $\tilde{S}O(r) = OF_x$ is called *spin group* $Spin(r)$.

Example. The tangent bundle of the 2-sphere is spin, but the Hopf bundle over S^2 is not, since OF, that is S^3 for the Hopf bundle, is simply connected.

Similarly – this an exercise in elementary topology,

an oriented bundle L of rank two over an oriented surface X is spin if and only if its *Euler class*, that is the *self-intersection number of* $X \subset L$ is *even*; if X is non-orientable, then L is spin if the *second Stiefel-Whitney class*, that is the self-intersection number mod 2 of $X \subset L$ *vanishes.* In either case L is spin if and only if the Whitney sum of L with the trivial line bundle $l \simeq X \times \mathbb{R}^1$ is trivial, $L \oplus l \simeq X \times \mathbb{R}^3$.

In general,

a bundle L over a manifold X of dimension $n \geq 3$ is spin, if and only if its restriction to all surfaces in X is spin, which is again equivalent to the vanishing of the second Stiefel-Whitney class $w_2(L)$.[110]

Half-spin Bundles. There exit two (remarkable) irreducible unitary representations of the group $Spin(r)$ for $r = 2k$ of complex dimensions 2^{k-1}, say $S^\pm(r)$. Accordingly, Riemannian spin manifolds, (i.e. with spin structures on them) X support two $Spin(n)$ bundles $\mathbb{S}^\pm$ with the fibers $S^\pm(r)$ that are associated with

[110] The value of $w_2(L) \in H^2(X; \mathbb{Z}_2)$ on a homology class $h \in H^2(X; \mathbb{Z}_2)$ is, almost by definition, equal to zero if and only if the restriction of L to surfaces in X that represent h is trivial.

Geometrically, the double cover $\tilde{F}_x \to F_x$ extends to F over the complement to a subvariety $\Sigma \subset X$ of codimension two, the homology class of which is Poincare dual to $w_2(X)$. This $\Sigma \subset X$ is waht stands on the way of applying Dirac theoretic methods to non-spin manifolds.

principal spin bundle $\tilde{S}O \to X$ for the double covering representing the spin structure on X. We let $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$ and call this $\mathbb{S}$ the *spin bundle*.[111]

The Dirac operator

$$\mathcal{D} : C^\infty(\mathbb{S}) \to C^\infty(\mathbb{S})$$

is a first order differential operator constructed in a canonical geometrically invariant way universally applicable to all X (see section).

This is an *elliptic selfadjoint* operator, which interchanges $C^\infty(\mathbb{S}^+)$ and $C^\infty(\mathbb{S}^-)$ where the operators

$$\mathcal{D}^+ : C^\infty(\mathbb{S}^+) \to C^-(\mathbb{S}^-) \text{ and } \mathcal{D}^- : C^\infty(\mathbb{S}^-) \to C^-(\mathbb{S}^+)$$

are mutually adjoint.

We explained already in section 1.6.1 how, following Lichnerowicz, that the ***Atiyah-Singer index theorem*** for the Dirac's $\mathcal{D}$ and the S-L-W-(B) identity

$$\mathcal{D}^2 = \nabla^2 + \frac{1}{4} Sc,$$

imply that

there are smooth *closed simply connected* manifolds X of all dimensions $n = 4k$, $k = 1, 2, \dots$, that admit no metrics with $Sc > 0$.

The simplest example of these for $n = 4$ is *the Kummer surface* X_{Ku} given by the equation

$$z_1^4 + z_2^4 + z_3^4 + z_4^4 = 0$$

in the complex projective space $\mathbb{C}P^3$.

In fact, all complex surfaces of *even degrees* $d \geq 4$ as well as their Cartesian products, e.g $X_{\mathsf{Ku}} \times \dots \times X_{\mathsf{Ku}}$ *admit no metrics with* $Sc > 0$.

Also we know that the Atiyah-Singer $\mathbb{Z}_2$-index theorem of 1971 allowed an extension of Lichnerowicz' argument to manifolds of dimensions $8k+1$ and $8k+2$, e.g. to exotic spheres in

Hitchin's theorem: there exist manifolds Σ homeomorphic (but no diffeomorphic!) to the spheres S^n, for all $n = 8k+1, 8k+2$, $k = 1, 2, 3\dots$, which admit no metrics with $Sc > 0$.

(What makes the *differential* structures of Hitchin's *topological* spheres Σ incompatible with $Sc > 0$ is that to these Σ are *not boundaries of spin manifolds.*)

The actual Lichnerowicz-Hitchin theorem says that if a certain topological invariant $\hat{\alpha}(X)$ *doesn't vanish*, then X *admits no metric with* $Sc > 0$, since, by the Atiyah and Singer index formulae,[112]

$$\hat{\alpha}(X) \neq 0 \Rightarrow Ind(\mathcal{D}_{|X}) \neq 0 \Rightarrow \exists \text{ harmonic spinor} \neq 0 \text{ on } X,$$

which is incompatible with the identity $\mathcal{D}^2 = \nabla^2 + \frac{1}{4} Sc$ for $Sc(X) > 0$

[111] In reality, $\mathbb{S}$ comes first and then splitting $\mathbb{S}^- \oplus \mathbb{S}^+$ follows, see section 3.3.3.

[112] The Dirac operator is defined only on *spin* manifolds; we postulate at the present moment that $\hat{\alpha}(X) = 0$ for non-spin manifolds X.

(In fact, if $n = dim(X) = 4k$, this $\hat{\alpha}(X)$ is a certain linear combination of the *Pontryagin numbers* of X, called *$\hat{A}$-genus* and denoted $\hat{A}[X]$.

Accidentally, since all *compact homogeneous* spaces $X = G/H$, except for tori, support metrics with $Sc > 0$, Lichnerowicz' theorem says that they *either non-spin or* $\hat{A}[X] = 0$.)

Conversely,

☞ if X is a *simply connected* manifold of *dimension* $n \geq 5$, and if $\hat{\alpha}(X) = 0$, then, an application of "thin surgery" (see section 1.3) to suitably chosen generators $O(n)$- and $Sp(n)$- cobordism groups in dimensions $n \geq 5$, where these generators carry metrics with $Sc > 0$, yields[113] that X admits a metric with positive scalar curvature.

Thus, for instance

all simply connected manifolds of dimension $n \neq 0, 1, 2, 4 \mod 8$ *admit metrics with* $Sc > 0$,[114] since $\hat{\alpha}(X) = 0$ is known to vanish for these n.[115]

Topology of Scalar Flat. By *Yau's solution of the Calabi conjecture,* the Kummer surface admits a metric with $Sc = 0$, even with $Ricci = 0$, but there is

no metrics with $Sc = 0$ *on Hitchin's exotic spheres* Σ.

In fact,

if a compact simply-connected scalar-flat manifold X *of dimension* ≥ 5 *admits no metric with* $Sc > 0$,[116] *then there are cohomology classes* $\alpha \in H^2(X)$ *and* $\beta \in H^4(X)$, *such that*

$$\langle \exp\alpha \smile \exp\beta \smile p_1(X)\rangle \neq 0,$$

where $p_1(X)$ *is the first Pontryagin number,* [Futaki(scalar-flat)1993], [Dessai(scalar flat) 2000].

And if X is non-simply connected then

a finite covering of X *isometrically splits into the product of a flat torus and the above kind simply connected manifold,*

as it follows from Cheeger-Grommol splitting theorem + Bourguignon-Kazdan-Warner perturbation theorem.

A Few Words on $n = 4$ *and on* $\pi_1 \neq 0$. If $n = 4$ then, besides vanishing of the $\hat{\alpha}$-invariant (which is equal to a non-zero multiple of the first Pontryagin number for $n = 4$), positivity of the scalar curvature also implies the vanishing of the *Seiberg-Witten invariants* (See lecture notes by Dietmar Salamon, [Salamon(lectures) 1999]; also we say more about it in section 3.16).

If X is a closed spin manifold of dimension $n \geq 5$ with the fundamental group $\pi_1(X) = \Pi$, then, again by an application of the thin surgery,

the existence/non-existence of a metric g *on* X *with* $Sc(g) > 0$ *is an invariant of the spin bordism class* $[X]_{sp} \in bord_{sp}(\mathsf{B}\Pi)$ in the classifying space $\mathsf{B}\Pi$,

where, recall, that (by definition of "classifying") the universal covering of $\mathsf{B}\Pi$ is contractible and $\pi_1(\mathsf{B}\Pi) = \Pi$. [117]

There is an avalanche of papers, most of them coming under the heading of "Novikov Conjecture", with various criteria for the class $[X]_{sp}$, and/or for the

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[113] [GL(classification) 1980], [Stolz 1992].

[114] If $dim(X) = 3$, this follows from Perelman's solution of the Poincaré' conjecture.

[115] As far as the exotic spheres Σ are concerned, these Σ admit metrics with $Sc > 0$ if and only if $\hat{\alpha}(\Sigma) = 0$, i.e. if Σ bound spin manifolds, which directly follows by the codimension 3 surgery of manifolds with $Sc > 0$ described in [SY(structure) 1979] and in [GL(classification) 1980]. Moreover, many of these Σ, e.g. all 7-dimensional ones, admits metrics with *non-negative sectional curvatures* but the full extent of "curvature positivity" for exotic spheres remains problematic (see [JW(exotic) 2008] and references therein.

[116] These X are Ricci flat, [Bourguignon (these) 1974], [Kazdan[complete 1982].

[117] See lecture notes [Stolz(survey) 2001].

corresponding homology class $[X] \in H_n$ (BΠ) (not) to admit g with $Sc(g) > 0$ on manifolds in this class, where these criteria usually (always?) linked to generalized index theorems for twisted Dirac operators on X with several levels of sophistication in arranging this "twisting".

Yet, despite the recent progress in this direction for dimensions 4 and 5[118] proving/disproving the following for $n \geq 4$ remains beyond the present day means. [119]

(Naive?) **Conjecture.**[120] If a closed oriented n-manifold X admits a continuous map to an *aspherical space* B,[121] such that the image of the rational fundamental homology class of $[X]_{\mathbb{Q}}$ in the rational homology[122] $H_n(\mathsf{B};\mathbb{Q})$ doesn't vanish, then X admits no metic g with $Sc(g) > 0$.

(We shall describe the status of this problem together with the *Novikov conjecture* in section 3.14.)

3.3 Unitary Connections, Twisted Dirac Operators and Almost Flat Bundles Induced by ε-Lipschitz Maps

We turn now to twisted Dirac operators $\mathcal{D}_{\otimes L}$ that act on tensor products $\mathbb{S} \otimes L$ for vector bundles $L \to X$ with linear (most of the time, unitary) connections ∇.

One can think of such a $\mathcal{D}_{\otimes L}$ as an *infinitesimal family* of $\mathcal{D}$-s parametrized by L, where the action takes place along $\mathbb{S}$ with no differentiation in the L-directions.

For instance if $L = (L, \nabla)$ is a trivial flat bundle, $L = X \times L_0$, where L_0 is a vector space (fiber), then $C^\infty(\mathbb{S} \otimes L) = C^\infty(\mathbb{S}) \otimes L_0$ and the $\mathcal{D}_{\otimes L}$ doesn't act on L at all:

$$\mathcal{D}_{\otimes L}(f \otimes l) = \mathcal{D}(f) \otimes l \text{ for all vectors } l \in L_0.$$

In general, the $\mathcal{D}_{\otimes L}$ differs from that in the flat case by a zero order term, which is, bounded by the curvature of L and, strictly speaking, is defined only locally, where the bundle L is topologically trivial. But exactly this impossibility of global comparison of $\mathcal{D}_{\otimes L}$ on $C^\infty(\mathbb{S} \otimes L)$ with $\mathcal{D}$ on $C^\infty(\mathbb{S}) \otimes L_0$ creates a correction term in the index formula.

This correction, unlike the background operator $\mathcal{D}$, carries no subtle topological information about X, such as $\hat{A}(X)$ for $n = 4k$, which is not a homotopy invariant for $n > 4$ and even less so about $\hat{\alpha}(X)$ for $n = 8k+1, 8k+2$, which is not even invariant under p.l. homeomorphisms and which is far removed from anything even remotely, geometric about X, while the topology (Chern classes) of L reflects the area-wise size of the metric g on X, which, in turn, influences *homotopy theoretic properties* of X linked to the fundamental group.

[118] See [Chodosh-Li(bubbles) 2020], [G(aspherical) 2020] and section 3.10.3

[119] The case $n = 3$ follows from the topological classification of compact 3-manifolds X with positive scalar curvature *these are connected sums of quotients of spheres S^3 and products $S^2 \times S^1$ by finite isometry groups* [GL(complete) 1983],[Ginoux(3d classification) 2013].)

[120] This, as many other our conjectures, is based on a limited class of examples with no idea of where to look for counter examples.

[121] That is the universal covering of B is contractible, hence, B is B(Π) for $\Pi = \pi_1(\mathsf{B})$.

[122] Bernhard Hanke pointed out to me that non-vanishing of this image in homology with *finite coefficients*, e.g. for finite groups Π, may also prohibit $Sc > 0$, but this remains obscure even on the level of conjectures.

The following definition gives you a fair idea of what kind of properties these are.

Profinite Hypersphericity. A Riemannian n-manifold X is *profinitely hyperspherical* if

given an $\varepsilon > 0$, there exists an orientable finite covering $\tilde{X} = \tilde{X}_\varepsilon$, which admits an ε-*Lipschitz* map between [123] $\tilde{X} \to S^n$ of *non-zero* degree.

This property of *compact* manifolds (the definition of this hypersphericity extends too open manifolds) doesn't depend on the Riemannian metric on X. Moreover

If X_1 is profinitely hyperspherical and X_2 admits a map of non-zero degree to X_1 then, obviously, X_2 is also profinitely hyperspherical; in particular, this property is a *homotopy invariant* of X.

Example. Manifolds X, which admit *locally expanding self-maps* $E : X \to X$, e.g. the n-torus $\mathbb{T}^n$, where the endomorphism $t \mapsto Nt$ locally expands the metric by N, are profinitely hyperspherical.

Indeed, such an E defines a *globally* expanding homeomorphism, call it $\hat{E}$, from X onto a finite covering $\tilde{X} = \tilde{X}(E)$, where the inverse map $\hat{E}^{-1} : \tilde{X} \to X$ contracts as much as E expands.

Therefore, the covering corresponding to the i-th iterate of E comes with an ε_i-Lipschitz map to X, where $\varepsilon_i \to 0$ for $i \to \infty$ and compositions of these with a map $X \to S^n$ of non-zero degree also have $deg \neq 0$, while their Lipschitz constants go to zero.[124]

Now, if you recall Atiyah-Singer index theorem for the twisted Dirac operator and $\mathcal{D}_{\otimes L}$ and the (untwisted) S-L-W-(B) formula $\mathcal{D}^2 = \nabla\nabla^* + \frac{1}{4}Sc$[125] you arrive at the following.

[Sc $\ngtr$ 0]: **Provisional Proposition**.[126] *Compact orientable*[127] *profinitely hyperspherical spin manifolds X of all dimensions n support no metrics with $Sc > 0$.*

Proof. This is obvious once said. Indeed, a *simple special case of the Atiyah-Singer index theorem* says that,

if a complex vector bundle L of rank k over a compact orientable *spin* Riemannian manifold X of dimension $n = 2k$, has *non-zero Euler* (*Chern*) *number*, that is the self-intersection index of the zero section $X \to\hookrightarrow \underline{L}$, then

the twisted Dirac $D_{\otimes L} : C^\infty(\mathbb{S} \otimes L) \to C^\infty(\mathbb{S} \otimes L)$ has non-zero kernel, for all linear connections in L, provided,

the number k is *odd*, and the restriction of L to the complement to a point in X is a *trivial bundle.*[128]

[123] A map between metric spaces, $f : X \to Y$, is ε-Lipschitz if $dist_Y(f(x_1), f(x_2)) \le \varepsilon dist_X(x_1, x_2)$ for all $x_1, x_2 \in X$. For instance, "1-Lipschitz" means "distance non-increasing". ε-Lipschitz for smooth maps f between Riemannian manifolds is equivalent to $||d(f(x)||\varepsilon$, $x \in X$.

[124] Further examples of this phenomenon and issuing topological obstruction to $Sc > 0$ for manifolds with *residually finite* fundamental groups are given in [GL(spin) 1980] under the heading of "enlargeability". Since the residual finiteness condition was eventually lifted, this terminology now applies to a broader class of manifolds, including spaces X the *universal covers* of which admit contracting self-maps of positive degrees, see section4.1.1

[125] This ∇ stands for the Levi-Civita connection in the spin bundle.

[126] This will be significantly generalized later on.

[127] If X is non-orientable, take an oriented double cover of it.

[128] These are minor technical conditions, the role of which is to avoid undesirable conse-

Then, by elementary algebraic topology,

the $2k$-sphere supports a complex vector bundle of rank k, say $\underline{L} \to S^{2k}$, which has *non-zero* Euler (Chern) number,

and

bundles $L = f^*(\underline{L}) \to X$ induced from $\underline{L}$ by continuous maps $f : X \to S^{2k}$ have their Euler numbers $e(L) = deg(f)e(\underline{L})$.

It follows that finite coverings $\tilde{X}_\varepsilon$ of X admit smooth ε-Lipschitz-maps $f_\varepsilon : \tilde{X}_\varepsilon \to S^n$ with arbitrary small ε and such that the twisted Dirac operators $\mathcal{D}_{\otimes L_\varepsilon}$ on $\tilde{X}_\varepsilon$ for $L_\varepsilon = f_\varepsilon^*(\underline{L})$, have non-zero kernels for all connections in L_ε.

Apply this to connections ∇_ε in L_ε induced by f_ε from a fixed smooth linear (unitary if you wish) connection $\underline{\nabla}$ in $\underline{L} \to S^{2k}$, let $\varepsilon \to 0$ and observe that, since the maps f_ε converge to constant ones on all unit balls in $\tilde{X}_\varepsilon$, the bundles $(L_\varepsilon, \nabla_\varepsilon)$ converge to trivial ones with trivial flat connections on all balls. Therefore the difference between the Dirac operator $\mathcal{D}_{\otimes L_\varepsilon}$ and $\mathcal{D}$ twisted with the trivial flat bundle L_{flat} of rank k becomes arbitrary small for $\varepsilon \to 0$, and the S-L-W-(B) formula applied to $\mathcal{D}_{L_{flat}}$ shows that $\inf_X Sc(X) = \inf_{\tilde{X}} Sc(\tilde{X}_\varepsilon) \leq 0$.

This completes the proof for $n = 4l + 2$ and the general case follows by (shamelessly) taking the product $X \times \mathbb{T}^{3n+2}$.

Well..., this is convincing but it is not quite a proof. We still have to define $\mathcal{D}_{\otimes L}$ and to make sense of the "difference" between the operators $\mathcal{D}_{\otimes L_\varepsilon}$ and $\mathcal{D}_{\otimes L_{flat}}$ that are defined in *different spaces*. We do all this below closer to the end of this section.

Why Spin? The essential new information delivered by $\mathcal{D}_{\otimes L}$ does not visibly depend on the spin structure (unlike to how it is with the Dirac operator $\mathcal{D}$ itself).[129]

However, one doesn't know how to get rid of the spin condition, in the cases where it appears irrelevant. For instance, there is no single known area-wise bound on the size of a non-spin manifold with a large scalar curvature.[130]

All in all, although "twisted Dirac" proofs are short and simple, their nature remains obscure.

Partly, this is why we explain below with such a care standard "trivial" properties of the "twist" $\mathcal{D} \rightsquigarrow \mathcal{D}_{\otimes L}$, hoping this may help us to visualize *something* behind this "trivial" that makes the Dirac's $\mathcal{D}$ work in geometry, "something", which is only tangentially related to the Dirac operator itself and, if untangled from $\mathcal{D}$ with its bondage to spin, would open up new possibilities.

---

quences of possible cancellation in the index formula (see section 4). For instance if X can be embedded or immersed into $\mathbb{R}^{2k+1}$, or if it admits a metric with positive scalar curvature then even k is allowed. (Observe in passing that these X are spin.)

[129]Sometimes, e.g. for lower bounds on the (area) norms of differentials of maps $X \times X_{\mathsf{kum}} \to S^n$, $n = dim(X)$, for metrics g on $X \times X_{\mathsf{kum}}$ with large scalar curvatures, the spin is irreplaceable.

[130]In truth, this applies only to *non-spin*$^{\mathbb{C}}$ manifolds, where spin$^{\mathbb{C}}$ means that the second Stiefel-Whitney class is equal to the mod 2 reduction of the Chern class of a complex line bundle $L \to X$.

Such bounds are available for spin$^{\mathbb{C}}$ manifolds. For instance (a special case of) *Min-Oo extremality/rigidity theorem* says that

if the scalar curvature a Riemannian metric g on on $\mathbb{C}P^m$ is (non-strictly) greater than that of the Fubini-Study metric, $Sc(g) \geq Sc(g_{FuSt})$, and $area_g(S) \geq area_{g_{FuSt}}(S)$ for all smooth surfaces $S \subset \mathbb{C}P^m$, than $g = g_{FuSt}$.

(The complex projective spaces $\mathbb{C}P^m$ are non-spin for even m, yet they are all spin$^{\mathbb{C}}$).

3.3.1 Recollection on Linear Connections and Twisted Differential Operators

A connection in a smooth fibration $L \to X$ is a *retractive homomorphism* from the tangent bundle $T(L)$ to the subbundle $T_{vert} = T_{ver}(L) \subset T(L)$ of the vectors tangent to the fibers of L.[131]

Denote this by

$$\hat{\nabla} : T(L) \to T_{vert} \subset T(L),$$

and observe that $\hat{\nabla}$ is uniquely defined by its kernel, that is what is called a horizontal subbundle, $T_{hor} = T_{hor}(L) \subset T(L)$ that is complementary to T_{vert} such that $T(L) = T_{vert} \oplus T_{hor}$.

If L is a trivia?l (split) fibration $L = X \times L_0$, then it comes with the trivial or split flat connection, where T_{hor} is the bundle of vectors tangent to the graphs of constant maps $X \to L_0$, $l \in L$.

A connection is called *flat* at $x_0 \in X$ if, over a neighbourhood $U \subset X$ of x, it is *isomorphic* to the trivial flat connection on $X \times L_{x_0}$, for the fiber L_{x_0} of L over x_0.

If the fibration L carries a fiber-wise geometric structure $\mathscr{S}$, say, linear, affine, unitary, etc, then "flat" signifies that the implied isomorphism, that is a fiber preserving diffeomorphism $L_{|U} \to U \times L_x$, preserves $\mathscr{S}$, i.e. it is fiber-wise linear, affine, unitary, etc.

A connection $\hat{\nabla}$ in L is called $\mathscr{S}$: linear, affine, unitary, etc if, for each $x \in X$, there exist a flat $\mathscr{S}$-connection $\hat{\nabla}_{x,flat}$ *adapted to* $\hat{\nabla}$ at x, i.e.such that the restriction of $\hat{\nabla}_{x,flat}$ to the fiber $L_x \subset L$, denoted $(\hat{\nabla}_{x,flat})_{|L_x}$ is equal to $\hat{\nabla}_{|L_x}$.

Twisting Differential s. A first order differential between (sections of) vector bundles (linear fibrations) K_1 and K_2 over a manifold X, is a linear map

$$D : C^\infty(K_1) \to C^\infty(K_2),$$

such that the value $Df(x) \in K_2$ depends only on the differential $df(x) : T_x(X) \to T_{f_x}(K_1)$ for all $x \in X$.

For instance, a linear connection in L defines a differential , denoted just ∇, from L to the bundle $Hom(T(X), L) = T^*(X) \otimes L$, that is the composition of the differential $df : T(X) \to T(L)$ with $\hat{\nabla} : T(L) \to T_{vert}$ combined with the canonical identifications of all (vertical) tangent spaces of the fiber L_x with L_x itself.

Such a ∇ uniquely determines (linear) $\hat{\nabla}$, it is also called "connection". where the values $\nabla f(\tau)$ at tangent vectors τ are written as (covariant) derivatives $\nabla_\tau f$.

Basic Example. If ∇ is the flat split connection in $X \times L_0$, then this is applies to sections $X \to X \times L_0$, that are the graph of maps $f : X \to L_0$, as the ordinary differential $df : T(X) \to L_0$.

If a section $f : X \to L$ vanishes at a point $x \in X$, then, clearly, $\nabla f(x) = \nabla_{flat} f(x)$ for all *nabla.*

It follows that the difference between two connections in L, $\nabla_1 - \nabla_2$, is a it zero order defined by a homomorphism $\Delta = \Delta_{1,2} : L \to Hom(T(X), L)$, that can be thought of as a $Hom(L, L)$-valued 1-form on X.

Thus any ∇ in a flat, e.g. split, bundle is $df + \Delta$.

[131] Here, "retractive" means being the identity on T_{vert}.

If ∇ is a *flat split connection*, in $L = X \times L_0$, then the twisted $D_{\otimes L} : C^\infty(K_1 \otimes L) \to C^\infty(K_1 \otimes L)$ is defined via the identity $C^\infty(K \otimes L_{split}) = C^\infty(K) \otimes L_0$, as it was explained above for the Dirac operator.

If ∇ is *flat*, then $D_{\otimes\nabla} = D_{\otimes(L,\nabla)}$ is defined on all neighbourhoods where this connection splits and local $\Rightarrow$ global by locality of differential s.

Finally, for *general* (L, ∇), the twisted $D_{\otimes\nabla}(\psi)$ for sections $\psi : X \to K_1 \otimes L$ is determined by its values at all points $x \in X$ that are defined as follows

$$D_{\otimes\nabla}(\psi)(x) = D_{\otimes\nabla_{x,flat}}(\psi)(x)$$

for flat connections $\nabla_{x,flat}$ adapted to ∇ at x.

Since the difference $\nabla - \nabla_{flat}$ is a *zero order* for all connections ∇ in flat split bundles $L = (X \times L_0 \nabla_{flat})$, the same is true for D twisted with ∇: the difference

$$\Delta_\otimes = D_{\otimes\nabla} - D_{\otimes\nabla_{flat}}$$

is a zero order – "vector potential" in the physicists' parlance.

A similar representation $D_{\otimes\nabla} = D_{\otimes\nabla_{flat}} + \Delta_\otimes$ for topologically non-trivial bundles L is achieved as follows.

Let $L^\perp \to X$ be a bundle complementary to L such that the Whitney sum of the two bundles topologically splits,

$$L \oplus L^\perp = L^\oplus \simeq X \times (L_0 \oplus L_0^\perp)$$

and let $\nabla^\perp$ be an arbitrary connection in $L \oplus L^\perp$ and

Define the connection $\nabla^\oplus = \nabla + \nabla^\perp$ in $L^\oplus$ by the rule

$$\nabla^\oplus_\tau(l + l^\perp) = \nabla_\tau l + \nabla^\perp_\tau l^\perp$$

and observe that the $\nabla^\oplus$-twisted operator $D_{\otimes\nabla^\oplus}$, that maps the space

$$C^\infty(K_1 \otimes L^\oplus) = C^\infty(K_1 \otimes L) \oplus C^\infty(K_1 \otimes L^\perp)$$

to

$$C^\infty(K_2 \otimes L^\oplus) = C^\infty(K_2 \otimes L) \oplus C^\infty(K_2 \otimes L^\perp)$$

respects this splitting:

$$D_{\otimes\nabla^\oplus} = D_{\otimes\nabla} \oplus D_{\otimes\nabla^\perp}.$$

Thus, if not the $D_{\otimes\nabla}$ itself, then its $\oplus$-sum with another is

$$D_{\otimes\nabla_{flat}} + \text{zero order}.$$

3.3.2 [Sc $\not>$ 0] for Profinitely Hyperspherical Manifolds, Area Decreasing Maps and Upper Spectral Bounds for Dirac Operators

Conclusion of the proof of provisional proposition [**Sc** $\not>$ **0**] *from 3.3.* Return to the bundles $L_\varepsilon = f^*(\underline{L}) \to X$ induced by smooth ε-Lipschitz maps $f : X \to S^n$, $n = dim(X) = 4l + 2$, with non-zero degrees and $\varepsilon \to 0$ from a complex vector bundle $\underline{L} \to S^n$, with the Euler number $e(\underline{L}) \neq 0$.

Let $\underline{L}^\perp \to S^{2k}$ be a bundle *complementary* to $\underline{L} \to S^{2k}$, i.e. the sum $\underline{L} \oplus \underline{L}^\perp$ is a trivial bundle, endow $\underline{L}$ and $\underline{L}^\perp$ with a connections $\underline{\nabla}$ and $\underline{\nabla}^\perp$ and let $\nabla^\oplus_\varepsilon$ be the connection on the (topologically trivial!) bundle

$$L^\oplus_\varepsilon = f^*(\underline{L} \oplus \underline{L}^\perp)$$

induced from $\underline{\nabla}^\oplus = \underline{\nabla} \oplus \underline{\nabla}^\perp$, where the latter is defined by the component-wise differentiation rule:

$$\underline{\nabla}^\oplus(\phi,\psi) = (\underline{\nabla}\phi, \underline{\nabla}^\perp\psi) \text{ for sections } (\phi,\psi) = \phi + \psi : S^n \to \underline{L} \oplus \underline{L}^\perp.$$

Then (see the proof of **[Sc $\ngtr$ 0]**) the twisted Dirac operator decomposes into the sum of (essentially) untwisted $\mathcal{D}$ and a zero order (vector potential)

$$\mathcal{D}_{\otimes \nabla^\oplus_\varepsilon} = \mathcal{D}_{\otimes \nabla_{flat}} + \Delta_\varepsilon$$

where ∇_{flat} is the flat split connection in the bundle $L^\otimes_\varepsilon$ with the splitting induced by f_ε from a splitting of $\underline{L} \oplus \underline{L}^\perp$, obviously (but most significantly), $\Delta_\varepsilon \to 0$ for $\varepsilon \to 0$.

Now, the (untwisted) S-L-W-(B) formula, applied to $\mathcal{D}_{\otimes \nabla_{flat}}$ says that

$$\mathcal{D}^2_{\otimes \nabla^\oplus_\varepsilon} = \nabla_{flat,\mathbb{S}} \nabla^*_{flat,\mathbb{S}} + \frac{1}{4} Sc + \Delta^\square_\varepsilon,$$

where $\nabla_{flat,\mathbb{S}}$ denotes the flat connection $\nabla_{flat,\mathbb{S}}$ in the twisted spin bundle associated with ∇_{flat}.

The correction term $\Delta^\square_\varepsilon$ in this formula is a first order differential (it depends on how you trivialise the bundle $\underline{L} \oplus \underline{L}^\perp$) which tends to 0 for $\varepsilon \to 0$,

$$\Delta^\square_\varepsilon \to 0 \text{ for } \varepsilon \to 0.$$

A priori, the ε-bound on the differential of f_ε doesn't make the coefficients of the $\Delta^\square_\varepsilon$ small, but an obvious smoothing allows an approximation of f_ε by maps their derivatives of which of *all orders converging to* 0.

Because of this, we may assume $\Delta^\square_\varepsilon \to 0$ in the *strongest conceivable sense*, while al is needed is that $\Delta^\square_\varepsilon \to 0$ becomes negligibly small compare to $\nabla_{flat,\mathbb{S}} \nabla^*_{flat,\mathbb{S}} + \frac{1}{4} Sc$, which implies *strict positivity* of the $\mathcal{D}^2_{\otimes \nabla^\oplus_\varepsilon} = \nabla_{flat,\mathbb{S}} \nabla^*_{flat,\mathbb{S}} + \frac{1}{4} Sc + \Delta^\square_\varepsilon$, for ε much smaller than the lower bound $\sigma = \inf_{x\in X} Sc(X,x) > 0$.

Thus, the condition $Sc(X) > 0$ leads to a contradiction with the index formula, which in this case, as we already know from the proof of **[Sc $\ngtr$ 0]** yields non-zero harmonic ∇_ε-twisted, hence $\nabla^\oplus_\varepsilon$ twisted, spinors, because the subbundle $\underline{L} \subset \underline{L} \subset$ is invariant under the parallel transport by the connection $\underline{\nabla}^\oplus = \underline{\nabla} \oplus \underline{\nabla}^\perp$, by the very definition of the sum of connections and this property is inherited by the induced connection $\nabla^\oplus_\varepsilon$.

This concludes the proof of **[Sc $\ngtr$ 0]** for $n = 4l + 2$ and, as we have already explained, the general case follows by stabilization $X \rightsquigarrow X \times \mathbb{T}^{3n+2}$.

Area Contraction instead of Length Contraction. Say that X is $\wedge^2$-*profinitely hyperspherical* if, instead of ε-Lipschitz property of maps $f_\varepsilon \tilde{X}_\varepsilon \to S^n$ of no-zero degree, we require that the second exterior power of the differential of f_ε is bounded by ε^2,

$$\| \wedge^2 df_\varepsilon(x) \| \le \varepsilon^2.$$

Geometrically, this means that f_ε decreases the areas of the smooth surfaces in X by factor ε^2, (This, obviously, is satisfied by ε-Lipschitz maps.)

It is clear, heuristically, that the Dirac operator *twisted with* ∇_ε in this case, similarly how it is for ε-Lipschitz maps, is *close to the untwisted* $\mathcal{D}$; this

rules out positive scalar curvature for $\wedge^2$-profinitely hyperspherical manifolds.

However, the above proof with the complementary bundle $L^\perp$ doesn't apply here; to justify heuristics, one has to pursue algebraic similarity between ∇ and the ordinary differential d a step further.

This can be done by pure thought, on the basis of general principles only, (no tricks like $L^\perp$) but writing down this "thought" turned out more space and time consuming than what is needed for (a few lines of) the twisted version of the S-L-W-(B) formula, as we shall see in section 3.3.4.

So, we conclude here with two remarks.

(i) It is unknown if "length contractive" is *topologically* more restrictive than "area contractive".

For instance one has no idea if there exist $\wedge^2$-profinitely hyperspherical manifolds which are *not* profinitely hyperspherical.

(ii) Representation of ∇-twisted differential operators by vector-potentials Δ in larger bundles has further uses, such as *Vafa-Witten's lower bounds on the spectra of Dirac operators*. For instance,

if a compact Riemannian spin n-manifold X admits a distance decreasing map to S^n of degree d, then the number N of eigenvalues λ of the Dirac on X in the interval $-C_n \leq \lambda \leq C_n$ satisfies $N \geq d$, where $C_n > 0$ is a universal constant.[132]

3.3.3 Clifford Algebras, Spinors, Atiyah-Singer Dirac Operator and Lichnerowicz Identity

The Dirac on $\mathbb{R}^n$ is a particular first order differential , which acts on the space of smooth $\mathbb{C}^N$-valued functions,

$$\mathcal{D} : C^\infty(\mathbb{R}^n, \mathbb{C}^N) \to C^\infty(\mathbb{R}^n, \mathbb{C}^N),$$

where $N = 2^{\frac{1}{2}n}$ for even n and $N = 2^{\frac{1}{2}(n-1)}$ for odd n and where essential properties of this $\mathcal{D}$ are as follows.

I. *Ellipticity.* The $\mathcal{D}$ is an *elliptic*, which means that the initial value (Cauchy) problem for the equation $Df = 0$ is *formally uniquely solvable* for *all* initial data on *all* smooth hypersurfaces in $\mathbb{R}^n$, where "formally" can be replaced by "locally" in the real analytic case.

Basic Example. The Cauchy-Riemann (system of two) equation(s) $D_{CR}f = 0$ for maps $f : \mathbb{R}^2 \to \mathbb{C}^1$, defines conformal orientation preserving maps $\mathbb{R}^2 \to \mathbb{C}$. These are called *holomorphic* if $\mathbb{R}^2$ is "identified" with $\mathbb{C}$, where the ambiguity inherent in this identification is responsible for spin.

D_{CR} *is elliptic*: real analytic functions locally uniquely extend from real analytic curves in $\mathbb{C}^1$ to holomorphic functions.

Let us describe ellipticity in linear algebraic terms applicable to all (systems of) partial differential equations of first order for maps between smooth manifolds, $f : X \to Y$. Such a system, call it S, is characterised by subsets in the spaces of linear maps between the tangent spaces of X and Y at all $x \in X$ and

[132] See §6 in [G(positive) 1996] for related spectral geometric inequalities.

$y \in Y$, denoted $\Sigma_{x,y} \subset Hom(T_x \to T_y)$, where $T_x = \mathbb{R}^n$, $n = dim(X)$, where $T_y = \mathbb{R}^m$, $m = dim(Y)$ and where f *satisfies* S if $df(x) \in \Sigma_{x,f(x)}$ for all $x \in X$.

Let $R_L : Hom(\mathbb{R}^n, \mathbb{R}^m) \to Hom(L, \mathbb{R}^m)$ denote the restriction of linear maps to $\mathbb{R}^m$ from $\mathbb{R}^n$ to linear subspaces $L \subset \mathbb{R}^n$, that is $R_L : h \mapsto h_{|L} : L \to \mathbb{R}^m$.

Call a smooth submanifold $\Sigma \subset Hom(\mathbb{R}^n, \mathbb{R}^m)$ *elliptic* if the map R_L *diffeomorphically* sends Σ *onto* $Hom(L, \mathbb{R}^m)$ for all *hyperplanes* $L \subset \mathbb{R}^n$.

Now, a PDE system S is called *elliptic* if the subsets

$$\Sigma_{x,y} \subset Hom(T_x \to T_y) = H_{n,m} = Hom(\mathbb{R}^n, \mathbb{R}^m)$$

are elliptic for $x \in X$ and $y \in Y$.

Put it another way, let $K_p \in H_{n,m}$, $p \in \mathbb{R}P^{n-1}$, be the family of m-dimension linear subspaces that are the kernels of the linear maps $R_{L_p} : H_{n,m} \to H_{n-1,m,p} = Hom(L_p, \mathbb{R}^m)$ parametrized by the projective space $\mathbb{R}P^{n-1}$ of hyperplanes $L = L_p \subset \mathbb{R}^n$. Then ellipticity says that

Σ *transversally* intersect K_p at a single point for all $p \in \mathbb{R}P^n$.

Finally, back to the linear case, observe that systems $Df = 0$ for maps

$$f : \mathbb{R}^n \to \mathbb{R}^m,\ x \in \mathbb{R}^n$$

are depicted by *linear* subspaces

$$\Sigma = \Sigma_x \subset Hom(T_x(\mathbb{R}^n), \mathbb{R}^m = T_0(\mathbb{R}^m)), x \in \mathbb{R}^n$$

and ellipticity says in these terms that

firstly, $dim(\Sigma) = n$

secondly

○ *the linear maps* $h : T_x(\mathbb{R}^n) \to \mathbb{R}^m$ *have* $rank(h) = n$ *for all non-zero* $h \in \Sigma$.

and finally

differential operators between sections of vector bundles over a smooth manifold X are elliptic if these properties are verified locally over small neighbourhoods of all points in X.

Exercises. (a) Twisting with ∇. Show that

D is elliptic $\Rightarrow$ $D_{\otimes\nabla}$ is elliptic:

twisting with connections doesn't hurt ellipticity.

(b) Symmetric but non-Elliptic. Figure out what makes the exterior differential

$$d : C^\infty(\bigwedge^k(T(X))) \to C^\infty(\bigwedge^{k+1}(T(X)))$$

on $(2k+1)$-dimensional manifolds *non-elliptic.*

II. *Symmetry and Spinors.* The Dirac operator $\mathcal{D}$ on $\mathbb{C}^\infty(\mathbb{R}^n, \mathbb{C}^N)$, is $Spin(n)$-*invariant*, where

$\bullet_1$ $Spin(n)$ denotes the double cover of the special orthogonal group $SO(n)$,

$\bullet_2$ the group $Spin(n)$ acts on $\mathbb{R}^n$ via the (2-sheeted covering map) homomorphism $Spin(n) \to SO(n)$,

$\bullet_3$ the action of $Spin(n)$ on $\mathbb{C}^k$, called *spin representation*, is *faithful*: it *doesn't factor* through an action of $SO(n)$,[133]

[133] The spin representation, as we shall explain below, is *irreducible* for odd n and it splits into *two irreducible half-spin representations* for even n. There are no *faithful* representations of $Spin(n)$ in lower dimensions (except for $n = 1, 2$), where, apparently, this faithfulness is necessitated by ellipticity of $\mathcal{D}$.

$\bullet_4$ "invariant" here means *equivariant* under the (diagonal) action of $Spin(n)$ on the space of maps $\psi : \mathbb{R}^n \to \mathbb{C}^N$, that is

$$g(\psi)(x) = g(\psi(g(x)),\ g \in Spin(n),^{134}$$

and "*equivariant*" says that

$$\mathcal{D}(g(\psi)) = g(\mathcal{D}(\psi)).^{135}$$

Cauchy-Riemann Example. The group $Spin(2)$ diagonally acts on on maps $f : \mathbb{R}^2 \to \mathbb{C}^1$, where all actions (representations) of $Spin(2) = \mathbb{T} = U(1) \subset \mathbb{C}^\times$ on $\mathbb{C}^1$ are possible: these are $t(z) = t^m z,\ m = ... -1, 0, 1, 2, ...$ (There are no such possibilities for for $n > 1$.)

The corresponding operators $\bar\partial = \bar\partial_m$ are all *locally non-canonically* isomorphic (this makes them often confused in the literature), but this m (spin quantum number), becomes the major feature of the $\bar\partial_i$ globally, where it controls its very existence and its index.

III. *Spin Representations and Clifford Algebras* $Cl_n = Cl(V)$.[136]The lowest dimension complex vector space, where $Spin(n)$, can linearly *faithfully* act is $\mathbb{C}^{2^k}$ for $k = \frac{1}{2}n$ for n even and $k = \frac{1}{2}(n-1)$ for odd n, where such an action (representation) is obtained by realizing $Spin(n)$ as a subgroups in the multiplicative semigroups of the *Clifford algebra*, denoted $Cl_n = Cl(\mathbb{R}^n) = Cl(\mathbb{R}^n, -\sum_1^n x_i^2)$.

Recall that Cl_n is an *unital*[137] *associative algebra* A over the field of real numbers with a distinguished *Clifford basis* that is linear subspace $V = V_{Cl} \subset Cl_n$ *endowed with a Euclidean structure*, that is represented by a *negative definite* quadratic form.[138]

We denote the *Clifford product* by $a_1 \cdot a_2$ an let "1" stand for the unit in A.

(There is nothing especially exciting about Cl_n understood as "just an algebra", especially if you tensor it with $\mathbb{C}$, which we do at the end of the day anyway. For instance, we shall see it presently, $Cl \otimes \mathbb{C}$ is isomorphic to a matrix algebra for even n and to the sum of two matrix algebras for odd n.

What gives to a particular favour to Cl_n is the distinguished linear subspace $V \subset Cl_n$, which, on the one hand, *generates all* of Cl_n, on the other hand, the matrices corresponding to all $v \neq 0$ in V, have maximal possible ranks, since all non-zero $v \in V$ are *invertible* in the multiplicative semigroup $CL_n^\times$. This "maximal rank property" is exactly what makes the Dirac operator *elliptic* and, because of this, so powerful in the Riemannian geometry.)

The fundamental feature of the pair (A, V) is that $A = Cl(V)$ is *functorially* determined by V:

isometric embeddings $V_1 \to V_2$ canonically extend to monomorphisms $A_1 \to A_2$.

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[134]To visualize this, think of the graphs $\Gamma_\psi \subset \mathbb{R}^n \times \mathbb{C}^N$ moved by the diagonal actions of $g \in Spin(n)$ on this product.

[135]This Dirac operator has "constant coefficients", which means is *invariant under parallel translations* t_y of $\mathbb{R}^n$ that act on our maps: $D(t_y(\psi)) = t_y(D(\psi))$ for $(t_y(\psi))(x) = \psi(x + y)$, $x, y \in \mathbb{R}^n$.

[136]The basic reading on this subject matter is the book [Lawson&Michelsohn(spin geometry) 1989] and a (very) brief outline of the main points is contained in [Min-Oo(K-Area) 2002], [Min-Oo(scalar) 2020].

[137]This means possessing a unit in it

[138]It is negative to agree with the Laplacian $\sum_i \partial_i^2$, which is a negative operator.

where this Clifford functor is uniquely characterised by the following two properties.

A. $V = V_{Cl}$ *is a Basis in* A. The subspace V generates A as an $\mathbb{R}$-algebra.

B. *Specification.* The algebra $Cl_1 = Cl(\mathbb{R}^1)$ is isomorphic to $(\mathbb{C}, i\mathbb{R})$, for $i = \pm\sqrt{-1}$.

(It is impossible to *mathematically*, distinguish i and $-i$; this unresolvable ±i-ambiguity is grossly amplified, at least psychologically,, when it comes to spinors. [139])

In simple words, the Clifford squares of all unit vectors $v \in V$ are equal to -1, or, equivalently,

$$v \cdot v = -||v||^2 = \langle v, v\rangle \text{ for all } v \in V.$$

A&B. *Anti-commutativity.* The Clifford product is *anti-commutative on orthogonal vectors.*

$$v_1 \cdot v_2 = -v_2, v_1, \text{ for } \langle v_1, v_2\rangle = 0.$$

Indeed, since $||v_1 - v_2||^2 = ||v_1 + v_2||^2$ for orthogonal vectors, bilinearity of the the Clifford product implies that

$$0 = (v_1 - v_2)^2 - (v_1 + v_2)^2 = -v_1 \cdot v_2 - v_2 \cdot v_1 + v_1 \cdot v_2 + v_2 \cdot v_1 = 2(v_1 \cdot v_2 + v_2 \cdot v_1).$$

Exercise. Show that

$$v_1 \cdot v_2 + v_2 \cdot v_1 = -2\langle v_1, v_2\rangle \text{ for all } v_1, v_v \in V.$$

IV. *Groups* $Pin(n)$ *and* $Spin(n)$ *and* G_n. The group $Pin(n)$ is defined in Cl_n-terms as the subgroup of the multiplicative semigroup of $Cl_n^\times \subset Cl_n$ multiplicatively generated by the unit vectors $v \in V \subset Cl_n$.

The subgroup $Spin(n) \subset Pin(n)$ consists of the products of *even numbers* of unit vectors from V. [140]

Existence & Uniqueness. Let us explain why the algebra $Cl(\mathbb{R}^n)$, if exists at all, is large enough to (multiplicatively) contain the group $Spin(n)$ that double covers the special orthogonal group $SO(n)$. [141]

Observe that the *Clifford relations* [142]

[Cl] $$e_i \cdot e_j = -e_j \cdot e_i \text{ and } e_i^2 = -1$$

[139] To be blameless, write $\pm i$ (even better, $\{\pm i, \mp i\}$) and never dare utter "left ring ideal" and "right group action", even in absence of left-handed (left-minded?) persons. (Defending such an action by *biological molecular homochirality* and *parity violation by weak interactions* is not recommended for being politically incorrect.)

Jokes apart, arbitrary terminological conventions presented as mathematical definitions sow confusion and undermine "rigor" in mathematics.

Who are the lucky ones who are able to tell if $f \circ g$ means $f(g(x))$ rather than $g(f(x))$ or vice versa?

Can *encoding formulas by Peano's integers*, e.g. in the proof of *Gödel's incompleteness theorem*, be accepted as "logically rigorous", unless you face the issue of "directionality" inherent in the decimal representation of integers?

[140] This parallels the definition of $SO(n) \subset O(n)$ as the subgroup consisting of products of *even numbers* of reflections of $\mathbb{R}^n$. In fact, $Spin(n)$ equals the connected component of the identity in $Pin(n)$ and $Pin(n)/Spin(n) = O(n)/SO(n) = \mathbb{Z}_2 = \{-1, 1\}$.

[141] To appreciate non-triviality of the problem, try to construct geometrically more than two, say three, anti-commuting linear isometric involutions represented by reflections around linear subspaces in some Euclidean space.

[142] This must be written in Clifford's unpublished note *On The Classification of Geometric Algebras* see [Diek-Kantowski (Clifford History)1995] for further references.

for an orthonormal frame $\{e_i\} \subset V$, $i = 1, ..., n$,

on the one hand, imply $v_1 \cdot v_2 + v_2 \cdot v_1 = -2\langle v_1, v_2\rangle$ for all $v_1, v_v \in V$, hence,

fully characterize Clifford's algebras,

on the other hand, define

a finite group G_n of order 2^{n+1} that is a central extension of $\mathbb{Z}_2^n$,

with an additional generator (central element) c of order 2 and the following relations,

$$[\mathbf{Cl_c}] \qquad ce_i = e_i c,\ c^2 = 1,\ e_i e_j = c e_j e_i \text{ and } e_i^2 = c.$$

where the central element c in G_n corresponds to $-1 \in Cl_n$.

Non-triviality of this G_n is apparent, since letting $c = 1$ defines a *surjective* homomorphism $G_n \to \mathbb{Z}_2^n$ with kernel $\mathbb{Z}_2$.

(What is not immediately apparent, is a pretty combinatorics of shuffling indices in $e_{i_1} e_{i_2} ... e_{i_m} \in G_n$, $i_1 < i_2 < ... < i_m$, under multiplication by e_k, which is rightly appreciated by people working on quantum computers.)

One look at G_n is sufficient to make it obvious that there is a *homomorphism from G_n to the multiplicative (semi) group $Cl_n^\times$* of the Clifford algebra (with the image in $Pin(n) \subset Cl^\times$), such that

the algebra homomorphism from the *real group algebra* $\mathbb{R}(G_n)$[143] to Cl_n associated with this group homomorphism $G_n \to Cl_n^\times$ is *surjective* and the kernel of this homomorphism is defined by the relation $c = -1$, that is

$$Cl_n = \mathbb{R}(G_n)/(c+1),^{144}$$

Amazingly, nowhere, except for a few papers on quantum computers, G_n is called "finite Clifford group",[145] while the authors of the only mathematical papers found on the web (unless I missed some) call G_n a "*Salingaros vee group.*"[146]

The structure this "vee group" $G_,$, which tells you everything about Cl_n, is transparently seen in the combinatorics of its multiplication table, where $g \in G$

[143] $\mathbb{R}(G)$ is the space of formal linear combinations $\sum_{g\in G} c_g g$ with the obvious product rule, where the identity element $id \in G$ serves as the unit of this algebra.

Alternatively, $\mathbb{R}(G)$ is defined as the algebra of linear operators $\mathbb{R}(G)$ on functions $\psi(g)$ that is generated by translations on the space of functions on G, for $\psi(g) \mapsto \psi(g'g)$, $g' \in G$.

The same space $\mathbb{R}(G) = G^{\mathbb{R}}$ of functions on G with the action of G by $\psi(g) \mapsto \psi(g'g)$ is called (not very inventively) the *regular $\mathbb{R}$-representation* of G, where just "regular representation" stands for regular $\mathbb{C}$-representation.

[144] Recall that $c \in G_n \subset \mathbb{R}(G_n)$ is the central involution in G_n and "1" is the unit in the algebra $\mathbb{R}(G_n)$ that is represented by the unit function, where $(c+1) \subset \mathbb{R}(G_n)$ denotes the ideal generated by $c+1 \in \mathbb{R}(G_n)$. (The quotient algebra $\mathbb{R}(G_n)/(c+1)$ has the same underlying linear space as the group algebra $\mathbb{R}(G_n/(c))$, for the normal subgroup $(c) \subset G_n$ generated by c, but multiplicatively $\mathbb{R}(G_n)/(c+1)$ is much different from the (commutative) group algebra of $G_n/(c) = \mathbb{Z}_2^n$.)

[145] The terms "Clifford group", sometimes "naive Clifford group", are reserved for the subgroup G of the multiplicative semigroup of Cl, the action of which on Cl by conjugation for $a \mapsto g \cdot a \cdot g^{-1}$ preserves V.

[146] See [AbVaWa(Clifford Salingaros Vee)2018] for more general definitions and references to the the original 1981-82 papers by Nikos Salingaros. (I don't know what is written in these papers, since these are not openly accessible on the web.)

Also, amazingly, no survey or tutorial on Clifford algebras I located on the web makes any use or even mentions G_n. Possibly, there is something about it in textbooks, but none seems to be openly accessible.

are written as *lexicographically ordered products* of e_i and (if it is there) c. Here are a few relevant properties of G_n.

All elements in G_n have orders 2 and/or 4.

The commutator subgroup $[G_n, G_n] = \{g_1 g_2 g_1^{-1} g_2^{-1}\}$ equals to the 2-element (central) subgroup $\{1, c\}$.

If n is even, it coincides with the center of G_n;

$$center(G_n) = [G_n, G_n] = \{1, c\}$$

If n is odd, the center has order 4. For instance $G_1 = \mathbb{Z}_4$; in general, the extra central element for $n = 2k + 1$ is the product $e_1 e_2, ..., e_n$.

If n is even, then the number $N_{conj}(G_n)$ of the conjugacy classes of G_n is $2^n + 1$ where 2^n of them comes from $\mathbb{Z}_2^n$ and the extra one is that of c. If n is odd, there are $2^n + 2$ classes, where centrality of $e_1 e_2, ..., e_n$ is responsible for the additional one.

V. *Representations of the Group G_n.* The space $\Psi_n = \mathbb{C}(G_n) = \mathbb{C}^{G_n}$ of *complex* functions on G splits into the sum $\Psi_n = \Psi_n^+ \oplus \Psi_n^-$, where Ψ_n^+ consists of *c-symmetric functions* $\psi(g)$ that are *invariant* under the action of the central $c \in G_n$, i.e. $\psi(g) = \psi(cg)$ and where the functions $\psi \in \Psi_n^-$ are *antisymmetric*, $\psi(cg) = -\psi(g)$.

The space Ψ_n^+ obviously identifies with the space $\mathbb{C}(\mathbb{Z}_2^n)$ of functions on the Abelian group $\mathbb{Z}_2^n$, where action of G_n factors through the homomorphism $G_n \to \mathbb{Z}_2^n$.

Since the commutator subgroup of G_n is equal to $\{1, c\}$, all 1-dimensional representations of G_n are contained in Ψ_n^+.

Frobenius

Now, the *number one theorem* in the representation theory of finite groups reads:[147]

the regular representation of G uniquely decomposes into the sum of sub-representations $G^{\mathbb{C}} \oplus_i R_i^2$, $i = 1, 2, ..., N = N_{irrd}(G)$, where each R_i^2 is (non-canonically) isomorphic to the sum of k_i-copies of an irreducible representation R_i of dimension k_i and where every irreducible representations of G is isomorphic to one and only one of R_i.

Accordingly, the group algebra of G (the same linear space $G^{\mathbb{C}}$, but now with the group algebra structure) *decomposes into the sum of matrix algebras*

$$\mathbb{C}(G) = \bigoplus_i End(\mathbb{C}^{k_i}).$$

This is an exercise in linear algebra. What is less obvious is that

The number $N_{irrd}(G)$ of mutually non-isomorphic irreducible complex representations of G is equal to the number of the conjugacy classes in G.

$$N_{irrd}(G) = N_{conj}(G) \text{ for all finite group } G.$$

Consequently,

[147]This must be attributed to Frobenius (1896), since it follows by his character theory, see `file:///Users/misha/Downloads/Curtis2001_Chapter_RepresentationTheoryOfFiniteGr.pdf` Unfortunately, this theorem has no name an can't be *instantaneously* found on Google.

the sum of the squares of the dimensions of the irreducible representations of G is equal to the order of the group G,

$$\sum_i k_i^2 = card(G).^{148}$$

If we apply this to G_n for $n = 2k$, we shall see that, besides the one dimensional representations, this group has a *single irreducible* one of dimension 2^k, call it S_n, which enters the regular representation with multiplicity 2^k.

Now, clearly,

the 2^k-multiple S_n-summand of the regular representation is exactly the space Ψ_n^- of antisymmetric functions ψ on G_n.

Equally clearly,

the space of antisymmetric functions $\psi(g) = -\psi(cg)$ on G_n (here we speak of $\mathbb{R}$-valued functions ψ) is G_n-equivariantly isomorphic to Cl_n.

VI. *Clifford Conclusion.* Since the Clifford algebra Cl_n is, as an algebra, generated by $G_n \subset Cl_n$, the representation S_n of G_n in $\mathbb{C}^{2^k}$, that is a multiplicative homomorphism $G_n \to End(C^{2^k})$, extends to an algebra homomorphism $Cl_n \to End(C^{2^k})$; hence, to

an irreducible representation of $Pin(N)$ in $\mathbb{C}^{2^k}$,

which extends (irreducible!) representation S_n of $G_n \subset Pin(n)$.

This is called *the spin representation* and still denoted S_n.

Why Clifford Algebra? Why algebras are needed here at all?

What we used for the construction of the spin representation S_n of $Pin(n)$ in $\mathbb{C}^{2^k}$ for even $n = 2k$ are the two following simple, not to say "trivial", but indispensable (are they?) algebra theoretic facts.

(i) The linear actions of $Pin(n)$ and G_n on the space Ψ_n^- (and also on Cl_n) generate the same subalgebras of operators on this space.

(ii) If an algebra A of operators on a linear space M, e.g. $M = \mathbb{C}^{N^2}$, is isomorphic to the (full matrix) algebra of endomorphisms of another space,

$$A \simeq End(L),$$

then M is A-equivariantly isomorphic to $End(L)$ for, say "left", action of the algebra $End(L)$ on itself.

(Also we were jumping back and forth between $\mathbb{R}$-linear and $\mathbb{C}$-linear spaces and actions, but with nothing non-trivial happening on the way.)

The correspondence $\Phi : L \rightsquigarrow A = End(L)$ *is a functor* from the category of vector spaces over $\mathbb{R}$ to the category of unital $\mathbb{R}$-algebras, but L can be reconstructed from $End(L)$ only *up to a homothety* $l \mapsto rl$, $r \in \mathbb{R}^\times$,where the projective space $P = L/\mathbb{R}^\times$ can be identified with the space of *maximal left ideals* in $End(L)$[149]

(Because of this ambiguity, one can't globally define the Dirac operator on a non-spin manifold X, because there is no vector bundle that would support $\mathcal{D}$.

[148]See https://projecteuclid.org/download/pdf_1/euclid.lnms/1215467411 and the character sections in https://web.stanford.edu/~aaronlan/assets/representation-theory.pdf and https://arxiv.org/pdf/1001.0462.pdf.

[149]Left ideals $I \subset End(L)$ corresponds to linear subspaces $L_I \subset L$, such that $a \in I \Leftrightarrow a_{|L_I} = 0$.

And although the the projectivized spin bundle $\mathcal{PS} \to X$ with a real projective space as the fiber is still there, this fibration *admits no continuous section* $X \to \mathcal{PS}$ – non-zero second Stiefel-Whitney class is an obstruction to that.)

VII. *Subgroup* $G_n^0 \subset G$ *and half-Spin Representations.* Let $\mathbb{Z}^n \to \mathbb{Z}_2$ be the (only) non-zero homomorphism, which is invariant under permutations of e_i, denote by $deg : G_n \to \mathbb{Z}_2 = \{-1, 1\}$ be the composition of this with the homomorphism $G_n \to \mathbb{Z}_2^n$ which sends $c \to 1$ and let G_n^0 be the *kernel of this "degree" homomorphism.*

In terms of Cl_n, this is the intersection of the subgroups G_n and $Spin(n)$ in $Pin(n)$,

$$G_n^0 = G_n \cap Spin(n) \subset Pin(N) \subset Cl_n.$$

Exercise. Show that G_{n+1}^0 is isomorphic to G_n.

Hint. Send $e_i \in G_n$, $i = 1, ..., n$, to the products $e'_{n+1}e'_i$ for $e'_1, ..., e'_{n+1} \in G_{n+1}$.

Let $\hat{e} = e_1 e_2 ... e_n$ and let us split the representation space $L = \mathbb{C}^{2^k}$ of S_n for even $n = 2k$ into ± 1-eigenspaces of $\hat{e}$, $L = L^+ \oplus L^-$

If n is even then this $\hat{e}$ anti-commute with all e_i, that is $\hat{e}e_i = ce_i\hat{e}$.

It follows that, for $n = 2k$,

all e_i that act via S_n on L send $\mathbb{L}^+ \leftrightarrow L^-$

and

the restriction of the representation S_n *on* $L = \mathbb{C}^{2^k}$ *from the group* G_n *to the subgroup* $G_n^0 \subset G_n$ *sends* $L^+ \to L^+$ *and* $L^- \to L^-$.

Furthermore, since the representation S_n is irreducible for G_n,

the representations $S_n^\pm$ on $L^\pm$ are irreducible for G_n^0.

Extend these representations by linearity to the subalgebra $Cl_n^0 \subset Cl_n$ generated by $G_n^0 \subset Cl_n$, observe that Cl_n^0 contains the group $Spin(n)$ and restrict from Cl_n^0 to $Spin(n)$. Thus, for $n = 2k$, we obtain

two faithful irreducible representations, called half-spin representations $S^\pm$ *of the group* $Spin(n)$ *of dimensions* 2^{k-1}.

Remark/Question. The above shows that a linear space of dimension $< 2^k$ can't have $2k$ anti-commuting anti-involutions. *Is there a direct geometric proof of this?*

(The answer must be known to some people.)

VIII. *Clifford's* $Spin(n)$ *Covers* $SO(n)$, What remains (for $n = 2k$) to show is that this $Spin(n)$, which is defined as the subgroup of the multiplicative group of the Clifford algebra Cl_n generated by products of even numbers of unit vectors $V \in V \subset Cl_n$, double covers the special orthogonal group $SO(n)$.

To do this we define an orthogonal (i.e. linear isometric) action of all of $Pin(n) \supset Spin(n)$ on the (n-dimensional Euclidean) subspace $V = V_{Cl} \subset Cl_n$ as follows.

Let $\alpha : Cl_n \to Cl_n$ be the automorphism that linearly extends $v \mapsto -v$ on $V \subset Cl_n$ and let

$$p(v) = \alpha(p) \cdot v \cdot p^{-1} \text{ for } v \in V \text{ and } p \in Pin(n).$$

It is clear that if p is a unit vector in V, then the transformation $v \mapsto p(v)$ sends V to itself by reflection in the hyperplane $p^\perp \subset V$ normal to p. (You can think of this $p \in Pin(n)$ as the square root of such a reflection.[150])

Since α is an *automorphism* of the Clifford algebra, the map from $Pin(n)$ to the group $O(n)$, regarded as the group of linear Euclidean isometries of $V = (V, \sum_i x_i^2)$, is a *homomorphism* of groups, which sends $Spin(n)$ onto this $SO(n)$.

To conclude, we need to show that the kernel of the homomorphism $Pin(n) \to O(n) \subset End(V)$ is equal to $\{1, -1\} \subset Cl(n)$, which is done by induction on n starting from $Pin(1) = \mathbb{Z}_4 = \{1, i, -1, i\}$ and $\alpha(i) = -1$, and using the following.
.

Lemma. If $\alpha(p) \cdot v \cdot p^{-1} = v$ for a unit vector $v \in V$, then p is contained in the subalgebra $Cl(v^\perp) \simeq Cl_{n-1}$ generated by the hyperplane $v^\perp \subset V$.

Proof. Decompose the Clifford algebra into sum of four linear subspaces,

$$Cl_n = A_0 \oplus v \cdot A_1 \oplus A_1 \oplus v \cdot A_0,$$

where $A_0 \subset Cl(v^\perp)$ is equal to the +1-eigenspace of α, i.e. where $\alpha(a) = a$, and $A_1 \subset Cl(v^\perp)$ is the -1-eigenspace.

Observe that all a_0 in A_0 are linear combinations of products of of *even numbers* of vectors from V, while all $a_1 \in A_1$ are combinations of *odd* products.

Now, by keeping track of parity of products we see that the relation $\alpha(p) \cdot v \cdot p^{-1} = v$ divides into two equalities,

$$(a_0 + v \cdot a_1') \cdot v = v \cdot (a_0 + v \cdot a_1') \text{ and } (a_1 + v \cdot a_0') \cdot v = -v \cdot (a_1 + v \cdot a_0'),$$

which imply that $a_1' = 0$ and $a_0' = 0$.

Indeed, since v *commutes* with a_0 and anti-commute with a_1,

$$(a_0 + v \cdot a_1') \cdot v = v \cdot (a_0 + v \cdot a_1') \Rightarrow v \cdot a_1' \cdot v = v \cdot v \cdot a_1' \Rightarrow -v \cdot v \cdot a_1' = v \cdot v \cdot a_1',$$

and $v \cdot v = -1 \Rightarrow a_1' = 0$.

Similarly, one shows that also $a_0' = 0$ and lemma follows.

Finally, we are through with *even* n:

the double cover group $Pin(n) \to O(n)$ for $n = 2k$ comes with a faithful irreducible complex representation $S_n = S_{2k}$ in $\mathbb{C}^{2^{2k}}$, called *spin representation.*[151]

The restriction of S_n to $Spin(n) \subset Pin(n)$, that is the double cover of $SO(n) \subset O(n)$, splits into the sum $S_n = S^+_{\frac{1}{2}n} \oplus S^-_{\frac{1}{2}n}$ of two complex conjugate[152] representations, called *half spin representations.*[153]

IX. *About Odd n.* A quick way to arrive at the spin representation S_{2k} of the group $Spin(n)$ in $\mathbb{C}^{2^k}$ for $n = 2k+1$ is by imbedding $Spin(n) \hookrightarrow Cl^\times_{n-1}$ and then restricting the spin representation $S_{n-1=2k}$ the Clifford algebra Cl_{n-1} to the so embedded $Spin(n) \subset Cl^\times_{n-1}$.

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[150] If you omit α, the resulting transformation square$v \mapsto pvp^{-1}$ becomes *minus reflection* in $p^\perp$. Thus, if n is odd, all of $P(n)$ ends up in $SO(n)$.

Since one wants $Pin(n)$ to cover the full orthogonal group $O(n)$ one brings in this α.

[151] There in no faithful representation of $Pin(n)$ in a lower dimensional space, since even the subgroup $G_n \subset Pin(n)$ admits no such representation.

[152] We didn't prove these are complex conjugate but this follows from their construction

[153] Arriving at this point took unexpectedly long – not a page or two as I had expected.

To achieve this, we start, somewhat paradoxically, with a (somewhat artificial) embedding $Cl_{n-1} \to Cl_n$ that sends Cl_{n-1} onto the *even part* $Cl_n^0 \subset Cl_n$, that is the +1-eigen space of the automorphism $\alpha : Cl_n \to Cl_n$ of the Clifford algebra induced by the central symmetry $v \mapsto -v$ of the Clifford base subspace $V = V_{Cl} \subset Cl_n$.

It is (nearly) obvious that Cl_n^0 is a *subalgebra* in Cl_n and that (this is slightly less obvious) this subalgebra is isomorphic to Cl_{n-1}^0.

To prove the latter, imbed Cl_{n-1} to Cl_n with the image Cl_n^0 as follows.

Map the orthogonal complement $v^\perp \subset V \subset Cl_n$ of a unit vector $v \in V$ back to Cl_n^0 by $e \mapsto v \cdot e$ for all $e \in v^\perp$ and show that this map extends to an *injective algebra homomorphism* $Cl_{n-1} = CL(v^\perp) \to Cl_n^0$.

All you need for this is an (easy) check up of the identities

$$(v \cdot e)^2 = -1 \text{ and } v \cdot e \cdot v \cdot e' = -v \cdot e' \cdot v \cdot e$$

for all $v, v' \in v^\perp$ (implicit in the above exercise about the homomorphism $G_n \to G_{n+1}^0$).

Finally, since that the group $Spin(n)$, by its very definition, is contained in $(Cl_n^0)^\times$ it goes to $Cl_{n-1}^\times$ by inverting the isomorphism $Cl_{n-1} \to Cl_n^0$. QED.

IX. *Spin Representation of $Pin(n)$ fo Odd n.* Just for completeness sake, let us explain why

the complexified Clifford algebra Cl_{2k+1}, which has dimension 2^{2k+1}, is isomorphic to the sum of of two matrix algebras $End(\mathbb{C}^{2^{k-1}})$.

Recall that the group G_{2k+1} has exactly two irreducible non-one-dimensional representations, where the sum of their dimensions is 2^k.

In fact both representation must have the same dimensions, because of another fundamental (also nameless?) theorem:

the dimensions of all irreducible representations of a finite group G divide the order order of G.[154])

Therefore the non-Abelian part of the group algebra of G_{2k+1}, hence the Clifford algebra Cl_{2k+1} is the sum of two matrix algebras of the same dimension. QED.

As a consequence, we get

two irreducible representations of the group $Pin(2k+1)$ of dimensions 2^{k-1}.

X. *Example: Pauli "Matrices".* The first interesting case of S_n is an irreducible 2-dimensional complex representation S_2 of the group G_2, hence of $Pin(2)$, where he latter is the non-trivial central $\mathbb{Z}_2$-extension of the circle group $\mathbb{T}^1 = U(1)$.

To obtain such a representation all you need is to find two *anti-commuting anti-involutions* σ_1, σ_2 of $\mathbb{C}^2$ corresponding to the generators of e_1, e_2 of the (sub)group $G_2 \subset Cl_2 \supset Pin(2)$.

This is kindergarten math: let $\underline{\sigma}_1, \underline{\sigma}_2$ be anti-commuting *involutions* of the real plane $\mathbb{R}^2$, namely reflections in two lines with the 45° between them. Their compositions, $\underline{\sigma}_1\underline{\sigma}_2$ and $\underline{\sigma}_2\underline{\sigma}_1$ are rotations by 90° in the opposite directions, thus $\underline{\sigma}_1$ and $\underline{\sigma}_2$ anti-commute:

$$\underline{\sigma}_1\underline{\sigma}_2 = -\underline{\sigma}_2\underline{\sigma}_1.$$

[154]See `https://math.stackexchange.com/questions/243221/proofs-that-the-degree-of-an-irrep-divides-the-order-of` for several proofs.

The *anti-involutions* $\sigma_1 = i\underline{\sigma}_1$ and $\sigma_2 = i\underline{\sigma}_2$, $i = \sqrt{-1}$, of $\mathbb{C}^2$ with $\sigma_3 = \sigma_1\sigma^2$ coming along are your Pauli guys.

$\hat{\otimes}$-*Remark.* This example can be amplified by taking tensor products, for

$$Cl_{m+n} = Cl_m \hat{\otimes} Cl_n,$$

where $\hat{\otimes}$ stands for $\mathbb{Z}_2$-*graded* tensor product, for which

$$(a \otimes b) \cdot (a' \otimes b') = (-1)^{deg(a')\deg(b)}(a \cdot a') \otimes (b \cdot b').$$

This allows a sleek construction of the spin representations but it doesn't make it more geometrical than the one via G_n.

X. *Clifford Moduli and Dirac operators.* It is convenient at this point to call a linear space L with an action S of the Clifford algebra $Cl(V)$ *"Clifford V-module"* and to write just S instead of $L = (L, S)$.

Also observe at this point that the actual action of $V \subset Cl(V)$ on such an S reduces to a single linear map $cl : V \otimes S \to S$, where the Clifford action is denoted by "$\cdot$",

$$cl(v \otimes s) = v \cdot s.$$

Now, recall, that such a map defines (and is defined by) a first order differential on the space of smooth maps $\psi : V \to S$, denoted $D : C^\infty(V, S) \to C^\infty(V, S)$, that is the composition of this cl with the differential $d : C^\infty(V) \to C^\infty(H)$ for $H = Hom(V, S)$ as we explained in the previous section.

Since $v^2 = -||v||^2$, $v \in V$, all $v \neq 0$ are *invertible* in the multiplicative semigroup $End^\times(S)$; thus,

the linear operators D are elliptic for all Cl_n-moduli S.

These D can be also defined with orthonormal frames $\{e_i\} \subset V$ by

$$D(\psi) = \sum_{i=1}^{n} e_i \cdot \partial_i \psi,$$

which shows that $D^2 = -\Delta^2 = -\sum_i \partial_i \partial_i$, since

$$D^2 = \sum_{i,j} e_i \partial_i e_j \partial_j = \sum_{i,j} e_i \cdot e_j \partial_i \partial_j = \sum_i e_i \cdot e_i \partial_i \partial_j + \sum_{i \neq j} (e_i \cdot e_j \partial_i \partial_j + e_j \cdot e_i \partial_j \partial_i) = -\sum_i \partial_i \partial_i.$$

or, where the symmetry is apparent, by integration over the unit sphere $\{||v|| = 1\} \subset V$,

$$D(\psi(v)) = const_n \int_{||v||=1} v \cdot \partial_v \psi(v) dv,$$

and if $V = \mathbb{R}^n$.

It follows by a simple symmetry consideration or by a one line computation that

$$D^2 = -\Delta = -\sum \partial_{e_i}^2.$$

Exercise. Prove, directly that

$$\int_{||w||=1} w \cdot \partial_w dw \int_{||v||=1} v \cdot \partial_v dv = const_n \int_{||v||=1} -\sum \partial_v^2 dv.$$ [155]

[155] I myself got lost in this calculation.

Dirac operator $\mathcal{D}$ on Spinors. This $\mathcal{D}$ is defined with the spinor representation S_{2k} in $\mathbb{C}^{2^k}$,

$$\mathcal{D} : \mathbb{S}_{2k} \to \mathbb{S}_{2k},$$

where the "spinors" are understood here as smooth maps $\psi : \mathbb{R}^n \to S_{2k}$ for $n = 2k$ or $n = 2k+1$.

If n is even, the spin representation splits into two adjoint representation, accordingly $\mathbb{S}_{2k} = \mathbb{S}^+_{2k} \otimes \mathbb{S}^-_{2k}$, where the action of the Clifford algebra interchanges $\mathbb{S}^+_{2k} \leftrightarrow \mathbb{S}^-_{2k}$. It follows that $\mathcal{D} = \mathcal{D}^+ \otimes \mathcal{D}^-$ for

$$\mathcal{D}^+ : \mathbb{S}^+_k \to \mathbb{S}^-_k \text{ and } \mathcal{D}^- : \mathbb{S}^-_k \to \mathbb{S}^+_k,$$

the operators $\mathcal{D}^+$ and $\mathcal{D}^-$ are *formally adjoint.*

XI. *$\mathcal{D}$ on Manifolds and Schrödinger-Lichnerowicz-Weitzenböck-Bochner Formula.* Let X be a Riemannian spin manifold of dimension n and let $\mathcal{S}_{2k}$ be the spin bundle associated with the principal spin bundle over X that is the double cover of the orthonormal frame bundle, where this cover is what defines the spin structure on X.

Let ∇ be the Riemannian Levi-Civita connection, which is, observe, simultaneously and coherently defined on all bundles associated with the tangent bundle. (It is irrelevant whether his is done via the principal $O(n)$-bundle or $Spin(n)$-bundle.)

We know (this applies to all bundles with connections, see section3.3.1) that this ∇ decomposes at each point $x \in X$ into the sum $\nabla = \nabla_{flat} + E_\nabla$, where $E_\nabla = E_{\nabla,x}$ a smooth endomorphism of $\mathcal{S}_{2k}$ over a (small) neighbourhood of $x \in X$, which *vanishes* at x.

This allows a "functorial transplantation" of the above $\mathcal{D} = \mathcal{D}_{flat}$ to an $\mathcal{D}_\nabla$ on the space $\mathbb{S}$ of sections of the bundle $\mathcal{S}_{2k}$, where $\mathcal{D}_\nabla$ infinitesimally agree with $\mathcal{D}$ at each point $x \in X$,

$$\mathcal{D}_\nabla = \mathcal{D}_{flat} = E_{\mathcal{D}},$$

for a smooth (locally defined) endomorphism $E_{\mathcal{D}} = E_{\mathcal{D},x}$ of S_{2k}, which *vanishes at x.*

If n is even, then, clearly, $\mathcal{S}_{2k} = \mathcal{S}^+_{2k} \oplus \mathcal{S}^-_{2k}$ and the operator $\mathcal{D}_\nabla$, denoted just $\mathcal{D}$ from now on, splits accordingly: $\mathcal{D} = \mathcal{D}^+ \otimes \mathcal{D}^-$ for

$$\mathcal{D}^+ : \mathbb{S}^+_k \to \mathbb{S}^-_k \text{ and } \mathcal{D}^- : \mathbb{S}^-_k \to \mathbb{S}^+_k,$$

where the operators $\mathcal{D}^+$ and $\mathcal{D}^-$ are *formally adjoint.*

Since the $\mathcal{D}^2_{flat} = \mathcal{D}^2_{flat,x}$, which is defined locally, is equal to $-\Delta = \nabla_{flat}\nabla^*_{flat}$ at each x, the square of $\mathcal{D} = \mathcal{D}_\nabla$, now *globally*, satisfies what is called "*Weitzenboeck identity*" (this applies to all "geometric operators")

$$\mathcal{D}^2 = \nabla\nabla^* + R_{\mathcal{D}},$$

where ∇^* stands for the differential *formally adjoint* to ∇ (this spinor ∇ acts from (sections of) $\mathcal{S}_{2k}$ to (sections of) the bundle $Hom(T(X), \mathcal{S}_{2k})$, where $R_{\mathcal{D}} = R_{\nabla,\mathcal{S},\mathcal{D}}$ is a selfadjoint endomorphism of the bundle $\mathcal{S}_2k$.

It would be nice to continue this line of this reasoning and see without calculation that, why this $R_{\mathcal{D}}$, is a multiplication by a scalar. Regretfully, I couldn't do this and have resort to the (standard) symbolic manipulations. [156]

[156] It doesn't feel right when you can't do mathematics solely in your mind: a piece of paper for this purpose is no more satisfactory than a digital computer.

To perform these we, observe that the bundle of the Clifford algebras $Cl(T_x(X))$ acts on $\mathcal{S}_{2k}$, where this action agrees with the covariant differentiation ∇ in $\mathcal{S}_{2k}$. Then we see that, for all orthonormal framed of tangent vectors e_i, $i = 1, ..., n$, the Dirac operator is

$$\mathcal{D} = \sum_i e_i \cdot \nabla_i \text{ for } \nabla_i = \nabla_{e_i}$$

and

$$\mathcal{D}^2 = \sum_{i,j} e_i \cdot \nabla_i e_j \cdot \nabla_j = \sum_{i,j} e_i \cdot e_j \cdot \nabla_i \nabla_j = \sum_{i=j} e_i \cdot e_j \nabla_i \nabla_j + \sum_{i \neq j} e_i \cdot e_j \cdot \nabla_i \nabla_j =$$

$$= -\sum_i \nabla_i \nabla_i + \sum_{i<j} e_i \cdot e_j \cdot (\nabla_i \nabla_j - \nabla_j \nabla_i) = \nabla\nabla^* + \sum_{i<j} e_i \cdot e_j \cdot R_{\mathcal{S}}(e_i \wedge e_j),$$

where $R_{\mathcal{S}}(e_i \wedge e_j)$ is the curvature of the bundle $\mathcal{S}_{2k}$ written as a 2-form on X with values in $End(\mathcal{S}_{2k})$.

The first term in this formula, $\nabla\nabla^*$ is the *Bochner Laplacian* in the bundle $\mathcal{S}_{2k}$ which a selfadjoint non-strictly positive .

This $\nabla\nabla^*$, regarded as a real operator, is characterized by the integral identity

$$\int_X \langle \nabla\nabla^* \phi(x), \psi(x) \rangle dx - \langle \nabla\phi(x), \nabla\psi(x) \rangle dx = 0$$

which is satisfied, whenever one of the two functions has a compact support.

The proof of this formula, which makes sense and is valid for all vector bundles with orthogonal connections, contains two ingredients, where the first *algebraic* one consists in finding

an invariant representation of the integrant as the differential of an $(n-1)$ form and the second ingredient is, of course, Green's formula.

In fact, all algebra needed in our is the following Leibniz formula for the Laplace Beltrami

$$\Delta \langle \phi(x), \phi(x) \rangle = \langle \nabla\nabla^* \phi(x), \phi(x) \rangle + \langle \phi(x), \nabla\nabla^* \phi(x) \rangle + 2 \langle \nabla\phi(x), \nabla\phi(x) \rangle.$$

This implies all positivity of $\nabla\nabla^*$ we need.

Next, turn to the curvature term $\mathcal{R} = \sum_{i<j} e_i \cdot e_j \cdot R_{\mathcal{S}}(e_i \wedge e_j)$ in the above Bochner – Weitzenböck formula for $\mathcal{D}^2$, that is an endomorphism $\mathcal{R} : \mathcal{S}_{2k} \to \mathcal{S}_{2k}$, which, being self adjoint as a real , is represented by a family of *symmetric* linear operators $\mathcal{R}_x : (\mathcal{S}_{2k})_x \to (\mathcal{S}_{2k})_x$, $x \in X$, in the fibers $(\mathcal{S}_{2k})_x \simeq S_{2k} = \mathbb{C}^{2^k} = \mathbb{R}^{2^{k+1}}$, while the curvature operators $R_{\mathcal{S}}(v_1 \wedge v_2)$ themselves are *antisymmetric*, for all bivectors $v_1 \wedge v_2 \in \bigwedge^2 T_x(x) = \bigwedge^2 \mathbb{R}^n$, since they represent the action of the Lie algebra of the group $Spin(n) \subset SO(2^{k+1})$ on $\mathbb{R}^{2^{k+1}}$.

In fact, a closer look shows[157] that

$$R_{\mathcal{S}}(v_1 \wedge v_2) = \frac{1}{2} \sum_{i<j} \langle R(v_1 \wedge v_2)(e_i), e_j \rangle e_i \cdot e_j$$

where $R(e_i \wedge e_j) : T(X) \to T(X)$ is the curvature of our connection ∇ as it acts on the tangent bundle of X.

[157] See formula 4.37 on p. p110 in [Lawson&Michelsohn(spin geometry) 1989].

(The presence of "$\frac{1}{2}$" agrees with the idea of the bundle $\mathcal{S}_{2k}$ being a "the square root" of the tangent bundle $T(X)$, hence having one half of the curvature of X, which is clearly seen for the Hopf complex line bundle $L \to S^2$, where $L \otimes_{\mathbb{C}} L$ is isomorphic to the tangent bundle $T(S^2)$ and, accordingly, $curv(L) = \frac{1}{2}curv(S^2)$.)

Everything up to this point was applicable to an arbitrary Euclidean vector bundle $T \to X$ of rank m with a spin structure, i.e. a double cover of the associate principal $SO(m)$-bundle and the action of bundle of the Clifford algebras $Cl(T)$ on the corresponding spin bundle with the fibers $\simeq \mathbb{C}^{2^l}$, for $m = 2l$ or $m = 2l + 1$, where the Dirac operator defined via an orthogonal connection in T enjoys all formulas we have presented so far.

But in the case of $T = T(X)$ the symmetries of the curvature tensor encoded by Bianchi identities allow the following simplification of $\mathcal{R}$.

Lichnerowitz' Identity.

$$\mathcal{R} = \sum_{i<j} e_i \cdot e_j \cdot R_{\mathcal{S}}(e_i \wedge e_j) = \frac{1}{2} \sum_{i<j,k<l} \langle R(e_k \wedge e_l)(e_i), e_j \rangle e_i \cdot e_j = \frac{Sc}{4};$$

Thus,

$$\mathcal{D}^2\phi(x) = \nabla\nabla^*\phi(x) + \frac{1}{4}Sc(X,s)\phi(x) \text{ for all sections } \phi : X \to \mathcal{S}_{2k}.$$

Why it is so. The action of the linear group $GL(n)$ on the space $RCT \simeq \mathbb{R}^{\frac{n^2(n^2-1)}{12}}$ of (potential) Riemannian curvature tensors splits into three irreducible representations $RCT = Sc \oplus Ri \oplus W$, where Sc is the trivial one dimensional representation, Ri the space of traceless symmetric 2-forms and W the space of Weyl tensors. Accordingly, every smooth n-manifolds X supports three (curvature) differential operators of the second order from the space G_+ of positive definite quadratic differential forms g on X to the space of sections of vector bundles over X associated with the tangent bundle $T(X)$ via one of these representations, such that

$\bullet_{lin}$ these operators are linear in the second derivatives of g;

$\bullet_{inv}$ these operators are equivariant under the action of the diffeomorphism group $diff(X)$ operator and where

these operators and their scalar multiples are the only ones with such quasi-linearity and invariance properties

On the other hand the $\mathcal{R}$ is also constructed in a $diff(X)$-equivariant manner but it operators on the spinor bundle S_{2k}, where the double cover of $GL(n)$ can't act.[158] This suggests that there is *no non-scalar intertwining* from the space of curvature tensors on X to the space of symmetric operators on S_{2k}, but since I didn't figure out how to prove this without a few lines of manipulations with Bianchi identities, let us accept this for a fact.[159]

3.3.4 Dirac Operators with Coefficients in Vector Bundles, Twisted S-L-W-B Formula and K-Area

Let $\mathcal{D}_{\otimes L}$ be the Dirac twisted with a complex vector bundle $L \to X$ with a unitary connection ∇^L on it. Then, as earlier, we have the general Bochner-

[158] Lemma 5.23. p 132 in [Lawson&Michelsohn(spin geometry) 1989].

[159] Or see "Proof of Theorem 8.8" on page 161 in [Lawson&Michelsohn(spin geometry) 1989].

Weitzenböck formula

$$\mathcal{D}^2_{\otimes L} = \nabla\nabla^* + \sum_{i<j} e_i \cdot e_j \cdot R_\otimes(e_i \wedge e_j),$$

where this $\nabla = \nabla^\otimes$ is the connection in the tensor product of the spinor bundle with L that is defined by the Leibniz rule,

$$\nabla^\otimes(s \otimes l) = \nabla^\mathcal{S} \otimes l + s \otimes \nabla^L(l);$$

hence, the curvature $\cdot R_\otimes$ of this connection, that is the commutator of the $\nabla^\otimes$-differentiations, also behaves by this rule:

$$R_\otimes(e_i \wedge e_j)(s \otimes l) = R_\mathcal{S}(e_i \wedge e_j)(s) \otimes l + l \otimes R_L(e_i \wedge e_j)(l)$$

which brings us to the following.

$[\mathcal{D}_\otimes]$ **Twisted S-L-W-B Formula:**

$$\mathcal{D}^2_{\otimes L}(\sigma \otimes l) = \nabla\nabla^*(\sigma \otimes l) + \frac{Sc(X)}{4}(\sigma \otimes l) + \sum_{i<j} e_i \cdot e_j \cdot \sigma \otimes R_L(e_i \wedge e_j)(l).$$

A basic application of this formula is the bound on the area-size of manifolds with $Sc \geq \sigma > 0$ expressed in terms of vector bundles over X.

$[K_{\star}]$ **Bound on K-Area by Scalar Curvature.** Let X be compact orientable Riemannian Manifold with positive scalar curvature and let $L \to X$ be a complex vector bundle with the unitary connection.

If the norms of the curvature operators $R_x(e_1 \wedge e_2) : T_x(X_x \to T_x(X)$ *of this connection are bounded by*

$$||R_x(e_1 \wedge e_2) : T_x(X_x \to T_x(X)|| \leq \kappa_n \cdot Sc(X, x)$$

for all $x \in X$*, all unit bivectors* $e_1 \wedge e_2$ *in the tangent spaces* $T_x(X)$ *and a universal strictly positive constant* $\kappa_n > 0$*, then,* provided X is spin*, all Chern numbers of the bundle* L *vanish.*

Proof. If some Chern number of L doesn't vanish, then an easy computation with Chern classes and the index formula shows[160] that there exists an associated bundle L', such that the curvature R' of the connectin in L' satisfies

$$||R'_x(e_1 \wedge e_2) : T_x(X_x \to T_x(X)|| \leq const_n \cdot ||R_x(e_1 \wedge e_2) : T_x(X_x \to T_x(X)||$$

and such that the *index of the twisted Dirac operator* on the spinor bundle tensored with L',

$$\mathcal{D}^+_{\otimes L'} : \mathbb{S}^+ \otimes L' \to \mathbb{S}^+ \otimes L',$$

doesn't vanish.

But if

$$||R'_x(e_1 \wedge e_2) : T_x(X_x \to T_x(X)|| < \frac{1}{4} \cdot \frac{2}{n(n-1)} \cdot Sc(X, x).$$

[160] For details and further applications see [GL(spin) 1980], §4-5 in chapter IV in [Lawson&Michelsohn(spin geometry) 1989], §4-5 in [G(positive) 1996], [Min-Oo(K-Area) 2002] and sections 3.3.4, 4.1, 4.1.4.

then, according to [$\otimes$] the $\mathcal{D}^2_{\otimes L'}$ is positive and the poof follows by contradiction.

At first sight this [$\star$] looks as an artifact of symbolic manipulations with curvatures of vector bundles, an insignificant generalization of the Lichnerowicz theorem, as devoid of an actual geometric information about X as this theorem is.

But, surpassingly, although the proof of [$\star$] is 90% the same[161] as that by Lichnerowicz, the information contents of the two statements are vastly different – almost nothing in common between them:

Lichnerowicz is 99% about *delicate smooth topological invariants* of manifolds with $Sc > 0$, while [$\star$] reveals raw geometric essence of $Sc(X) \geq \sigma > 0$, which, as it becomes a *positive curvature* condition, *limits the size* of X.[162]

Below is a specific instance of this.

Rough Area (non)-Contraction Corollary. Given a *compact* Riemannian manifold $\underline{X}$, there exists a positive constant $\kappa = \kappa_{\underline{X}} > 0$, which restricts how much manifolds X with $Sc \geq \frac{1}{\kappa}$ *can be area-wise greater than* $\underline{X}$, which is expressed by *a bound on a possible decrease of areas* of surfaces in X under "topologically significant" maps $X \to \underline{X}$.

In precise language,

[$\bigstar$] *let X be an oriented Riemannian manifold with $Sc(X) > 0$ and $f : X \to \underline{X}$ a smooth map, such that the norm of the second exterior power of the differential of f,*

$$\wedge^2 df : \bigwedge^2 T(X) \to \bigwedge^2 T(\underline{X}),$$

is *bounded by the reciprocal of the scalar curvature of X times $\kappa_{\underline{X}}$,*

$$\| \wedge^2 df(x) \| < \frac{\kappa_{\underline{X}}}{Sc(X,x)}, \text{ for all } x \in X.$$

Then, provided X is spin, *the image h of of the fundamental homology class of X in the homology of $\underline{X}$, that is*

$$h = f_*[X] \in H_n(\underline{X}),\ n = dim(X),$$

is torsion.

Proof. By basic topology (a corollary to a theorem by Serre), an *even dimensional non-torsion* homology class h in $\underline{X}$ is "detected" by a complex vector bundle: that is a $\underline{L} \to \underline{X}$, such that some characteristic cohomology class $\underline{c}$ of $\underline{L}$, doesn't vanish on h

$$\underline{c}(h) \neq 0.$$

If $h = f_*[X]$, then $f^*(\underline{c})[X]$, which serves as a *characteristic number* of the induced bundle $L = f^*(\underline{L}) \to X$, is equal to $\underline{c}(h)$; hence it *doesn't vanish* either.

Now, arguing as in the proof of **[Sc $\not> 0$]** for profinitely hyperspherical manifolds (see section 3.3), let $\underline{\nabla}$ be a unitary connection in $\underline{L}$ and observe that the

[161] The proof of [$\star$], unlike that of Lichnerowicz' theorem, needs only 10% of the power of the Atiyah-Singer theorem – the easy part of it: non-trivial variability of the index of $\mathcal{D}_{\otimes L}$ with variations of (the Chern classes of) L, rather than a more subtle aspect of the formula which involves $\hat{A}$-genus of X.

[162] Positivity of the sectional (and Ricci) curvature, imposes bounds the *first and the second derivatives* of the growths of balls in respective manifolds.

norm of the curvature R of the induced connection in L, which is, after all, is a 2-*form*, is bounded by the curvature $\underline{R}$ of $\underline{\nabla}$,

$$||R_x|| \leq || \wedge^2 df(x)|| \cdot ||\underline{R}_x||.$$

Thus, if $n = dim(X)$ is even, the proof follows from [☆] and the odd case reduces to the even one by taking the products of both manifolds with the circle.

Remarks and Exercises. **(a)** We use the word "K-*area*" to express the idea that

if X contains "homologically significant" families of surfaces with *small areas*, then K-cohomology classes of X can't be represented by bundles with connections, which have small curvatures

and where

the norm of $\wedge^2 df$ measures by how much f *contracts/expands these areas.*[163]

Yet, we shall eventually switch to an uglier but more appropriate word "K-cowaist$_2$".

(b) Let X and Y be closed oriented surfaces with Riemannian metrics on them and let $f_0 : X \to Y$ be a continuous map of degree d. Show that f_0 is homotopic to a smooth *strictly area decreasing* map f, i.e. where $||\wedge^2 df(x)|| < 1$ for all $x \in X$, if and only if $area(X) > d \cdot area(Y)$.

(c) The principal case in the above corollary, which yields most topological applications, [164] is where $\underline{X}$ is the n-sphere S^n and where the non-torsion condition amounts to *non-vanishing of the degree* of $f : X \to S^n$.

In fact, as one knows by a theorem of Serre, the multiple of every cohomology class h in $\underline{X}$ with $h \smile h = 0$ can be induced from the the fundamental class of S^n by a smooth map $\underline{X} \to S^n$, the general case of this corollary, for all dimensions, can be (with a minor effort) reduced $\underline{X} = S^n$.

(d) We call this corollary "rough", since the (lower) bound on $\kappa_{\underline{X}}$ its proof delivers is far from optimal;

Optimal bounds, however, are available, albeit only in a few cases, including $\underline{X} = S^n$ as we shall see in the following sections.

Questions. **(A)** *Is the spin condition in* [$\star$] *redundant?*

Or the opposite is true: if an orientable non-spin n-manifold X admits a metric g_0 with $Sc(g_0) > 0$, then it carries metrics g_ε, for all $\varepsilon > 0$, with $Sc(g_\varepsilon) \geq 1$, for which allow smooth maps $f_\varepsilon : (X, g_\varepsilon) \to S^n$ with $deg(f_\varepsilon) \neq 0$, and $|| \wedge^2 df|| \leq \varepsilon$?

(B) *Can the torsion conclusion in* [$\star$] *be replaced by "p-torsion for some particular p, preferably for $p = 2$ and, in lucky cases, even by just $f_*[X] \neq 0$?*

(It is not even clear if this can be done with a bound on $||df||$ rather than on $\wedge^2||df||$, where there is a chance for a successful use of minimal hypersurfaces.)

3.4 Sharp Lower Bounds on *sup*- and *trace*-Norms of Differentials of Maps from Spin manifolds with $Sc > 0$ to Spheres.

There is no single numerical invariant faithfully representing the size of X, but there are several ways of comparison the sizes of different manifolds.

[163] See [G(positive) 1996], [Min-Oo(K-Area) 2002] and sections 3.3.4, 4.1.4 for more about this K-area.

[164] See [GL(spin) 1980], [GL(complete) 1983], [Lawson&Michelsohn(spin geometry) 1989].

In the case, where two Riemannian metrics are defined on the same background manifold, say g and $\underline{g}$ on $\underline{X}$, one compares these at a point $\underline{x}$ by simultaneously diagonalizing them and recording the ratios of their values on the vectors e_i from the common orthonormal frame $\{e_1, e_2, ..., e_n\} \subset T_{\underline{x}}(\underline{X})$, that are the numbers

$$\lambda_i(\underline{x}) = \lambda_i(\underline{g}/g, \underline{x}) = \frac{||e_i||_{\underline{g}}}{||e_i||_g}.$$

In terms of these numbers, the inequalities $\lambda_i(\underline{x}) \leq 1$, $\underline{x} \in \underline{X}$, say that $g \geq \underline{g}$, while the inequalities $\lambda_i\lambda_j(\underline{x}) \leq 1$ convey that g is (only) *area wise* (non-strictly) *greater* than $\underline{g}$, where, of course, the former implies the latter.

Another way to compare the metrics is by using the *trace of* $\underline{g}$ *relative to* g, denoted

$$trace(\underline{g}/g) = \sum_1^n \lambda_i, \;\; n = dim(\underline{X}),$$

where the inequality

$$\frac{1}{n} trace(\underline{g}/g) \leq 1$$

expresses the idea of g being greater than $\underline{g}$.

This "trace-wise greater" is less restrictive, yet, moderately so, than the "ordinary greater" $g \geq \underline{g}$, for

$$g \geq \underline{g} \Rightarrow g \underset{tr}{\geq} \underline{g} \Rightarrow g \geq \frac{1}{n^2}\underline{g}.$$

(Notice that $\lambda_i(\underline{g}/c^2g) = \frac{1}{c}\lambda_i(\underline{g}/g)$.)

A more relevant for us is the "area trace"

$$trace_{\wedge^2}(\underline{g}/g) = \sum_{i \neq j} \lambda_i\lambda_j$$

where "trace area-wise greater" inequality reads

$$\frac{1}{n(n-1)} trace_{\wedge^2}(\underline{g}/g) \leq 1,$$

which is related to the "untraced area-wise greater" ratio by the relations

$$[g \underset{\wedge^2}{\geq} \underline{g}] \Rightarrow [g \underset{tr_{\wedge^2}}{\geq} \underline{g}] \Rightarrow \left[g \geq \frac{1}{n(n-1)}\underline{g}\right].$$

3.4.1 Area Inequalities for Equidimensional Maps:Extremality and Rigidity

In order to apply the above to Riemannian metrics g and $\underline{g}$ on *different* manifolds X and $\underline{X}$ we relate them by a smooth map, say $f : X \to \underline{X}$, where the principal case is of $dim(X) = dim(\underline{X}) = n$ and where, to make sense of what follows, the map f must be "*homotopically onto*", that is *not homotopic to a map into a proper subset in* $\underline{X}$.

If both manifolds are *orientable* – they are assumed compact without boundaries at this point – this is equivalent to *non-vanishing of the degree* $deg(f)$ of the map, [165]

If non-orientability is easily taken care of by just passing to orientable double covers, what does cause a problem is the *spin condition*, the relevance of which the following two geometric theorems remains problematic.

$[X_{spin} \to \bigcirc]$ **Spin-Area Convex Extremality Theorem.** Let $\underline{X} \subset \mathbb{R}^{n+1}$ be a smooth compact convex hypersurface and let $\underline{g}$ be the Riemannian metric on $\underline{X}$ induced from $\mathbb{R}^{n+1}$. Let $X = (X, g)$ be a compact orientable Riemannian n-manifold with $Sc \geq 0$ and let $f : X \to \underline{X}$ be a smooth map of *non-zero degree.*

Let $g^\circ = Sc(g) \cdot g$ and $\underline{g}^\circ = Sc(\underline{g}) \cdot \underline{g}$ be the corresponding Sc-normalized metrics

If X is spin *and n is* even, *then the map f can't be strictly area decreasing, that is the metric g° is not area-wise greater, than the induced metric $f^*(\underline{g}^\circ)$ on X.*

Put it another way,

there necessarily exists a point $x \in X$, where the norm of the second exterior power of the differential of f is bounded from below by the scalar curvature of X as follows

$$Sc(\underline{X}, f(x)) \cdot || \wedge^2 df(x)|| \geq Sc(X, x),$$

which, in terms of $\lambda_i^\circ = \lambda_i(f^*(\underline{g}^\circ))/g^\circ)$, reads

$$\max_{x \in X, i \neq j} \lambda_i^\circ(x)\lambda_j^\circ(x) \geq 1.$$

In the simplest case, where $\underline{X}$ is the unit sphere $S^n \subset \mathbb{R}^{n+1}$, this theorem can be refined as follows.

$[X_{spin} \to \bigcirc]$ **Spherical Trace Area Extremality Theorem**. Let X be a compact orientable Riemannian spin manifold of dimension n and $f : X \to S^n = \underline{X}$ be a map with $deg(f) \neq 0$.

Then *f can't be trace area-wise strictly decreasing with respect to the Sc-normalized metrics $g^\circ = Sc(g) \cdot g$ on X and $\underline{g}^\circ = Sc(\underline{g}) \cdot \underline{g}^\circ = n(n-1)ds^2$, which, in terms of the exterior power of f, says that there is a point $x \in X$, where the trace-norm of the second exterior power of the differential of f is bounded from below by the scalar curvature of X as follows*

$$2|| \wedge^2 df(x)||_{trace} \geq Sc(X, x),$$

that is

$$\frac{1}{2n(n-1)} \sum_{i \neq j} \lambda_i^\circ(x)\lambda_j^\circ(x) \geq 1 \textit{ for } \lambda_i^\circ = \lambda_i(f^*(\underline{g}^\circ))/g^\circ).$$

Remarks (a) Neither $[X_{spin} \to \bigcirc]$ nor $[X_{spin} \to \bigcirc]$ seem obvious even, where X is also a convex hypersurface in $\mathbb{R}^{n+1}$.

Question. Are there counterparts of $[X_{spin} \to \bigcirc]$ and/or of $[X_{spin} \to \bigcirc]$ for symmetric function s_k of the principal curvatures $\alpha_1, \alpha_2, ..., \alpha_n$ of convex hypersurfaces X and $\underline{X}$? (We shall return to this question in (b) of 3.5.)

[165] The implication $[deg(f) \neq 0] \Rightarrow [f$ is homotopically onto$]$, which is obvious by the modern standards, is by no means trivial. For instance, "homotopically onto" for the identity map of the n-sphere is equivalent (one line kindergarten argument) to the Brauer fixed point theorem for the $(n+1)$-ball.

(b) The condition $n = 2k$, which is unneeded for $[X_{spin} \to \bigcirc]$, probably is also redundant for $[X_{spin} \to \bigcirc]$.

(c) These two theorem will be later generalized in several directions.

(d_1) One may allow non-compact, and sometimes even non-complete manifolds X with suitable conditions on maps f, in order to have their degrees being properly defined.

(e_2) In the case, where $dim(X) = dim(\underline{X}) + 4l$, the condition $deg(f) \neq 0$ can be replaced by $\hat{A}[f^{-1}(x)] \neq 0$ for a generic point $x \in X$ of a smooth map $f : X \to \underline{X}$. (c_3) Instead of a convex hypersurface in $\mathbb{R}^{n+1}$, one may take a more general Riemannian manifold for $\underline{X}$, namely one with *a non-negative curvature operator* and – this is, probably, unnecessary – *with non-zero Euler characteristic.*

(f) Who is extremal? These two extremality theorems can be thought of as *properties of* X, saying that "*large scalar curvature makes* X *small*".

From another perspective, *these theorems are about* $\underline{X}$, saying that $\underline{X}$ *can't be enlarged without making its scalar curvature smaller at some point.*

This suggest two avenues of generalizations that we shall explore in the following sections.

1. Widen the class of manifolds X and maps $f : X \to \underline{X}$, which satisfy the above or similar theorems and, regardless of the scalar curvature, study invariants of manifolds X responsible for existence/non-existence of metrically contracting, yet topologically significant, maps from X to "standard" manifolds $\underline{X}$ such as the spheres, for instance.

2. Find further instances of *extremal* manifolds $\underline{X} = (\underline{X}, \underline{g})$ with $Sc(\underline{g}) > 0$, i.e. where no Sc-normalized metric g can be greater the so normalized $\underline{g}$,

$$Sc(g) \cdot g \not> Sc(\underline{g}) \cdot \underline{g}$$

and study properties of such metrics.[166]

A few Words about the Proofs. [167]. The logic here is the same as in the proof of the rough area (non)-contraction corollary from the previous section, where the sharpness of the bound on $\wedge^2 df$ is achieved by a choice of the bundle $\underline{L} \to \underline{X}$ with a non-zero top Chern class with a connection $\underline{\nabla}$ with minimal possible curvature, that allows the necessary strong bound on the "twisted curvature" term $\sum_{i<j} e_i \cdot e_j \cdot \sigma \otimes R_L(e_i \wedge e_j)(l)$ in the Schrödinger-Lichnerowicz-Weitzenböck-Bochner formula for the Dirac operator on X tensored with induced connection $\nabla = f^*(\underline{\nabla})$ in the bundle $L = f^*(\underline{L}) \to X$,

$$\mathcal{D}^2_{\otimes L}(\sigma \otimes l) = \nabla\nabla^*(\sigma \otimes l) + \frac{Sc(X)}{4}(\sigma \otimes l) + \sum_{i<j} e_i \cdot e_j \cdot \sigma \otimes R_L(e_i \wedge e_j)(l).$$

The natural choice of $\underline{L}$ – this was suggested by Blaine Lawson 40 years

[166]See [Sun-Dai(bi-invariant)2020] for the proof of the extremality of *bi-invariant metrics* on compact Lie groups in the class of *left invariant* metrics.

[167]For detailed poofs the above mentioned results see [Llarull(sharp estimates) 1998], [Min-Oo(Hermitian) 1998], [Min-Oo(K-Area) 2002], [Goette-Semmelmann(symmetric) 2002], [Goette(alternating torsion) 2007], [Listing(symmetric spaces) 2010]; also we say a bit more about this in sections4, 4.1.

ago –[168]) is one of the *Bott generator bundles*, that are the $\frac{1}{2}$-spinor bundles $\underline{L}^{\pm} = \mathcal{S}^{\pm}(\underline{X})$ (with $rank_{\mathbb{C}}(\underline{L}) = 2^{k-1}$ for $n = 2k$), which, being the "moral square roots" of the tangent bundle $T(\underline{X})$, have their curvatures equal to the one half of that of $T(\underline{X})$. (This is clearly seen for $n = 2$ where $\underline{L}^{+}$ is the Hopf complex line bundle over S^2.

What makes $\underline{L}^{\pm}$ promising candidates for S-L-W-B-extremality, is the fact that $\underline{L}^{\pm}$-twisted Dirac operator on the manifold $\underline{X}$ itself *does have harmonic spinors* but only *barely so*: these spinors are *parallel* as they correspond to *constant functions* and/or to *constant multiples of the Riemannian volume n-form* on $\underline{X}$.

The extremality property of $\underline{L}^{\pm}$ was confirmed by Llarull in the case of $\underline{X} = S^n$ and – this was by no means expected – by Goette and Semmelmann for manifolds $\underline{X}$ with positive curvature operators, while the possibilities of Sc-normalization and of tracing $\wedge^2 df$, were suggested by Listings. (Although there is *no technical novelties in the proofs* of the Sc-normalised and traced modifications of $[X_{spin}\to\bigcirc]$ and $[X_{spin}\to\bigcirc]$ these significantly *widen the range of applications* of these extremality theorems.)

Besides facing algebraic complexity of the "twisted curvature" one has to ensure the existence of non-zero $L^{\pm}$-twisted harmonic spinors on X for $L^{\pm} = f^{*}\underline{L}^{\pm}$.

The index formula guarantees this for $n = 2k$ and, under an additional condition on f, also for $n = 4l+1$, but in general the existence of such spinors for all metrics on X and all n remains problematic.

$\bigcirc_{\blacksquare}$. *The Proof of $[X_{spin}\to\bigcirc]$ for odd $n = dim(X)$.* Given a map $X \to S^n \subset S^{n+1}$, radially (and obviously) extend it to the map $X \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \to S^{n+1}$ with the bottom and the top of the cylinder $X \times \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ sent to the poles of S^{n+1}, $X \times \left\{\mp\frac{\pi}{2}\right\} \to \mp 1$.

One can proceed three ways from this point.

1. Endow $X\times\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with the (spherical suspension) warped product metric $\hat{g}$ with the same warping factor as that for the spherical cylinder $S^{n+1}\smallsetminus\{-1,+1\}$ and observe that, say in the case of $Sc(X) \geq n(n-1) = Sc(S^n)$, this metric has greater scalar curvature than that of S^{n+1}.

Then, by an easy argument, an ε-small C^0-perturbation of this metric ε-near the boundary extends, for all $\varepsilon > 0$, to a complete metric $\hat{g}_{\varepsilon}$ on the infinite cylinder $X \times (-\infty, +\infty)$, such that $Sc(\hat{g}_{\varepsilon}) \geq n(n+1) - \varepsilon$ and such that the geometry of $X \times (-\infty, +\infty), g_{\varepsilon}) \geq n(n+1) - \varepsilon$ is cylindrical for $|t| \geq \frac{\pi}{2} + \varepsilon$ infinity with the scalar curvature $\geq n(n+1)+1$.

Thus the untraced inequality $[X_{spin}\to\bigcirc]$ applies to the product $S^n \times S^1(R)$ $R \geq 2$ obtained by closing this cylinder at infinity and letting $\varepsilon \to 0$.

2. Apply the traced inequality $[X_{spin}\to\bigcirc]$ to maps $X^n \times S^1(R) \to S^{n+1}$, where $S^n \times S^1(R)$ comes with with the product metric, and let the radius of the circle $R \to \infty$. (This is, essentially, how it was done in [Llarull(sharp estimates) 1998].)

3. Regard a map $X^n \times S^1 \to S^{n+1}$ of non-zero degree as a family of maps $f_s : X \to S^{n+1}$ and use the spectral flow index theorem for the family of operators

[168] I recall this well, since I was taken by surprise by the properties of this bundle, which has the minimal curvature (one half of that of the tangent bundle of the sphere) among all unitary bundles with non-trivial Euler class.

on $X = X \times s$ parametrized by S^1.[169]

.

Exercise. Fill in the details in (1) and (2).

Question Is there a more direct (K^1-theoretic?) proof of the inequality $[X_{spin} \to \bigcirc]$ for odd n with no direct reference to S^{n+1} and desirably of $[X_{spin} \to \bigcirc]$ as well for, odd n, e.g. by a spectral flow argument?

Infinite Dimensional Remark. Both, spherical suspension in 1 and the cylindrical one in 2, when repeated N-times times and can be interpreted in the limits for $N \to \infty$ as properties of

1^∞ infinite dimensional manifolds X^∞ with $Sc(X^\infty) \geq Sc(S^\infty)$;

2^∞ $Sc(X^\infty) \geq Sc(S^n \times \mathbb{R}^{\infty - n}$;

inequalities are implemented in both cases by certain special Fredholm-type maps $X^\infty \to S^\infty$.

Conversely, one can prove an infinite dimensional version of $[X_{spin} \to \bigcirc]$ for limits of the above maps, say for

Fredholm maps from a Hilbertian manifold X to the Hilbertian sphere, $f : X \to S^\infty$,such that $deg(f \neq 0$ and such that there exists a sequence of equatorial spheres

$$S^{N_1} \supset S^{N_2} \supset \ldots \supset S^{N_i} \supset \ldots \supset S^\infty,$$

where the union $\bigcup_i S^{N_i}$ is *dense* in S^∞ and such that the pullbacks $X_i = f^{-1}(S^{N_i}) \subset X$ are smooth submanifolds of dimensions N_i, the scalar curvatures of which with the induced metrics satisfy $Sc(X_i) - N_i(N_i - 1) - 0$ for $i \to \infty$.

Infinite Dimensional Questions. What is the most general/natural infinite dimensional inequality $[X_{spin} \to \bigcirc]$?

Is there a direct proof of such an inequality with no use of finite dimensional approximation?

Are there natural Hilbertian and/or non-Hilbertian spaces X to which such an inequality may apply?

Stability Remark. Probably, (I haven't thought trough this) the reduction argument *even* $\rightsquigarrow$ *odd* implies certain stability of harmonic spinors on $(2m-1)$-manifolds X twisted with spherical spinors, that are section of the induced bundle $f^*(\mathbb{S}(S^{2m-1}))$ by maps $f : X \to S^{2m-1}$ with $deg(f) \neq 0$.

Another (seemingly unrelated) instance of *stability of harmonic spinors (seemingly) independent of the index theorem* is present in

Witten's argument in his proof of the Euclidean positive mass theorem as well in Min-Oo's proof of the hyperbolic one.

Probably, there are many examples of stable (twisted) harmonic spinors on *compact manifolds*, where this stability is not not predicted, at least not directly, by the index theorem.[170]

[169] Such argument was used in [Vafa-Witten(fermions) 1984] for lower bounds on spectral gaps for the Dirac operator, succinctly exposed in [Atiyah(eigenvalues) 1984] and applied in §6 in [G(positive) 1996] to spectral bounds for the Laplace operators on odd dimensional Riemannian manifolds.

Also spectral flow for Dirac operators combined with a *refined Kato inequality* is used in [Davaux(spectrum) 2003] for the proof of sharp upper bounds on the scalar curvatures of Riemannian metrics on compact manifolds which admit hyperbolic metrics.

[170] To make sense of this one has to properly specify the meaning of "stability" not to run into (counter) a example, see *Harmonic Spinors and Topology* by Christian Bár,`https://link.springer.com/chapter/10.1007/978-94-011-5276-1_3`

Area Rigidity Problem: Examples and Counter Examples. Given a smooth convex hypersurface $\underline{X} \subset \mathbb{R}^{n+1}$ and let $\underline{g}$ be the induced Riemannian metric on $\underline{X}$.

Describe (all) Riemannian$n$-manifolds $X = (X, g)$ along with smooth maps $f : X \to \underline{X}$, such that

$$Sc(g, f(x)) \le Sc(\underline{g}, x) \cdot \| \wedge^2 df(x)\|$$

at all $x \in X$ and also X and f where

$$Sc(g, f(x)) = Sc(\underline{g}, x) \cdot \| \wedge^2 df(x)\|.$$

In the "ideal rigid" case, at least for $Sc(\underline{X}) > 0$, one wants all such maps to be *locally isometric with respect to the Sc-normalised metrics* $g^\circ = Sc(g) \cdot g$ and $\underline{g}^\circ = Sc(\underline{g}) \cdot \underline{g}$. (This, if I am not mistaken, is the same as *local homothety* with respect to the original metrics: the induced Riemannin metrics $f^*(\underline{g})$ on X are constant multiples of g, i.e. $g = \lambda \cdot f^*(\underline{g})$)

But the true picture is more interesting than this "ideal". Here is what one can say in this regard.

(A) If n=2 then the equality $Sc(\underline{g}, f(x)) = Sc(g, x) \cdot \| \wedge^2 df(x)\|$ says that f is *locally area preserving* with respect to g° and $\underline{g}^\circ$; hence, the space of such maps is (at least) as large as the group of area preserving diffeomorphisms of the disc.

(B) If $n \ge 3$, then locally area preserving maps are locally isometric and, in fact,

"Ideal rigidity", i.e. the implication

$$Sc(\underline{g}, f(x)) \le Sc(g, x) \cdot \| \wedge^2 df(x)\| \Rightarrow g = \lambda \cdot f^*(\underline{g}),$$

was proven by Mario Listing under the following assumptions:[171]

• $\underline{X}$ is a closed *strictly* convex hypersurface of dimension $n \ge 3$, where this "strictly" signifies that all principal curvatures are > 0 (rather than non-existence of straight segments in $\underline{X}$);

• X is a closed connected orientable *spin* manifold and $deg(f) \ne 0$.

Now let us look at *non-strictly convex* hypersurfaces of dimensions $n \ge 3$.

(C) Let a hypersurface $\underline{X} \subset \mathbb{R}^{n_0+m}$ be the product

$$\underline{X} = \underline{X}_0 \times \mathbb{R}^m$$

where $\underline{X}_0 \subset \mathbb{R}^{n_0}$ is a smooth hypersurface. Then all (self) maps

$$f = (f_0, f_1) : \underline{X} \to \underline{X}_0 \times \mathbb{R}^m = \underline{X},$$

such that $\|df_1\| \le 1$, satisfy $Sc(\underline{g}, f(x)) = Sc(g, x) \cdot \| \wedge^2 df(x)\|$.

If $m \ge 2$, there are no *closed* Euclidean hypersurfaces displaying such non-rigidity (unless I am missing obvious Euclidean examples)[172] but this non-rigidity, of cylinders, i.e. for $m = 1$, can be cast into a compact form; also this can be done to conical hypersurfaces as follows.

[171] See theorem 1 in [Listing(symmetric spaces) 2010], and compare with Theorem 4.11 in [Llarull(sharp estimates) 1998].

[172] There are these in $\mathbb{R}^{n_0} \times \mathbb{T}^m$.

(D) Let $C \subset \mathbb{R}^{n+1}$ a smooth convex cone and let $\underline{X} \subset C$ be a smooth closed convex hypersurface, such that the intersection $\underline{X} \cap \partial C$ contains a conical annuls A in the boundary of C pinched between two spheres,

$$A = \{a \in \partial C\}_{R_1 \le ||a|| \le R_2}.$$

Thus, the boundary of $\underline{X} \subset C$ consists of three parts:

the *side boundary* that is the intersection $\underline{X} \cap \partial C$;

bottom $\underline{X}_1 \subset \underline{X}$ that lies on the R_1-side in the interior of C, i.e. $||\underline{x}_1|| < R_1$, for $\underline{x}_1 \in \underline{X}_1$,

top of $\underline{X}_2 \subset \underline{X}$ that lies on the R_2-side in the interior of C, i.e. $||\underline{x}|| > R_2$ for $\underline{x}_2 \in \underline{X}_2$.

Scale up the top of $\underline{X}$ and set:

$$X = (\underline{X} \setminus \underline{X}_2) \cup \lambda X_2, \ \lambda > 1.$$

This X admits an obvious (infinite dimensional) family of diffeomorphisms $f : X \to \underline{X}$, that

fix the bottom,

return back the top by $x \to \lambda^{-1} x$,

send all straight radial segments in the side boundary of X to themselves,

satisfy the equality $Sc(\underline{g}, f(x)) = Sc(g, x) \cdot || \wedge^2 df(x)||$.

Probbaly, (C) and (D) give a fair picture of possible kinds of *not-quite-rigid* $\underline{X}$ with $Sc(\underline{X}) > 0$ *in the class of convex* X, but it is not so clear for the class of all X with $Sc(X) > 0$

3.4.2 Area Contracting Maps with Decrease of Dimension

The lower bounds on the norms $|| \wedge^2 df||$ for equividimensional maps $f : X \to \underline{X}$ with *non-zero degree* generalize to maps, where $dim(X) > dim(\underline{X})$ with an appropriate generalization of the concept of degree.

For example, the proofs of the rough Area (non)-contraction property (section 3.3.4) and of both its above refinements $[X_{spin} \overset{\to}{} \bigcirc]$ and $[X_{spin} \to \bigcirc]$, which say that such norms can't be too small at all points in X,

$$|| \wedge^2 df(x)|| \nless \frac{Sc(X,x)}{Sc(\underline{X}, f(x)||} \text{ and } || \wedge^2 df(x)||_{trace} \nless 2\frac{Sc(X,x)}{n(n-1)} \text{ correspondingly,}$$

extend with (almost) no change to maps $f : X^{n+4l} \to \underline{X}^n$ with *non-zero* $\hat{A}$-*degrees*, which means non vanishing of the $\hat{A}$-genera of the pullbacks $f^{-1}(\underline{x}) \subset X^{n+4l}$ of generic points $\underline{x} \in \underline{X}^n$. [173]

For instance:

$[X_{spin} \overset{\hat{A}}{\to} \bigcirc]]$ **$\hat{A}$-Extremality Theorem**. Let X be a compact orientable Riemannian spin manifold of dimension $n + 4l$ and $f : X \to \underline{X} = S^n$ be a smooth map, such that the $\hat{A}$-genus of the f-pullback of a regular point from S^n doesn't vanish,

$$\hat{A}[f^{-1}(\underline{x}_0)] \neq 0, \ \underline{x}_0 \in S^n.$$

.

[173] This is done in [GL(spin)1980], [Llarull(sharp estimates) 1998], [Goette-Semmelmann(symmetric) 2002] and in [Goette(alternating torsion)2007] for bounds on $||\wedge^2 df||$, but the corresponding lower bound on $||\wedge^2 df||_{trace}$ is missing from [Listing(symmetric spaces) 2010]; however, as I see it, there in no problem with this either.

Then there exists a point $x \in X$, where the the trace-norm of the second exterior power of the differential of f is bounded from below by the scalar curvature of X as follows,

$$2\|\wedge^2 df(x)\|_{trace} \geq Sc(X,x).$$

Since

$$2\|\wedge^2 df(x)\|_{trace} = \sum_{i\neq j} \lambda_i(x)\lambda_j(x) \leq n(n-1) \max_{i\neq j} \lambda_i(x)\lambda_j(x) = n(n-1)\|\wedge^2 df(x)\|,$$

this implies that if $Sc(X) \geq n(n-1) = Sc(S^n)$, then the map f *can't be strictly area decreasing.*

Generalization to $\hat{\alpha}$. The above remains true with $\hat{\alpha}$ instead of $\hat{A}$, e.g. where the pullback of a regular point $f^{-1}(\underline{x}_0) \subset X$ is diffeomorphic to Hitchin's exotic sphere Σ^n for $n = 8k+1, 8k+3$.[174]

Question. Does the conclusion of the above theorem remain true if the nonvanishing of $\hat{A}[f^{-1}(\underline{x}_0)]$ is replaced by the following

the pullbacks $(f')^{-1}(\underline{x})$, for all smooth maps $f' : X^{n+m} \to \underline{X}^n$ homotopic to f and

all f'-non-critical $x \in \underline{X}^n$, *admit no metrics with $Sc > 0$.*

This is beyond the present day techniques, already for manifolds X^{n+m} homeomorphic to $S^n \times Y^m$, where Y^m is SYS-manifold.

But if Y^m is *the torus* or, more generally an *enlargeable manifold*, e.g. if it admits *a metric with non-positive sectional curvature*, then Dirac theoretic techniques on *complete manifolds* (see sections 3.14.2, 4.1) delivers the proof of the following.

$\times\mathbb{R}^m$**- Stabilized Mapping Theorem.** Let X^{n+m} be a complete orientable Riemannian spin manifold with $Sc(X^{n+m}) \geq \sigma > 0$ [175] and let $\underline{X}^n$ be a smooth convex hypersurface in $\mathbb{R}^{n+1}$. Let $f_1 : X^{n+m} \to \underline{X}^n$ and $f_2 : X^{n+m} \to \mathbb{R}^m$ be smooth maps, where f_2 is a proper[176] *distance decreasing* map and where the "product map",

$$(f_1, f_2) : X^{n+m} \to \underline{X}^n \times \mathbb{R}^m$$

has *non-zero degree.*

Then, if n is even, there exists a point $x \in X$, where

$$\|\wedge^2 df(x)\| \geq \frac{Sc(X,x)}{Sc(\underline{X}, f(x)).}$$

Furthermore, if $\underline{X}^n = S^n$, one can allow odd n *and replace the above inequality by the stronger one:*

$$2\|\wedge^2 df(x)\|_{trace} \nless Sc(X,x).^{177}$$

[174] Such a Σ^n is homeomorphic to the ordinary sphere S^n, but *doesn't bound a spin manifold.*

[175] In view of [Zhang(Area Decreasing) 2020], one can, probably, relax this to $Sc(X^{n+m}) \geq 0$.

[176] This is the usual "proper": pullbacks of compact subsets are compact.

[177] See[Cecchini(long neck) 2020], [Cecchini-Zeidler(generalized Callias) 2021], [Cecchini-Zeidler(Scalar&mean) 2021] for more general results applicable to manifolds X^{n+m} with boundaries and to all closed manifolds Y^m, the non-existence of metrics with $Sc > 0$ on which follows from non-vanishing of *Rosenberg index.*

There is a particularly useful corollary of this theorem, where $X^{n+m} = Y^n \rtimes \mathbb{T}^m$ is *a* $\mathbb{T}^{\rtimes}$*-extension* of a manifold Y^n, that is the product $Y^n \times \mathbb{T}^m$ with a warped product metric $dy^2 + \phi(x)^2 dt^2$ and where the map $f : X \times \mathbb{T}^m$ factors as $Y^n \times \mathbb{T}^m \to Y^n \to \underline{X}$ for the coordinate projection $Y^n \times \mathbb{T}^m \to Y^n$

For instance, such a $\mathbb{T}^{\rtimes}$-stabilized mapping theorem for $m = 1$ together with the *μ-bubble separation theorem* (sections 3.7, 5.4), yield *a sharp area mapping inequality* for a class of *manifolds X with boundaries*, e.g. for $X = Y \times [-1, 1]$.

3.4.3 Parametric Area Inequalities for Families of Maps

Introduce parameters wherever possible is a motto of modern mathematics; Grothendieck concept of *topos* – a category of sets parametrized by a "topological site" – is the most general manifestation of this.

The first instance of this in the present context is an application of the index theorem to the

family of flat complex line bundles L_p over the torus parametrized by the dual (*Picard*) torus

and thus showing that the torus $\mathbb{T}^{2m}$ with an *arbitrary Riemannian metric g* supports a *non-zero harmonic spinor* twisted with a *flat* unitary bundle; hence, *no metric g on the torus may have $Sc(g) > 0$* by the (untwisted) S-L-W-B formula. [178]

Today, this idea is expressed in terms of elliptic operators $\mathcal{E}_{\otimes A}$ with *coefficients in C^*-algebras A*, which, for commutative A, are algebras of continuous functions on toplogical spaces P parametrizing families of operators $\mathcal{E}_p$, $p \in P$.

Closer home, we want to determine a homotopy bound on a *space of maps* $f : X \to \underline{X}$ in terms of $\inf Sc(X)$ and the the norms $\| \wedge^2 df \|$ of these maps.

Here is an instance of what we are looking for.

$[X \times P \to \bigcirc]$ **Sharp Parametric Area Contraction Theorem.** Let X be an orientable spin manifold of dimension n, let P be an m-dimensional orientable pseudomanifold, let g_p, $p \in P$, be a C^2-continuous family of smooth Riemannian metrics and let $f : X \times P \to S^{n+m}$ be a continuous map, where all maps $f_p = f_{|X_p} : X = X_p = X \times \{p\} \to S^{n+m}$ are C^1-smooth.

Then there exists a point $(x, p) \in X$, where the g_p-trace norm of the exterior square of the differential of $f_p(x)$ is bounded from below by

$$2\|trace(\wedge^2 df_p)\| = \sum_{i \neq j}^{n} \lambda_i(x)\lambda_j(x) \geq Sc(g_p, x)$$

for some $(x, p) \in X \times P$.

Consequently,

the inclusion $\mathcal{I}_{\{g\}}$ of the space $\mathcal{F}_{\{g\}}$ of pairs (g, f), where g is a Riemannian metric on X and $f : X \to S^{n+m}$ f is a smooth map, such that

$$2\|trace(\wedge^2 df_p)\| = \sum_{i \neq j}^{n} \lambda_i(x)\lambda_j(x) < Sc(g_p, x) \textit{ for all } (x, p) \in X \times P,$$

[178] This idea goes back to George Lusztig's paper *Novikov's higher signature and families of elliptic s* where it is used for a proof of the homotopy invariance of "torical" Pontryagin classes.

to the space of all continuous maps $X \to S^{n+m}$,

$$\mathcal{I}_{\{g\}}\mathcal{F}_{\{g\}} \hookrightarrow \mathcal{F}_{cont}(X, S^{n+m}),$$

is contractible.

Outlines of two Proofs. 1. Apply the parametric index theorem to the Dirac operators on X_p twisted with bundles $L_p \to X_p$ induced from the same bundle $\underline{L} = \mathcal{S}^{\pm}(S^{n+m}) \to S^{n+m}$ that was used in the proof of the area contraction theorems in section 3.4.1 and confirm curvature estimates needed for the twisted S-L-W-B formula.

(If n is odd, one has to argue as in $\bigcirc_{\blacksquare}$ in the proof of $[X_{spin} \to \bigcirc]$ for odd n in section 3.4.1.)

2. Reduce the parametric problem to the non-parametric trace extremality theorem $[X_{spin} \to \bigcirc]$ from section 3.4.1 applied to maps $X^{n+m} \to S^{n+m}$.

To do this, assume P is a manifold[179] and let h_λ, $\lambda \geq 0$, be a family of Riemannian metric on P such that $g_\lambda \geq \lambda \cdot h_0$ and $Sc(h_\lambda) \leq \lambda^{-1}$ and send $\lambda \to \infty$. Then, due to *additivity of trace*, application of $[X_{spin} \to \bigcirc]$ yields $[X \times P \to \bigcirc]$.

Remarks.(a) If instead of the trace norm of *df* we had used the sup-norm, this argument would give you a non-sharp inequality, namely with the extra constant $\frac{(n+m)(n+m-1)}{n(n-1)}$.

(b) *Non-product families.* Let $\{X_p\}$ be a continuous family of compact connected orientable Riemannian n-manifolds parametrized by an orientable N-psedomanifold $P \ni p$, that is $\{X_p\}$ is represented by a fibration $\mathcal{X} = \{X_p\} \to P$ with the fibers X_p.

Let $f : \mathcal{X} \to S^{n+N}$, where $n = dim(X_p)$ and $N = dim(P)$, be continuous map the *restrictions* of which to all X_p are, smooth *area non-decreasing*, e.g. 1-Lipschitz maps, the differentials of which are continuous in $p \in P$, and let the degree of f be *non zero.*

If the fiberwise tangent bundle $\{T(X_p)\}$ of $\mathcal{X}$ is spin, then the above mentioned parametric index theorem to the Dirac operators on X_p implies that

the infimum of the scalar curvatures of all X_p satisfies

$$\inf_{x \in X_p, p \in P} Sc(X_p, x) \leq n(n-1).$$

Moreover, in the extremal case of $\inf_{x \in X_p, p \in P} Sc(X_p, x) = n(n-1)$, one can show that *some of X_p is isometric to S^n.*

(If P is a smooth manifold, such that $\mathcal{X}$ is spin, then all this can be proved with the index theorem on $\mathcal{X}$.)

(c) *Maps to Fibrations.* Let $\underline{\mathcal{X}} \to P$ be a sphere bundle with the fibers $S_p^{n+N} = S^{n+N}$ and $f : \mathcal{X} \to \underline{\mathcal{X}}$ a fiberwise map,

$$f = \{f_p : \mathcal{X}_p \to S_p^{n+N}\}.$$

Then, with a suitable defined condition "$deg(f) \neq 0$",the above inequality on the scalar curvatures of the fibers X_p remains valid.

[179] In the general case, by using a Thom's theorem, replace P by a manifold P' mapped to P with non-zero degree

To see this, reduce (c) to (b) as follows.

Let $\underline{\mathcal{X}}^\perp \to P$ be the complementary S^m-bundle, that is the *join bundle* $\underline{\mathcal{X}} * \underline{\mathcal{X}}^\perp$ with the fibers $S_p^{n+N+m+1} = S_p^{n+N} * S^m$ is *trivial*, and observe that the map f canonically suspends to a fiberwise map

$$X * \underline{\mathcal{X}}^\perp \to \underline{\mathcal{X}} * \underline{\mathcal{X}}^\perp,$$

which, due to the triviality of the fibration $\underline{\mathcal{X}} * \underline{\mathcal{X}}^\perp$, defines a map

$${}^*f : \mathcal{X} * \underline{\mathcal{X}}^\perp \to S^{n+N+m+1}.$$

Since the scalar curvatures of the fibers $\mathcal{X}_p * \underline{\mathcal{X}}_p^\perp$ are bounded from below by the curvature of $S^{n+N+m+1}$ (see exercise [*] in section 1.1) one can use (b), where, as in the reduction of the odd dimensional case of maps $X \to S^n$ to n even in ○■ in section 3.4.1, the fibers $\mathcal{X}_p * \underline{\mathcal{X}}_p^\perp$ and thus the space $\mathcal{X} * \underline{\mathcal{X}}^\perp$ must be completed by slightly perturbing the metric and then extending it cylindrically at infinity with (arbitrarily) large scalar curvature.

Exercises. (c_1) Use the trace norm on $\wedge^2 df$ and reduce (c) to (b) with the fiberwise version of ○■.

(c_2) Directly define "$deg(f)$" and prove (c) with the parametric index theorem.

(d) *Families of Non-Compact Manifolds.* The above generalizes to families of complete manifolds X_p and maps $f : \mathcal{X} \to S^{n+N}$, which are (locally) *constant at infinities* of all X_p (degrees are well defined for such maps f), where, the parametric relative index theorem, according to [Zhang(area decreasing) 2020], applies whenever all X_p have (not necessarily uniformly) positive scalar curvatures and where the conclusion concerns the scalar curvatures of X_p on the support of the differential df on the manifolds X_p

$$\inf_{x \in supp(df_{|X_p}), p \in P} \frac{Sc(X_p, x)}{2\| \wedge^2 df_{|X_p}(x)\|_{trace}} \le 1,$$

(e) *Foliations.* There is a further generalizations of (b) to smooth foliations n-dimensional leaves on compact orientable $(n+N)$-dimensional manifolds $\mathcal{X}$, with smooth Riemannian metrics on them X.i

Namely, let $\mathcal{X} \to S^{n+N}$ be a smooth map of non-zero degree.

If either the manifold $\mathcal{X}$ is spin or the tangent bundle to the leaves is spin, then there exists a point $x \in \mathcal{X}$,such that the scalar curvature of the leaf $X = X_x \subset \mathcal{X}$ passing trough x at x is related to the differential of f restricted to X by the inequality

$$Sc(X, x) \le 2\| \wedge^2 df_{|X}(x)\|_{trace}.$$

This is proven with $n(n-1)\|df\|^2$ instead of $\| \wedge^2 df_{|X}(x)\|_{trace}$ by Guangxiang Su [Su(foliations) 2018] and extended to complete manifolds in [Su-Wang-Zhang(area decreasing foliations) 2021] by sharpening the arguments by Alain Connes and Weiping Zhang. (The proofs in these papers, if I red them correctly, allows a use of $\| \wedge^2 df_{|X}(x)\|_{trace}$ rather than $\|df\|^2$.

Examples. Most natural (homogeneous) foliations with non-compact leaves support no metrics with $Sc > 0$ by Alain Connes' theorem, but their products with spheres S^i, $i \ge 2$ carry lots of such metrics, to which Su's theorem applies.

Questions. Does this theorem remain valid for foliations with smooth fibres but only C^k-continuous in the transversal direction, such for instance, as stable/unstable foliations of Anosov systems?

(Notice in this regard that another Connes' theorem, which generalizes Atiyah L_2-index theorem and applies to foliations with *transversal measures,* needs these foliations to be only C^3-continuous in the transversal direction, compare with discussion in sections $9\frac{2}{3}$, $9\frac{3}{4}$ in [G(positive)1996].)

What is the comprehensive inequality that would include all of the above from (b) to (e)?

Families with Singularities. Is there a meaningful version of the above for families X_p, where some X_p are singular, as it happens, for instance, for Morse functions $\mathcal{X} \to \mathbb{R}$?

Notice in this regard that Morse singularities, are, essentially, conical, where positivity of $Sc(X_p)$ for singular X_p in the sense of section 5.4.1 can be enforced by a choice of a Riemannian metric in $\mathcal{X}$. [180]

Conversely, positivity of $Sc(X_p)$, for all X_p including the singular ones, probably, yields a smooth metric with $Sc > 0$ on $\mathcal{X}$.

And it must be more difficult (and more interesting) to decide if/when a manifolds with $Sc > 0$ admits a Morse function, where all, including singular, fibers have positive scalar curvatures or, at least, positive operators $-\Delta + \frac{1}{2}Sc$.

3.4.4 Area Multi-Contracting Maps to Product Manifolds and Maps to Symplectic Manifolds

A guiding principle in the scalar curvature geometry reads:

If certain geometric and/or topological properties of Riemannian manifolds X_i, $i = 1, 2, ..., k$ imply that $\inf Sc(X_i) \le \sigma_i$, then such a property of Riemannian manifolds X *homeomorphic the products* $\times_i X_i = X_1 \times ... \times X_k$ implies that $\inf Sc(X) \le \sum_i \sigma_i$.

1. Topological non-Existence Example. If X_1 and X_2 admit no complete metrics with $Sc > 0$, and if X_2 is compact, then in many, probably, not in all cases the product $X_1 \times X_2$ admits no such metric either, (this seems to fail for SYS-manifolds).

A a prominent instance of this – here and everywhere with scalar curvature – is X_2 equal to the N-torus $\mathbb{T}^N$.

2. Length Contraction Example. Let $\underline{X}_i$ $i = 1, ..., k$, be orientable (spin) length extremal Riemannian manifolds with $Sc(\underline{X}_i) \ge 0$, which means that all smooth maps of *non-zero* degrees from orientable (spin) Riemannian n_i-manifolds X_i with $Sc(X_i) > 0$ to $\underline{X}_i$

$$f_i : X_i \to \underline{X}_i,$$

satisfy

$$\inf_{x_i \in X_i} \frac{Sc(X_i, x_i)}{Sc(\underline{X}_i, f(x_i)) \|df_i(x_i)\|^2} \le 1.$$

Then – this is expected in many cases – the Riemannian manifold $\underline{X} = \times_i \underline{X}_i$ is also (spin) length extremal. (This is, probably, true for all *known* examples of *spin* length extremal manifolds $\underline{X}_i$.)

[180] These are cones over $S^k \times S^{n-k-1}$, $n = dim X_p$, where the scalar curvature of such a cone can be made positive, unless $k \le 1$ and $n - k - 1 \le 1$.

Moreover all smooth maps from orientable (spin) Riemannian manifolds X to the product $\underline{X} = \times_i \underline{X}_i$ defined by a k-tuple of maps $X \to X_i$,

$$\Phi = (\phi_1, ..., \phi_k) : X \to \bigtimes_i^k \underline{X}_i,$$

which have non-zero degree should satisfy the following stronger inequality,

$$\min_{i=1,...,k} \left(\inf_{x \in X} \frac{Sc(X,x)}{Sc(\underline{X}_i, \phi_i(x)) ||d\phi_i(x)||^2} \right) \leq 1.$$

And in the ideal world one expects even more:

$$\left(\inf_{x \in X} \frac{Sc(X_i, x_i)}{Sc(\underline{X}, \Phi(x)) \left(\sum_{i=1}^k ||d\phi_i(x)|| \right)^2} \right) \leq k^2.$$

One also expects this product property for *area* rather than *length*, that is with the norm of the exterior power of the differentials, $|| \wedge^2 d\phi_i(x)||$ instead of $||d\phi_i(x)||^2$, which is (partly) justified by what follows.

Rough Multi-Area non-Contraction Inequality. Let $\underline{X}$ be a compact Riemannian manifold decomposed into product of Riemannian manifolds of positive dimensions,

$$\underline{X} = \underline{X}_1 \times ... \times \underline{X}_i \times ... \times \underline{X}_k, \ dim(\underline{X}_i) \geq 1,$$

let X be a compact orientable spin manifold of dimension $n \leq dim(\underline{X})$ and let $X \to \underline{X}$ be a smooth map defined by a k-tuple of maps to $\underline{X}_i$,

$$f = (f_1, ..., f_i, ...f_k) : X \to \underline{X} = \underline{X}_1 \times ... \times \underline{X}_i \times ... \times \underline{X}_k.$$

If the image of the fundamental homology class under f,

$$f_*[X] \in H_n(\underline{X})$$

is non-torsion, *then the scalar curvature of X is bounded by the area contraction by f, as follows*

$$\min_i \inf_{x \in X} \frac{Sc(X,x)}{|| \wedge^2 df_i(x)||} \leq \sigma,$$

where the constant σ depends on $\underline{X}$ but not on X.[181]

Proof. Since $f_*[X]$ is non-torsion, there exist cohomology classes $h_i \in H^{n_i}(\underline{X}_i; \mathbb{Q})$, $\sum_i n_i = n$, such that the cup product $h^* \in H^n(\underline{X})$ of their lifts to $\underline{X}$ doesn't vanish on $f_*[X]_{\mathbb{Q}})$.

By multiplying $\underline{X}_i$, where k_i are odd, by circles and multiplying X by the product of these circles, we reduce the situation to the case, where all k_i as well $n = dim(X) = \sum_i n_i$ are *even.*

Then, by the rational isomorphism between the K-theory and ordinary cohomology,

[181]If $\underline{X}$ is infinite dimensional, e.g. this is the Grassmann manifold of m-planes in the Hilbert space, then σ may depend on $n = dim(X)$.

there exist complex vector bundles $\underline{L}_i \to \underline{X}_i$, such that the Chern character of the tensor product $\underline{L} \to \underline{X}$ of the pull-backs of $\underline{L}_i$ to $\underline{X}$ doesn't vanish on $f_*[X]_{\mathbb{Q}}$ either.

It follows, that the *index of the Dirac operator* on X with values in the f-induced bundle $L^* = f^*(\underline{L})$ – we assume that X is spin and the this is defined – or in some associated bundle $L^\star \to X$ *doesn't vanish.* (This is elementary algebra as in the definition f the K-area.)

Endow the bundles L_i with unitary connections and observe, as we did earlier, that the norm of the curvature of the corresponding connection in $L^\star \to X$ is bounded by a constant C which depends only on $\underline{X}$ and on the norms $\|\wedge^2 df_i\|$, but not in any other way on X and on f.

Therefore, by the twisted Schrödinger-Lichnerowicz-Weitzenböck-Bochner formula the index of $\mathcal{D}_{\otimes L^\star}$ would vanish for $Sc(X) >> C$ and the proof follows.

Rank 1 Corollary. *If $Sc(X) > 0$ and (the differentials of) all maps f_i have ranks ≤ 1 then $f_*[X]_{\mathbb{Q}} = 0$.*

This follows from the inequality $\sigma(0,0,...,0) \leq 0$ and the definition of $\underline{\sigma}$.

For instance, this shows again that

continuous maps from orientable Riemannian spin manifolds X with $Sc(X) > 0$ to T^m send the fundamental homology classes $[X] \in H_n(X)$ to zero in $H_n(T^m)$, since tori are products of circles and maps to circles have ranks ≤ 1.

(Maps f with all their components f_i of rank one, may be themselves smooth embeddings $X \to \underline{X}$.)

Sharp Multi-Area Inequalities. Let $\underline{X}_i$, $i = 1, ..., k$, be compact orientable Riemannian manifolds, either with *non-negative curvature s* or Hermitian ones with positive Ricci curvatures. Let X be a compact orientable manifold and let

$$f = (f_1, ..., f_k) : X \to \underline{X} = \bigtimes_{i=1}^{k} \underline{X}_i$$

be a map a positive degree. Let $\|\wedge^2 df_i\|$ stands either for the *norm* of the second exterior power of the differential of the map $f_i : X \to \underline{X}_i$ or , in the case where $\underline{X}_i$ is the sphere S^{n_i}, it for the *averaged trace* of $\wedge^2 df_i$ defined as earlier:

$$\frac{1}{n(n-1)}\|trace(\wedge^2 df_i(x))\| = \frac{1}{n(n-1)}\sum_{\mu\neq\nu}^{n} \lambda_\mu(x)\lambda_\nu(x).$$

(The latter is non-greater than the former.)

Conjecture. There exists a point $x \in X$, such that

$$(\bigstar) \qquad Sc(X,x) \leq Sc(\underline{X}, f(x)) \cdot \sum_i \|\wedge^2 df_i(x)\|.$$

1. Start with enumerating the cases, where this conjecture *was proved* for maps from *spin manifolds* X to *unsplit into products* manifolds $\underline{X}$, i.e. for $k = 1$.

1.A. *$\underline{X}$ is the n-sphere S^n.*

The main computation and reduction of the case $n = 2m-1$ to $n = 2m$ via the map $X \times \mathbb{T}^1 \to S^{2m}$ was performed in [Llarull(sharp estimates) 1998]. Then the scale invariant trace form of Llarull's inequality was established in [Listing(symmetric spaces) 2010] for even n, and as we explained in section

3.4.1 the trace form of the area inequality allows an automatic reduction $n = 2m-1 \rightsquigarrow n = 2m$.

1.B. $\underline{X}$ *is a Hermitian symmetric space with* $Ricci(\underline{X}) > 0$. This was proved for symmetric $\underline{X}$ in [Min-Oo(Hermitian) 1998] and extended to all Hermitian spaces with $Ricci(\underline{X}) \geq 0$ and $Ricci(\underline{X}, x_0) > 0$ at some point in [Goette-Semmelmann(Hermitian) 2002].

1.C. $\underline{X}$ *has non-zero Euler characteristic.* Proved in [Goette-Semmelmann(symmetric) 2002] and brought to the scale invariant form in [Listing(symmetric spaces) 2010].

2. *Stabilization by* $\mathbb{T}^N$. Whenever the inequality (★) is established for manifolds X_o of dimension n_o and maps $X_o \to \underline{X}_o$ by confronting the index theorem with the twisted Schrödinger-Lichnerowicz-Weitzenböck-Bochner formula (there is no known alternative for this) then this argument also applies to maps $f : X \to \underline{X}_o \times \mathbb{T}^N$, $dim(X) = n_o + N$.

To show this, recall that the N-tori $\mathbb{T}^N$ for N even, support (almost flat) unitary bundles $\underline{L}_\varepsilon$ for all $\varepsilon > 0$, (and similar families of flat bundles a la Lusztig) with

(a) *non-zero Chern characters* man and, at the same time with

(b) *curvature operators with norms* $\leq \varepsilon$.

Now, suppose that (★) follows with the Dirac $\mathcal{D}$ on X_o twisted with the bundle $L_o \to X_o$ induced from a bundle $\underline{L}_o \to \underline{X}_o$ by a map $f_o : X_o \to \underline{X}_o$.

Then observe that the same argument applies to $\mathcal{D}$ on on X twisted with the bundle $L \to X$ induced by a map $f : X \to \underline{X}_o \times \mathbb{T}^N$ of non-zero degree from the tensor product $\underline{L}_o \otimes \underline{L}_\varepsilon \to \underline{X}_o \times \mathbb{T}^N$ by letting $\varepsilon \to 0$.

Indeed, (a)&$(deg(f) \neq 0)$ imply non-vanishing of $index(D_{\otimes L})$,while (b) guaranties the same bound on the L-curvature term in the twisted S-L-B-W formula for $\varepsilon \to 0$, as in the L_o-curvature for $D_{\otimes L_o}$.

Remark. As we mentioned above, one can use families of *flat* bundles over $\mathbb{T}^N$, (or more generally, suitable Hilbert moduli over the C^*-algebra of $\pi_1(\mathbb{T}^N)$) which have a advantage of giving (slightly) sleeker proofs of rigidity theorems.

3. The above argument, probably, applies to general manifolds $\underline{X}_1$ with bundles $\underline{L}_1 \to \underline{X}_1$ instead of $\underline{L}_\varepsilon \to \mathbb{T}^N$, where an essential point is checking that the curvature contribution to the S-L-B-W formula from the induced bundle $L = f^*(\underline{L}_0 \otimes \underline{L}_1) \to X$ for maps $f : X \to \underline{X}_o \times \underline{X}_1$ is bounded by the sum of the corresponding contributions from $f_o^*(\underline{L}_o)$ and $f_1^*(\underline{L}_1)$ for maps $f : X_o \to \underline{X}_o$ and $f_1 : X_1 \to \underline{X}_1$.

We suggest the reader will verify this, while we turn ourselves to a special case, where the necessary linear algebraic computation has been already done.

4. *Maps to Products of 2-Spheres and to Symplectic Manifolds.* Let

$$\underline{X} = \bigtimes_{i=1}^{k} S_i^2,$$

$\underline{S}_i^2$ are spheres with smooth Riemannian metrics, let X be a compact orientable Riemannian manifold of dimension $2k$ and let

$$f = (f_1, ..., f_k) : X \to \underline{X}$$

be a smooth map.

Let $\underline{\omega}_i$ be the area forms of S_i^2, thus, $\int_{\underline{S}_i^2} \underline{\omega}_i = area(\underline{S}_i^2)$, and let ω_i be the 2-forms on X induced from $\underline{\omega}_i$ by $f_i \to S_i^2$.

Observe that $\| \wedge^2 df_i(x)\| = \|\omega_i(x)\|$ equals the maximal area dilation by f_i at x of surfaces $S \ni x$ in X.

f has non-zero degree, then there exist a point $x \in X$, where the scalar curvature of X is bounded un terms of $\| \wedge^2 df_i(x)\|$ as follows,

$$(\bigstar_2) \qquad Sc(X,x) \le 8\pi \sum_i \frac{\| \wedge^2 df_i(x)\|}{area(\underline{S}_i^2)},$$

where the equality holds if and only if X is the product of Euclidean spheres $X = \times_{i=1}^k S^2(r_i)$ with no restrictions on their radii r_i and on the Riemannian metrics in $\underline{S}_i^2$.

Proof. Start by observing that the right hand side of $(\bigstar_2)$ doesn't depend on the choice of Riemannian metrics on $\underline{S}_i^2$ and we may assume all $\underline{S}_i^2$ isometric to the unit sphere $S^2 = S^2(1)$.

Let $\underline{L} \to \underline{X} = (S^2)^k$ be the tensor product of the pullbacks of the Hopf bundle over S^2 under the k projections $\underline{X} \to S^2$ and observe that the curvature form of this (complex unitary line) bundle $\underline{L} \to X$ is:

$$curv(\underline{L}) = \frac{1}{2}\sum_i \omega_i.$$

Therefore, for all $x \in X$, the diagonal decomposition of form ω_x in an orthonormal basis in the tangent space $T_x(X)$, orthonormal basis (τ_i, θ_i), $i = 1, ..., k$,

$$\omega = \sum_i \lambda_i \tau_i, \wedge \theta_i, \ \ \lambda_i \ge 0$$

satisfies

$$\sum_i \lambda_i \le \sum_i \| \wedge^2 df_i(x)\|.$$

It follows (theorem1.1 in [Hitchin(spinors) 1974]) that if

$$Sc(X,x) > 8\pi \sum_i \frac{\| \wedge^2 df_i(x)\|}{area(\underline{S}_i^2)},$$

then X supports *no non-zero harmonic spinors* twisted with L.

On the other hand the top term in the Chern character of L is *non-zero* and the index theorem says that X *does support* such a spinor, and, as everywhere in this kind of argument, the proof follows by contradiction.

Symplectic Manifolds and $\underline{\omega}$-Extremality. The above argument equally applies to maps of non-zero degree between $2k$-dimensional orientable manifolds, $f : X \to \underline{X}$, where $\underline{X}$ is endowed with a closed 2-form $\underline{\omega}$, such that

• the cohomology class $\underline{c} = \frac{1}{2\pi}[\underline{\omega}] \in H^2(\underline{X}; \mathbb{R}$ is *integral*: $\int_S [\underline{\omega}] \in 2\pi\mathbb{Z}$ for all closed oriented surfaces in $\underline{X}$ (the basic example is one half of the area form on S^2);

• the product of $\exp c = 1 + c + \frac{c^2}{2} + ... + \frac{c^k}{k!}$ where $c = f^*(\underline{c}) \in H^2(X)$ for the cohomology homomorphism $f^* : H^2(\underline{X}) \to H^2(X)$, with the *Todd class* $\hat{A}(X)$

(a polynomial in Pontryagin classes of X, see section 4) doesn't vanish on the fundamental homology class of X

$$(\exp c) \smile \hat{A}[X] \neq 0.$$

(For instance $c^k \neq 0$ and $\underline{X}$ is *stably parallelizable*, which, by *Hirsch immersion theorem*, is equivalent to the existence of a smooth immersion $\underline{X} \to \mathbb{R}^{2k+1}$, while $c^k \neq 0$.)

$\kappa_\star$-*Invariant.* Let $\underline{X} = (\underline{X}, \underline{\omega}, \underline{h})$ be a smooth manifold, where:

$\underline{\omega}$ is a *differential 2-form* on $\underline{X}$, e.g. a symplectic one, i.e. where $\underline{\omega}$ is closed, the dimension of $\underline{X}$ is even and $\underline{\omega}^m$, $m = \frac{dim(\underline{X})}{2}$, nowhere vanishes on $\underline{X}$

and where $\underline{h} \in H_n(\underline{X})$ is a *distinguished homology class.*

Define $\kappa_\star(\underline{X})$n as the infimum of the numbers $\kappa > 0$, such that all smooth maps of from all closed orientable Riemannian *spin*[182] manifolds of dimension n to $\underline{X}$,

$$f : X \to \underline{X},$$

which send the fundamental homology class of $X to \underline{h}$,

$$f_*[X] = \underline{h},$$

satisfy

$$\inf_{x \in X} Sc(X, x) \leq 4 \cdot \kappa \cdot trace(\omega(x)),$$

where $\omega = f^*(\underline{\omega}))$ is the f-pullback of the form $\underline{\omega}$ and

$$trace(\omega(x)) = \sum \lambda_i$$

for the above g-diagonalization of ω.

(See §$5\frac{4}{5}$ in [G(positive)1996] and section3.4 in [Min-Oo(scalar) 2020] for integral versions of this invariant.)

A Riemannian manifold X is called $\underline{\omega}$-*extremal* if it admits a smooth map $f : X \to \underline{X}$, such that $f_*[X] = \underline{h}$ and

$$Sc(X, x) = 4 \cdot \kappa_\star \cdot trace(\omega(x)), \text{ for all } x \in X.$$

The above proof of ($\bigstar_2$) actually shows that the product of spheres $\underline{X} = (S^2)^k$ is $\underline{\omega}$-*extremal* for the sum of the area forms $\underline{\omega}_i$ of the S^2-factors of $\underline{X}$,

$$\underline{\omega} = \sum_i \underline{\omega}_i,$$

where $\underline{h} \in H_{2k}(\underline{X})$ is the *fundamental* class $[\underline{X}]$, where $\kappa = \frac{1}{2}$ and where any *symplectomorphism* $X = \underline{X} \to \underline{X}$ can be taken for f.

Remarks. (a) The above is a reformulation of a special case of *area extremality*[183] theorems from [Min-Oo(Hermitian) 1998], [Bär-Bleecker(deformed algebraic) 1999] and [Goette-Semmelmann(Hermitian) 1999], where the authors

[182] This can be relaxed to properly formulate $spin^c$.

[183] *Area extremality* of a Riemannian manifold $X = X(g)$ (essentially) means that all metrics g' with $Sc(g') > Sc(g)$ on X must have $area_{g'}(S) < area(S_g)$ for some surface $S \subset X$. If X is a Kähler manifold then ω-extremality (obviously) implies area extremality for the Kähler form ω of X.

establish the $\underline{\omega}$-extremality of several classes of *Kähler manifolds* including compact Hermitian symmetric spaces, Kähler manifolds X with $Ricci(X) > 0$ and also of certain complex algebraic submanifolds $X \hookrightarrow \underline{X} = \mathbb{C}P^N$, with the Fubini-Study form $\underline{\omega}$ on $\mathbb{C}P^N$.

(b) Besides multi-area contraction inequalities there are similar multi-length inequalities, such as the multi-width $\square^n$-inequality from section **??**, where the (stronger) multi-area contraction inequality doesn't apply.

Conjecture All (most?) $\underline{\omega}$-extremal manifolds are *Kählerian*, or closely associated with with *Kählerian* or similar manifolds, such, e.g. as Kälerian$\times\mathbb{T}^m$.

Admission. I don't even see, why the forms $\underline{\omega}$ in all extremal cases must be closed but not, say, "maximally non-closed", such as generic ones.

Question. Are there further sharp inequalities between (norms of) differentials df_i for maps

$$f = (f_1, ...f_i, ...f_k) : X \to \underline{X} = \bigtimes_{i=1}^{k} S^{n_i}, \ \ \sum_i n_i = n,$$

with $deg(f) \neq 0$ and (the lower bound on) $Sc(X)$ besides

$$\inf_{x \in X} \frac{Sc(X,x)}{Sc(\underline{X}, f(x)) \cdot \sum_i \| \wedge^2 df_i(x)\|} \leq 1$$

from the above ($\bigstar$) and/or its $\| \wedge^2 df_i(x)\|_{trace}$ counterpart?

Namely, what are conditions on numbers σ and $b_1, ...b_i, ...b_k$, such that there exists a compact orientable (spin) manifold X of dimension $n = \sum_i n_i$ with $Sc(X) \geq \sigma$ and a smooth map $f = (f_1, ...f_i, ...f_k) : X \to \underline{X} = \times_{i=1}^{k} S^{n_i}$ with $deg(f) \neq 0$, such that $\| \wedge^2 df_i(x)\| \leq b_i$ for all $x \in X$?

3.5 Sharp Bounds on Length Contractions of Maps from Mean Convex Hypersurfaces

The Atiyah-Singer theorem, when applied to the *double* $\mathbb{D}(X)$ *of a compact manifold* X with boundary, delivers a non-trivial geometric information on X as well as on the boundary $Y = \partial X$.

For instance, if $mean.curv(Y) > 0$, then, as we explained in section 1.4, the natural, continuous, metric g on $\mathbb{D}(X)$ can be approximated by C^2-metrics g' by smoothing g along the "Y-edge" *without a decrease of the scalar curvature* in a rather canonical manner. Here is an instance of what comes this way.

[$Y_{spin} \to \bigcirc$] **Mean Curvature Spin-Extremality Theorem.** Let X be a compact Riemannian manifold of dimension n with orientable *mean convex boundary*[184] Y and let $\underline{Y} \subset \mathbb{R}^n$ be a smooth compact convex hypersurface.

Let h and $\underline{h}$ denote the Riemannian metrics in Y and $\underline{Y}$ induced from their ambient manifolds and let $h^\natural$ and $\underline{h}^\natural$ be their MC-normalizations (see section **??**),

$$h^\natural(y) = mean.curv(Y,y)^2 \cdot h(y) \text{ and } \underline{h}^\natural(y) = mean.curv(Y,y)^2 \cdot \underline{h}(y).$$

[184] This mean the mean curvature of the boundary is non-negative, where the sign convention is such that boundaries of convex domains in $\mathbb{R}^n$ are mean convex.

Then, provided the manifold X is spin*, all λ-Lipschitz maps $f : Y \to \underline{Y}$ with $\lambda < 1$ are contractible.*

In other words,

if a smooth (Lipschitz is OK) map $f : Y \to \underline{Y}$ has a non-zero degree, then there exists a point $y \in Y$, where the norm of the differential of f is bounded from below as follows:

$$||df(y|| \geq \frac{mean.curv(Y, y)}{mean.curv(\underline{Y}, f(y))}.^{185}$$

If $\underline{Y} = S^{n-1}$, this extremality, as in the case of the scalar curvature, can can be sharpened with a use of the trace norm of the differential df ..., except that I have not verified the computation and leave the following "theorem" with a question sign.

[Y_{spin} →○] **Mean Curvature Trace Extremality Theorem(?)**[186]. Let X be a compact orientable Riemannian spin manifold of dimension n with orientable boundary Y and $f : Y \to S^{n-1} = \underline{y}$ be a map with $deg(f) \neq 0$.

Then f *can't be trace-wise strictly decreasing with respect to the MC-normalized metrics $h^\natural = mean.curv(Y)^2 h$ for the Riemannian metric h on Y induced from $X \supset Y$ and $\underline{h}^\natural = (n-1)^2 ds^2$ on S^{n-1}*, that is there is a point $x \in X$, where *the trace-norm of the differential of f is bounded from below by the mean curvature of X as follows:*

$$\frac{1}{(n-1)} \sum_{i=1}^{n-1} \lambda_i^\natural(y) \geq 1 \textit{ for } \lambda_i^\natural = \lambda_i(f^*(\underline{h}^\natural))/h^\natural),$$

which means that the trace-norm of df with respect to the original (non-normalized) metrics satisfies:

$$\frac{1}{(n-1)} ||df(y)||_{trace} \geq \frac{mean.curv(\underline{Y}, f(y))}{mean.curv(Y, y)}.$$

The simplest and most interesting common corollary of these two theorems is the following.

❂ (*Seemingly Elementary*) **Example.** If the mean curvature of a smooth hypersurface $Y \subset \mathbb{R}^n$ is *bounded from below by* $n-1$, that is the mean curvature of the unit sphere $S^{n-1} \subset \mathbb{R}^n$, then *all λ-Lipschitz map $f : Y \to \mathbb{R}^n$, where $\lambda < 1$, are contractible.*[187]

(If "Lipschitz" is understood with respect to the *Euclidean distance function* on X, rather than the larger one which is associated with the *induced Riemannian metric*, the proof easily follows from *Kirszbaum theorem.*)

[185] Here we agree that $\frac{0}{0} = \infty$.

[186] Probably, the quickest way to remove "?", at least for even n, is by adapting/refining the argument from [Lott(boundary) 2020]and/or from [Bär-Hanke(boundary) 2021].

[187] It is impossible not to ask oneself what happens for $\lambda = 1$, i.e. where f is distance non-increasing. You bet, such an f is either contractible, or it is an *isometry*. Indeed (almost) all our extremality theorems are accompanied by rigidity results in the equality cases, as we shall see later on.

But it is non-trivial to formulate and hard to solve the *stability problem*: what happens to geometries of hypersurfaces $X_\varepsilon \subset \mathbb{R}^n$ with $mean.curv(X_\varepsilon) \geq n-1$ and to $(1+\varepsilon)$-Lipschitz maps to S^{n-1} with non-zero degrees, when $\varepsilon \to 0$.

About the Proof of $\bigcirc$. Let X lie in a (slightly larger) Riemannian n-manifold $X_+ \supset X$ without boundary, let $Y_\varepsilon^{n+l-1} \subset X_+ \times \mathbb{R}^l$ be the boundary of the ε-neighbourhood of $X \subset X_+ \times \mathbb{R}^l$ and let us similarly, define $\underline{Y}_\varepsilon^{n+l-1} \subset \mathbb{R}^{n+l} = \mathbb{R}^n \times \mathbb{R}^l$ as the boundary of the ε-neighbourhood of $\underline{X} \subset \mathbb{R}^{n+l}$ = for $\underline{X} \subset \mathbb{R}^n$ with boundary Y.

Observe – this needs a little computation as in section 1.4 – that the lower bounds on the scalar curvatures of the "interesting parts"

$$Y_{\varepsilon\varepsilon}^{n+l-1} \subset Y_\varepsilon^{n+l-1} \text{ and } \underline{Y}_{\varepsilon\varepsilon}^{n+l-1} \subset \underline{Y}_\varepsilon^{n+l-1}$$

which are ε-close to the original $Y \subset Y_\varepsilon^{n+l-1}$ and $\underline{Y} \subset \underline{Y}_\varepsilon^{n+l-1}$, are perfectly controlled by their mean curvatures, while their complements, being flat in the ambient manifolds, have the same scalar curvatures as X and $\underline{X}$, where the latter is equal to zero.

Then extend $f : Y \to \underline{Y}$ a map

$$f_\varepsilon : Y_\varepsilon^{n+l-1} \to \underline{Y}_\varepsilon^{n+l-1},$$

such that the "interesting part"of Y_ε^{n+l-1} goes to that of Y_ε^{n+l-1} and the complement of one to the complement of the other and such that the "interesting part" of this extensions is done in a most economical manner along normal geodesics to $Y \subset Y_{\varepsilon\varepsilon}^{n+l-1}$ and to $\underline{Y} \subset \underline{Y}_{\varepsilon\varepsilon}^{n+l-1}$.

If we do it with a proper care then, for a small enough ε and l with the same parity as n, we shall be able to apply the spin-area convex extremality theorem [$X_{spin} \to \bigcirc$] from the section 3.4.1 to the map f_ε, which that would need a preliminary smoothing of the manifolds Y_ε^{n+l-1} and $\underline{Y}_\varepsilon^{n+l-1}$ by tiny C^1-perturbations (these manifolds themselves are only C^1-smooth), where, while while smoothing the hypersurface $\underline{Y}_\varepsilon^{n+l-1}$ convex, smoothing of $\underline{Y}_\varepsilon^{n+l-1}$ must *keep the flat part flat.*

Because of the latter, the point $y_\varepsilon \in \underline{Y}_\varepsilon^{n+l-1}$, where

$$Sc(\underline{Y}_\varepsilon^{n+l-1}, f(y\varepsilon) \cdot \| \wedge^2 df(\varepsilon)\| \geq Sc(Y_\varepsilon^{n+l-1}, y\varepsilon),$$

provided by [$X_{spin} \to \bigcirc$] must be necessary located in the "interesting region" $Y_{\varepsilon\varepsilon}^{n+l-1}$; then the needed inequality for the mean curvature of Y will be satisfied by the point $y \in Y$ nearest to y_ε.

Remark about [$Y_{spin} \to \bigcirc$]. To carry out the above argument one needs a generalization of of the spherical trace inequality [$X_{spin} \to \bigcirc$] from the previous section to manifold $\underline{X}$ that don't have full $O(n+1)$-symmetry of S^n.

In the present case the relevant metric $\underline{g}$ is $O(n)$ invariant and one needs a separate bounds on the two parts of the trace norm of $\wedge^2 df$:

the first part comes from $\frac{n-1(n-2)}{2}$ bivectors $e_i \wedge e_j$ with e_i and e_j, $i, j = 1, ..., n-1$, tangent the S^{n-1}-spherical $O(n)$-orbits and the second one from the $n-1$ remaining $e_i \wedge e_n$ with the vector e_n normal to these orbits.

This is an instance of a more general principle:

to achieve the sharpest inequality, one should choose the norm for measuring df in accordance with the the symmetries of the manifold $\underline{X}$.

We shall see later on other instances of this "principle", e.g. for maps to products of spheres in section 3.4.4.

On Non-spin Manifolds and on $\sigma < 0$**.** Conjecturally, if the boundary $Y = \partial X$ of a compact orientable Riemannin n-manifold X with $Sc \geq -n(n-1)$

admits a smooth map f with non-zero degree to the boundary of the R-ball in the hyperbolic n-space with sectional curvature -1,

$$f : Y \to \partial B(R) \subset \mathbf{H}^n(-1),$$

and if

$$mean.curv(Y) \geq n-1 \text{ and } ||df|| \leq 1,$$

then the map f is an *isometry*. Moreover, f extends to an isometry $X \to B(R)$.[188]

We shall prove a partial result in this direction with a use of *stable capillary μ-bubbles*, which may also apply to maps to more general hypersurfaces in $\mathbf{H}^n(-1)$ (see section 5.8.1), but it remains unclear how to approach the trace-norm version of this conjecture.

Questions and Exercises. (a) Is there an elementary proof of this inequality for $n \geq 4$?

(b) Besides the lower bound on the mean curvature, that is the sum of the principal curvatures, $\sum_i \alpha_i$, the "size" of a hypersurface Y is bounded by the scalar curvature $\sum_{i \neq j} \alpha_i \alpha_j$ and also - this is obvious by the product of the principal curvatures $\prod_i \alpha_i$.

Are there similar inequalities for other elementary symmetric functions of α_i.

(If $Y \subset \mathbb{R}^n$ is *convex*, i.e. all $\alpha_i \geq 0$, then $\prod_i \alpha_i$ minorizes the rest of elementary symmetric functions, which gives a trivial proof of ❂ and similar inequalities for other symmetric functions for distance decreasing maps from convex hypersurfaces to S^n.)

the above theorems for *convex* hypersurfaces $Y\mathbb{R}^n$.)

But it is unclear if, for instance, there is a bound on this radius in terms of $\sum_{i>j>k} \alpha_i \alpha_j \alpha_k$ for $n \geq 5$ when this sum is positive.)

(d) Let $Y_0 \subset \mathbb{R}^n$ be a smooth compact cooriented submanifold with boundary $Z = \partial Y_0$, such that

the mean curvature of Y_0 with respect to its coorientation satisfies

$$mean.curv(Y) \geq n-1 = mean.curv(S^{n-1}).$$

Show that

every distance decreasing map

$$f : Z \to S^{n-2} \subset \mathbb{R}^{n-1}$$

is contractible,

where "distance decreasing"refers to the distance functions on $Z \subset \mathbb{R}^n$ and on $S^{n-2} \subset \mathbb{R}^{n-1}$ coming from the ambient Euclidean spaces $\mathbb{R}^n$ and $\mathbb{R}^{n-1}$.

Hint. Observe that the maximum of the principal curvatures of Y_0 is ≥ 1 and show that *the filling radius* of $Z \subset \mathbb{R}^n$ is ≤ 1.[189]

(e) *Question.* Does contractibility of f remains valid if the distance decreasing property of f is defined with the (intrinsic) spherical distance in S^{n-2} and with

[188] Granted f *is* an isometry (with respect to the induced Riemannin metrics in $\partial X \subset X$ and $\partial B(R) \subset B(R)$), an isometry $X \to B(R)$ follows from *Min-Oo's hyperbolic rigidity theorem* from section 3.13.

[189] This means that Z is homologous to zero in its 1-neighbourhood.

the distance in $Z \subset Y_0$ associated with the *intrinsic metric* in $Y_0 \supset Z$, where $dist_{Y_0}(y_1, y_2)$ is defined as the infimum of length of curves in Y_0 between y_1 and y_2?

(f) Formulate and prove the mean curvature counterparts of the theorems $[X_{spin} \overset{\hat{A}}{\to} \bigcirc]]$, $\times\mathbb{R}^m$ and $[X \times P \to \bigcirc]$ for maps $X^{n+m} \to \underline{X}^n$ and $X^n \to \underline{X}^{n+m}$ from sections 3.4.1 and 3.5, either by the above $Y^{n+l-1}_{\varepsilon\varepsilon}$-construction or by generalizing Lott's argument for manifolds with boundaries.

(h) *Question.* Is there a version (or versions) of the mean curvature extremality theorems for maps to products of convex hypersurfaces in the spirit of area multi-contracting maps in section 3.4.4

3.6 Riemannian Bands with $Sc > 0$ and $\frac{2\pi}{n}$-Inequality.

We saw in the previous sections how a use of twisted Dirac operators leads to geometric bounds, including certain sharp ones, on the size of compact Riemannian spin manifolds. Such bounds usually (always) extend to non-compact complete manifolds, but until recently no such result was available for non-complete manifolds and/or for manifolds with boundaries. [190]

On the other hand minimal hypersurfaces were used in [GL(complete)1983] for obtaining rough bounds for non-complete manifolds; below, we shall see how such hypersurfaces (and μ-bubbles in general) serve for getting *sharp* geometric inequalities of this kind.

Bands, sometime we call them *capacitors*, are manifolds X with two distinguished disjoint non-empty subsets in the boundary $\partial(X)$, denoted

$$\partial_- = \partial_- X \subset \partial X \text{ and } \partial_+ = \partial_+ X \subset \partial X.$$

A band is called *proper* if $\partial_\pm$ are unions of connected components of ∂X and

$$\partial_- \cup \partial_+ = \partial X.$$

The basic instance of such a band is the segment $[-1, 1]$, where $\pm\partial = \{\pm 1\}$.

Furthermore, *cylinders* $X = X_0 \times [-1, 1]$ are also bands with $\pm\partial = X_0 \times \{\pm 1\}$, where such a band is proper if X_0 has no boundary.

Riemannian bands are those endowed with Riemannian metrics and

the width of a Riemannian band $X = (X, \partial_\pm)$ is defined as

$$width(X) = dist(\partial_-, \partial_+),$$

where this distance is understood as the infimum of length of curves in V between ∂_- and ∂_+.

We are mainly concerned at this point with *compact* Riemannian bands X of dimension n, such that

$\square\!\!\Vert_{Sc \not> 0}$ *no* closed embedded hypersurface $Y \subset X$, which *separates* ∂_- from ∂_+, admits a $\mathbb{T}^1$-stabilization $Y^{\rtimes}$ with positive scalar curvature, i.e. no complete

[190] Several such results have appeared in the papers [Zeidler(bands) 2019], [Zeidler(width) 2020] and [Cecchini(long neck) 2020],[Cecchini-Zeidler(generalized Callias) 2021], [Cecchini-Zeidler(Scalar&mean) 2021], [Guo-Xie-Yu(quantitative K-theory) 2020], which we briefly discuss letter on .

(warped product) metric on the product $Y\times\mathbb{T}^1$ of the form $dy^2+\phi(y)^2dt^2$ has $Sc(h^{\rtimes})>0$.

(Since Y is compact, the existence of this (warped product) metic $h^{\rtimes}$ with $Sc(h^{\rtimes})>0$ is equivalent to the existence of a metric h with $Sc(h)>0$ on Y itself, since the conformal Laplacian $-\Delta+areaextremality3n-24(n-1)Sc$ is more positive that the $-\Delta+\frac{1}{2}Sc$ implied by positivity of $Sc(h^{\rtimes})$.)

Representative Examples of compact bands with this property are:

$\bullet_{\mathbb{T}^{n-1}}$ *toric bands* which are homeomorphic to $X=\mathbb{T}^{n-1}\times[-1,1]$;

$\bullet_{SYS}$ manifolds X homeomorphic to a Schoen-Yau-Schick manifolds times $[-1,1]$;

$\bullet_{\hat{\alpha}}$ these, called $\hat{\alpha}$ *bands,* are diffeomorphic to $Y\times[-1,1]$, where the Y is a closed spin $(n-1)$-manifold with non-vanishing $\hat{\alpha}$-invariant (see 3.2 the IV above);

$\bullet_{\mathbb{T}^{n-1}\times\hat{\alpha}}$ these are bands *diffeomorphic to products* $X_{n-k}\times\mathbb{T}^k$, where $\hat{\alpha}(X_{n-k})\neq 0$.

(A characteristic non-compact example with a similar property is

$\bullet_{\times\mathbb{R}\setminus\{\mathbb{Z}\}}$: X is homeomorphic to the product $\mathbb{T}^{n-2}\times\mathbb{R}\times[-1,1]$ minus a discrete subset.)[191]

$\mathbf{\frac{2\pi}{n}}$**-Inequality.** Let X be a *proper compact Riemannian bands X of dimension n with $Sc(X)\geq\sigma>0$.*

If no closed hypersurface in X which separates ∂_- from ∂_+ admits a metric with positive scalar curvature, then

$$\left[\circledcirc_{\pm}\leq\tfrac{2\pi}{n}\right]\qquad width(X)=dist(\partial_-,\partial_+)\leq 2\pi\sqrt{\frac{(n-1)}{n\sigma}}=\frac{2\pi}{n}\cdot\sqrt{\frac{n(n-1)}{\sigma}}$$

In particular if $Sc(X)\geq Sc(S^n)=n(n-1)$, then

$$width(X)\leq 2\pi\sqrt{\frac{(n-1)}{n\sigma}}=\frac{2\pi}{n}.$$

Moreover, the equality holds in this case only for warped products $X=Y\times\left(-\frac{\pi}{n},\frac{\pi}{n}\right)$[192] *with metrics φ^2h+dt^2, where the metric h on Y has $Sc(h)=0$ and where*

$$\varphi(t)=\exp\int_{-\pi/n}^{t}-\tan\frac{nt}{2}dt,\quad -\frac{\pi}{n}<t<\frac{\pi}{n},$$

as in section 2.4.

About the Proof. If a hypersurface $Y\subset X$, which separates ∂_- from ∂_+ contains a descending chain (flag) of closed oriented hypersurfaces,

$$Y\supset Y_{-1}\supset\ldots\supset Y_{-i}\supset\ldots,$$

[191] The property $\sqsupset\!\!\Vert_{Sc\not>0}$ for *toric bands* and for SYS-bands follows from the Schoen-Yau codimension 1 descent theorem (see section 2.7), in the case $\bullet_{\hat{\alpha}}$ this is the Lichnerowicz-Hitchin theorem (section 3.2) and $\bullet_{\mathbb{T}^{n-1}\times\hat{\alpha}}$ is a corollary to theorem 2.1 in [GL(spin) 1980], while a "complete" version of this property for the non-compact $(\mathbb{T}^{n-2}\times\mathbb{R}\times[-1,1])\setminus\{\mathbb{Z}\}$ is an example, where theorem 6.12. from [GL(complete) 1983] applies. (See sections 4.7, 5.10 for more about these and more general examples.

[192] Here, since X is non-compact, the width is understood as the distance between the two ends of X.

where where each $Y_{-i} \subset X$ is equal to a transversal intersection of $Y_{-(i-1)}$ with a smooth closed oriented sub-band $H_i \subset X$, of codimension one,

$$H_i \cap X_{-(i-1)} = X_{-i}$$

and where Y_{-i} represent *non-zero* classes in the homology $H_{n-1-i}(X)$, then one can proceed by the inductive Schoen-Yau's kind of descent method (see sections 2.7) with minimal hypersurfaces

$$...X_{-i} \subset X_{-(i-1)} \subset ... \subset X_{-1} \subset X,$$

where these X_{-i} are $\mathbb{T}^{\rtimes}$-symmetrised as in the $[\rtimes_\varphi]^N$-symmetrization theorem in section 2.8 where X_{-i} in our band X have "free" (pairs of) boundaries contained in $\partial_\mp(X_{-(i-1)})$, and such that the intersections $X_{-i} \cup Y$ are homologous to Y_{-i}.

This argument delivers the sharp version of $\frac{2\pi}{\mathbf{n}}$ for *over-toric bands*, i.e. those which admit maps $X \to \mathbb{T}^{n-1}$, $n = dim(X)$, with non-zero degrees of their restriction to $\partial_\mp$, but when it comes to SYS-bands, one gets only a weaker lower bound on $width(X)$, that is by $\frac{4\pi}{n}$, instead of $\frac{2\pi}{n}$.

The same weakening of $\frac{2\pi}{\mathbf{n}}$ takes place if separating hypersurfaces $Y \subset X$, are *enlargeable*, e.g. if the interior of X, assumed compact, admits a complete metric with non-positive sectional curvature. And if separating Y are *SYS times enlargeable*, one has to be content with $\frac{8\pi}{n}$.[193]

In section 3.7, we present a more efficient argument, where, instead of working with chains of minimal hypersurfaces, we show in one step that if $width(X) \geq 2\pi\sqrt{\frac{(n-1)}{n\sigma}}$, then a certain *stable μ-bubble $Y_{st} \subset X$, which separates Y_- from Y_+, supports a metric with $Sc > 0$.*

Besides, the sharp $\frac{2\pi}{\mathbf{n}}$ for wide class of *spin* bands was recently proven by Zeidler, Cecchini and Guo-Xie-Yu with new index/vanishing theorems on Dirac operators with potentials on manifolds with boundaries.[194]

Remarks.(a) If hypersurfaces separating ∂_- from ∂_+ in X are enlargeable, e.g. if X is homeomorphic to $\mathbb{T}^{n-1} \times [0,1]$, then a non-sharp version of $\frac{2\pi}{n}$-inequality,

$$dist(\partial_-, \partial_+) \leq 2^n\pi\sqrt{\frac{(n-1)}{n\sigma}}$$

follows from theorem 12.1 in[GL (complete)1983].

(b) One might think that the sharp $\frac{2\pi}{n}$-inequality, must be obvious for domains in the unit sphere S^n homeomorphic to $\mathbb{T}^{n-1} \times [-1,1]$ and for bands with *constant sectional curvatures* in general; to my surprise, I couldn't find a direct proof of it even for X is homeomorphic to $\mathbb{T}^{n-1} \times [0,1]$.

3.6.1 Quadratic Decay of Scalar Curvature on Complete Manifolds with $Sc > 0$.

***QD-Exercise**. Quadratic Decay Property.* Let X be a complete non-compact Riemannian n-manifold and $X_0 \subset X$ a compact subset, such that *there is no*

[193]This is worked out in §2-6 of [G(inequalities) 2018].

[194]See [Zeidler(bands) 2019], [Zeidler(width) 2020], [Cecchini(long neck) 2020] and the most recent [Guo-Xie-Yu(quantitative K-theory) 2020],[Cecchini-Zeidler(generalized Callias) 2021], [Cecchini-Zeidler(Scalar&mean) 2021].

domain $X_1 \subset X$, which contains X_0 and the boundary ∂X_1 of which (assumed smooth) *admits a metric* with $Sc > 0$, e.g. X is homeomorphic to $\mathbb{T}^{n-2} \times \mathbb{R}^2$.

Show that there exists a constant $R_0 = R_0(X, x_0)$, such that

the minima of the scalar curvature of X on concentric balls $B(R) = B_{x_0}(R) \subset X$ around a point $x_0 \in X$, satisfy

$$\min_{x \in B(R)} Sc(X, x) \leq \frac{4\pi^2}{(R - R_0)^2} \text{ for all } R \geq R_0.$$

Hint. Apply $\frac{2\pi}{n}$-inequality to the annuli between the spheres or radii R and R for a suitable constant c.

(Compare this with the quadratic decay theorem in section 1 of [G(inequalities) 2018] and see [Wang-Xie-Yu(decay) 2021] for estimates of the scalar curvature decay rates by contractibility radius and the diameter control of the asymptotic dimension and observe that, if X is homeomorphic to $\mathbb{T}^{n-2} \times \mathbb{R}^2$, than the quadratic decay with the constant $2^{n+1}\pi^2$ follows from [GL(complete 1983].)

Critical Rate of Decay Conjecture. There exists a universal *critical constant* c_n, conceivably, $c_n = \frac{4\pi^2(n-1)}{n}$, such that:

[a] if a smooth manifold X admits a complete metric g_0 with $Sc(g_0) > 0$, then, for all $c < c_n$, it admits a complete metric g_ε, with $Sc(g_\varepsilon) > 0$ and at most *c-sub-quadratic* scalar curvature decay,

$$Sc(g_\varepsilon, x) \geq \frac{c}{dist(x, x_0)^2} \text{ for a fixed } x_0 \in X \text{ and all } x \in X \text{ with } dist(x, x_0) \geq 1;$$

and

[b] if X admits a complete metric g_0 with $Sc(g_0) > 0$ and c-sub-quadratic for $c > c_n$ scalar curvature decay,

$$Sc(g_\varepsilon, x) \geq \frac{c_n}{dist(x, x_0)^2} \text{ for } dist(x, x_0) \geq 1,$$

then it admits a complete metric with $Sc \geq \sigma > 0$.

Moreover,

for all continuous functions $\omega = \omega(d)$, there exists a complete metric g_ω on X, such that

$$Sc(g_\omega, x) \geq \omega(dist(x, x_0)) \text{ for a fixed point } x_0 \text{ and all } x \in X.$$

Here is a related ***compactness conjecture***, which expresses the following idea:

The existence of a complete metric with $Sc \geq \sigma > 0$ on an X is detectible by topologies of compacts parts V of X:

if, for all compact subsets $V \subset X$ and all constants $\rho > 0$, there exists a (non-complete) metric on X with $Sc \geq 1$, such that the closed ρ-neighbourhood $U_\rho(V) \subset X$ is *compact*, then X *admits a complete Riemannian metric with $Sc \geq 1$.*

3.7 Separating Hypersurfaces and the Second Proof of the $\frac{2\pi}{n}$-Inequality

The main ingredient in the proof of the general $\frac{2\pi}{n}$-Inequality is the following.

||| **μ-Bubble Separation Theorem.** Let X be an n-dimensional, Riemannian band, possibly non-compact and non-complete.

Let

$$Sc(X,x) \geq \sigma(x) + \sigma_1, \ ,$$

for a continuous function $\sigma = \sigma(x) \geq 0$ on X and a constant $\sigma_1 > 0$, where σ_1 is related to $d = width(X) = dist_X(\partial_-, \partial_+)$ by the inequality

$$\sigma_1 d^2 > \frac{4(n-1)\pi^2}{n}.$$

(If scaled to $\sigma_1 = n(n-1)$, this becomes $d > \frac{2\pi}{n}$.)

Then there exists a smooth hypersurface $Y \subset X$, which separates ∂_- from ∂_+, and a smooth positive function ϕ on Y, such that the scalar curvature of the metric $g_\phi = g_\phi^{\times} = g_{Y_{-1}} + \phi^2 dt^2$ on $Y \times \mathbb{R}$ is bounded from below by

$$Sc(g_\phi, x) \geq \sigma(x).$$

Derivation of $\frac{2\pi}{n}$-Inequality from |||. If a band X with $Sc \geq \sigma > 0$ has $width(X) = dist(\partial_-, \partial_+) > 2\pi\sqrt{\frac{(n-1)}{n\sigma}}$, then ||| implies the existence of a separating hypersurface Y and a function $\phi(y)$, such that $Sc(g_\phi^{\times}) \geq \varepsilon$ for a *small* $\varepsilon > 0$.

About the Proof of |||. If X is compact and $n \leq 7$, we take a *μ-bubble Y_{min}* for Y, that is the minimum of the functional

$$Y \mapsto vol_{n-1}(Y) - \mu[Y, \partial_-]$$

defined in the space of separating hypersurfaces $Y \subset X$, where $[Y, \partial_-] \subset X$ denotes the region in X between Y and $\partial_- \subset \partial X$ and where the key point is to choose μ suitable for this purpose.

What is required of μ is that

• the boundaries $\partial_\pm$ must serve as barriers for our variational problem and thus ensure the existence of Y_{min};

• positivity of the second variation should imply the positivity of the $\Delta + Sc(Y_{min}) - \sigma$ on Y.

This is achieved with μ, that is modeled after the measure $\underline{\mu}$ on $\mathbb{T}^{n-1} \times [-1,1]$, (the density of) which is equal the mean curvatures of the hypersurfaces $\mathbb{T}^{n-1} \times \{t\}$ with respect to the warped product metric $\varphi^2 h + dt^2$ for

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2} dt, \ \ -\frac{\pi}{n} < t < \frac{\pi}{n}.$$[195]

$|||_{\circlearrowleft}$ **Separation with Symmetry.** *If the Riemannian band is isometrically acted upon by a compact group G, then the separating hypersurface $Y \subset X$ and the function ϕ on Y can be chosen invariant under this action.*

Proof. Use the multi-dimensional Morse lemma (see section 2.9); alternatively, apply more elementary uniqueness/symmetry property of the lowest

[195] This is the same $\varphi(t)$ that was used in section 12 in [GL(complete) 1983] for proving a rough lower bound on the norms of the differentials of smooth maps of non-zero degrees from *non-complete* Riemannian manifolds X with $Sc(X) \geq 1$ to S^n for $n = dim(X) \leq 7$.

eigenfunction of the (linear elliptic) second order variational (linear elliptic) $\Delta_Y + s$ on a hypersurface Y, which minimizes the functional $vol_{n-1}(Y) - \mu[Y, \partial_-]$ among G-invariant separating hypersurfaces $Y \subset X$.

Remark. In our case, the group G is the torus $\mathbb{T}^k$, which freely acts on X, and the equivariant μ-bubble problem (trivially) reduces to the ordinary one on the quotient space $X/\mathbb{T}^k$.

To make use of this for the next step of $\mathbb{T}^{\rtimes}$-symmetrization, one only needs to check – this is an exercise to the reader – that the corresponding warped product with $\mathbb{T}^{k+1}$ will have the same scalar curvature as one gets by doing this in X itself.

Compact/Non-compact. If X is non-compact, then, as usual, we exhaust X by compact submanifolds with boundaries, proceed as in the compact case (these compact bands an not proper, part of their boundary is not contained in $\partial X = \partial_- \cup \partial_+$, but this causes no problem) and then pass to the limit. This is routine.

Example of Corollary. Let $X = (X, g)$ be an n-dimensional manifold with uniformly positive scalar curvature, $Sc(X) \geq \sigma > 0$, and let $f : X \to \underline{X} = \mathbb{R}^{n-m}$ be a smooth proper (infinity to infinity) 1-Lipschitz (i.e. distance non-increasing) map.

Then the homology class of the pullback of the generic point, $f^{-1}(\underline{x}) \subset X$, is representable by a compact submanifold $Y \subset X$, such that the product $Y \times \mathbb{T}^m$ admits a $\mathbb{T}^m$-invariant (warped product) *metric $h^{\rtimes}$ ($h = g_{|Y}$) with $Sc(h^{\rtimes}) > 0$.*

Consequently, Y itself admits a metric with $Sc > 0$.

Singularity Problem for $dim(X) > 7$ and the Second Proof of the μ-Bubble Separation Theorem. By the standard theorems of the geometric measure theory, the minimizing μ-bubble $Y \subset X$ exists for all n but it may have singularities of codimension 7.[196]

(The first instance of this is the vertex of the famous cone from the origin over $S^3 \times S^3 \subset S^7 \subset \mathbb{R}^8$.)

If $n = 8$, then (a minor generalization of) Natan Smale's *generic regularity theorem* takes care of things, but if $n \geq 9$ one needs to adapt *Lohkamp's minimal smoothing* results and/or techniques to our case. My, rather superficial, understanding of Lohkamp works suggests that this is possible, but it can't be safely applied unless everything is written out in full detail.

I feel more comfortable at this point with generalizing theorem 4.6 from the Schoen-Yau paper [SY(singularities) 2017], where it used in the inductive descent method with *singular minimal* hypersurfaces, to our minimizing μ-bubbles.

Such a generalization feels plausible and, if it's true, this must be obvious to Schoen and Yau. (I guess, the same can be said about what Lohkamp thinks about generalization of his theorem to μ-bubbles.)

Granted this, one gets the sharp $\frac{2\pi}{\mathbf{n}}$-inequality for SYS-bands, and in fact for all bands X which satisfy ❑, where the poof of non-existence of metrics of positive scalar curvatures on separating hypersurfaces $Y \subset X$ is obtained by

[196]The most general existence theorem of this type applicable to all codimensions is in the technically difficult Almgren's 1986 paper "*Optimal Isoperimetric Inequalities*".

The existence and regularity theorem we need in codimension one are easier, they follow by the usual technique of integer currents and regularity theorems, see [Ros(isoperimetric) 2001]), [Morgan(isoperimetric)(2003); the arguments from this papers which are applied there to the more traditional formulation of the isoperimetric problem, can be carried over to our μ-bubble setting with no problem; alternatively, one can use the language of Caccippoli sets.

exclusively by inductive decent with no appeal to Dirac operators and related invariants, such as the $\hat{A}$-genus and the $\hat{\alpha}$-invariant.[197]

Minimal Hypersurfaces in Non-compact Bands. An essential advantage of μ-bubbles over minimal hypersurfaces is that the former are easier to "trap" them and prevent from fully sliding away to infinity than the former.

For instance if X is a complete non-flat manifold with positive sectional curvature which is conical at infinity then it contains no complete (even locally) volume minimizing hypersurfaces, but it contains lots of stable complete (and compact) μ-bubbles.

However, a version of the $\frac{2\pi}{\mathbf{n}}$ can be proven for non-compact complete bands by reduction to "large" compact *non-proper* bands X, where the boundary is divided into three parts

$$\partial X = \partial_+ \cup \partial_- \cup \partial_{side}$$

where $\partial_+ = \partial_+(X)$ and $\partial_- = \partial_-(X)$ are disjoint with controlled lower bound on the distance between them, while $\partial_{side} = \partial_{side}(X)$, which intersects both ∂_+ and ∂_- is supposed to be far away from the bulk of the intended minimal hypersurfaces in X.

Example. Let $\underline{X}$ be the cylinder $B^{n-1}(R) \times [-1,+1]$, where $B^{n-1}(R)$ the Euclidean R-ball of dimension $n-1 \geq 2$ and where $\partial_{side}(\underline{X}) = S^{n-3}(R)^{n-1}(R) \times [-1,+1]$ for the equatorial sphere $S^{n-3}(r) \subset S^{n-2}(r) = \partial B^{n-1}(r)$.

Let $\underline{\partial}_r \subset \underline{X}$, $r > R$ be the r-cylinder concentric to $\partial_{side}(X)$, that is

$$\underline{\partial}_{side(r)} = S^{n-2}(R)^{n-1}(R) \times [-1,+1].$$

Minimal hypersurfaces $Y = Y_r \subset X$ we shall meet in X will be similar to those $\underline{Y} \subset \underline{X}$, which have their boundaries contained in $\underline{Z}_r = \partial_+(\underline{X}) \cup \partial_-(\underline{X}) \cup \underline{\partial}_{side(r)}$ and which represent non-zero homology classes in $H_{n-1}(\underline{X}, \underline{Z}_r$.

Namely, let X be a compact orientable non-proper n-dimensional band. Let $f = (f_1, f_2) : X \to \underline{X} = B^{n-1}(R) \times [-1,+1$ be a smooth map which sends $\partial_\mp(X) \to \partial_\mp(\underline{X})$ and $\partial_{side}(X) \to \partial_{side}(\underline{X})$ and such that

$\bullet_1$ the map $f_1 : X \to B^{n-1}$ is 1-Lipschitz;

$\bullet_2$ the map $f_2 : X \to [-1,+1]$ is λ-Lipshitz for $\lambda > 0$

$\bullet_3$ the map f has *non-zero* degree.

Observe that

$\bullet_1$ implies that

$$dist(\partial_{side}(X), \partial_{side(r)}(X) \geq R - r;$$

$\bullet_2$ makes

$$width(X) = dist(\partial_-(X), \partial_+(X)) \geq d = d_\lambda = \frac{2}{\lambda};$$

$\bullet_3$ shows that if an oriented hypersurface $Y \subset X$ with $\partial Y \subset Z_r$, represents a *non-zero* homology class in $H_{n-1}(X, Z_r)$, then it necessarily intersects $\partial_{side(r)}(X)$.

In fact, Y intersects every $(n-3)$-dimensional submanifold $Z' \subset Z_r$ (observe that $dim(Z_r) = n-2$ for generic maps f) which separates $\partial_-(Z_r) = Z_r \cap \partial_-(X)$ from $\partial_+(Z_r) = Z_r \cap \partial_+(X)$.

[197] This is complementary to what can obtained by Dirac operators methods of Zeidler and Cecchini.

3.7.1 Paradox with Singularities

Singularities must enhance the power of minimal hypersurfaces and stable μ-bubbles rather than to reduce it, since the large curvatures of hypersurfaces $Y \subset X$ (these curvatures are infinite at singularities) *add* to the positivity of the second variation .

Thus, for instance, if a $Y \subset X$, where $dim(X) = 8$ and $Sc(X) \geq -1$, is a stable minimal hypersurface with a singularity at $y_0 \in Y$ and if no smooth submanifold in the homology class of Y admits a metric with $Sc > 0$, e.g. X is homeomorphic to the torus and Y is non-homologous to zero, then scalar curvature of X can't be non-negative outside a small neighbourhood of $y_0 \in X$.

Yet, there is no known argument for $dim(X) \geq 9$ fully implementing this idea.

On $n = dim(X) = 8$. If $dim(X) = 8$ then stable minimal hypersurfaces and μ-bubbles $Y \subset X$ have isolated singularities which can be removed by small generic perturbation as in [Smale(generic regularity) 2003] as follows.

Theorem. Let $Y_0 \subset X$ be a cooriented compact isolated volume minimizing hypersurface and let et $X_t = [X_0, Y_t] \subset X$ be the bands between Y_0 and hypersurfaces Y_t, which are positioned close to Y_0 on their "right sides" in X, and which minimize the function $Y \mapsto vol_{n-1}(Y) - t \cdot vol[Y_0, Y]$ for $0 \leq t \leq \delta$ for a small $\delta > 0$.

If $n = 8$, then submanifolds Y_t are non-singular for an open dense set of $t \in [0, \delta]$.

Outline of the Proof. The key (standard) facts one needs here are as follows.

1. *Monotonicity.* If the sectional curvature of X is bounded by $\bar{\kappa}^2$, then the volume of intersections of m-dimensional minimal subvarieties $Y \subset X$ with r-balls $B_{y_0}(r) \subset X$ centered at $y_0 \in Y$ satisfy

$$\frac{dr^{-m} vol_m(Y \cap B_x(r)}{dr} \leq const_n r\bar{\kappa}. \text{ for all } r \leq r_0 = r_0(X, y_0) > 0.$$

2. *Corollary.* The densities of (singularities of) minimal $Y \subset X$ are *semicontinuous:*

if be a sequence of pointed manifolds with uniformly bounded geometries, (X_i, x_i), Haussdorf converges to (X, x) and if minimal subvarieties $Y_i \subset X_i$, which contain the points x_i, current-converge to $Y \subset X$, then

$$\limsup dens(Y_i, x_i) \leq dens(X, x),$$

where, recall,

$$dens(Y, x) = \lim_{r \to 0} r^{-m} vol_m(Y \cap B_x(r)), mboxm = dim(Y).$$

3. *Weak Compactness*: The set $\mathcal{Y}_A$ of minimal subvaraities $Y \subset X$ with volumes bound by a constant A is compact in the current topology for all $A < \infty$.

4. *Codimension one Intersection Property.* Minimal codimension one cones $C_1, C_2 \subset \mathbb{R}^n$ *necessarily intersect* by the *maximum principle.*

5. *Split Cone Property.* Let $C \subset \mathbb{R}^n$ be a minimal cone. Then either the density of this cone at the apex $0 \in C$ is maximal $+\varepsilon$,

$$dens(C, 0) \geq dens(C, c) + \varepsilon \text{ for all } 0 \neq c \in C \text{ and some } \varepsilon = \varepsilon(C) > 0,$$

or the cone split, i.e. $C = C_{-1} \times \mathbb{R}^1$ for a minimal cone $C_{-1} \subset \mathbb{R}^{n-1}$.

Now, turning to the proof, let all Y_t have singularities at some points $y_t \to y_0 \in Y_0$, $t \to 0$, and assume without loss of generality, this is possible due to 2, that

$$dens(Y_t, y_t) = dens(Y_0, y_0).$$ [198]

Let λ-scale these Y_t at y_0, thus making λY_0, converge to a minimal cone, call it $Y_0' \subset T_{y_0}(X)\mathbb{R}^n$, and let Y_t' be what remains of the limits of other Y_t.

Since these Y_t' don't intersect Y_0', none of Y_t' is conical, which is only possible if the singularities of Y_t slide tangentially along Y_0 for $t \sim \lambda^{-1}$ by the distance $c(t)$, such that $c(t)/\lambda \to \infty$ for $\lambda \to \infty$. It follows that if all Y_t were singular, these singularities would accumulate in the limit to a (one dimensional or larger) singularity of Y_0' of constant density equal to that of $dens(Y_0, y_0)$. Therefore, the cone Y_0' splits, since $n = 8$, it is non-singular and the proof follows by contradiction.

On $n = dim(X) \geq 8$. (a) Schoen-Yau in their desingularization argument apply descent by warped $\mathbb{T}^{\rtimes}$-symmetrised/stabilized minimal hypersurfaces

$$X = X^n \supset Y^{n-1} \supset ... \supset Y^{n-i} \supset .... \supset Y^2,$$

where minimization and $\mathbb{T}^{\rtimes}$-stabilization (essentially) apply to non-singular parts of these Y and where the main difficulty, as far as I can see, is to show that Y^{n-i} can't be eventually sucked in the singularity of Y^{n-1},[199] and where the outcome of this process - the surface Y^2 – is non-singular.[200]

(b) The main desingularization result by Lohkamp in [Lohkamp(smoothing) 2018], is

approximation theorem of volume minimizing codimension one cones $C^{n-1} \subset \mathbb{R}^n$ by smooth minimal hypersurfaces (generalizing Smale's result in the case of cones) with the following

Splitting Corollary. sf Let X be a compact orientable Riemannian manifold with $Sc(X) > 0$. Then all homology classes in $H_{n-1}(X)$ are representable by hypersurfaces $Y \subset X$, which support metrics with $Sc > 0$.

Remarks (a) As far as the topology of compact manifolds with $Sc > 0$ this result is more general than that by Schoen and Yau.

For instance it implies that

products of Hitchin's spheres and connected sums of tori with non-spin manifolds admit no metrics with $Sc > 0$.

Nor alternative proof of this kind of results is available.

(b) As far as I understand,[201] Lohkamp's smoothing allows applications of our μ-bubble arguments to manifolds of all dimensions n, with possible exceptions for *rigidity theorems* for *non-compact* manifolds.

[198]Our Y_t are μ-bubble rather than minimal, but this makes no difference at this point.

[199]If $n \leq 9$, this problem for overtorical X can be handled with Dirac operators, as in section 5.3 in [G(billiards) 2014].

[200]Schoen and Yau articulate their main results (theorems 4.5 and 4.6 in [SY(singularities) 2017]) for compact SYS-manifolds, although the basic arguments of their paper are essentially local and and apply to a wider class of manifolds.

[201]My understanding of the results by Lohkamp as well as those by Schoen and Yau is limited, since I haven't mastered the proofs from [SY(singularities) 2017]) and from [Lohkamp(smoothing) 2018].

(c) The above 1-5 seems to suffice for smoothing conical singularities (am I missing hidden subtleties?) but it is unclear to me how Lohkamp's splitting corollary for $n \geq 9$ follows from it.

3.7.2 $\mathbb{T}^{\rtimes}$-Stabilized Scalar Curvature and Geometry of Submanifolds of Codimensions One, Two and Three

Besides $\frac{2\pi}{\mathbf{n}}$, there are other immediate applications of the separation theorem |||.

[1] Compact Exhaustion Corollary. Let X be a complete Riemannian manifold with $Sc(X) \geq \sigma > 0$.

Then X can be exhausted by compact domains U_i with smooth boundaries $Y_i = \partial U_i$

$$U_1 \subset U_2 \subset ... \subset U_i \subset ... \subset X, \ \bigcup_i U_i = X,$$

such that U_{i+1} is contained in the ρ-neighbourhood of U_i for all $i = 1, 2, ...$ and and where all Y_i admit $\mathbb{T}^{\rtimes}$-extension $Y_i \rtimes \mathbb{T}^1$ with

$$Sc(Y_i \rtimes \mathbb{T}^1) \geq \frac{\sigma}{2}.$$

Poof. Let $S(10), S(20), ...S(10i), ... \subset X$ be concentric spheres around a point $x_0 \in X$, let Y_i be hypersurfaces in the annuli $[S(10i), S(10(i+1))]$ between these spheres, which separate $S(10i)$ from $S(10(i+1))$ and which enjoy the properties supplied by |||. Then take the domains in X bounded by Y_i for U_i.

[2] Codimension 2 Corollary. Let X be a (possibly non-compact) connected orientable n-dimensional Riemannian manifold with boundary, let $\underline{X}$ be a compact connected orientable surface with boundary and with an arbitrary metric compatible with topology and let $\Psi : X \to \underline{X}$ be a smooth *distance decreasing* map which sends the boundary ∂X to $\partial \underline{X}$.

If $Sc(X) \geq \sigma + \sigma_1$, $\sigma, \sigma_1 > 0$ and the inradius of $\underline{X}$ is bounded from below by

$$inrad(\underline{X}) = \sup_{\underline{x} \in \underline{X}} dist(\underline{x}, \partial \underline{X}) > \frac{2\pi}{\sqrt{\sigma}},$$

then X contains an oriented codimension two (possibly disconnected) submanifold $Y \subset X$, which, if X is non-compact, is *properly* embedded to X *and which is homologous* for the homology group $H^{ncpt}_{n-2}(X)$ with *infinite* supports in the case of non-compact X) *to the pullback $\Psi^{-1}(\underline{x}) \subset X$ of a generic point $\underline{x} \in \underline{X}$, and such that Y with the induced Riemannian metric from X admits a $\mathbb{T}^2$-extension, that is the product $Y \times \mathbb{T}^2$ with the metric $g_\phi = dy^2 + \phi^2(dt_1^2 + dt_2^2)$, such that*

$$Sc(g_\phi) \geq \sigma_1.$$

Proof. Let $X_1 \subset X$ be the |-hypersurface that, according to |||, separates the boundary of X from the f-pullback of the (small disc around) the point $\underline{x} \in \underline{X}$ furthest from the boundary (as in the proof of $\mathbb{T}^{\rtimes}$-stabilized Bonnet-Myers diameter inequality [BMD] in section 2.8 and apply $\frac{2\pi}{\mathbf{n}}$ to the infinite cyclic covering of $X_1 \rtimes \mathbb{T}^1$ induced by the natural cyclic covering of $\underline{X}$ minus this point.

[2′] Codimension 2 Sub-Corollary. Let X be a closed orientable n-dimensional Riemannian manifold with $Sc(X) \geq \sigma > 0$, let $\underline{X}$ be a closed surface with an arbitrary metric compatible with topology and let $\Psi : X \to \underline{X}$ be a smooth *distance decreasing* map.

If no *closed oriented codimension two submanifold* $Y \subset X$ homologous to the pullback $\Psi^{-1}(\underline{x}) \subset X$ *of a generic point* $\underline{x} \in \underline{X}$ *admits a metric with* $Sc > 0$*, then the diameter of the surface* $\underline{X}$ *is bounded in terms of* σ *as follows.*

$$diam(\underline{X}) < \frac{2\pi}{\sqrt{\sigma}}.$$

Proof. Let $\underline{x}_0, \underline{x}_1 \in \underline{X}$ be mutually furthest points and apply the above to the pullback X_- of the complement $\underline{X}_-$ to a small disc in $\underline{X}$ around $\underline{x}_0$.

[3] Area non-Contraction Corollary. Let X be a proper compact orientable Riemannian band of dimension $n+1$, let $\underline{X} \subset \mathbb{R}^{n+1}$ be a smooth convex hypersurface and let $f : X \to \underline{X}$ be a smooth map the restriction of which to $\partial_- \subset \partial X$ (hence, to ∂_+ as well) has *non-zero* degree.

If X *is spin and if* n *is even,*[202] *then there exists a point* $x \in X$*, where the exterior square of the differential of* f *is bounded from below in terms of* $d = width(X) = dist(\partial_-, \partial_+)$ *and the scalar curvature* $Sc(X, x)$ *as follows.*

$$Sc(\underline{X}, f(x)) \cdot \| \wedge^2 df(x)\| \geq Sc(X, x) - \frac{4(n-1)\pi^2}{nd^2}.$$

Furthermore, *if* $\underline{X} = S^n$*, then, now for odd as well as for even* n*, the trace norm of* $\wedge^2 df$ *satisfies:*

$$2\| \wedge^2 df(x)\|_{trace} \geq Sc(X, x) - \frac{4(n-1)\pi^2}{nd^2}.$$

Proof. Apply the $\mathbb{T}^m$*-stabilized area/mapping extremality theorem* (3.4.1, 3.4.4) for $m = 1$ to $Y \rtimes \mathbb{T}^1$ where $Y \subset X$ is the separating hypersurface from III.

Exercises. (**a**) *Codimension 3 Linking Inequality.* Let X be a closed orientable n-dimensional Riemannian manifold with $Sc(X) \geq \sigma > 0$, let $\underline{X}$ be the 3-sphere with an arbitrary metric compatible with topology and let $f : X \to \underline{X}$ be a smooth *distance decreasing* map. Show that

if no *closed oriented codimension three submanifold* $Y \subset X$ homologous to the pullback $f^{-1}(\underline{x}) \subset X$ *of a generic point* $\underline{x} \in \underline{X}$ *admits a metric with* $Sc > 0$*, then the distances between all pairs of embedded circles* $S_1, S_2 \subset \underline{X}$ *with* non zero linking numbers *between them satisfy:*

$$dist(S_1, S_2) < \frac{2\pi}{\sqrt{\sigma}}.$$

Hint. Use the argument from the proof of the codimension 2 corollary **[2]** and consult [Richard(2-systoles) 2020][203]

[202] As we have said already several times, these conditions must be redundant.

[203] Our codimension 2 area bounds, including this exercise, are motivated by Richard's bound on systoles of 4-manifolds with $Sc > \sigma$ proved in this paper.

(**b**) *Area non-Contraction in Codimension 3.* Let X, $\underline{X}$ and $f : X \to \underline{X}$ be as in (**a**), let $\underline{X}_1 \subset \mathbb{R}^{n-2}$ be a smooth closed convex hypersurface and let $f_1 : X \to \underline{X}_1$ be a smooth map, such that the "product" of the two maps,

$$(f, f_1) : X \to \underline{X} \times \underline{X}_1,$$

has *non-zero degree*. Show that

if X is spin and n is odd (thus, $dim(\underline{X}_1)$ even) then there exists a point $x \in X$, where the exterior square of the differential of f is bounded from below in terms of $d = width(X) = dist(\partial_-, \partial_+)$ and the scalar curvature $Sc(X, x)$ as follows.

$$Sc(\underline{X}, f(x)) \cdot \| \wedge^2 df(x)\| \geq Sc(X, x) - \frac{4(n-1)\pi^2}{nd^2},$$

for d equal the supremum of the distances between pairs of linked circles in $\underline{X}$.

3.7.3 On Curvatures of Submanifolds in the unit Ball $B^N \subset \mathbb{R}^N$

(The earlier versions of this section contained errors.)

Here is our

Problem. Given a closed smooth n-manifold X and a number $N > n$,

evaluate the minimum of the curvatures of smooth immersion of X to the unit N-ball,

$$f : X \hookrightarrow B^N = B^N(1) \subset \mathbb{R}^N.$$

We shall briefly describe in this section what is known and and what is unknown about this problem and refer to section 3 and 7 in [G(inequalities) 2018] and to [G(growth of curvature) 2021] for more general discussion and for the proofs.

Six Examples of Immersed and Embedded Manifolds with Small Curvatures

Just to clear the terminology, we agree that a smooth map $f : X \to Y$ is an *immersion* if the differential $df : T(X) \to T(Y)$ is *injective* on all tangent spaces $T_x(X) \subset T(X)$.

An immersion f of a compact manifold is an *embedding* if it has no double points, $f(x) \neq f(y)$ for $x \neq y$.

If Y is a *Riemannian* manifold, e.g. $Y = \mathbb{R}^N$, then the curvature of this f, denoted

$$curv_f(X) = curv_f(X \hookrightarrow Y) = curv(X \hookrightarrow Y) = curv(X),$$

is the *supremum of the Y-curvatures of all geodesics* in X, where "geodesic" is understood with respect to the Riemannian metric in X induced from Y.

1. *Clifford Embeddings.* Here, $X = X^n$ is the product of m spheres of dimensions n_i, $\sum_{i=1}^m n_i = n$, all of the radius $r = \frac{1}{\sqrt{m}}$,

$$X = S^{n_1}\left(\frac{1}{\sqrt{m}}\right) \times ... \times S^{n_i}\left(\frac{1}{\sqrt{m}}\right) \times ... \times S^{n_m}\left(\frac{1}{\sqrt{m}}\right)$$

and
$$f_{Cl} : X \hookrightarrow S^{N-1} \subset B^N(1) \subset \mathbb{R}^N,\ N = m + \sum_i n_i,$$
is the obvious embedding, that is the $\frac{1}{\sqrt{m}}$-scaled Cartesian product of the imbeddings $S^{n_i}(1) \subset \mathbb{R}^{n_i+1}$.

Clearly,
$$curv_{f_{Cl}}(X \hookrightarrow B^N) = \sqrt{m}$$
and the curvature of X in the unit sphere is
$$curv_{f_{Cl}}(X \hookrightarrow S^{N-1}) = \sqrt{m-1}.$$

Two natural questions arise:

Can the products of spheres be immersed to the unit ball with smaller curvatures?

Are there non-spherical, immersed or embedded, submanifolds $X \hookrightarrow B^N(1)$ with $curv(X) < \sqrt{2}$?

A definite answer is available only for immersions $X^n \to S^{n+1}$ by a *theorem of Jian Ge.* [204]

[$\bigcirc \times \bigcirc$] *Clifford's are the only codimension one immersed non-spherical submanifolds X in the spheres with curvatures $curv(X \hookrightarrow S^{n+1}) \leq 1$.*

But if $m \geq 3$ then there are immersions of *non-spherical* n-manifolds to S^{n+m-1} with smaller curvature.

2. *Veronese embeddings* of projective spaces to spheres,
$$f_{Ver} : \mathbb{R}P^n \to S^{\frac{(n+1)(n+2)}{2}-2} \subset B^{\frac{(n+1)(n+2)}{2}-1} = B^{\frac{(n+1)(n+2)}{2}-1}(1)$$
satisfy

$\left[\sqrt{\frac{n-1}{n+1}}\right]$
$$curv_{f_{Ver}}\left(\mathbb{R}P^n \hookrightarrow S^{\frac{(n+1)(n+2)}{2}-2}\right) = \sqrt{\frac{n-1}{n+1}} < 1.$$
and
$$curv_{f_{Ver}}\left(\mathbb{R}P^n \hookrightarrow B^{\frac{(n+1)(n+2)}{2}-1}\right) = \sqrt{\frac{n-1}{n+1}+1} < \sqrt{2}.$$

Conjecturally, these have the minimal curvatures among *all non-spherical* n-submanifolds in the unit spheres and unit balls, where the minimum for all n is achieved (only conjecturally) by Veronese's projective plane in unit 4-sphere, where

$\left[\frac{1}{\sqrt{3}}\right]$,
$$curv_{f_{Ver}}(\mathbb{R}P^2 \hookrightarrow S^4) = \frac{1}{\sqrt{3}} = 0.577350...$$
and
$$curv_{f_{Ver_2}}(\mathbb{R}P^2 \hookrightarrow B^5) = \frac{2}{\sqrt{3}} = 1.15470... .$$

3. The $\frac{1}{\sqrt{l}}$-scaled Cartesian power of the Veronese map
$$F = \frac{1}{\sqrt{l}} \cdot f_{Ver}^{\times l} : X^{2l} = \underbrace{\mathbb{R}P^2 \times ... \times \mathbb{R}P^2}_{l} \to S^{4l-1} \subset B^{4l}(1)$$

[204]See [Ge(linking) 2021] and $\lozenge$ in this section.

competes with the Clifford embedding, for

$$curv_F(X^{2l} \hookrightarrow B^{4l}) = \sqrt{l} \cdot \sqrt{\frac{l}{3} + 1} < \sqrt{2l}.$$

4. If $N \geq (1+\Delta)^n$, say for $\Delta > 10$ then *all n-manifolds X* admits immersions

$$f : X \hookrightarrow S^N$$

with

$$curv_f(X) \leq C_\Delta,$$

where $C_\Delta < \sqrt{2}$ for all n and where

$$C_\Delta \to \sqrt{2 - \frac{6}{n+2}} \text{ for } \Delta \to \infty$$

with the rate of convergence which, a priori, may depend on n.

It is unclear if the "true" C_∞ is, actually, *smaller* than $\sqrt{2 - \frac{6}{n+2}}$ and it is also unclear what happens to C_Δ for Δ close to zero.

5. It easily follows from the above that

if the dimension n_m of the last factor in a product of spheres

$$X^n = \bigtimes_{i=1}^{m} S^{n_i}, \quad \sum_{i=1}^{m} n_i = n,$$

is much greater then the remaining ones, say, roughly,

$$n_m \geq \exp\exp \sum_{i-1}^{m-1} n_i,$$

then X^n admits an immersion

$$f : X^n \hookrightarrow B^{n+1}(1)$$

such that

$$curv_f(X^n) < 2\sqrt{3}.$$

This is *smaller* than Clifford's $\sqrt{m}$ starting from $m = 12$.

It is unclear, however, if these X^n admit *embeddings* to the unit ball with $curv(X^n \hookrightarrow B^{n+1}) \leq 100$, for example.

6. There are *no topological bounds on curvatures* of *immersed* submanifolds of a *given dimension n*:

if an X^n admits a smooth immersion to $\mathbb{R}^N$, then it also admits an immersion to the unit ball with $curv(X^n \hookrightarrow B^N) < const_n$.

But all we can say about this constant is, roughly, that

$$0.1n < const_n < 10n^{\frac{3}{2}}.$$

Imbeddings, at least these with codimension one, are different from immersions in this regard.

For instance, if $X = X^n$ is disconnected and contains m *mutually non-diffeomorphic* components, then, clearly,

$$curv_f(X \hookrightarrow B^{n+1}) \geq const_n m, \ const_n \geq \frac{1}{(10n)^n},$$

for all embeddings $f : X \hookrightarrow B^{n+1}(1)$.

It is also not hard to construct similar *connected* X for $n \geq 6$ and, probably, for all $n \geq 3$.

Conceivably the same is possible for imbeddings with higher codimensions k, at least for $k << n$, where one expects that, say for $k < \frac{n}{3}$ and a given, *arbitrarily large*, constant $C > 0$, there exists

a connected n-dimensional submanifold $X \subset \mathbb{R}^{n+k}$, such that all imbeddings $X \hookrightarrow B^{n+k}(1)$ satisfy

$$curv(X \hookrightarrow B^{n+k}) \geq C.$$

But it should be noted that

all connected orientable surfaces embed to the unit ball B^3 with curvatures ≤ 100

and

the *connected sums* X of copies of products of spheres with any number of summands admit *embeddings*

$$f : X \hookrightarrow B^{n+1}(1), \ n = dim(X),$$

with

$$curv_f(X) \leq 100 n^{\frac{3}{2}}.$$

Questions. Do all smooth n-manifold admit embeddings to the unit $2n$-ball with

$$curv(X^n \hookrightarrow B^{2n}) \leq 100?$$

Do the products of spheres

$$X = \bigtimes_{i=1}^{m} S^{n_i}, \text{ where all } n_i \geq 2, \text{ e.g. } X = (S^2)^m,$$

embed to $B^N(1)$, $N = 1 + \sum_i n_i$ with $curv(X) \leq 100$?

LOWER BOUNDS ON $curv(X)$.

A. It is obvious that

immersed n-manifolds $X \hookrightarrow B^N(1)$ with $curv(X) \leq 1 + \delta$ for a small $\delta > 0$ keep close to an equatorial N-sphere in $S^n \subset S^{N-1} = \partial B^N$; thus, they are diffeomorphic to S^m.

In fact, it is is not hard to show, that

$\delta = 0.01$*, is small enough for this purpose,*

while, conjecturally, this must hold for

$$\delta < \frac{2}{\sqrt{3}} = 1.15470...$$

with the Veronese surface being the extremal one.

B. Also conjecturally,

$[\bigcirc \times \bigcirc]_?$ the inequality $curv_f(X) < \sqrt{2}$ for codimension one immersions $f : X \to B^{n+1}$ must imply that X is diffeomorphic to S^n (with the equality for non-spherical X achieved by the Clifford embeddings).

This is apparently unknown even for $n = 2$..

C. Let X be an n-dimensional $\nexists$-PSC manifold, i.e. *admitting no metric with* $Sc > 0$, e.g. Hitchin's sphere or a connected sum of n-tori.

Then a simple application of Gauss's Theorema Egregium,[205] shows that

immersions $f : X \to S^N$ *satisfy*

$$curv_f(X) \geq \sqrt{\frac{n-1}{N-n}}$$

and

$$curv_f(X) \geq \sqrt{1 - \frac{1}{n}}.$$

for all n *and* N.

Here, observe, it is as it should be: no contradiction with the above 4, for

$$1 - \frac{1}{n} \leq 2 - \frac{6}{n+2}$$

for all $n \geq 2$ with the the equality for $n = 2$.

D. If $X = X^n$ is $\nexists$-PSC, then all immersions $f : X \to B^N = B^N(1)$ satisfy

$$curv_f(X) \geq \frac{1}{C_\circ}\sqrt{\frac{n-1}{N-n} + 1}$$

where $C_\circ > 0$ is a universal constant that is defined as

the minimal possible increase of curvatures of curves

under smooth immersions $B^N \to S^n = S^N(1)$. More precisely, $C_\circ$ is the infimum of the numbers $C > 0$, for which

there exits an immersion $g : B^N \subset S^N$, *such that all curves* $S \subset B^N$ *with curvatures*

$$curv_{B^N}(S) \leq \sqrt{1 + \kappa^2}$$

are sent to curves with curvatures

$$curv_{S^N}(g(S)) \leq C\kappa.$$

This $C_\circ$, most probably, is assumed by *a radial* (i.e. $O(n)$-equivariant) map g and then it must be easily computable; without computation, one can get

$$C_\circ < 4.^{206}$$

[205] Compare with [Guijarro-Wilhelm(focal radius) 2017].

[206] A natural candidate for g is a *projective map*, where $curv_{S^n}(g(S) \leq const_g curv_{B^n}(S)$ for *all* curves $S \subset B^n$. But since we are essentially concerned only with what happens to curves with $curv > 1$, the best g doesn't have to be projective – it might be conformal, for example.

E. **Conjecture + Theorem.** If If $X = X^n$ is $\nexists$-PSC, then conjecturally all immersions $f : X \to B^N = B^N(1)$ satisfy

$\left[\frac{n}{N-n}\right]$
$$curv_f(X) \geq const\frac{n}{N-n}.$$

E$_1$. *It is esay to see* in this regard that the $\frac{2\pi}{n}$-inequality yields this conjecture for $N = n+1, n+2$:

if $N = n+1$, then
$$curv_f(X) \geq \frac{N}{2\pi} = \frac{n+1}{2\pi}.$$
and if $N = n+2$, then
$$curv_f(X) \geq \frac{N}{4\pi} = \frac{n+2}{4\pi}.$$

Here we must recall that our proof of the $\frac{2\pi}{n}$-inequality in section 3.6 is unconditional only for $N \leq 8$, where these inequalities are not especially informative. And if $N \geq 9$, our proof relies on not formally published "desingularization" results by Lohkamp and by Schoen-Yau.

Fortunately, there are now two Dirac theoretic proofs for a large class of $\nexists$-PSC manifolds of all dimensions, including n-tori $\mathbb{T}^n$ and connected sums of these for, example.[207]

E$_2$. If X is *enlargeable* e.g. the connected sum of the n-torus with another closed manifold, then a minor generalization of the Schoen-Yau "desingularization" theorem allows a proof of the following version of $\left[\frac{n}{N-n}\right]$ for $N = n+3$:

$$curv(X \hookrightarrow B^N) \geq const_3 N,$$

where, roughly, $const_3 > \frac{1}{16\pi}$.

Also, granted a more serious (but realistic) generalization of the Schoen-Yau result or a version of Lohkamp's theorem, one can prove a similar inequality for $N = n+4$.
$$curv(X \hookrightarrow B^N) \geq const_4 N$$
with $const_4 > \frac{1}{400\pi}$.

Finally, assuming that one can "go around singularities" of stable μ-bubbles, and that (this is more serious)

the filling radii of n-manifolds Y with $Sc(Y) \geq \sigma > 0$ satisfy

$$filrad(X) \leq 100\frac{n}{\sqrt{\sigma}},$$

one can show for all n and $k = N - n$ that

$$curv(X \hookrightarrow B^N) \geq const_k N$$

where one needs $const_k$ about $\frac{1}{500^{500}k}$.

F. All of the above equally applies to immersions of *products of enlargeable manifolds X_0 with spheres*, say to

$$f : X = X_0^{n_0} \times S^{n_1} \to B^{n_0+n_1+k},$$

[207]See [Cecchini-Zeidler(generalized Callias) 2021] and [Guo-Xie-Yu(quantitative K-theory) 2020].

where we conjecture that

$\left[\frac{n_0}{n_1+k}\right]$ $$curv_f(X \subset B^{n_0+n_1+k}) \geq const\frac{n_0}{n_1+k}$$

and where the case $n_1 + k \leq 4$ is within reach. (Notice that $\left[\frac{n_0}{n_1+k}\right]$ implies $\left[\frac{n}{N-n}\right]$.)

Four Questions

I. Are there lower bound on $curv_f(X)$ unrelated to the scalar curvature?

II. What is the minimal dimension $N = N(n)$ such that all n-manifold can be immersed to the unit N-ball with curvatures $\leq 1\,000$?

III What is the minimal $C = C(n)$ such that the n-torus can be immersed to the unit $(n+1)$-ball with

$$curv(\mathbb{T}^n \hookrightarrow B^{n+1}) \leq C?$$

IV Can the Cartesian n-th power of the 2-sphere be immersed to the unit $(2n+1)$-ball

$$X = \underbrace{S^2 \times \ldots \times S^2}_{n} \hookrightarrow B^{2n+1}$$

with

$$curv(X \hookrightarrow B^{2n+1}) \leq 100?$$

Looking back on the above examples, questions and conjectures, one may be disconcerted by their chaotic irregularity. But this only highlights the patchiness of our present-day knowledge of the basic geometry of submanifolds in Euclidean spaces.

◊ *Wide bands with sectional curvatures* ≥ 1. Let a proper compact Riemannian band Y (see 3.6) of dimension $n+1$ admit an immersion to a complete $(n+1)$-dimensional Riemannian manifold Y_+ with sectional curvature

$$sect.curv(Y_+) \geq 1,$$

and let the width of Y with respect to the induced Riemannian metric satisfy

$$width(Y) = dist(\partial_- Y, \partial_+ Y) > \frac{\pi}{2}.$$

Then

Y contains a subband $Y_- \subset Y$ of width $d = width(Y) > \frac{\pi}{2}$, which is homeomorphic to the spherical cylinder $S^n \times [0,1]$.

Acknowledgement. A similar result for $n = 3$ is proved in [Zhu(width) 2020], while our argument below follows that of Jian Ge from [Ge(linking) 2021], who sent me his preprint prior to publication.

Proof. Let Y_- be the intersection of the d-neighbourhoods of the $\partial_\mp$-boundaries of Y,

$$Y_- = U_d(\partial_-) \cap U_d(\partial_+),$$

and observe that the $\partial_\mp$-boundaries of this Y_- are *concave* for $\kappa \geq 1$ and $d > \frac{\pi}{2}$. Therefore, $\partial_\mp$ are diffeomorphic to S^{n-1} and the immersions

$$\partial_\mp \to Y_+$$

extend to immersions of n-balls, such that the *locally convex* boundaries of these are equal to $\partial_\mp$ (with their coorientations opposite to those in Y_-). [208]

It follows, that if Y_+ is simply connected, then then the immersion $Y_- \to Y_+$ is one-to-one and the complement $Y_+ \setminus Y_-$ consists of two convex balls with distance $> \frac{\pi}{2}$ between them.

Hence, $diam(Y_+) > \frac{\pi}{2}$ and Y_+ is homeomorphic to S^{n+1} by the *Grove-Shiohama diameter theorem*; consequently, Y_- is homeomorphic to $S^n \times [0,1]$. QED.

Remark. (a) The conclusion of the theorem, probbaly, holds if $sect.curv(Y_-) \geq 1$ and $sect.curv(Y_-) \geq 0$, since the proof of the diameter theorem seems to work in this case.

(b) It also doesn't seem difficult to prove the rigidity theorem a la *Berger-Gromoll-Grove* in case of an *open band with width*$(Y) = \frac{\pi}{2}$, where the only alternatives to the homeomorphism of Y to $S^n \times (0,1)$ should be as follows:

• Y is isometric the open $\frac{\pi}{4}$-neighbourhood of a Clifford submanifold

$$S^{n_1} \times S^{n_2} \subset S^{n+1} \ n_1 + n_2 = n;$$

•• Y_+ is isometric to the *projective space* over complex numbers, quaternion numbers or Cayley numbers and Y is isometric to the open $\frac{\pi}{2}$-ball minus the center in such an Y_+.

In fact, the poof of this rigidity seems quite easy in the case of the interest (the above $[\bigcirc \times \bigcirc]$), where Y is equal to the (normal) $\frac{\pi}{4}$-neighbourhood of a hypersurface $X \subset S^{n+1}$ with $curv(X) \leq 1$.

Questions. (i) Is the manifold $Y_+ \supset Y$ indispensable? Do there exist "non-obvious" bands with $sect.curv \geq 1$ and with $width \geq \frac{\pi}{2}$?

(ii) Given a closed n-manifold X, e.g. a product of spheres, $X = \times_i S^{n+1}$, what is the supremum of the widths of the Riemannian bands Y homeomorphic to $X \times [0,1]$ with $sect.curv(Y) \geq 1$?

3.8 Multi-Width of Riemannian Cubes

Let g be a Riemannian metric on the cube $X = [-1,1]^n$ and let d_i, $i = 1,2,...,n$, denote the g-distances between the pairs of the opposite faces denoted $\partial_{i\pm} = \partial_{i\pm}(X)$ in this cube X, that are the length of the shortest curves between ∂_{i-} and ∂_{i+} in X.

$\square^n$-**Inequality**. *If* $Sc(g) \geq n(n-1) = Sc(S^n)$, *then*

$$\square_\Sigma \qquad \sum_{i=1}^{n} \frac{1}{d_i^2} \geq \frac{n^2}{4\pi^2}$$

[208] Recall that *a closed immersed locally convex* hypersurface in a complete Riemannian manifold of dimension $n \geq 3$ with sectional curvatures > 0 bounds an immersed ball.

In particular,

$\square_{\min}$ $$\min_i dist(\partial_{i-}, \partial_{i+}) \leq \frac{2\pi}{\sqrt{n}}.$$

(On the surface of things, this inequality is *purely geometric* with no topological string attached. But in truth, the combinatorics of the cube fully reflects toric topology in it.)

$\frac{2\pi}{n}$***-Corollary.*** *If X is a proper orientable non-compact band with $Sc(X) \geq n(n-1)$, which admits a proper 1-Lipschitz map $f : X \to \mathbb{R}^{n-1}$, such that the restriction of f to the $\partial_\pm$-components of the boundary, of X,*

$$\partial_-(X), \partial_+(X) \to \mathbb{R}^{n-1},$$

have non-zero degrees,(these two degrees are mutually equal) then

$$width(X) = dist(\partial_-, \partial_+) \geq \frac{2\pi}{n}.$$ [209]

The proof of $\square_\Sigma$ proceeds by inductive dimension descent with $\mathbb{T}^\rtimes$-symmetrization with the use of the "separation with symmetry" theorem $|||_{\circlearrowleft}$ from section 5.4. [210]

Generalization. We shall apply this argument in 5.4 to more general "cube-like" manifolds X, such as products of surfaces with square-like decompositions of their boundaries and also to products $Y_{-m} \times [-1,1]^{n-m}$, where this yields inequalities mediating between $\square_\Sigma$ and the $\frac{2\pi}{n}$-inequality.

$\square^2$*-Example.* Let Z be a compact connected orientable surface with non-empty connected boundary where this (circular) boundary $S = \partial Z$ is decomposed into four segments meeting at their ends,

$$S = S_{1+} \cup S_{2+} \cup S_{1-} \cup S_{2-}.$$

Let g be a Riemannian metric on $Z \times \mathbb{T}^{n-2}$ with $Sc(g) \geq \sigma > 0$.

Then the g-distances between the products of the pairs of the opposite (i.e. non-intersecting) segments in S by the torus $\mathbb{T}^{n-2}$, denoted $\partial_{i\pm} = S_{i\pm} \times \mathbb{T}^{n-2} \subset Z \times \mathbb{T}^{n-2}$, $i = 1, 2$, satisfy:

$[2\sqrt{2}]$ $$\min_{i=1,2} (dist_g(\partial_{1-}, \partial_{1+}), dist_g(\partial_{2-}, \partial_{2+})) \leq 2\sqrt{2}\pi \cdot \sqrt{\frac{n-1}{n}} \cdot \sqrt{\frac{1}{\sigma}}.$$

Proof. Pass to $Z \times \mathbb{R}^{n-2}$ for the universal covering $\mathbb{R}^{n-2} \to \mathbb{T}^{n-2}$ and apply the $\square^n$-inequality to $Z \times [-d, d]^{n-2}$ for $d \to \infty$.

Hemi-spherical Corollary. Let X be a Riemannian manifold with $Sc(X) \geq n(n-1) = Sc(S^n)$, which admits a λ_n-Lipschitz, (i.e. $dist(f(x)f(y)) \leq \lambda_n dist(x,y)$) homeomorphism onto the hemisphere S^n_+,

$$f : X \to S^n_+.$$

[209] A proof of this for bands with locally bounded geometries can be performed using minimal hypersurfaces rather than μ-bubbles as it is (briefly and sloppily) indicated in section 11.7 in [G(inequalities 2018].

[210] A Dirac theoretic proof of this inequality is given in the recent paper [Wang-Xie-Yu(cube inequality) 2021].

Then

$$\lambda_n \geq \frac{\arcsin \beta_n}{\pi \beta_n} > \frac{1}{\pi} \text{ for } \beta_n = \frac{1}{\sqrt{n}}.$$

Proof. The hemisphere S^n_+ admits an obvious cubic decomposition with the (geodesic) edge length $2\arcsin \frac{1}{\sqrt{n}}$ and $\square_{\min}$ applies to the pairs of the f-pullbacks of the faces of this decomposition.

Remarks and Exercises. (a) This lower bound on λ_n improves those in §12 of [GL(complete) 1983] and in §3 of [G(inequalities) 2018].

Moreover the *sharp* inequality for Lipschitz maps to the punctured sphere stated in the next section implies that $\lambda_n \geq \frac{1}{2}$ for all n.

But it remains *problematic* if, in fact, $\lambda_n \geq 1$ for all $n \geq 2$.

(b) Show that $\lambda_2 \geq 1$.

(c) The proof of the inequality $\square_\Sigma$ in section **??** applies to *proper* ((boundary→boundary) λ-*Lipschitz* maps with *non-zero degrees* from all compact connected orientable manifolds X to S^n_+, while the proof via punctured spheres needs X to be spin.

(d) Show that the Riemannian metrics with sectional curvatures ≥ 1 on the square $[-1,1]^2$ satisfy

$\square^2_{\min}$.
$$\min_{i=1,2} dist(\partial_{i-}, \partial_{i+}) \leq \pi.$$

(e) Construct iterated warped product metrics g_n on the n-cubes $[-1,1]^n$ with $Sc(g_n) = n(n-1)$, where, for $n=2$, both d_i, $i=1,2$, are equal to π and such that

$$d_i > 2\arcsin \frac{1}{\sqrt{n}}, \ \ i = 1, ..., n, \ \text{ for all } n = 3, 4, ..., \ .$$

(f) Show, that $\square_{\min}$ is equivalent to the *over-torical* case of $\frac{2\pi}{n}$*-Inequality.* modulo constants. Namely,

(i). If a Riemannian n-cube X has $\min_i dist(\partial_{i-}, \partial_{i+}) \geq d$, then it contains an n-dimensional Riemannian band $X_\circ \subset X$, where $dist(\partial_- X_\circ, \partial_+ X_\circ) \geq \varepsilon_n \cdot d$, $\varepsilon_n > 0$, and where $X_\circ$ admits a continuous map to the $(n-1)$ torus, $f_\circ : X_\circ \to \mathbb{T}^{n-1}$, such that all closed hypersurfaces $Y_\circ \subset X_\circ$ which separate $\partial_- X_\circ$ from $\partial_+ X_\circ$ are sent by $f_\circ$ to $\mathbb{T}^{n-1}$ with *non-zero degrees.*

(ii). Conversely, let X_o be a band, where $dist(\partial_- X_o, \partial_+ X_o) \geq d)$ and which admits a continuous map to the $(n-1)$ torus, such that the hypersurfaces $Y_o \subset X_o$, which separate $\partial_- X_o$ from $\partial_- X_o$, are sent to this torus with *non-zero degrees.*

Then there is a (finite if you wish) covering $\tilde{X}_o$ of X_o, which contains a domain $X_\square \subset \tilde{X}_o$, where this domain admits a continuous proper map of degree one onto the d-cube $f_\square : X_\square \to (0,d)^n$, such that the n coordinate projections of this map, $(f_\square)_i : X_\square \to (0,d)$, are distance decreasing.

3.9 Extremality and Rigidity of Punctured Spheres

Let $\underline{X}$ be the unit sphere S^n minus two opposite points $\pm x_0 \in S^n$ and let $\underline{g} = g_{sphe}$ denote the spherical metric (of constant curvature $+1$) restricted to this $\underline{X} = S^n \setminus \{\pm x_0\} \subset S^n$.

Double Puncture Extremality/Rigidity Theorem. *If a smooth metric g on $\underline{X}$ satisfies*

$$g \geq \underline{g} \text{ and } Sc(g) \geq n(n-1) = Sc(\underline{g}),$$

then $g = \underline{g}$.

This is shown by applying the spin-area extremality theorem $[X_{spin}\to\bigcirc]$ from section 3.4.1 (one needs here only the spherical case of it but sharpened by rigidity in the case of equality) to the $\mathbb{T}^1$-symmetrization of a certain stable μ-bubble, $Y \subset S^n \smallsetminus \{\pm x_0\}$, which separates the punctures $\pm x_0 \in S^n$.

(See section 5.5 for the proof of this for general *spin* manifolds with $Sc \geq n(n-1)$ *properly mapped* to $S^n \smallsetminus \{\pm x_0\}$ with $deg \neq 0$, where, recall, the details of this proof for $n \leq 8$ are yet to be worked out.)

Remark. If the above metric g on $\underline{X} = S^n \smallsetminus \{\pm x_0\}$ is *complete*, one can prove that the inequalities $g \geq \underline{g}$ and $Sc(g) \geq n(n-1)$ imply that $g \geq \underline{g}$ for the complements $\underline{X} = S^n \smallsetminus \Sigma$ for certain subsets Σ larger than $\{\pm x_0\}$.

For instance, Llarull's inequality implies this for all *finite subsets* $\Sigma \subset S^n$ and a similar (purely index theoretic) argument yields this for

piecewise smooth 1-dimensional subsets (graphs) $\Sigma \subset S^n$, such that the *monodromy transformations* of the principal tangent $Spin(n)$-bundle (that is double cover of the orthonormal tangent frame-bundle over all closed curves in Σ are *trivial* (e.g. Σ is contractible).

But if one makes *no completeness assumption*, our proof is limited to Σ being either empty, or consisting of a single point or of a pair of opposite points.

Exercise. Prove with the above that no metric g on the hemisphere $(S^n_+, \underline{g})$ can satisfy the inequalities $g \geq 4\underline{g}$ and $Sc(g) > n(n-1)$. Then directly show that if $n = 2$ then the inequality $g \geq \underline{g}$ and $Sc(g) \geq 2$ imply that $g = \underline{g}$.

Questions. (a) Does the implication

$$[g \geq \underline{g}]\&[Sc(g) \geq n(n-1)] \Rightarrow g = \underline{g}$$

ever hold for $S^n \smallsetminus \Sigma$ apart from the above cases?

(b) Can the sphere S^n with k-punctures carry a metric g, such that $[Sc(g) \geq n(n-1)]$ and such that the g-distances between these punctures are all $\geq 10^{nk}$?[211]

3.10 Slicing and Sweeping 3-Manifolds and Bounds on their Widths and Waists .

If $n \geq 4$, then then all known bounds on the size of n-manifolds X with $Sc(X) \geq \sigma > 0$ are expressed by *non-existence* of "topologically complicated but geometrically simple" maps from these X to "standard manifolds" $\underline{X}$.

But if $n = 3$ then

complete 3-Manifolds X with scalar curvature $Sc(X) \geq \sigma > 0$ are known to satisfy the following properties **A, B, C**

A. Uryson's 1-Width Estimate. Let X be a complete Riemannian 3-manifold with $Sc(X) \geq \sigma > 0$.

[211] The negative answer was recently delivered by Cecchini's *long neck theorem*, see section 3.14.3.

Then there exists a continuous map $f : X \to P^1$, where P is a 1-dimensional polyhedral space (topological graph), such that the diameters of the pullback of all points are bounded by

$$[width_{3/1}] \qquad diam(f^{-1}(p)) \leq \frac{24\pi}{\sqrt{\sigma}}.$$

A′. *Moreover, if the rational homology group $H_1(X,\mathbb{Q})$ vanishes, then the diameters of the connected components of the levels of the distance function*

$$x \mapsto dist(x_0, x)$$

are bounded by $\frac{8\pi}{\sqrt{\sigma}}$. for all $x_0 \in X$.

We prove a $\mathbb{T}^{\rtimes}$-stabilized version of **A′** for manifolds with mean convex boundaries in the next section and then derive **A**, also in the $\mathbb{T}^{\rtimes}$-stabilized form needed for applications, for all 3-manifolds X with $Sc(X) \geq 6$.[212]

A″. Corollary. *The filling radius of a complete* 3*-manifold X with $Sc(X) \geq \sigma$ is bounded by*

$$fil.rad(X) \leq C \cdot \frac{1}{\sigma} \text{ for } C \leq 24\pi.$$

(We shall show in the next section that $C \leq 8\pi$.)

Exercises. (a) Map the unit sphere $S^n \subset \mathbb{R}^{n+1}$ onto the cone $P^1 = P^1_n$ over the vertex set of the regular simplex inscribed into S^n, such that the diameters of the pullbacks of all points are $\leq \pi - \delta_n$ for $\delta_n > 0$.

(Probably, The Uryson 1-width of S^n is realized by such a map.)

(b) Let X be a compact Riemannin 3-manifold with $Sc(X) \geq \sigma > 0$ and let $f : X \to X$ be a continuous map. Show that if the 1-dimensional homology of X is torsion, e.g. zero, then there exists a point $x \in X$, such that $dist(x, f(x)) \leq 6\pi\sqrt{\frac{2}{\sigma}}$.

Hint. See (E'_5) in Appendix 1 in [G(filling) 1983].

B. Topological $\bigsqcup S^2$-Sweeping Theorem. A compact 3 manifold admits a Riemannian metric with with $Sc > 0$ *if and only if there exists a finite covering $\tilde{X} \to X$ and a Morse function $\tilde{f} : \tilde{X} \to \mathbb{R}$, such that the pullbacks $\tilde{f}^{-1}(t) \subset X$ of all non-critical $t \in \mathbb{R}$ are disjoint unions or spheres S^2.*

About the Proof. This is a reformulation of the classification theorem for compact manifolds X with $Sc > 0$, which says, in effect, that these X

admit finite coverings $\tilde{X}$ diffeomorphic to connected sums of $S^2 \times S^1$,

and which follows from *non-existence of aspherical components* in the prime decompositions of manifolds with $Sc > 0$, and *Perelman's solution of Thurston conjecture.* [213]

B′. ***S^2-Sweeping Complete Manifolds.*** If X is a complete oriented 3-manifold with $Sc(X) \geq \sigma > 0$, then, instead of a finite covering $\tilde{X}$, one constructs

a 3-polyhedron $\hat{X}$, a proper piecewise smooth locally finite-to-one map $\hat{\Phi} : \hat{X} \to X$ and a proper piecewise linear positive function $\hat{f} : \hat{X} \to \mathbb{R}_+$, such that

(i) the map $\hat{\Phi}$ sends a (non-compact) homology class from the *rational* 3-dimensional homology group of $\hat{X}$ with infinite supports to the fundamental class of X;

[212] **A′** is proven in [GL(complete)1983], where the condition $H_1(X,\mathbb{Q}) = 0$ was erroneously omitted. Also see sections (E)-(E'_2) in Appendix 1 in [G(filling). 1983].

[213] See [Gl(complete) 1983] and [Ginoux(3d classification) 2013].

(ii) the connected components of the pullbacks $\hat{f}^{-1}(t) \subset \hat{X}$ for all $t \in \mathbb{R}$ are either single points or joints of 2-spheres.

(This is suggestive of what can be expected for $n > 3$.)

C. **Sharp Area Slicing Inequality.** Let X be a Riemannian 3-manifold diffeomorphic to S^3 or to a connected sum of several $S^2 \times S^1$.

If $Sc(X) \geq 6$, then X admits a Morse function f, the non-singular levels of which are disjoint union of spheres, where the areas of all these spheres are bounded by 4π.

About the Proof. We already know the all stable minimal surfaces Y in X have areas bounded by $\frac{4\pi}{3}$ by Schoen-Yau's rendition of the second variation inequality (sections 2.5 and 2.7 + the Gauss-Bonnet theorem. Furthermore, this inequality combined with *Hersch's upper bound on the first non-zero eigenvalue* of the Laplace on surfaces Y diffeomorphic to S^2 with $area(Y) \geq 4\pi = area(S^2)$, that is

$$\lambda_1(Y) \geq 2 = \lambda_1(S^2)$$

implies that minimal surfaces $Y \subset X$ with Morse index 1 have their areas bounded by 4π. [214]

Then "the almost extremal $\bigsqcup S^2$-Morse slicing" $f : X \to \mathbb{R}$, that almost minimizes the area of the maximal pullback sphere is the required one. [215]

C. **Liokumovich-Maximo Area+Diameter Slicing Inequality.** Let X be a compact Riemannian 3-manifold with $Sc \geq 6 = Sc(S^3)$. *Then X admits a Morse function, the connected components Σ of all nonsingular levels $f^{-1}(t) \subset X$, $t \in \mathbb{R}$ satisfy*:

(i) $area(\Sigma) \leq 64\pi$),

(ii) $diam(\Sigma) \leq \frac{40\pi}{\sqrt{6}}$,

(iii) $genus(\Sigma) \leq 13$.

Corollary. *X admits a map $F : X \to \mathbb{R}^2$, such that the lengths of the pullbacks of all points are bounded by a universal constant $C \leq 100$.*

Consequently, *X contains a stationary geodesic net of length $\leq C$.*

For the proof we refer to [Lio-Max (waist inequality) 2020].

All *known* manifolds with $Sc \geq \sigma > 0$ satisfy counterparts of these **A, B, C** for all dimensions n, which suggests the following conjectures.

Topological S^2-Sweeping Conjecture. Let X be a complete, e.g. compact, orientable n-manifold, $n \geq 3$ with $Sc(X) \geq \sigma > 0$.

Then there exists an n-polyhedron $\hat{X}$, a proper piecewise smooth locally finite-to-one map $\hat{\Phi} : \hat{X} \to X$ and a proper piecewise linear map $\hat{f} : \hat{X} \to P^{n-2}$, where P^{n-2} is an $(n-2)$-dimensional polyhedral space (pseudomanifold maybe?), such that

[214] See J. Hersch, *Quatre propriétés isopérimétriques de membranes sphériques homogènes*, C.R. Acad. Sci.Paris Sér. A-B 270 (1970), A1645-A1648 and [Marques-Neves(min-max spheres in 3d) 2011].

[215] See [Lio-Max (waist inequality) 2020], where this is proved using the mean curvature flow.

Probably, this can be also proved by the *Sacks-Uhlunbeck direct minimization*, where bubbling creates disconnectedness of the levels of f. (Apparently, if X diffeomorphic to S^3 contains no stable minimal surfaces, then it admits a Morse function with two critical points and areas of all levels bounded by 4π. But, in general high disconnectedness of the levels of f is inevitable, even for X diffeomorphic to S^3.)

(i) the map $\hat{\Phi}$ sends a (non-compact if X is non-compact) homology class from the *rational* n-dimensional homology group of $\hat{X}$ (with infinite supports if X and $\hat{X}$ are non-compact) to the fundamental class of X;[216]

(ii) the connected of the pullbacks $\hat{f}^{-1}(t) \subset \hat{X}$ for all $t \in \mathbb{R}$ are either single points or joints of 2-spheres.

Corollary. If a compact orientable manifold X admits such an $\hat{\Phi} : \hat{X} \to X$, then all continuous maps from X to aspherical spaces induce *zero homomorphisms* on $H^n(X)$.

(This remains unknown for manifolds X with $Sc(X) > 0$ of dimensions $n \geq 4$, but a weaker property – *non-contractibility of the universal covering of* X – is confirmed by the Chodosh-Li theorem we prove in the next section.)

Reversed S^2-Slicing Conjecture. Let a complete (e.g.compact) smooth manifold X admits a (proper in the non-compact case) piecewise linear map with respects a smooth triangulation of X to a pseudomanifold P^{n-2}, such that the connected components of the pullbacks of the points are either single points or joints of 2-spheres.

Then X admits a metric with $Sc > 0$.

(Possibly, one should add an extra condition on singularities of such a map.)

Width/Waist Conjecture. All complete n-manifolds X with $Sc(X) \geq n(n-1)$ admit continuous maps to polyhedral spaces of dimension $n-2$, say, $F : X \to P^{n-2}$, such that

$$diam(F^{-1}(p)) \leq const_n \text{ and } vol_{n-2}(F^{-1}(p)) \leq const'_n \text{ for all } p \in P^{n-2}.$$

Observe in this regard the following.

• The existence of a proper map $F : X \to P^{n-1}$ with $diam(F^{-1}(p)) \leq C$ would imply that $fill.rad(X) \leq C$, that remains unknown if $n \geq 4$ even for universal coverings of compact n-manifolds with $Sc > 0$.

• The bound $fill.rad(X) \leq C < \infty$ implies that the balls in X *can't be all contractible*; moreover,

given a continuous function $R(r) \geq r$, $r \geq 0$, and a number r_0, there exits a ball of radius $r \geq r_0$ in X, which is non-contractible in the concentric $R(r)$-ball.

This *uniform non-contractibility property*, remains conjectural for $n \geq 6$ but we prove it for $n = 4, 5$ in the next section.

The only known result of this kind, which implies a (sharp) bound on the injectivity radius of manifolds with $Sc \geq \sigma$ is

Green-Berger Integral Scalar Curvature Inequality. *Among all compact manifolds X with given $vol(X)$ and the integral $\int_X Sc(X, x)dx$, the round spheres maximize the average distance between conjugate points on geodesics.*[217]

[216] This condition ensures "homotopical surjectivity" of this map, that is non-existence of its (proper in the non-compact case) homotopy to a map into a subset $Y \subsetneq X$. I am not certain if another such condition is relevant here.

[217] M. Berger, *Lectures on geodesics in Riemannian geometry*, Tata Institute of Fundamental Research, 1965.

Berger's proof of this applies to complete non compact *amenable* manifolds X with $Sc(X) \geq n(n-1)$, e.g. with *subexponential volume growth*, thus providing a bound $inj.rad \leq C_n < \infty$. But I don't see offhand how to prove such a bound for non-amenable X.

3.10.1 Filling Radii of 3-Manifolds, Hyperspherical Radii, Enlargeability and Uniform Asphericity

Recall that the Uryson k-width $width_m(X)$ of a metric space X is the infimum of the numbers $d \geq 0$, such that X admits a continuous map a m-dimensional polyhedral space P^m, such that that the diameters of the pullbacks of all points $p \in P^m$ are bounded by d.

Lemma (A). Let a *proper*[218] locally contractible metric space X be covered by closed locally contractible subsets X_i, $i \in I$, such that

$\bullet_1$ there is *no triple intersections* between X_i;

$\bullet_2$ the *connected components* $Y = Y_{ijk} \subset X_i \cap X_j$ of all double intersections are locally contractible,[219] and their *rational homology* $H_1(Y_{ijk};\mathbb{Q})$ *vanish*;

$\bullet_3$ the diameters of all these Y are *bounded by*

$$diam(Y_{ijk}) \leq \delta_{ijk} < \infty.$$

Then
$$width_1(X) \leq \sup_{ijk}(2\delta_{ijk} + width_1(X_i)).$$

Proof. Let $\chi_i : X_i \to P_i^1$ be a continuous map, such that the pullbacks $\chi_i^{-1}(p)$, $p \in P_i^1$, are bounded by

$$diam(\chi_i^{-1}(p)) \leq d_i \text{ for all } p \in P_i^1$$

and observe that the union U_p of the χ_i-pullback $\chi_i^{-1}(p)$ of $p \in P_i^1$ with the components $Y_{ijk} \subset X_i$ for which $\chi(Y_{ijk}) \ni p$ satisfies:

$$diam(U_p) \leq diam(\chi_i^{-1}(p)) + 2\sup_{j,k}\delta_{ijk} \leq d_i + 2\sup_{j,k}\delta_{ijk} \text{ for all } p \in P_i^1.$$

Since $H_1(Y_{ijk};\mathbb{Q}) = 0$ the χ_i-maps from Y_{ijk} onto their images in P_i^1 are *contractible*. Therefore,

given an arbitrary small neighbourhood $U_i' \subset X_i$ of the union $\bigcup_{j,k} Y_{ijk} \subset X_i$, there exists a map $\chi_i' : X_i \to P_i^1$ homotopic to χ_i , such that

• χ_i' is *constant* an all $Y_{ijk} \subset X_i$;

• χ_i' is *equal to* χ_i *outside* U_i', where we may assume that $U_i' = \bigcup_{j,k} U_{ijk}'$ for the connected components $U_{ijk}' \supset Y_{ijk}$ of U_i';

• the image $\chi_i'(U_{ijk} \subset P_i^1$ is contained in the image $\chi_i(U_{ijk})$.

It follows that

$$diam(\chi_i')^{-1}(p) \leq diam(U_p) + \varepsilon_i \leq d_i + 2\sup_{j,k}\delta_{ijk} + \varepsilon_i,$$

where ε_i can be made arbitrarily small for small U_i.

Finally, we glue the graphs P_i^1 and P_j at the points $p_{ijk}\chi_i'(Y_{ijk} \in P_i^1$ and $p_{jik}\chi_j'(Y_{jik} \in P_j^1$ and let $P^1 = \bigcup_i P_i^1$ be the resulting graph.

Then the obvious map

$$\chi' : X = \bigcup_i X_i \to P^1 = \bigcup_i P_i^1$$

[218] Closed bounded subsets are compact.

[219] Probably, "locally contractible" is an unnecessary precaution, but in our case X_i are manifolds with boundaries, where the intersections $X_i \cap X_j$ are unions of connected components of common boundaries of X_i and X_j.

satisfies

$$diam((\chi')^{-1}(p)) \le \sup_i diam((\chi'_i)^{-1}(p)) \le \sup_{ijk}(2\delta_{ijk} + width_1(X_i) + \varepsilon_i)$$

which concludes the proof for $\sup_i \varepsilon_i \to 0$.

Remark (A′). The condition $H_1(Y_{ijk};\mathbb{Q}) = 0$ looks strange here but I don't know if it can be omitted.

On the other hand, the 2-width of X can be bounded by

$$[width_2] \qquad width_2(X) \le \max(\sup_i width_1(X_i), \sup_{i,j,k} diam(Y_{ijk}))$$

with no such condition, where the relevant 2-polyhedron where $X = \bigcup_i X_i$ goes is obtained from the cones $C_{jk}(X_i)$ by gluing the apexes of $C_{jk}(X_i)$ with these of $C_{ik}(X_j)$.

Next – this is a provisional definition adapted to our present purpose – define *maximal* $\square$*-width* of a metric space S *homeomorphic to the circle*, denoted $max.width_\square(S)$, as the maximum of the numbers D, such that S admits a decomposition into four segments, with the same combinatorial arrangement as that of the four faces of the square $[-1,1]^2$ and such that the distances between both pairs of opposite (i.e. non-intersecting) segments are *bounded from below by* D.

Lemma (B). Let a circle in a proper metric space, say $S \subset X$, with the induced metric satisfies

$$max.width_\square(S) \ge D.$$

Then there exists a pair of non-negative 1-Lipschitz functions on X that define a *proper* map

$$\Psi_\square = (\psi_1, \psi_2) : X \to \mathbb{R}^2_+,$$

such that

S is sent by $\Psi_\square$ to the complement of the interior of the square $[0,D]^2 \subset \mathbb{R}^2_+$, where the induced homology homomorphism

$$\mathbb{Z} = H_1(S) \to H_1(\mathbb{R}^2_+ \setminus (0,D)^2) = \mathbb{Z}$$

is an isomorphism.

Proof. Let

$$S = S_{1+} \cup S_{2+} \cup S_{1-} \cup S_{2-},$$

where

$$dist(S_{1+}, S_{1-}) \ge D \text{ and } dist(S_{2+}, S_{2-}) \ge D$$

and observe that the map defined by

$$\psi_i(x) = dist(x, S_{1+}),\ i = 1,2,$$

is the required one.

Corollary (B′). The rational filling radius of S in X is bounded from below by

$$fil.rad(S, X; \mathbb{Q}) \ge \frac{1}{2} width_\square(S).$$

This means that *no multiple of the curve S bounds in the ρ-neighbourhood* $U_\rho(S) \subset X$ for $\rho < width_\square(S)$, or, more formally, the homology boundary homomorphism $H^2(U_\rho(S), S) \to H_1(S) = \mathbb{Z}$ *vanishes* for these (small) ρ.

These (A) and (B) and (B′), albeit useful, are boringly trivial, but the following one is mildly amusing.

Lemma (C). Let X be a locally contractible *path metric* space [220] and let $\gamma(x) = dist(x, x_0)$ be the distance function on X to some point $x_0 \in X$.

If all embedded circles $S \subset X$ have

$$\max.width_\square \leq D,$$

then the diameters of all connected components of the levels $\gamma^{-1}(t) \subset X$, $t \geq 0$ are bounded by $3D$.

Proof. Let $\gamma(x) = dist(x, x_0)$ for some point $x_0 \in X$ and let us show that the diameters of all connected components of the levels $\gamma^{-1}(t) \subset X$, $t \geq 0$ are bounded by $3D$. Indeed, assume without loss of generality that $t \geq \frac{3}{2}D$ and let x_1, x_2 be two points in a *connected component* of $\gamma^{-1}(t)$.

Let $\widetilde{x_1, x_2}$ be a segment joining x_1 with x_2 in a small neighbourhood of this component and let $[x_1, x_0]$ and $[x_2, x_0]$ be almost shortest segments between x_0 and x_i, $i = 1, 2$, where we assume without loss of generality that the union S of these three segments

$$S = \widetilde{x_1, x_2} \cup [x_1, x_0] \cup [x_2, x_0] \subset X$$

makes a topological circle.

Let $[x_i, x'_i] \subset [x_i, x_0]$ be subsegments of length $D + \varepsilon$ and let

$$\widetilde{x'_1, x'_2} = [x'_1, x_0] \cup [x'_2, x_0] \subset [x_1, x_0] \cup [x_2, x_0]$$

be the union of the complementary segments. Now,

$$dist(\widetilde{x_1, x_2}, \widetilde{x'_1, x'_2}) > D$$

while

$$dist([x_1, x'_1], [x_2, x'_2]) \geq dist(x_1, x_2) - 2D - \varepsilon',$$

which, due to our assumption $max.width_\square(S) \leq D$, implies for $\varepsilon, \varepsilon' \to 0$ that $dist(x_1, x_2) \leq 3D$. QED.

Corollary (C′). *The inequality*

$$\max.width_\square \leq D$$

for all embedded circles $S \subset X$ implies that the Uryson 1-width of X is bounded by:

$$width_1(X) \leq 3D.$$

Proof. Factor the map $\gamma : X \to \mathbb{R}$ as

$$X \overset{\alpha}{\to} P^1_0 \overset{\beta}{\to} \mathbb{R},$$

[220] The distances between pairs of points are equal to the infima of lengths of curves between them.

such that the levels of α are equal to the connected components of γ, and approximate (this is trivial) the (1-dimensional!) space P_0^1 by a polyhedral P^1 as it is done for this purpose in the proof of corollary 10.11 in [GL(complete) 1983].

Corollary (C″). If the first cohomology of X vanishes then the above graph P^1 is a *tree.*

Consequently, X *is quasiisometric to a tree* in this case.

Proof. Slightly modify α to make the levels of α path connected and observe that continuous onto maps with path connected fibers are surjective on the fundamental groups.

Corollary (D) If all closed curves immersed to X bounds in their ρ-neighbourhoods, then the *the Uryson 1-width of X is bounded by:*

$$width_1(X) \le 6\rho.$$

Corollary (E). *If X has infinite 1-width, then it contains closed curves S with arbitrarily large maximal $\square$-widths.*

Example (E′). Universal coverings $\tilde{X}$ of compact spaces X with *non-virtually free* fundamental groups $\pi_1(X)$ contains circles with arbitrarily large maximal $\square$-widths.[221]

Lemma (F). Let $X = (X,g)$ be a complete Riemannian 3-manifold with a boundary and let $X^{\rtimes} = X \rtimes \mathbb{T}^m = (X \times \mathbb{T}^m, g^{\rtimes} = g + \phi^2 dt^2$ be a $\mathbb{T}^{\rtimes}$ extension of X with *mean convex* boundary and such that $Sc(g^{\rtimes}) \ge \sigma > 0$.

Then all immersed circles $S \subset X$, homologous to zero (e.g. contractible ones) *have their rational filling radii bounded by*

$$fil.rad(S, X'\mathbb{Q}) < \frac{2\pi}{\sqrt{\sigma}}.$$

Proof. Since S is homologous to zero, it bounds an orientable surface in X; let $Z \subset X$ be such a surface with boundary $\partial = S$, which minimizes the $g^{\rtimes}$*-area* of Z that is the $(m+2)$-volume of the hypersurface $Y \times \mathbb{T}^m \subset X^{\rtimes}$, that is equal to the area of Y with respect to the *conformal metric* $\psi(x) \cdot g(x)$ on X, where $\psi(x) = vol(\{x\} \times \mathbb{T}^m) = (2\pi)^m \phi^m(x)$.

Since the minimal hypersurface $Z \times \mathbb{T}^m \subset X^{\rtimes}$ is stable it admits a $\mathbb{T}^1$ extension with the scalar curvature bounded from below by that of $X^{\rtimes}$ and the proof follows from codimension 2 corollary to $\frac{2\pi}{n}$-inequality. (See **[2]** in section 3.7.2, where the surface is denoted $\underline{X}$ rather than Z.)

This, together with the above corollary (D) yields the following.

Proposition (F′). Let $X = (X,g)$ be a complete Riemannian 3-manifold with a boundary, such that the homology group of X is torsion.

If X admits a $\mathbb{T}^{\rtimes}$-extension $X^{\rtimes} = X \rtimes \mathbb{T}^m = (X \times \mathbb{T}^m, g^{\rtimes} = g + \phi^2 dt^2)$ with *mean convex* boundary and with $Sc(g^{\rtimes}) \ge \sigma > 0$, then the *first Uryson width of X is bounded by*

$$width_1(X) \le \frac{12\pi}{\sqrt{\sigma}}.$$ [222]

[221] Compare with Corollary in the section 1.2 C in [G(foliated) 1991].

[222] Our present proof of this inequality follows that of corollary 10.11 in [GL(complete) 1983]

3D Classification Corollary (F″). A compact 3-manifold admits a metric with $Sc(X) > 0$ *only if* it contains *no aspherical connected summand* in its Kneser-Milnor prime decomposition. ("If" is also true by Perelman's theorem.)

Proof. In view of (C″), the universal covering of X is quasiisometric to a tree, and then an application of Stallings' theorem shows that the fundamental group is virtually free. QED,.

On Domination. The proof of this classification theorem in [Gl(complete), 1983], which used the index theorem, albeit less straightforward, yields more:

If a closed orientable 3-manifold X contains an *aspherical manifold in its prime decomposition*, then it *can't be dominated* by a complete manifold $\hat{X}$ *with* $Sc > 0$: all maps between such orientable manifolds, $\hat{X} \to X$, have *degrees zero*.

Theorem (G). *If a complete orientable Riemannian* 3*-manifold* $X = (X, g)$ *with a boundary admits a* $\mathbb{T}^{\rtimes}$*-extension* $X^{\rtimes} = X \rtimes \mathbb{T}^m = (X \times \mathbb{T}^m, g^{\rtimes} = g + \phi^2 dt^2)$ *with mean convex boundary and with* $Sc(g^{\rtimes}) \geq \sigma > 0$*, then the first Uryson width of* X *is bounded by*

$$width_1(X) < \frac{36\pi}{\sqrt{\sigma}}.$$

Proof. Decompose X into a union of submanifolds with common boundaries.

$$X = \bigcup_i X_i,$$

where two such submanifolds intersect by a stable $g^{\rtimes}$*-minimal* surface (as in lemma (F)) and such that all connected stable $g^{\rtimes}$-minimal surfaces in all X_i are homologous to the connected components of the boundaries of these X_i.

We know that all these surfaces are spherical and there diameters are bounded by $\frac{12\pi}{\sqrt{\sigma}}$.

It is also clear that the rational homology groups $H_1(X_i, \mathbb{Q})$ vanish and the proof follows from lemma (A) and proposition (F′).

Corollary (G′). If X is compact with no boundary, then its *absolute filling radius* is bounded by

$$fillrad(X) < \frac{18\pi}{\sqrt{\sigma}}.$$

This means that

there exists an orientable 4*-dimensional pseudomanifolds* V *with boundary and a metric* $dist_V$ *on* V*, such that the boundary* ∂V *is* isometric to V *and*

$$dist_V(v, \partial V) < \frac{18\pi}{\sqrt{\sigma}}.$$

Proof. If a piecewise linear map[223] $\chi : X \to P^1$ has $diam(\chi^{-1}(p) \leq d$, $p \in P^1$, then it the cone $F = C_\chi$ of χ is a pseudomanifold, which carries an obvious metric which has the required properties.

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where it is stated in the non $\mathbb{T}^{\rtimes}$-stable form and where the H_1-torsion condition, albeit implicitly used in the argument, was erroneously omitted.

[223]This is understood with respect to some smooth triangulation of X map. Notice that our χ in the definition of width was assumed *continuous* rather than piecewise , linear, but it can be approximated by piecewise linear ones.

Remark (G″). It follows from [$width_2$] in Remark (A′) that

$$fillrad(X) < \frac{6\pi}{\sqrt{\sigma}},$$

Probably,

$$fillrad(X) < \frac{\pi}{2}\sqrt{\frac{6}{\sigma}}.$$

Enlargeability in Dimension 3. Let us conclude this section by proving that

the universal coverings of compact aspherical 3-manifolds X are enlargeable.

Notice that "enlargeability" (defined below) is, a priori, stronger, than non-existence of a metric with $Sc > 0$; this remains conjectural for higher dimensional aspherical manifolds.

Also notice that in dimension 3, one can easily prove this property for manifolds of each of the 7 non-elliptic Thurston's geometries separately, and then (this is also easy) show that enlargeability is stable under JSJ-decomposition.

Our point here is to furnish a direct elementary proof. [224]

Definitions of the Hyperspherical Radius, Hypersphericity, Enlargeability and Uniform Lipschitz Asphericity. The *hyperspherical radius* $Rad_{S^n}(X)$, of a *closed* orientable Riemannian n-manifold X as *the supremum of the radii $R > 0$ of n-spheres, such that X admits a non-contractible 1-Lipschitz, i.e. distance non-increasing, map $f : X \to S^n(R)$.*

More generally, if X is an *open* manifold, this definition still make sense for maps $f : X \to S^n$, which are *locally constant at infinity*,[225] i.e. outside compact subsets in X. Similarly, if X allowed a boundary, then f should be constant on all components of this boundary.

Notice that a (locally constant at infinity if X is open) map f from an *orientable* n-manifold X to the sphere S^n is *contractible* (in the space of locally constant at infinity maps in the open case) if and only if f has *zero degree.*

In view of that, we define $Rad^{deg1}_{S^n}(X) \leq Rad_{S^n}(X)$ as the supremum of R for 1-Lipschitz, maps $f : X \to S^n(R)$ of degrees 1.

A manifold X is called *hyperspherical* if $Rad_{S^n}(X) = \infty$ and X is *enlargeable* if it admits coverings with arbitrary large hyperspherical radii.

A manifold X is called *deg1-hyperspherical* if, $Rad^{deg1}_{S^n}(X) \leq Rad_{S^n}(X) = \infty$, or, in different terms if X λ -Lipschitz dominates (the fundamental homology class of) the unit sphere S^n for all $\lambda > 0$.

A metric space S is called *uniformly Lipschitz k-aspherical* if λ-Lipschitz maps from the unit sphere S^k to X, are extendable to $\Lambda(\lambda)$-Lipschitz maps from the unit ball B^{k+1} that bounds S^K, i.e. $\partial B^{k+1} = S^k$, for all $d > 0$, where $\Lambda(d) = \Lambda_X(d)$ is a continuous (control) function.

[224] The concepts of enlargeability and related "bad ends" are discussed in [GL(complete) 1983] around theorem 8.1, in [Lawson&Michelsohn(spin geometry) 1989] around theorem IV.6.18] and also in [G(positive) 1996] in §§$9\frac{1}{4}$, $9\frac{3}{11}$, where similar properties are proved for "multiple largeness"; later this appears in [G(inequalities) 2018], section 4, under the name of *iso-enlargeability.*

[225] On can drop 'locally" if X is connected at infinity.

Example. If X has bounded geometry,[226] e.g. X is a covering of a compact locally contractible space, and if X is k-aspherical, i.e. $\pi_k(X) = 0$, then it is uniformly Lipschitz k-aspherical.

Exercise . Construct complete uniformly contractible surfaces, which are not Lipschitz uniformly 1-aspherical.

Also construct complete uniformly contractible Riemannian 3-manifolds where the sectional is asymptotically non-positive: $\kappa(X,x) \leq \varepsilon(dist(x,x_0))$, where $\varepsilon(d)$ is a positive function which goes to 0 for $d \to \infty$.

Lemma (I). Let X be a complete orientable Riemannian n-manifold, and let $Y_i \subset X$ be smooth closed connected orientable codimension 2 submanifolds with trivial normal bundles, (e.g. $H_{n-2}(X) = 0$) and let $U_i = U_{\rho_i} \supset Y$ be the ρ_i-neighbourhoods of Y_i in X where $\rho_i \to \infty$ for $i \to \infty$.

Let X be 2-aspherical and Lipschitz uniformly 1-aspherical, let Y_i admit $deg1$-hyperspherical coverings $\tilde{Y}_i$.

If the inclusion homomorphisms $\pi_1(Y_i) \to \pi_1(U_{\rho_i})$ are injective and if the manifolds Y_i are not rationally homologous to zero in U_{ρ_i}, then X is $deg1$-hyperspherical.

Proof. Let $U_i' \subset U_i$ be a small tubular neighbourhood of Y, where, observe the boundary $\partial U_i'$ topologically splits as $Y \times S^1$.

Since X is 2-connected and the class $[Y] \in H_{n-2}(Y)$ goes to a non-zero class in $H_{n-2}(U_i;\mathbb{Q})$, the linking number between closed curves in the complement $U_1 \smallsetminus Y$ with Y defines a homomorphism from $\pi_1(U_1 \smallsetminus Y)$ to $\mathbb{Z}$, which *doesn't vanish on the class of the circles* $\{y\} \times S^1 \subset \partial U_i' \subset U_i \smallsetminus Y$, and *vanish on* $\pi_1(Y) = \pi_1(Y \times \{s\}$.

It follows that the inclusion homomorphism $\pi_1(\partial U_i') \to \pi_1(U_1 \smallsetminus Y$ is *injective.*

Now, let $\widetilde{U_i \smallsetminus Y}$ be the covering of $U_1 \smallsetminus Y$, the restriction of which to $\partial U_i' = Y \times S^1$ equals to $\tilde{S}^1 \times \tilde{Y}_i$ for $\tilde{S}^1 = \mathbb{R}^1$ and the above $deg1$-hyperspherical covering $\tilde{Y}_i$ of Y_1 and show, this is easy, [227] that

since $Rad^{deg1}_{S^{n-2}}(\tilde{Y}_i) = \infty$, the $deg1$ hyperspherical radius of $\widetilde{U_i \smallsetminus Y}$ bounded from below only by ρ_i, say as follows,

$$Rad^{deg1}_{S^n}(\widetilde{U_i \smallsetminus Y_i}) \geq \frac{1}{10}\rho_i.$$

Finally, since X is Lipschitz uniformly 1-aspherical the covering map

$$\widetilde{U_i \smallsetminus Y_i} \to \partial(U_i \smallsetminus Y_i \subset X$$

is one-to-one "deeply inside" $\widetilde{U_i \smallsetminus Y_i}$, i.e. sufficiently far from Y_i and $\partial(U_i \smallsetminus Y_i)$, ("how deeply"or har "far" depends on the (control) function $\Lambda(\lambda)$) and the proof follows.

Corollary (I′). *Compact orientable aspherical* 3*-manifolds X are enlargeable.*

Indeed, as we know, the universal coverings $\tilde{X}$ of these X contain closed curves with arbitrarily large maximal □-widths; these can be taken for the above Y_i.

[226]Probably, a lower bound on the sectional curvature suffices.
[227]See §§5, 6 in [GL(complete) 1983] and IV. 6 in [Lawson&Michelsohn(spin geometry) 1989].

Theorem (J). *If a compact orientable 3-manifold X contains an aspherical summand in its prime decomposition then no non-zero multiple of (the fundamental class of) X can be dominated by a complete orientable manifold $\hat{X}$ with* $Sc(\hat{X}).0$.

In simple words,

all continuous maps $\hat{X} \to X$ constant at infinity have zero degrees.

If X is enlargeable and $\hat{X}$ dominates X, then $\hat{X}$ is also enlargeable, and one knows that enlargeable *spin* n-manifolds support no metrics with $Sc > 0$ for all n (theorem 6.12 in [GL(complete) 1983]). Since all 3-manifolds are spin, the proof follows.

3.10.2 Geometry and Topology of Complete 3-Manifolds with $Sc > 0$

Start with a simple proof of the following result by Laurent Bessèeres, Gérard Besson, and Sylvain Maillot [Be-Be-Ma(Ricci flow) 2011].

(A) Theorem. *Complete 3-manifolds X with $Sc(X) \geq \sigma > 0$ are infinite connected sums of spacial space forms S^3/Γ and copies of $S^2 \times S^1$.*

Proof. By the compact exhaustion corollary from section 3.7.2, X decomposes into infinite connected sum of *compact* manifolds X_i, and theorem (J) from the previous section implies that there is no aspherical summands in these X_i. Then the conclusion follows by Perelman's theorem.

Remarks. (i) The argument in [Be-Be-Ma(Ricci flow) 2011] depends on a generalization of Perelman's arguments to non compact manifolds. Also, Gérard Besson recently told me that Jian Wang found a proof of **(A)** with minimal surfaces.

(ii) A close look at proof of **(A)** shows that X decomposes into a union of compact submanifolds $X_i \subset X$, such that

• X_i intersect with X_j, for all $i \neq j$, over the common components of their boundaries;

• the boundaries of X_i are union of spheres the areas and the intrinsic diameters of which are bounded by a constant depending only on σ;

• the diameters of all X_i are also bounded by a constant depending only on σ.

(A′) Corollary. *No non-torsion homology class $\underline{h} \in H_3(\underline{X})$ in an aspherical space can be dominated by a complete 3-manifold X with $Sc(X) \geq \sigma > 0$.*

Proof. Given a map $X \to \underline{X}$, homotop it to f' constant on representatives of all non-contractible 2-spheres in X and thus reduce the problem to the, case where X is a single spherical space form

Alternatively, argue algebraically and use the fact that all finitely generated subgroups in $\pi_1(X)$ are virtually free.

(B) ***Generalization to*** **Sc > 0**. *If a* 3*-manifold X admits a complete metric with $Sc > 0$ then all finitely generated subgroups in $\pi_1(X)$ are virtually free.*

Proof. Let $\tilde{X} \to X$ be a covering with non-virtually free fundamental group and let $\bar{X} \subset \tilde{X}$ by the compact *Scott core* of $\tilde{X}$. Then, by the loop theorem, the boundary of $\bar{X} \subset \tilde{X}$ is incompressible and the proof follows from theorem 6.12 in [GL(complete) 1983].

Alternatively, one can prove that X contains a (compact or complete) *stable minimal surface*, which is *non-simply connected*, while one knows (see the proof of Wang's theorem below) that such surfaces don't exist in 3-manifolds with $Sc > 0$.

Remarks,Examples and Open Problems for $\mathbf{Sc > 0}$. **(a)** The apparent *irreducible*, i.e. non-trivially indecomposable into connected sum, example of an open manifold, which admits a complete metric with $Sc > 0$ is $\mathbb{R}^2 \times S^1$ with the (radial warped product) metric

$$g = dr^2 + \varphi(r)^2 d\theta^2,\ t \in [0,\infty), \theta \in [0, 2\pi], \text{ with } \varphi(r) = r^{2\alpha},$$

where the scalar curvature

$$Sc(g)(r) = -\frac{2\varphi''(r)}{\varphi(r)} = \alpha(\alpha - 1)\frac{1}{r^2}$$

is positive for $1 < \alpha < 2$ with quadratic decay for $r \to \infty$.

By the above, $\mathbb{R}^2 \times S^1$ admits no metric with $Sc \geq \sigma > 0$; moreover, the the curvature must decay at least as $\frac{4\pi^2}{r^2}$ according to QD-exercise in section 3.6.1.

(a′) If $n \geq 4$, there are similar complete warped metrics with $Sc > 0$ on $\mathbb{R}^2 \times X^{n-2}$ for all (compact and open) manifolds X^{n-2}.

(b) It is *unknown* (unless I am missing something obvious) if open handle bodies of all genera, hence, the interiors all compact 3-manifold $\bar{X}$ with boundaries, which have a virtually free fundamental groups $\pi_1(\bar{X})$, admit complete metrics with $Sc > 0$.

(b′) If X is an n-manifold for $n \geq 4$, (maybe one should assume $n \geq 5$), which contracts to its codimension 2 skeleton, e.g. a contractible one, then, *conjecturally,* it *admits a complete metric with* $Sc > 0$.

However, no such metrics are known, for instance, in the interiors of compact manifolds the boundaries of which admit metrics with negative sectional curvatures < 0.

The following result by Jian Wang shows that obstructions to the complete metrics with $Sc > 0$ on X may resides in the complexity of the *proper homotopy type* of X.

(C) Theorem. *Complete* contractible 3*-manifolds with* $Sc > 0$ *are simply connected at infinity* (see [Wang(Contractible) 2019] and [Wang(topological characterization) 2021] in this volume).

Idea of the Proof. Recall that the first contractible 3-manifold $X = X_{Wh}$ *not simply connected at infinity*, which was discovered by Whitehead in 1935, is equal to the union of an infinite increasing sequence of solid tori,

$$X_{Wh} = \bigcup_k T_i,\ T_1 \subset T_2 \subset ... \subset T_k \subset .... \subset X_{Wh},$$

where the *boundary* of T_k, $k \geq 2$, is *not contractible in* T_2 for all $k \geq 2$.

Wang shows in this case that, given an arbitrary complete Riemannian metric on X_{Wh}, there exist connected stable minimal surfaces $\Sigma_k \subset T_k$ of genus zero with boundaries $\partial\Sigma_k \subset T_k$, such that the number of connected of the intersections $\Sigma_k \cap T_1$ goes to infinity for $k \to \infty$.

Then, in the limit, he obtains a *connected stable* minimal surface $\Sigma = \Sigma_\infty \subset X_{Wh}$ of genus zero with a *complete* induced metric, such that the intersection $\Sigma_k \cap T_1$ has *infinite* area; this, in the case of $Sc(X > 0)$, contradicts to the

Fischer-Colbrie&Schoen (Gauss-Bonnet-Cohn-Vossen) *inequality*

$$\int_\Sigma Sc(X,\sigma)d\sigma \leq 2\pi\chi(\Sigma).$$

Hence

the Whitehead manifold admits no complete metrics with $Sc > 0$.

(C′) ***Possible Generalizations.*** Wang's argument applies (as far as I understand it) to connected sums $X = X_{Wh}\#X_1\#X_2...$ with other 3-manifolds and shows that these$X$ admit no complete metric with $Sc > 0$.

Conceivably, Wang's argument can be also applied to manifolds X, which *dominate* the fundamental homology class $[X_{Wh}]$ (with infinite support).

If so, then by the (non-compact $\mathbb{T}^\rtimes$-stabilized version of the) Schoen-Yau inductive descent argument , the products $X_{Wh}\times\mathbb{T}^m$ (and probably, the products of X_{Wh} with enlargeable manifolds in general) admit no complete metrics with $Sc > 0$ either. (If $m > 5$, one has to appeal to Lohkamp's desingularization theorem.)

In fact, contractibility of manifolds in Wang's theorem doesn't seem that essential.

Conjecturally, if an orientable 3-manifold can be exhausted by compact submanifold $V_1 \subset V_2 \subset ... \subset V_i \subset ... \subset X$, such that all components of the complements $X \smallsetminus V_i$ are aspherical with infinitely generated fundamental groups and the inclusion homomorphisms $\pi_1(X\smallsetminus V_{i+1}) \to \pi_1(X\smallsetminus V_i)$ for all $i = 1,2,....$, are injective (maybe, its enough to assume that the images of the inclusion homomorphisms $\pi_1(X\smallsetminus V_{i+1}) \to \pi_1(X\smallsetminus V_1)$ are infinitely generated), then X *admits no complete metric with* $Sc(X) > 0$.

Moreover,

no non-zero multiple of the fundamental homology class $[X_+]$ *can be dominated, by a complete manifold with Sc* > 0, *that is, no complete orientable 3-manifold* $\hat{X}$ *with* $Sc(\hat{X}) > 0$ *admits a proper map to a* $\underline{X}_+$ *with non-zero degree.*

Example. Let a connected orientable manifold X decompose into a countable union of compact aspherical submanifolds with aspherical boundaries, $X = \cup_i X_i$, such that

• every two X_i intersect (if at all) over several connected components of their boundaries, where these intersections are denoted $Y_{ij} = X_i \cap X_j = \partial X_i \cap \partial X_j$.

• the inclusion homomorphisms $\pi_1(Y_{ij}) \to \pi_1(X_i)$ are injective and their images have infinite indices in the fundamental groups $\pi_1(X_i)$. e.g. as in the Whitehead manifold, where X_i are (the closures of $T_{i+1}\smallsetminus T_i$.

Then the above conjecture implies that for $n = 3$ *no manifold* X_+, *which contains* X *as a submanifold, admits a complete metric with* $Sc > 0$.

Remark about $n > 3$. For all we know, the n-manifolds X (minus the boundaries) and $X_+ \supset X$ in this example don't admit complete metric with $Sc > 0$ enlargeable for all n, but this can be proved at the present moment only in special cases, for instance, if some manifold $Y_{ij} \subset \partial X_i$ is *enlargeable*, (e.g. if $dim(X) = 4$, since compact aspherical 3-manifolds are enlargeable, see the previous section) and if the inclusion homomorphism $\pi_1(Y_{ij}) \to \pi_1(X_+)$ is *injective* (see section 4.7).

Attaching Cylinders to Stable Hypersurfaces. Let $X = (X, g)$ be a complete, e.g. compact, Riemnnian manifold with a boundary,

Notice that completeness of X, i.e. compactness of closed bounded subsets, implies completeness of the boundary with respect to the Riemannian distance function $dist_g$ in $X \supset Y$.

Let $Y \subset \partial X \subset X$ be a connected component of the boundary and let $Y \rtimes_\phi \mathbb{R}_+$ be the warped product with the metric $h_\phi = h + \phi^2 dt^2$ for the Riemannian metric h on Y induced from g on X and a smooth positive function $\phi = \phi(y)$.

Observe that the boundary $Y \times 0 \subset Y \rtimes_\phi \mathbb{R}_+$ is isometric to $Y \subset X$ and let

$$X_O = X \sqcup_Y Y \rtimes_\phi \mathbb{R}_+$$

be obtained by attaching $Y \subset X$ to $Y \times 0 \subset Y \rtimes_\phi \mathbb{R}_+$ by this isometry.

This X_O,which homeomorphic to the complement $X \setminus Y$ carries a natural continuous Riemannian metric which is complete of X and hence, Y, are complete.

Now, if the $Y \subset X$ is mean convex, and if the scalar curvatures of both manifold are positive, $Sc(g) > 0$ and $Sc(h_\phi) > 0$, then the metric on X_O can be approximated by smooth metrics with $Sc > 0$, since $Y \times 0 \subset Y \rtimes_\phi \mathbb{R}_+$ is *totally geodesic* in $Y \rtimes_\phi \mathbb{R}_+$ (see section 1.4). This yields the following.

(D) Proposition. Let $X = (X, g)$ be a complete Riemannian manifold with $Sc > 0$ and let $Y \subset X$ be a cooriented stable minimal hypersurface. The the complement $X \setminus Y$ admits a complete metric G with $Sc(G) > 0$, which is equal to g outside a given neighbourhood $U \supset Y$ intersected with $X \setminus Y$.

Let us apply this to 3-dimensional manifolds X, where Y is a topological 2-sphere and where we benefit from the following homotopy theorem of due to Laurent Bessières, Gérard Besson, Sylvain Maillot, and Fernando Coda Marques.

(D) Theorem. *The space of complete Riemannian metrics of bounded geometry and uniformly positive scalar curvature on an orientable 3-manifold is path-connected.*

It follows the the metric G on $X \setminus Y$ can be homotoped outside a given compact subset to the standard cylindrical metric $ds^2 + dt^2$ on $S^2 \times \mathbb{R}_+$, and then extended to the ball B^3 keeping the curvature positive all along.

The one can attach the unit 3-ball to the sphere $S^2 \times \{t_0\}$ and smooth the resulting C^1 metric with $Sc > 0$.

Remark. One may use the compact case of **(D)**, namely for $S^2 \times S^1$, where this was earlier proven in [Marques(deforming $Sc > 0$)2012].

Besides, one doesn't need here the full power of the Ricci flow, since the relevant deformation proceeds in the space of S^1-invariant metics, which are moreover, of the forms $g + \phi^2 dt^2$, and where the 3-D equations reduce to 2-dimensional ones for pairs (g, ϕ) of Riemannian metrics g and functions ϕ on S^2.

(E) Corollary. Let X be a complete 3-manifold with $Sc(X) > 0$. Then there exists a complete (disconnected) manifold $X^\sim$, such that

• *all connected $X_i^\sim$ of $X^\sim$ are "simple":*

the complement to a embedded 2-sphere S^2 or to a properly (infinity → infinity) embedded plane $\mathbb{R}^2$ in $X_i^\sim$, for all i, is disconnected and at least one *of the two components is homeomorphic to S^3 with finitely or countably many punctures;*

• *The complement to a finite or countable set of disjoint complete stable connected minimal surfaces Σ_{min} in in X is isometric to an open subset in $X^{\sim}$, where all these Σ_{min} are simply connected, and if $Sc(X) \geq \sigma > 0$ they are all compact, hence spherical.*

Proof. Let Σ be an *"essential"* embedded 2-sphere or a properly embedded plane in a complete 3-manifold X, i.e. such that the complement $X \setminus \Sigma$ is either connected or none of the two components is homeomorphic to S^3 with punctures. Then either X contains an essential stable minimal sphere or an essential stable minimal plane. [228]

If the scalar curvature of X is uniformly positive, i.e. $Sc(X) \geq \sigma > 0$, then all these minimal surfaces are spherical and we attach 3- balls to them as above.

In general, where $Sc(X) > 0$, we attach 3-balls to the spherical cutting surfaces and cylinders to the planar ones. QED.

Question Is there a version of the above for, say compact, 4-manifolds with $Sc > 0$?

Remark. The first things one needs is a "natural filling" of the spherical space forms S^3/Γ by 4-manifolds (may be singular ones?) with $Sc > 0$, something in the spirit of discs bundles over S^2 that fill in the diagonal lens spaces $S^3/\mathbb{Z}_k$.

3.10.3 Non-Existence of Uniformly Contractible and Aspherical 4- and and 5-manifolds with $Sc > 0$

Recall that a metric space X is *uniformly contractible* if there exists a function $R(r) \geq r$, Recall that a metric space X is *uniformly contractible* if there exists a function $R(r) \geq r$, called *contractibility control function* such that the r-balls $B_x(r) \subset X$ around all $x \in X$, are contractible in the concentric balls $B_x(R(r))$.

For instance, if X is bounded then "uniformly contractible"="contractible".

Also obviously, but more interestingly, the same applies to spaces X that with *cobounded*, (e.g. compact) isometry groups: there is a constant d, such that,

for every two points $x_1, x_2 \in X$, there exists isometry $I : X \to X$ such that $dist(x_1, I(x_2)) \leq d$.

In particular,

universal coverings of compact aspherical manifolds are uniformly contractible.

An essential property of these X is a

bound on the filling radii of cycles $Y \subset X$ in terms of the absolute filling radii of these cycles.

In fact, a standard *induction by skeletons extension* argument shows the following.

★ Let W be a polyhedral space, $Y \subset W$ a polyhedral subspace and let $\varphi : W \to X$ be a continuous map.

If X is uniformly contractible, then φ extends to a continuous map $\Phi : W \to X$, such that the distances from the points $\Phi(w)$ to the image of φ are bounded

[228] It is, certainly well known. I apologize to the author for not being able to find his/her article.

Notice, however, that this fact is easy in our case, where $Sc(X) > 0$, since all complete minimal surfaces necessarily are either spherical or planar in these X.

by

$$dist(w, \varphi(Y) \leq D(dist(v, Y)),$$

where $D(d)$ is a continuous function that depends only on the contractibility control function $R(r)$ of X. [229]

The following immediate corollary to ★ will be used below for manifolds X of dimension $n = 4$.

★$_1$ ***Codimension 1 Filling Lemma.*** Let X be an n-dimensional orientable pseudomanifold a proper metric and let $U_1 \subset U_2 \subset ... \subset U_i \subset ... \subset X$ be an exhaustion of X by compact sub-pseudomanifolds with boundaries.

For instance, X can be a *complete Riemannian manifold* exhausted by compact domains with smooth boundaries.

X is uniformly contractible, then the absolute filling radii of the boundaries of U_i tend to infinity:

$$filrad(\partial U_i) \to \infty \text{ for } i \to \infty.$$

Proof. Let $S \subset X$ be an infinite path, i.e. a curve, issuing at a point $x_0 \in U_1$ and tending to infinity. Since S has *non-zero* intersection indices with the boundaries of all X_i, the boundary ∂U_i can bound in its D-neighbourhood only if $D < dist(x_0, \partial U_i)$. Since $dist(x_0, \partial U_i) \to \infty$ so does D and the proof follows.

Now we recall that, according to compact exhaustion corollary (**[1]** section 3.7.2), that complete Riemannian manifolds X with $Sc(X) > \sigma > 0$ it can be exhausted compact smooth domains U_i, the boundaries $Y_i = \partial U_i$ of which admit $\mathbb{T}^{\rtimes 1}$-extension $Y_i \rtimes \mathbb{T}^1$ with

$$Sc(Y_i \rtimes \mathbb{T}^1) \geq \frac{\sigma}{2}.$$

If $dim(X) = 4$ and $dim(Y_i) = 3$. then all these Y_i have their filling radii bounded by

$$fillrad(X) < \frac{18\pi}{\sqrt{\sigma/2}}$$

by corollary (G′) from 3.10.1.(corrected!!!!!!!!!!)Hence,

[4D]A. *complete uniformly contractible 4-manifolds X can't have $Sc(X) \geq \sigma > 0$.*

This, applied to the universal coverings of compact manifolds yields

[4D]B. Chodosh-Li 4D Theorem. *No compact aspherical 4-manifold admits a metric with $Sc > 0$.*

[4D]C. ***Generalization-Exercise.*** Let $\underline{X}$ be an orientable *uniformly rationally acyclic* four dimensional pseudomanifold, which means that the rational homology inclusion homomorphisms between the balls around all points $x \in X$,

$$H_i(B_x(r); \mathbb{Q}) \to H_i(B_x(R); \mathbb{Q})$$

[229] Consult [G(filling) 1983], [G(aspherical) 2020] for basics on filling and uniform contractibility and see [Katz(systolic geometry) 2017], [Guth (waist) 2014], [Wenger(filling) 2007], [DFW(flexible) 2003], [Dranishnikov(asymptotic) 2000]. [Dranishnikov(macroscopic) 2010], [Dranishnikov (large scale) 1999]. [Dra-Kee-Usp(Higson corona) 1998] and section 7 for related topics.

vanish for all $i = 1, 2, ...$ and $R \geq R_{\mathbb{Q}}(r)$ for some (acyclicty control) function $R_{\mathbb{Q}}(r)$.

Let X be a complete orientable Riemannian manifold and $f : X \to \underline{X}$ be a proper 1-Lipschitz map with *non-zero* degree.

Show that there exist constants $C, R_0 > 0$, such that

the minima of the scalar curvature of X on concentric balls $B(R) = B_{x_0}(R) \subset X$ around a point $x_0 \in X$, satisfy

$$\min_{x \in B(R)} Sc(X, x) \leq \frac{C}{R^2} \text{ for all } R \geq R_0.$$

Hint. Adapt the above proof to maps $X \to \underline{X}$ similarly to how this is done in [G(aspherical) 2020].

Remark. This implies that if X is a compact orientable 4-manifold with $Sc(X > 0)$, then continuous maps from X to *aspherical 4-dimensional pseudo-manifolds* send the rational fundamental homology class of X to zero. But this remains unknown for maps from these X to aspherical spaces in general.

Let us prove another corollary to ★ needed which will be used in dimension $n = 5$.

★$_2$ ***Codimension 2 Filling Lemma.*** Let X be an n-dimensional orientable pseudomanifold where the *singular locus of* X, (where it is *not* a manifold) has *codimension* 3, i.e. the links of the codimension 2 faces are connected and let X be endowed with a proper path metric with respect to which X is *uniformly contractible.*

Then, for all $R > 0$, there exits a proper piecewise linear 1-Lipschitz map $\Psi = \Psi_R : X \to \mathbb{R}^2$, such that

($\star$) *all orientable codimension 2-sub-pseudomanifolds Y, which are contained in the the pullback $\Psi^{-1}(B(R))$ of the R-ball $B(R) = B_{\mathbf{0}}(R) \subset \mathbb{R}^2$ and which are*

homologous in $\Psi^{-1}(B(R))$ to the pullbacks
$\Psi^{-1}(t) \subset X$, of regular points $t \in B(R)$ of Ψ, [230]

have their absolute filling radii bounded from below by

$$fillrad(Y) \geq r(R),$$

*where $r(R)$ is a continuous function (*which depends on the contractibility control function of X), *such that*

$$r(R) \to \infty \text{ for } R \to \infty.$$

Proof. The uniform contractibility of pseudomanifolds X implies that their Uryson 1-width are infinite

$$width_1(X) = \infty;$$

otherwise, the hypersurfaces in X would have bounded width, hence their filling radii as well in contradiction with ★$_1$.

[230] These points are dense in $\mathbb{R}^2$ and their pullbcks $\Psi^{-1}(t) \subset X$ are compact sub-pseudomanifolds in X, which, for $t \in B(R)$, are all mutually homologous in $\Psi^{-1}(B(R))$, i.e. represent the same class in the group $H_{n-2}(\Psi^{-1}(B(R)))$.

Next, by lemma (C) from the previous section, there exists closed curves $S \subset X$ with arbitrary large maximal $\square$-widths D, and let $\Psi_\square : X \to \mathbb{R}^2$ be the corresponding maps delivered by lemma (C),which, recall, is $\sqrt{2}$-Lipschitz and which sends S onto the boundary of the square $[0, D]^2 \subset \mathbb{R}^2$ with degree 1.

Scale this map by $\frac{1}{\sqrt{2}}$, shift it to move the center of the square to the origin $\mathbf{0} \in \mathbb{R}^2$ and take the resulting map for Ψ.

Since X is contractible, the curves $S = S_D$ bound orientable surfaces $\Sigma = \Sigma_D$ which have non-zero intersection indices with Y. Therefore, if r is much smaller then D, yet goes to infinity along with D, then the filling radii of $Y \subset \Psi^{-1}(B)$ also tend to infty. Q.E.D.

Now, if X is a Riemannian n-manifold with $Sc(X) \geq \sigma > 0$, then, by codimension 2 corollary [2] from section 3.7.2, it contains submanifolds $Y \subset \Psi^{-1}$ which admit $\mathbb{T}^\rtimes$ extensions $Y^\rtimes = Y \rtimes \mathbb{T}^2$ with $Sc(Y^\rtimes) \geq \frac{\sigma}{2}$, which for $n = 5$ and $dim(Y) = 3$, have filling radii uniformly bounded by (G′) from the previous section; hence,

[5D]A. *complete uniformly contractible 5-manifolds X can't have $Sc(X) \geq \sigma > 0$.*

Accordingly, one has the following

[5D]B. **$5D$-Non-asphericity Theorem.** *No compact aspherical 5-manifold admits a metric with $Sc > 0$.*

Remarks, Generalizations, Problems. (a) Albeit these [5D]A&B *imply* [4D]A&B (for $X^4 \rightsquigarrow X^5 = X^4 \times \mathbb{R}^1$) the mapping versions $X \to \underline{X}$ of them, [4D]C and [5D]C,[231] are *formally independent*, due to the codimension 3 condition on singularities of $\underline{X}$ for $dim(\underline{X}) = dim(X) = 5$ that is needed for a homological definition of the linking numbers between curves $S \subset X$ and codimension two sub-pseudomanifolds $Y \subset X$.

(b) This kind of "dual linking" appears in section 1.2 of [G(foliated) 1991] for the purpose of "trapping" minimal foliations and also in §$9\frac{3}{11}$ [G(positive) 1996], where Y is a circle, for the proof of enlargeability and the Novikov conjecture for 3-manifolds.

In the present context, Chodosh and Li use it for their proof of non-asphericity of 4-manifolds with $Sc > 0$ (enlargeability and the Novikov conjecture remain problematic for $n \geq 4$), where Y is a surface and the bound on $filrad(Y)$ (in terms which I don't quite understand) was derived in the first version of [Chodosh-Li(bubbles) 2020] from the area bound due to Zhu.

Then Chodosh-Li's linking idea, combined with the $\mathbb{T}^\rtimes$-stabilized bound on widths of 3-manifolds, was applied to $n = 5$ in [Chodosh-Li(bubbles) 2020] (I didn't quite follow how this is done in their paper) and in [G(aspherical) 2020], where it is proved that

complete uniformly rationally acyclic (e.g. the universal coverings of compact aspherical) Riemannian manifolds $\underline{X}$ of dimension 5 can't be 1-Lipschitz dominated[232] *with degrees $\neq 0$ by Riemannian manifolds X with $Sc(X) \geq \sigma > 0$.*

(c) To extend the linking argument to $n = 6$ one needs, as it is explained in [G(aspherical) 2020], either a proof of a *universal bound on the filling radii of 4-manifolds with $Sc \geq \sigma > 0$* (this remains conjectural), or the existence of closed *surfaces Σ (instead of curves S) in uniformly contractible manifold X,*

[231] The statement and the proof of this is left to the reader.

[232] See section 1.5 for the definition.

with filling radii $fillrad(\Sigma, X) \geq \rho$, *for all* $\rho > 0$, i.e. non-homologous to zero in their ρ-neighbourhoods in X. (This remains problematic even for the universal covers of compact manifolds.)

At the present moment, one has only limited results for $n \geq 6$ available along these lines, e.g.

(d) non existence of metrics with $Sc > 0$ on closed aspherical manifolds X of dimension $n \geq 5$, the *fundamental groups of which contain subgroups isomorphic to* $\mathbb{Z}^{n-4}$, see section 7.5.

(e) A closer look at the above argument shows the following.

Let $\underline{X}$ be an orientable n-pseudomanifold with a proper (bounded subsets are compact) metric. If $\underline{X}$ is uniformly contractible (uniformly rationally acyclic will do) and if either $n = 4$ or if $n = 5$ and the singularity of $\underline{X}$ has codimension 3 (or more).

Then, if complete Riemannian n-manifold X *with* $Sc(X) \geq \sigma > 0$ *admits a proper* 1*-Lipschitz map* $f : X \to \underline{X}$ *then* $\inf_{x \in X} Sc(X, x) \leq 0$.

Moreover,

the scalar curvature of X*, assuming it is positive, can't decay subquadratically, or even slow quadratically: one can't have*

$$Sc(X, x) \geq \frac{C}{dist(x, x_0)^2}$$

for a fixed point $x_0 \in X$*, all* $x \in X$ *with* $dist(x, x_0) \geq 1$ *and a positive constant* $C = C(\underline{X})$.

Corollary. *Compact aspherical manifolds of dimensions 4 and 5 with punctures admit no complete metrics with* $Sc \geq \sigma > 0$.

Question *Are there complete metrics with* $Sc > 0$ *on these punctured manifolds?*

[5E]. Classification of Non-aspherical 4- and 5-Dimensional Manifolds with $Sc > 0$. The classification theorem for 3-manifolds with positive scalar curvatures was generalized in [Chodosh-Li-Liokumovich (classification) 2021] as follows.

● Let X be a closed connected Riemannian n-manifold with *infinite* fundamental group $\pi_1(X)$ and such that the higher homotopy groups $\pi_2(X)$,..., $\pi_{n-2}(X)$ *vanish.*

If X *admits a metric with* $Sc > 0$, then, assuming $n = 4$ or $n = 5$, a finite covering of X is homotopy equivalent to the *connected sum of several copies of* $S^{n-1} \times S^1$.

Let us extend the above non-asphericity arguments to classification and prove ● by observing the following.

1. The (obvious induction by skeleta) proof of the above ★ actually shows that

if a compact polyhedral space, e.g. a compact manifold X has *trivial* homotopy groups $\pi_2(X), ..., \pi_k(X)$, then the filling radii R of all m-dimensional submanifolds (and subpseudomanifolds, if you wish) of dimensions $m \leq k$ in the universal covering of X, say $Y \subset \tilde{X}$, are bounded in terms of their absolute filling radii r,

$$R \leq D(r) \text{ for } R = filrad(Y \subset X) \text{ and } r = filrad(Y)$$

and where $D = D_X = D_{X,k}$ is the iterated contractibility control function for l-dimensional subpolyhedra in $\tilde{X}$, for $l = 2, 3, k$.

2. Notice that,

unless the fundamental group of a compact manifold X is virtually free,

the conclusion of the codimension 2 filling lemma $\bigstar_2$ holds for $\tilde{X}$: that is,

there exists of a proper piecewise linear 1-Lipschitz map $\Psi = \Psi_R : \tilde{X} \to \mathbb{R}^2$, such that all orientable codimension 2-sub-pseudomanifolds Y, which are contained in the the pullback $\Psi^{-1}(B(R))$ of the R-ball $B(R) = B_0(R) \subset \mathbb{R}^2$ and which are homologous in $\Psi^{-1}(B(R))$ to the pullbacks $\Psi^{-1}(t) \subset X$, of regular points $t \in B(R)$ of Ψ, have their absolute filling radii bounded from below by

$$fillrad(Y) \geq r(R),\ r(R) \to \infty \text{ for } R \to \infty.$$

In fact, according to example (E′) in 3.10.1.

universal coverings of compact spaces X with *non-virtually free* fundamental groups $\pi_1(X)$ do contain circles with *arbitrarily large maximal $\square$-widths*

and the presence of such "large circles" was all we used in the proof of $\bigstar_2$. (In truth, we also needed these circles to be *homologous to zero*, which is automatic for $\tilde{X}$ is simply connected.)

Now, if $Sc(X) \geq \sigma > 0$, then, as earlier, the "large" codimension 2 cycle $Y \subset \tilde{X}$ delivered by the codimension 2 filling lemma can be represented by a submanifold Y' which is positioned close to Y and such that a $\mathbb{T}^{\rtimes}$-stabilization of it has $Sc \geq \frac{\sigma}{2}$.

Since, as we know,[233] the inequality $Sc(Y' \rtimes \mathbb{T}^2) \geq \sigma/2$ implies that the filling radius of Y' is bounded by $filrad(Y') \leq const/\sqrt{\sigma}$, it follows by contradiction that

($\star$) *the fundamental group $\pi_1(X)$ is virtually free.*

We conclude the proof with the following elementary topological lemma.

($\ast$) If a closed orientable manifold $\hat{X}$ of dimension n has *free fundamental group* and *zero* $\pi_2(X), ..., \pi_k(X)$, $k = n-2$, then it is homotopy equivalent to a connected sum of copies of $S^{n-1} \times S^1$. QED.

Exercises. (a_1) Show that the conclusion of ($\ast$) remains valid for $n < 2k$, e.g. for $k = 3$ and $n = 6$.

(If $n = 5$, then these manifolds are actually *diffeomorphic* to connected sums of $S^4 \times S^1$, see [Gadgil-Seshadri(isotropic)2008], [Kreck-Su](5-manifolds) 2017] and references therein.)

(a_2) Show that ($\ast$) also holds for orientable *pseudomanifolds* $\hat{X}$, the singular loci of which have codimensions 3.

(a_3) Formulate and prove a counterpart of (a_1) for pseudomanifolds with singular loci of codimensions l.

(b) Generalize $\bullet$ by replacing "*X admits a metric with $Sc > 0$*" with "*X can be dominated by a complete Riemannin manifold with $Sc \geq \sigma > 0$*".

Generalize further to manifolds X, *the universal coverings $\tilde{X}$ of which admit* such *Lipschitz dominations*, i.e. such that

[233] If $n = 4$ this follows from the $\mathbb{T}^*$-stable Bonnet-Myers diameter inequality proved in section 2.8 and if $n = 5$ this is stated in corollary (G′) in section 3.10.1.

there exist complete orientable Riemannin manifolds X' with $Sc(X') \geq \sigma > 0$ and quasi-proper (e.g. proper) 1-Lipschitz maps $X' \to \tilde{X}$ with non-zero degrees.

Then do the same for *pseudomanifolds* X, the singular loci of which have codimension 3.

3.11 Asymptotic Geometry with $Sc > 0$, Positive Mass Theorem and Penrose Inequality

Let us show that complete Riemannian n-manifold X with $Sc(X) \geq 0$ can't grow faster the Euclidean space $\mathbb{R}^n$, which is regarded as the cone over the unit sphere S^{n-1} with the Euclidean metric represented (in polar coordinates) is $g_{\mathbb{R}^n} = dr^2 + r^2 ds^2$.

1. ***Conical Example.*** Let Y be a Riemannian manifold of dimension $(n-1) \geq 2$ with $Sc(Y) \leq (n-1)(n-2) = Sc(S^{n-1})$ and let g be the Riemannian metric on $X = Y \times [0, \infty)$ asymptotic to a conical, one, namely

$$g = g(y,r) = dr^2 + \lambda r^2 dy^2 + g_o(y,r),$$

where $g_o(y,r) = o(1)$, or, in words, $g_o(y,r)$ converges to 0 for $r \to \infty$.

This, in the scale invariant terms, means that the differential dI of the identity map

$$I : (X, g) \to (X, dr^2 + \lambda r^2 dy^2)$$

converges to isometry for $t \to \infty$,

$$||dI(y,r)I| \to \text{ and also } ||(dI)^{-1}(y,r)I| \to 1,$$

that is I is $\lambda(r)$*-bi-Lipschitz for* $\lambda(r) \to 1$.

Granted the above, if $Sc(X) \geq 0$*, then* $\lambda \leq 1$.

Proof. The condition $g_o(y,r) = 0(r^2)$ is equivalent to the C^0-convergence of the ε scaled metric g to the background conical (scale invariant) metric

$$\varepsilon^2 g(y,r) \to dr^2 + \lambda r^2 dy^2 \text{ for } \varepsilon \to 0.$$

Hence, $Sc(dr^2 + \lambda r^2 dy^2) \geq 0$ by the C^0-closure theorem (section 3.1.3). But if $\lambda > 1$, then, for $dim(X) \geq 3$, the conical metric $dr^2 + \lambda r^2 dy^2$ on X has $Sc(dr^2 + \lambda r^2 dy^2) < 0$. QED.

2. ***Asymptotically Schwarzschild.*** Recall (see section2.6) that *the scalar curvature (of the space slice of the) Schwarzschild metric with mass* m,

$$g_{Sw_m} = g_{Sw} = \left(1 + \frac{2m}{r}\right)^4 g_{Eucl}.$$

is zero and that:

if $m > 0$, the metric g_{Sw_m} is defined on $\mathbb{R}^3$ minus zero, and it is complete,

if $m = 0$, this is the flat Euclidean metric;

if $m < 0$, then this metric is defined only for $r > m$ with a singularity ar $r = m$.

For all m, the metric g_{Sw_m} is asymptotically Euclidean (conical), where, g_{Sw_m} grows slightly slower then the Euclidean metric one and if $m < 0$ it growth slightly faster.

This seen by rewriting this metric as

$$g_{Sw_m} = \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2 ds^2$$

and computing the mean curvature of the r-spheres with respect to the Schwarzschild metric (see [Brewin(ADM) 2006]),

$$mean.curv(S_{Sw_m}) = \frac{2}{r}\left(1 - \frac{2m}{r}\right)^{\frac{1}{2}} = \frac{2}{r} - \frac{2m}{r^2} + O\left(\frac{1}{r^3}\right),$$

where $\frac{2}{r}$ is equal to the Euclidean mean curvature of the sphere $S^2(r)$.

Observe that the difference between the Euclidean and Schwarzschild metrics and their first derivatives in the Euclidean coordinates satisfy

$$g_{Sw_m} - g_{Eucl} = \frac{2m}{r} dr^2 - O\left(\frac{1}{r^2}\right)$$

and

$$\partial^1(g_{Sw_m} - g_{Eucl}) = \partial^1 g_{Sw_m} = \frac{\partial g_{Sw_m}}{dr} = \frac{2m}{r^2} dr^2 + O\left(\frac{1}{r^3}\right)$$

Now we are going to estimate the scalar curvature of a metric $g = g(s,r)$ that is asymptotic to g_{Sw_m}, where we start with the following.

Observation. Let a g_0 be a Riemannian metric in a neighbourhood $U_0 \subset \mathbb{R}^n$ of the origin $0 \in \mathbb{R}^n$, let $S \subset U_0$ be a smooth hypersurface passing through the origin and let g be another smooth Riemannian metric in U_0, which is ε-close to g_0 with its first Euclidean derivatives at the origin,

$$||g_0(0) - g(0)|| + ||\partial^1(g_0(0) - g(0))|| \le \varepsilon \le 1.$$

Then

the mean curvatures of S at the origin with respect to the two metrics satisfy:

$$|mean.curv_g(S,0) - mean.curv_{g_0}(S,0)| \le C\varepsilon,$$

where the constant $C > 0$ depends only on the g_0 and its fist derivatives at the origin, i.e.

$$C \le C_n(1 + ||g_0(0) - g_{Eucl}|| + ||\partial^1(g_0(0)||)$$

for a universal constant C_n.

Proof. Check it for metrics $g_0 = a_0^2 dx_i^2 + b_0^2 dy^2$ and $g = a^2dx^2 + b^2dy^2$ for smooth positive functions a_0, b_0, a, b in the (x,y)-plane and for the parabola $S = \{y = cx^2\}$; then reduce the general case to this special one.

3. Positive Mass Corollary. Let $g_0 = g_0(s,r)$ be a warped product metric on the cylinder $S^{n-1} \times [2m, \infty)$ written as

$$g_0 = (1 - \alpha(r))dr^2 + r^2 ds^2,$$

where $0 < \alpha(r) < m$ is a smooth function with $\sup_r \frac{d\alpha(r)}{dr} < \infty$. Let g_i be a sequence of smooth Riemannian metrics defined in neighbourhoods

$$V_i \subset S^{n-1} \times [2m, \infty)$$

of r_i-spheres

$$S_i = S^{n-1}(r_i) = S^{n-1} \times \{r_i\} \subset S^{n-1} \times [2m, \infty),$$

where $r_i \to \infty$ with $i \to \infty$, such that the differences between g_i and g_0 and their first derivatives measured with respect to the Euclidean metric $dr^2 + r^2 ds^2$ are asymptotically bounded as follows,

$$\frac{\|g_i(s, r_i) - g_0(r_i, s)\|}{\alpha(r_i)} \underset{i\to\infty}{\to} 0 \text{ and } \frac{\|\partial^1 g_i(s, r_i) - \partial^1 g_0(r_i, s)\|}{\alpha(r_i)} \to 0.$$

If a complete orientable (possibly disconnected) Riemannian spin n-manifold X contains closed smooth embedded hypersurfaces Σ_i which admits isometries $f_i \to \Sigma_i \to S_i$, which preserves their mean curvatures,

$$mean.curv(\Sigma_i, x) = mean.curv(S_i, f_i(x),$$

then X can't have non-negative scalar curvature

$$\inf_{x\in X} Sc(X, x) < 0.$$

Proof. Observe that the mean curvatures of S_i with respect to g_0 are

$$mean.curv_{g_0}(S_i) = (1 - \alpha(r))^{-\frac{1}{2}}.$$

Then, according to the above observation, mean curvatures of Σ_i are strictly bounded away from below by those of the Euclidean r_i-spheres,

$$(mean.curv_{g_i}(\Sigma_i)) > \frac{n-2}{r_i^2}$$

and the proof follows from the mean curvature spin-extremality theorem (○) in section 3.5) applied to spheres.

Example. If $\alpha(r) = \left(1 - \frac{2m}{r}\right)^{-1}$ the above applies to to complete manifolds with asymptotically Schwarzschild's metrics g and shows that positivity of the scalar curvature of g makes $m \geq 0$.

Worrisome Remark. Our assumptions on the asymptotics of g are suspiciously weak compared to the commonly used in the literature, e.g. by Schoen and Yau in [SY(positive mass) 1979]. This makes me wonder if I haven't make a silly mistake in my interpretation of "derivatives of metrics". [234]

4. C^0- *Variation.* Let the above neighbourhoods V_i be the annuli (bands) between the spheres of radii r_i and $c_i r_i$,

$$V_i = S^{n-1} \times [r_i, c_i r_i] \subset S^{n-1} \times [2m, \infty), \ c_i > 1.$$

If the difference $\|g_i(s, r) - g_0(r)\|$ divided by the width of such a band becomes *sufficiently small* on V_i

$$\frac{\|g_i(s, r_i) - g_0(r_i, s)\|}{(c_i - 1) r_i} \leq \varepsilon_i,$$

[234] Only while preparing these notes, I attempted to penetrate the meaning of these mysterious "derivatives", the geometry of which still remains above my understanding. Probably, if these have any meaning, it should reside with physics (which I don't know) rather than with geometry.

then, regardless of any bound on the derivatives of g_i, such a band (V_i, g_i) contains a μ-bubble $S_{i,min} \subset V_i$ for a (density) function $\mu_i(s,r)$, which is close to the g_0-mean curvature of the spheres $S^{n-1}(r)$

$$|\mu_i(s,r) - mean.curv_{g_0}(S^{n-1}(r)| \leq \delta_i$$

For instance if g_0 is the Schwarzschild metric with mass $m < 0$, and if ε_i is very small, e.g. $o(r_i^{-4})$, $r_i \to \infty$, then the argument, as in the proof of the approximation corollary in §5$\frac{5}{6}$ from[G(positive1996],[235] based on Llalrull's inequality, shows that $\inf Sc(g) \leq 0$, and it is not hard to show that the mean curvature extremality theorem allows the same conclusion with $\varepsilon_i = o(r_i^{-1})$.

About Rigidity. Our argument, unlike these by Schoen-Yau and by Witten, is poorly adapted to the case, where g is asymptotically close, even when it very close, to the Euclidean metric, that is Schwarzschild with the mass $m = 0$. Apparently, rigidity of this kind is hard to derived from the geometry of finite object without passing to the infinite limit at an earlier stage of the argument.

Remark/ Questions. What happens to twisted harmonic spinors (best seen in Lott's rendition of the mean curvature spin-extremality theorem) that lie at the bottom of our argument in the limit for $r \to \infty$?

They don't seem to converge in an obvious way to Witten's spinors, but do they?

Does the positive mass rigidity holds for C^0-perturbations of the Euclidean metric?

Although no available techniques is capable to prove this even for very fast decay of $||g - g_0||$, we formulate the following.

Euclidean C^0-Rigidity Conjecture. If a smooth Riemannian metric g on $\mathbb{R}^n$

(a) satisfies

$$||g(x) - g_{Eucl}(x)|| = o\left(\frac{1}{||x||}\right), \ x \to \infty,$$

or

(b) if the identity map

$$(\mathbb{R}^n, g) \to (\mathbb{R}^n, g_{Eucl})$$

is λ-bi-Lipschitz for some $\lambda < \infty$ and the difference of the two distance functions

$$dist_g(x_1, x_2) - dist_{g_{Eucl}}(x_1, x_2)$$

is bounded, on $\mathbb{R}^n \times \mathbb{R}^n$, then either

$$\inf_{x \in \mathbb{R}^n} Sc(g(x)) < 0,$$

or g is Riemannian flat.

(If g satisfies (a) and is everywhere C^0-close to the Euclidean g_{Eucl}, one may try the Hamilton-Ricci flow.)

Admission. It is unclear, not even conjecturally, how close these sufficient rigidity conditions (a) and (b) are close to necessary ones..

[235] This argument was motivated by trying to geometrically understand Min-Oo's hyperbolic positive mass theorem.

5. *On History and Recent Developments.* The following special case of the positive mass conjecture (unsolved by that time) was emphasized by Robert Geroch in his expository article [Geroch(relativity) 1975] for geometers.

The Euclidean metric on $\mathbb{R}^n$ admits no compactly supported perturbations with increase of the scalar curvature.

Moreover,

If a metric g on $\mathbb{R}^n$ with $Sc(g) \geq 0$ is equal to g_{Eucl} outside a compact subset in $\mathbb{R}^n$, then $(\mathbb{R}^n, g)$ is isometric to $(\mathbb{R}^n, g_{Eucl})$.

This, of course, "trivially" follows from non-existence of non-flat metrics with $Sc \geq 0$ on tori, since compactly supported perturbations of the flat metric on $\mathbb{R}^n$ yield similar perturbations of flat tori. [236]

Originally Schoen and Yau directly proved a stronger *positive mass/energy theorem*, that claims positivity of the *ADM-mass*,[237] which means that the

total (i.e. integral) mean curvature of the Euclidean spheres $S^2(R)$ with respect to g, is bounded, for large $R \to \infty$, by $8\pi R$. [238]

Two years later, Schoen and Yau extended their argument, based on non-compact minimal surfaces, to manifolds of dimensios $n \leq 7$, while Witten suggested a proof applicable to spin manifolds of all dimensions.

Witten's argument, that uses perturbations of invariant (non-twisted) harmonic spinors on $\mathbb{R}^n$, was worked out in details by Bartnik and it was adapted by Min-Oo to hyperbolic spaces.

Later, Lohkamp found a (relatively) simple reduction of the general, and technically more challenging, case of the positive mass theorem to that of compactly supported perturbations, thus reducing the problem to $Sc \ngtr 0$ on tori.

Most recently, the positive mass theorem was extended to a class of incomplete manifolds. (See [Lesourd-Unger-Yau(arbitrary ends) 2021], where there are references to the earlier work by these authors.) [239]

Problems. What are other (homogeneous?) Riemannian spaces that admit no (somehow) localised deformations with increase of the scalar curvatures?

What are most general asymptotic conditions on such deformations that would allow their localization?

6. Penrose Inequality. Recall that the Schwarzschild metric with mass $m > 0$,

$$g_{Sw_m} = \left(1 + \frac{2m}{r}\right)^4 g_{Eucl},$$

defined in the 3-space minus the origin, is invariant under the the (conformal)

[236] The reduction to tori is *amazingly* simple, where this "amazing" brings it far from "trivial".

[237] In their paper [SY(positive mass) 1979] the authors refer to some earlier results, e.g. to Jang, P.S.: J. Math. Phys. 1, 141 (1976), but its hard to say what's in there since it is not openlc available on line.

[238] This interpretation of the ADM-mass is explained in [Brewin(ADM) 2006], where the autor referres to Brown and York for the origin of this idea.

[239] We don't even attempt to convey the basics of physics and mathematics behind the positive mass/energy idea, with dozens(hundreds?) papers dedicated to it, besides the early ones we mentioned: [SY(positive mass) 1979], [Witten(Positive Energy) 1981], [Bartnik(asymptotically flat) 1986], [Min-Oo(hyperbolic) 1989], [Lohkamp(hammocks) 1999]; we refer to the survey [Herzlich(mass) 2021] and to *Positive energy theorem* in Wikipedia for an overview of this subject matter.

reflection of $\mathbb{R}^3$ around the sphere $S^2(\rho) \subset \mathbb{R}^3$ of radius $\rho = \frac{m}{2}$, that is

$$(s, r) \mapsto \left(s, \frac{\rho^2}{r}\right).$$

This show that he Schwarzschild metric is complete and that the sphere $S^2(\rho)$ is totally geodesic in geometry of g_{Sw}, with area

$$area_{g_{Sw_m}}(S^2(\rho) = \pi\rho^2 \left(1 + \frac{\rho}{\rho}\right)^4 = 16\pi m^2.$$

In 1973 Penrose formulated in [Penrose(naked singularities) 1973] a conjecture concerning black holes in general relativity with an evidence in its favour, that would, in particular imply the following.

Special case of the Riemannian Penrose Inequality. Let X be complete Riemannian 3-manifolds with compact boundary $Y = \partial X$, such that

• X is isometric at infinity to the Schwarzschild space of mass m at one of its two ends at infinity;

• the scalar curvature of X is everywhere non-negative: $Sc(X) \geq 0$;

• the boundary Y of X has zero mean curvature;[240]

• no minimal surface in X separates a connected component of Y from infinity.

Then the area of $Y = \partial X$ is bounded by the mass of the Schwarzschild space as follows.[241]

$$area(Y) \leq 16\pi m^2.$$

.

This, in a greater generality was proven by Hubert Bray in [Bray(Penrose inequality) 2009].

3.12 Extensions and Fill-ins with $Sc > 0$

The positive mass/energy results from in the previous two sections concerning asymptotically flat and asymptotically hyperbolic spaces, as well as sharp bounds on the size of mean convex hypersurfaces from section 3.5 are solutions of special cases of the following two general problems.

A. ***Extension Problem for $Sc \geq \sigma$.*** Let X be a smooth manifold with a boundary $Y = \partial X$, , let h be a Riemannin metric on Y and let $\sigma(x)$ and $\mu(y)$ be smooth functions on X and on Y.

What are necessary and what are sufficient conditions for the existence of a complete (if X is non-compact) Riemannin metric g on X, which extends h,

$$g_{|Y} = h,$$

with respect to which the mean curvature of $Y \subset X$ is equal to μ,

$$mean.curv_g(Y) = \mu,$$

[240]It suffices to assume that the the boundary is *mean convex*, i.e. its mean curvature relative to the normal field pointing outward is positive.

[241]This version of the Penrose conjecture is taken from the modern literature. It is unclear, at least to the present author, when, where and by whom an influence of positivity of scalar curvature in 3D on geometry of surfaces, which was, probably, known to physicists since the early 1970s (1960s?) was explicitly formulated in mathematical terms for the first time.

and such that

$$Sc(X,x) \geq \sigma(x)?$$

B. ***Fill-in Problem for*** $Sc \geq \sigma$***.*** Let $Y = (Y,h)$ be a Riemannian manifold and $\mu(y)$ be a smooth function on Y.

Under what condition(s) does there exist, for a given number σ, a complete Riemannian manifold $X = (X,g)$ with $Sc(g) \geq \sigma$ with boundary $\partial X = Y$, such that

$$g_{|Y} = h \text{ and } mean.curv_g(Y) = \mu,$$

and where, if Y is compact, one may (or may not) require that X is also compact?

3.12.1 Construction of Extensions of Metrics with $Sc > 0$

Prior to enlisting known obstruction to extensions and fill-ins with $Sc > 0$ in the next section, let is describe known instances of existence of such extensions and formulate several questions.

Remarks. (a) One could, instead of the *Bartnik data* (h,μ) on Y, prescribe a germ $g_\circ$ of a Riemannin metric on an infinitesimal neighbourhood of Y in X, since, by the proof Miao's gluing lemma in section 1.4, a metric g_0 on X with the same Bartnik data on Y several as $g_\circ$ can be deformed to g with the same germ at Y as $g_\circ$ without decease of the scalar curvature.

(b) Following the general logic of the scalar curvature problems, one is concerned not only with the shear existence of metrics g, (manifolds $X = (X,g)$ in the case **B**), but with the space of all g which have given Bartnik data on Y and $Sc(g) \geq \sigma$.

Also, one may ask for a metric g with some its metric invariant(s) (e.g. the hyperspherical radius) bounded from below.

(c) If one drops μ from Bartnik Data $(h(y)\mu(y)$ then one expect no constraint for $Sc(g)$ on X at all, where a recent definite result in this regard, due to Yuguang Shi, Wenlong Wang, Guodong Wei in [SWW(total mean) 2020] (responding to "embarrassing question" from an earlier version of this manuscript) is as follows.

SWW Extension Theorem. *All smooth Riemannin metrics h on the boundary $Y = \partial X$ of a compact n-manifold X, extend to metrics g on X with $Sc(g) > 0$.*

The main technical ingredient of the proof is the following,

SWW Lemma. Let h_0 and h_1 be smooth Riemannin metrics on a compact Riemannin manifold Y, and let M_1 be a constant.

If $h_1 > h_0$, then then there exists a smooth metric $g_\circ$ on the cylinder $Y \times [0,1]$ with $Sc > 0$, which *extends h_0 and $\lambda \cdot h_1$,*

$$g_\circ|_{Y\times\{0\}} = h_0 \text{ and } g_\circ|_{Y\times\{1\}} = doth_1.$$

and such that

the mean curvature of the 1-end of the cylinder is bounded from below by M_1,

$$mean.curv_{g_\circ}(Y \times \{1\} \subset Y \times [0,1]) \geq M_1.$$

Derivation of the theorem from the lemma. By the *h-principle for open manifolds*, there exists a Riemannin metric g_1 on X with $Sc(g_1) > 0$.

Let $g_\circ$ be the metric on the cylinder $Y \times [0,1]$ delivered by the lemma for $h_0 = h$ and $h_1 = g_{|Y\times\{1\}}$, such that the $g_\circ$-mean curvature of the 1-end $Y\times\{1\} = Y$ of the the cylinder is greater than the minus g_1-curvature of the boundary $\partial X = Y$

$$mean.curv_{g_\circ}(Y,y) = mean.curv_{g_1}(Y,y)$$

Multiply the metric g_1 by $\lambda \geq 1$ from the lemma and isometrically attach the cylinder to $(X, \lambda \circ g_1)$. By Miao's gluing lemma 1.4, the (continuous Riemannian) metric on

$$X \sqcup_{Y\times\{1\}} Y \times [0,1]$$

can be approximated by a smooth metric g on $X \sqcup_{Y\times\{1\}} Y \times [0,1]$ with $Sc(g) > 0$ and, by an obvious identification $X = X \sqcup_{Y\times\{1\}} Y \times [0,1]$, the proof of the theorem follows.

On the Proof of the Lemma.[242] The metric $g_\circ$ is constructed in [SWW(total mean) 2020] in the form

$$g_\circ = g_u = (1-t)h_0 + th_1 + u^2 dt^2,$$

where the needed function $u = u(y,t)$ is obtained as a solution of a (non-linear parabolic) equation expressing $Sc(g_u)$ in terms of the function u and its first an second derivatives.)

σ-Remark. The metric $g_\circ$ delivered by the argument in [SWW(total mean) 2020] can be chosen with *arbitrarily large scalar curvature*

$$Sc(g_\circ) \geq \sigma \text{ for a given } \sigma > 0.$$

σ-**Corollary.** *All smooth Riemannin metrics h on the boundary $Y = \partial X$ of a compact n-manifold X, extend to metrics g on X with $Sc(g) > \sigma$ for all $\sigma > 0$.*

Proof. By the h-principle, one gets g_1 on X with $Sc(g_1) > \sigma$, where then the induced metric h_1 on $Y = \partial X$ can be made greater than a given h by the *(Nash)-Kuiper stretching construction.*

The following elementary proposition also yields SWW theorem (albeit only with small σ) via the gluing argument from [SWW(total mean) 2020] .

Weak SWW Lemma. There exists positive constants $\delta_\nu > 0$, for $\nu > 0$ and a family of smooth positive monotone increasing functions λ_ν on the segments $[0, \delta_\nu]$

$$\lambda_\nu(t),\ 0 \leq t \leq \delta_\nu,\ \nu \geq 0,\ \lambda_\nu(0) = 1$$

with the following property.

Let $\underline{h}_t$, $0 \leq t \leq 1$, be a smooth family of Riemannian metrics on a compact manifold Y. Then the scalar curvature of the metric

$$g_\nu = \lambda_\nu^2(t) \cdot \underline{h}_t + dt^2 \text{ on } X_\nu = Y \times [0, \delta_\nu]$$

[242] I want to thank Yuguang Shi who explained to me several points in the proof of this lemma and pointed out an error in my first version of the proof of the "weak lemma".

becomes arbitrarily large for large ν,

$$Sc(g_\nu) \geq \sigma = \sigma(h_t, \nu) \to \infty \text{ for } \nu \to \infty$$

and also the mean curvature of the δ_ν-boundary of X becomes large

$$mean.curv_{g_\nu}(Y \times \{\delta_n u\}) \geq M = M(h_t, \nu) \to \infty \text{ for } \nu \to \infty.$$

Proof. Recall the function

$$\varphi_\nu(t) = \exp \int_{-\pi/\nu}^{t} -\tan \frac{\nu t}{2} dt, \quad -\frac{\pi}{\nu} < t < \frac{\pi}{\nu},$$

from section 2.4, let

$$\lambda^\circ_\nu(t) = \frac{\varphi_\nu(t)}{\varphi_\nu(t_0)}, \ t \in [t_0, t_1],$$

and

$$\lambda_\nu(t) = \lambda^\circ_\nu(t + t_0), \ t \in [0, t_1 - t_0].$$

Now an elementary computation[243] shows that if

$$t_0 = -\tfrac{\pi}{\nu} + \tfrac{1}{\nu^3} \text{ and } t_1 = -\tfrac{\pi}{\nu} + \tfrac{1}{\nu^2},$$

then the family $\lambda_\nu(t)$, $t \in [0, \delta_\nu = t_1 - t_0]$ is the required one.

Remark. The lower bounds on the scalar curvature of g and on the mean curvature of $Y \times \{1\}$ in the sublemma depend only on the lower bounds on the scalar curvatures of the metrics h_t on Y and on the mean curvatures of the submanifolds $Y_t = Y \times [0,t] \subset [0,t]$ with respect to the metric $\underline{g} = h_t = dt^2$ on $Y \times [0,1]$. Thus the sublemma remains valid for non-compact manifolds, where the scalar curvatures of the metrics h_t and on the mean curvatures of the submanifolds $Y_t = Y \times [0,t] \subset [0,t]$ are bounded from below.

Corner Corollary to SWW Theorem.[244] Let $X = (X, g_0)$ be a smooth manifold with corners. Then X admits a metric g with $Sc(g) > 0$ and such that all codimension 1 faces are mean convex and all dihedral angles are bounded from above by given positive numbers.

Proof. It is obvious that there exists a Riemannin metric with $Sc > 0$ in a small neighbourhood $U \subset X$ of the boundary $\partial X \subset X$ with respect to which ∂X is mean convex with arbitrarily small dihedral angles. Then the theorem applies to a domain $X_0 \subset X$ with smooth(!) boundary $\partial X_0 \subset U$ and the proof is concluded with Miao's gluing lemma.

Exercise. Let $R : G_+(X) \to H(Y)$ be the restriction map, $g \mapsto g_{|Y}$, from the space $G_+(X)$ of metrics g on X with $Sc(g) > 0$ to the space $H(Y)$ of (all) Riemannian metrics h on Y. shows that *R is a Serre fibration,*

---

[243]The formulas one needs, collected in sections 2.1, 2.3,2.2, 2.4 are: Riemannian variation formula: $\frac{dh_t}{dt} = 2A_t^*$, Second Main Formula: $\frac{dA_t}{dt} = -A^2(Y_t) - B_t$, and Gauss' theorema egregium, while the relevant computation is sufficient to perform for the case of $h_t = h$, where h is a flat metric (as in example (c) in **??**) and then argue by continuity. In fact, a similar computation free argument can be applied to the metric with constant curvature 1 on S^2.

[244]A version of this was suggested in in section 6 in [G(boundary) 2019] as an approach to "Unproven (non-extendability with $Sc > 0$) Corollary", which we will prove by a different argument in section 5.8.1.

(It is not so clear if the Serre fibration property remains satisfied if $G_+(X)$ is replaced by a subspace $G_+(X, U_0, g_0)$ of metrics that are equal to a given g_0 away from a small neighbourhood $U_0 \subset X$ of $Y \subset X$. [245])

(Naive?) Questions. Let X be a compact manifold with a boundary.

(1) Does, assuming $n = dim(X) \geq 3$, (it may be safer to assume $n \geq 5$) the manifold X admits a Riemannin metric g such that

$$Sc(g) \geq \sigma \text{ and } mean.curv_g(\partial X) \geq M_-$$

for *given* $\sigma_+ > 0$ *and* $M_- < 0$*?*

Observe the following in this regard.

(i) If n=2, such a g seems to exist for *all* $\sigma_+ > 0$ and $M_- < 0$ *only* if X is homeomorphic to the disc, cylinder or the Möbius band.

(ii) It is obvious that g exists for all $\sigma_+ > 0$ and $M_- < 0$ if X contracts to the $(n-2)$-dimensional polyhedral subset $P \subset X$.

(iii) It is unclear if such metrics exist, for all $\sigma_+ > 0$ and $M_- < 0$, on the n-torus minus an open ball and/or on an X homeomorphic to a compact hyperbolic manifold with a totally geodesic boundary.

(iv) If such g don't always exist, then the supremum of $\frac{\sigma_+}{|M_-|}$, for which such a g does exist on an X, makes a non-trivial topological invariant of X, which, one can only dream of this , would assume several different values at certain X.

(v) This may be too good to be true, but this invariant does make sense for *Riemannian* manifolds $Y = (Y, h)$, where the above metric g must extend h and where the maximum of the ratios $\frac{\sigma_+}{|M_-|}$, where such a g exists is an interesting (is it?) invariant of (Y, h), evaluation of which may be possible for specific manifolds Y, such as compact symmetric spaces, for instance.

(2) Let X be a compact orientable manifold with two boundary components, say $\partial X = Y_0 \sqcup Y_2$ and let h_0 and h_1 Riemannian metrics on Y_0 and on Y_1 and let $f : Y_1 \to Y_0$ be a smooth strictly distance decreasing map ($\|df\| < 1$) of degree 1 (e. g. a diffeomorphism) and let $M_0 < 0$ and $M_1 > 0$ be two numbers such that $M_0 + M_1 < 0$.

Does the pair of metrics (h_0, h_1) extend to a metric g on X with $Sc(g) > 0$ and such that the g-mean curvatures of Y_0 and Y_1 are bounded from below by M_0 and M_1 respectively?

3.12.2 Obstructions to Fill-ins with $mean.curv \geq M$ and $Sc \geq \sigma$

I. BMN-Counter Example. Motivated by Min-Oo's conjecture to the contrary, Simon Brendle, Fernando C. Marques and Andre Neves constructed in [Bre-Mar-Nev(hemisphere) 2011] a C^2-small perturbation of the standard Riemannian metric on the hemisphere S^n_+, $n \geq 3$, that enlarges its scalar curvature while keeping unchanged the metric and the (zero) second fundamental form on the boundary sphere $S^{n-1} = \partial S^n_+$.

II. BM-Non-Perturbation Theorem Brendle and Marques proved in [Brendle-Marques(balls in S^n)N 2011] that small balls in S^n admit no such perturbations and conjectured that

there is a critical radius $r_n > 0$, such that

[245] It is easy to see if you replace $H(Y)$ by the quotient space $H(Y)/\mathbb{R}$ for the action of the multiplicative group $\mathbb{R}$ on metrics by $r : H \mapsto r \cdot h$.

if a compact Riemannin manifold X with a boundary has $Sc \geq n(n-1)$, and if the mean curvature $mean.curv\partial X$ is bounded from below by that of the r-ball $B^n(r) \subset S^n$, $r \leq r_n$, then X is isometric to this ball.

III. STEMW Total Mean Curvature Rigidity Theorem. Michael Eichmair, Pengzi Miao and Xiadong Wang generalized an earlier result by Yuguang Shi and Luen-Fai Tam[246] and proved the following.

Let $\underline{X} \subset \mathbb{R}^n$ be a star convex domain, e.g. a convex one, such as the unit ball, for example, and let X be a compact Riemannian manifold, the boundary $Y = \partial X$ of which is isometric to the boundary $\underline{Y} = \partial \underline{X}$.

If $Sc(X) \geq 0$ and if the total mean curvature of Y is bounded from below by that of $\underline{Y}$,

$$\int_Y mean.curv(Y, y)dy \geq \int_{\underline{Y}} mean.curv(\underline{Y}, \underline{y})d\underline{y},$$

then X is isometric to $\underline{X}$.

This is proven in the above cited papers by extending g (from a small neighbourhood of Y in X) to a complete asymptotically flat metric g_+ on $X_+ \supset X$ with $Sc(X_+) \geq 0$, where Y serves as the boundary of the closure of $X_+ \setminus X \subset X_+$, and such that

ADM-$mass(g_+) < 0$ *for* $\int_Y mean.curv(Y, y)dy > \int_{\underline{Y}} mean.curv(\underline{Y}, \underline{y})d\underline{y}$

and then applying the positive mass theorem, where, originally this was for $n \leq 7$. But this restriction, due to possible singularities on minimal hypersurfaces, may be now removed in view of the recent results by Lohkamp and Schoen-Yau.

Conjecture. Let X be a compact Riemannian manifold with $Sc \geq \sigma$. Then the integral mean curvature of the boundary $Y = \partial X$ is bounded by

$$\int_Y mean.curv(Y, y)dy \leq const,$$

where this $const$ depends on σ and on the (intrinsic) Riemannian metric on Y induced from that of $X \supset Y$.

IV. Non-Fill for Euclidean Hypersurfaces. It is shown in [SWWZ(fill-in) 2019], [SWW(total mean) 2020] among other things that a pointwise version of STEMW holds for *non-spin* Riemannian n-manifolds $X = (X, g)$ with boundaries $Y = \partial X$ which admit *smooth topological embeddings* to $\mathbb{R}^n$:

(A) if $Sc(X) \geq 0$, then the lower bound on the mean curvature of Y is bounded in terms of topology of Y and (geometry of) g,

$$\inf_{y \in Y} mean.curv(Y, y) \leq const(Top(Y), g).$$

Furthermore,

(B) if Y is diffeomorphic to S^{n-1} and the induced Riemannian metric $g_{|Y}$ on Y is *homotopic in the set of metrics on Y with $Sc > 0$ to one with constant sectional curvature*, then

$$\int_Y mean.curv(Y, y)dy \leq const'_n(g_{|Y}),$$

that confirms the above conjecture in a special case.

[246] See [EMW(boundary) 2009] and [Shi-Tam(positive mass) 2002]

This is proven by extending g from a small neighbourhood of Y in X to a complete asymptotically flat metric g^+ with $Sc \geq 0$, where Y serves as the boundary of the closure of $X_+ \smallsetminus X \subset X_+$ and such that the ADM mass of g^+ is negative provided *the mean curvature of (or its integral over) Y is sufficiently large.* Then the the positive mass theorem applies. ´

Remark. Probably, by incorporating Lohkamp reduction of the positive mass theorem to the flat at infinity case (see section 3.11) one can make g_+ *flat*, rather than only *asymptotically flat with mass* ≤ 0 at infinity, where this may generalize to manifolds Y that are not necessarily diffeomorphic to S^{n-1}.

V. Pointwise non-Fill-in for Compact Y. Pengzi Miao found a simple derivation of the following version of (A) from the SWW extension theorem for *all* Y [Miao(nonexistence of fill-ins) 2020]:

$$\inf_{y \in Y} mean.curv(Y, y) \leq const(Top(X), g).$$

Proof. Given $X = (X, g)$ with $Sc(g) \geq 0$ and $mean.curv_g(Y) \geq \mu_+$, $Y = \partial X$ let X' be a connected sum of X with the n-torus and let g' be a metric with $Sc > 0$ such the restriction of g' to $Y = \partial X' = \partial X$ is equal to the restriction of $g^{|Y}$, where the existence of such a g' is guaranteed by the SWW extension theorem.

Observe that the supremum $\mu'_* = \sup_{g'} \inf_{x' \in Y} mean.curv_{g'}(Y, x')$ depends on the topology X and on the restriction of g to Y.

Also observe that the manifold $X \sqcup_Y X'$ obtained by gluing X and X' along Y admits no metric with $Sc > 0$ by the Schoen-Yau theorem.

But if $\mu_+ + \mu'_* > 0$, the natural continuous metric $g\&g'$ on $X \sqcup_Y X'$ can be smoothed with $Sc > 0$ by Miao gluing theorem; hence, $\mu_+ \leq -\mu'_*$. QED.

VI. Another derivation of pointwise non-fill-in theorem from SWW extension theorem is with a use of the Corner Corollary from the previous section. Indeed, if the mean curvature of ∂Y is sufficiently large, one can modify the Riemannin metric on X (by attaching an external color to X along $Y \partial X$) keeping $Sc \geq 0$ and creating cubical corner structure on the boundary with dihedral angles $< \frac{\pi}{2}$, as in "Unproven Corollary" from section 6 in [G(boundary) 2019].

Then, by the reflection argument from section 3.1.1, the problem reduced to Schoen-Yau theorem on non-existence of metrics with $Sc > 0$ on manifolds which admits maps with non-zero degrees to tori.

VII. In the case of spin manifolds X a more precise non-fill inequality follows from the mean curvature spin-Extremality theorem in section 3.5, and a μ-bubble approach to the non-spin case is indicated in section 5.8.1.

Questions. Let X be compact n-manifold with boundary, let $Y_i \subset \partial X$ be be the connected components of the boundary. (For instance, X is the n-torus minus two open balls and $\sigma = 1$.)

(a) Given numbers σ and M_i, when does there exist a Riemannin metric g on X, such that $Sc(X) \geq \sigma$ and the mean curvatures of Y_i are bounded from below my M_i,

$$mean.curv_g(Y_i) \geq M_i?$$

(b) Let all Y_i be diffeomorphic to the sphere S^{n-1} and let, besides σ and M_i, we are given positive numbers κ_i.

When does there exist a Riemannin metric g on X, such that $Sc(X) \geq \sigma$, the induced metrics $g_{|Y_i}$ have constant sectional curvatures κ_i and

$$mean.curv_g(Y_i) \geq M_i?$$

(c) Let now Riemannin metrics g_i on Y_i be given. When does there exist a Riemannin metric g on X, such that $Sc(X) \geq$, $g_{|Y_i} - g_i$ and

$$mean.curv_g(Y_i) \geq M_i?$$

3.13 Manifolds with Negative Scalar Curvature Bounded from Below

If a "topologically complicated" closed Riemannian manifold X, e.g. an aspherical one with a *hyperbolic fundamental group*, has $Sc(X) \geq \sigma$ for $\sigma < 0$, then a certain "growth" of the universal covering $\tilde{X}$ of X is expected to be *bounded from above* by $const\sqrt{-\sigma}$ and accordingly, the "geometric size" – ideally $\sqrt[n]{vol(X)}$– must be *bounded from below* by $const'/\sqrt{-\sigma}$.

If $n = 3$ this kind of lower bound are easily available for areas of stable minimal surfaces of large genera via Gauss Bonnet theorem by the Schoen-Yau argument from [SY(incompressible) 1979].

Also Perelman's proof of the geometrization conjecture delivers a sharp bound of this kind for manifolds X with hyperbolic $\pi_1(X)$ and similar results for $n = 4$ are possible with the Seiberg-Witten theory for $n = 4$ (see section 3.16).

No such estimate has been established yet for $n \geq 5$ but the following results are available.

Ono-Davaux Hyperbolic Spectral Inequality.[247] Let X be a closed Riemannian manifold and let $\tilde{X} \to X$ be some Galois covering of X, e.g the universal covering, such that all smooth functions $f(\tilde{x})$ with compact supports on $\tilde{X}$ of X satisfy

$$\int_{\tilde{X}} f(\tilde{x})^2 d\tilde{x} \leq \frac{1}{\tilde{\lambda}_0^2} \int_{\tilde{X}} ||df(\tilde{x})||^2 d\tilde{x}.$$

(The maximal such $\tilde{\lambda}_0 \geq 0$ serves as the lower bound on the spectrum of the Laplace on the universal covering $\tilde{X}$ of X).

If $\tilde{X}$ is spin and if one of the following two conditions (A) or (B) is satisfied, then

$$[Sc/\tilde{\lambda}_0] \qquad \inf_{x \in X} Sc(X, x) \leq \frac{-4n\tilde{\lambda}_0}{n-1}.$$

Condition (A). The dimension of X is $n = 4k$ and the $\hat{\alpha}$-invariant from section 3.2 (that is a certain linear combination of Pontryagin numbers called $\hat{A}$-*genus*) doesn't vanish.

Condition (B). The manifold $\tilde{X}$ is *hypereuclidean*: it properly Lipschitz dominate the Euclidean space, i.e. $\tilde{X}$ is orientable and it admits a proper distance decreasing map to $\mathbb{R}^n$ with *non-zero* degree.

Idea of the Proof. By *Kato's inequality* (and/or by *the Feynman-Kac formula*, see 6.1.2), the lower bound on $\tilde{\lambda}_0$ implies a similar bound on the Bochner

[247] See [Ono(spectrum) 1988], [Davaux(spectrum) 2003].

Laplacian ∇^2 on $\tilde{X}$, hence, a corresponding bound on the (untwisted) Dirac operator expressed by the SLW(B)-formula $\mathcal{D} = \nabla^2 + \frac{1}{4}Sc$.

This, confronted with the L_2-index theorem, yields *Condition (A)* and *Condition (B)* for n even follows by similar argument for $mathcalD$ on $\tilde{X}$ twisted with suitable almost flat bundles, while the sharp inequality for odd n needs a an odd dimensional version of the L_2-index theorem and a delicate analysis of the spectral flow for a family of Dirac operators (see [Davaux(spectrum) 2003]).

Remarks. (a) The inequality $[Sc/\tilde{\lambda}_0]$ is sharp: if X has constant negative curvature -1, then

$$-n(n-1) = Sc(X) = \frac{-4n\tilde{\lambda}_0}{n-1}$$

for $\tilde{\lambda}_0 = \frac{(n-1)^2}{4}$, that is the bottom of the spectrum of $\mathbf{H}^n_{-1} = \tilde{X}$.

(b) The rigidity sharpening of $[Sc/\tilde{\lambda}_0]$ is proved in [Davaux(spectrum) 2003] in the case A and it seems that a minor readjustment of the argument from this paper would work in the case B as well.

(c) Since the spectrum of the Laplacian is Lipschitz continuous under C^0-deformations of Riemannian metrics, the Ono-Davaux hyperbolic spectral inequality implies, for instance, that

if a metric g on a compact n-manifold X is λ-bi-Lipschitz, $\lambda \geq 1$, to a metric g_0 with sectional curvatures $\kappa \leq -1$, then

$$\inf_{x \in X} Sc(g,x) \leq -\frac{const_n}{\lambda^2},\ const_n > 0.$$

On the other hand, the spectrum of Δ drastically drops down, if for instance one takes connected sums of X with spheres S^n attached to X by long narrow "necks" by means of the thin surgery with only a minor perturbation of the infimum of the scalar curvature.

Non-Amenable Hypereuclidean Manifolds with $Sc \geq \sigma < 0$. Probably, the above bound on the scalar curvature in case (B) remains true for all complete Riemannin manifold $\tilde{X}$,with no spin assumptions and with no action of any (deck transformation) group on it.

Below is a result in this direction, which we formulate in geometric rather than analytic terms.

A Riemannian n-manifold X is called (uniformly) *α-non-amenable* if all compact smooth domains $U \subset X$ satisfy the *linear isoperimetric inequality with constant α,*

$$vol(U) \leq \alpha \cdot vol_{n-1}(\partial U).$$

It is easy to see that

if a *complete α-non-amenable* Riemannin n-manifold X has *bi-Lipschitz bounded local geometry,* i.e. all δ-balls in X are λ-bi-Lipschitz homeomorphic to the Euclidean δ-ball for some positive numbers $delta$ and λ depending on X, then X can be it exhausted by compact smooth domains V_i,

$$V_1 \subset V_2 \subset ... \subset V_i \subset ... \subset X$$

such that boundaries $Y_i \partial V_i$ satisfy

$$mean.curv(Y_i) \geq \alpha - \varepsilon_i, \text{ where } \varepsilon_i \to 0 \text{ for } i \to \infty.$$

Indeed, let $\mu_i(x)$ be a sequence of smooth functions on X, such that
$\star$ *all μ_i is very large* at a point $x_0 \in X$,
$\star$ all $\mu_i(x) < \alpha - \epsilon_i$ for x very far from x_0,
$\star$ the gradients of all $\mu_i(x)$ is very small at all $x \in X$,
$\star$ $\mu_i(x) \to \alpha$ for $i \to \infty$ and $x \to \infty$.

Then our conditions on X imply the existence of μ_i-bubbles $Y'_i = \partial V'_i \subset X$, where V_i *exhaust* X and where Y'_i can be smoothed to the required $Y_i = \partial V_i$ (compare with 1.5(C) in [G(Plateau-Stein) 2014]).

This, combined with *multi-width mean curvature inequality* from section 5.8.1, yields the following.

Rough Negative bound on Sc(X). Let X be an α-non-amenable *hypereuclidean* Riemannian n-manifold. *If $n \le 7$, then the infimum of the scalar curvature of X is bounded by α as follows*

$$\inf_{x\in X} Sc(X,x) \le -const_n \alpha^{\frac{2(n-1)}{n}}$$

for some $const_n > 0$.

Let us indicate an application of this to manifolds X discretely a cocompactly acted upon by , countable groups Γ, e.g. to universal coverings of compact manifolds, where Γ *uniformly no-amenable*, where

a finitely generated group Γ is uniformly non-amenable if there exists an $\underline{\alpha} = \underline{\alpha}(\Gamma) > 0$, such that, for all symmetric finite generating subset $\Delta \subset \Gamma$ the cardinalities of all finite subsets $V \subset \Gamma$ are bounded by the cardinalities of their Δ-boundaries,

$$card(V) \le \underline{\alpha} \cdot card(\Delta \cdot S \setminus S.$$

Example. Non-virtually solvable subgroups of the linear group $GL(N,\mathbb{C})$ are uniformly non-amenable (see [Breuillard-Gelander(non-amenable) 2005] and references therein)

To use this, we observe that if a Riemannin n-manifold is discretely and isometrically acted upon by such a Γ with compact quotient X/Γ, then the isoperimetric constant $\alpha = \alpha(X)$ is bounded from below in terms of $\underline{\alpha}(\Gamma$ and a bound on the local Lipschitz geometry of X.

Namely, given numbers $\lambda > 0$, d and $\underline{\alpha} > 0$, there exists $\alpha = \alpha_n(\lambda,, d\underline{\alpha}) > 0$, such that if all unit balls in X are λ bi-Lipschitz homeomorphic to the unit Euclidean ball, and the diameter of the quotient space is bounded by $diam(X/\Gamma) \le$,

then, by an easy argument,

the inequality $\underline{\alpha}(\Gamma) \ge \underline{\alpha}$ implies that X is α-non-amenable.

Remark/Conjecture. The conditions on the Lipschitz geometry and the diameter are unpleasantly restrictive. Conjecturally all one needs is a *bound on the volume of X/Γ.*

The last theorem in tis section formulated below, was, historically, the first result on the geometry of $Sc \ge \sigma$ for *negative σ*.

Thus,

if X is hypereuclidean, then the infimum of the scalar curvature $Sc(X)$ is bounded by a strictly negative constant which depends only on Γ and the bound on the local Lipschitz geometry of X.

Exercise. Show that the universal covering $\tilde{X}$ of the n-torus X with an arbitrary Riemannian metric can be exhausted by *over-cubical* domains $V_i \subset \tilde{X}$ *with corners*, i.e. such that they admit face preserving (*corner proper* in terms of section3.18) maps

$$f_i : V_i \to [0,1]^n$$

with *degree* 1 and such that all $(n-1)$-faces of all V_i have *positive* mean curvatures and the *dihedral angles* of all V_i along the $(n-2)$-faces *are* $\le \frac{\pi}{2}$.

Hint. Cut the manifold X by a minimal hypersurface in the homology class of $\mathbb{T}^{n-1} \subset \mathbb{T}^n \underset{homeo}{\simeq} X$, then cut the resulting band by a sub-band homologous to $\mathbb{T}^{n-2} \times [0,1]$ etc. If $n \le 7$ this terminates in a cubical $V_1 \subset \tilde{X}$, and by applying the same procedure to finite coverings of X with fundamental subgroups $i \cdot \mathbb{Z}^n \subset \mathbb{Z}^n = \pi_1(X)$ we obtain an exhaustion of $\tilde{X}$ by over-cubical V_i with *minimal* $(n-1)$-faces and all *dihedral* angles $\pi/2$.

These V_i may have, however, not very smooth faces and an extra work is needed to smooth them.

And if $n \ge 8$, such V_i , come, in general, with more serious singularities, but one can smooth them keeping the $(n-1)$-faces mean convex and the dihedral angles $\le \pi/2$, as it is done in [G(Plateau-Stein) 2014].

Remark/Conjecture. It is not impossible (but unlikely) that *all contractible* manifolds $\tilde{X}$ which admit *cocompact isometric group actions* also admit similar over-cubical exhaustions, where this seem quite realistic for *enlargeable* X.

Also other "large" manifolds $\tilde{X}$ without any group actions, e.g. complete simply connected manifolds with non-positive sectional curvatures admit such exhaustions or, at least, contain arbitrarily large man convex overtorical domains with dihedral angles $\le \pi/2$.

Question. What are possible values of dihedral angles of large *non-over-cubical* domains with corners in various manifolds?

For instance, it seem not hard to show in this regard that the 2-plane with a metric *bi-Lipschitz homeomorphic to the hyperbolic plane* can be exhausted by convex k-gons, for all $k = 2, 3, 4, ...$ with all *angles* $\le \varepsilon$ *for all* $\varepsilon > 0$.

Also it seems not impossible that, for *all convex polyhedral domains* $P \subset \mathbb{R}^n$, the above universal covering $\tilde{X} \to X \underset{homeo}{\simeq} \mathbb{T}^n$ can be exhausted by mean convex "*over P-domains*" V_i (admitting face respecting maps $V_i \to P$ with degrees 1), such that the *dihedral angles of all V_i are bounded by the corresponding angles in P,*

$$\angle_{kl}(V_i) \le \angle kl(P),$$

and where, moreover, unless X is Riemannin flat, one can find/construct such V_i with $\angle_{kl}(V_i) < \angle kl(P)$.

The last theorem in this section we state below was, historically, the first result on ıgeometry of $Sc \ge \sigma$ for $\sigma < 0$.[248]

Min-Oo Hyperbolic Rigidity Theorem. Let X be a complete Riemannian manifold, which is isometric at infinity (i.e. outside a compact subset in X) to the hyperbolic space $\mathbf{H}^n_{-1}$.

If $Sc(X) \ge -n(n-1) = Sc(\mathbf{H}^n_{-1})$, *then* X *is isometric to* $\mathbf{H}^n_{-1}$.

[248]Strictly speaking, the first, for alI know, *topological-geometric* constraint on $Sc \ge \sigma < 0$ appears in [Ono(spectrum) 1988], but his argument resides within the realm of $Sc \ge 0$.

About the Proof. The original argument in [Min-Oo(hyperbolic) 1989], which generalizes Dirac-theoretic Witten's proof of the positive mass/energy theorem for asymptotically Euclidean (rather than hyperbolic) spaces, (see section 3.11) needs X to be *spin*.

But granted spin, Min-Oo's proof allows more general asymptotic (in some sense) agreement between X and $\mathbf{H}^n_{-1}$ at infinity.

If one wants to get rid of spin, one can use minimal hypersurfaces or μ-bubbles, where it is convenient, to pass to a quotient space $\mathbf{H}^n_{-1}/\Gamma$, where Γ is a *parabolic* isometry group isomorphic to $\mathbb{Z}^{n-1}$, and where the quotient $\mathbf{H}^n_{-1}/\Gamma$ is the *hyperbolic cusp-space*, that is $\mathbb{T}^{n-1}\times\mathbb{R}$ with the metric $e^{2r}dt^2+dr^2$.[249]

Then one applies the rigidity theorem for the flat metrics on tori with $Sc\geq 0$ to $\mathbb{T}^1$-*symmetrised stable μ-bubbles* in manifolds X *isometric to* $\mathbf{H}^n_{-1}/\Gamma$ *at infinity*, where these bubbles *separate the two ends* of X and where $\mu=(n-1)dx$. Thus one shows that

n-manifolds X with $Sc(X)\geq -n(n-1)$, which are isometric to $\mathbf{H}^n_{-1}/\Gamma$ at infinity, are, isometric to $\mathbf{H}^n_{-1}/\Gamma$ everywhere.[250]

Finally, a derivation of a Min-Oo's kind *hyperbolic positive mass theorem* without the spin condition from the rigidity theorem follows by an extension of the Euclidean Lohkamp's argument from to the hyperbolic spaces, due to *Andersson, Cai, and Galloway.*[251]

*Questions.*Can one put the index theoretic and associated Dirac-spectral considerations on equal footing with Witten's and Min-Oo's kind of arguments on stability of harmonic spinors with a given asymptotic behavior under deformation/modifications of manifolds away from infinity?

Can Cecchini kind long neck argument(s) be extended to $\sigma<0$?[252]

Three conjectures

[$\#_{-n(n-1)}$] *Let X be a closed orientable Riemannian manifold of dimension n with $Sc(X)\geq -n(n-1)$.*

Then the following topological invariants of X must be bounded by the volume of X, and, even more optimistically, (and less realistically), where the constants are such that the equalities are achieved for compact hyperbolic manifolds with sectional curvatures -1.

Namely, granted [$\#_{-n(n-1)}$] one expects the following.

1. ***Simplicial Volume Conjecture***: There exist orientable n-dimensional *pseudomanifolds* $X^\triangle_i$ and continuous maps $f^\triangle_i: X^\triangle_i\to X$ with degrees

$$deg(f^\triangle_i)\underset{i\to\infty}{\to}\infty,$$

[249]The logic of what we do here is similar to the proof of rigidity of $\mathbb{R}^n$ by passing to $\mathbb{T}^n=\mathbb{R}^n/\mathbb{Z}^n$ and thus, reducing the problem to the scalar curvature ≥ 0 rigidity of the flat tori.

[250]Instead of using μ-bubbles as in §5$\frac{5}{6}$ of [G(positive) 1996], one can proceed here by inductive descent with $\mathbb{T}^\rtimes$-symmetrised minimal hypersurfaces with free boundaries, as in the proof of the $\frac{2\pi}{n}$-inequality indicated in section 3.6; see section **??** for this and for more general results of this kind.

[251]See [Lohkamp(hammocks) 1999] and [AndMinGal(asymptotically hyperbolic) 2007].

[252]Notice that the long neck proofs in [Cecchini(long neck) 2020] and in [Cecchini-Zeidler(generalized Callias) 2021], similarly to these in [Min-Oo(hyperbolic) 1989], depend on Dirac operators with potentials.

such that the numbers N_i of *simplices* in the triangulations of $X_i^{\vartriangle}$ and the degrees $deg(f_i^{\vartriangle})$ are related to the volume of X by the following inequality:

$$N_i \leq C_n^{\vartriangle} \cdot deg(f_i^{\vartriangle}) \cdot vol(X).$$

2. ***The L-Rank Norm Conjecture:*** There exist, for all sufficiently large $i \geq i_0 = i_0(X)$, smooth orientable n-dimensional *manifolds* X_i° and continuous maps $f_i : X_i^{\circ} \to X$, with degrees

$$deg(f_i^{\vartriangle}) \underset{i\to\infty}{\to} \infty,$$

such that the minimal possible numbers N_i of *the cells* in the cellular decompositions of X_i° and the degrees of the maps $f_i^{\vartriangle}$ are related to the volume of X by the following inequality:

$$N_i \leq C_n^{\circ} \cdot deg(f_i^{\circ}) \cdot vol(X).$$

3. **Characteristic Numbers Conjecture.** if, additionally to **[$\#_{-n(n-1)}$]**, the manifold X is *aspherical*, then *the Euler characteristic* $\chi(X)$ and *the Pontryagin numbers p_I of X are bounded by*

$$|\chi(X)|, |p_I(X)| \leq C_n^{\square} \cdot vol(X).$$

Remarks. (i) Conjecture 1 makes sense for an X, in so far as X has *non-vanishing* simplicial volume $\|X\|_{\vartriangle}$, e.g. if X admits a metric with *negative sectional curvature* or a locally symmetric metric with *negative Ricci curvature.* [253]

(ii) The L-rank norm $\|[X]_L\|$ is defined in §$8\frac{1}{2}$ of [G(positive) 1996] via the Witt-Wall L-groups of the fundamental group of X.

This $\|[X]_L\|$ is known to be *non-zero* for compact locally symmetric spaces with non-zero Euler characteristic as it follows from [Lusztig(cohomology) 1996].[254]

In fact, all *known* manifolds X with $\|[X]_L\| \neq 0$ admit maps of non-zero degrees to locally symmetric spaces with non-zero Euler characteristics.

And nothing is known about zero/non-zero possibility for the values of the L-rank norm for manifolds with negative sectional curvatures of odd dimensions > 3.

(Vanishing of $\|[X]_L\|$ for all 3-manifolds X trivially follows from the Agol-Wise theorem on virtual fibration of hyperbolic 3-manifolds over S^1.)

Question. What are realtions between the $\|X\|_{\vartriangle}$ and $\|[X]_L\|$? Are there natural invariants mediating between the two?

(It is tempting to suggest that $\|X\|_{\vartriangle} \geq \|[X]_L\|$, since being a triangulation is (by far) more restrictive than being just a cell decompositions, but since $\|[X]_L\|$,

[253] See [Lafont-Schimidt(simplicial volume) 2017] and the monograph [Frigero(Bounded Cohomology) 2016] for the definition and basic properties of the simplicial volume.

[254] In the simplest case, where X is the product of k closed surfaces $S_1, S_2, ..., S_k$ with negative Euler characteristics, non-vanishing of $\|[X]_L\|$ is proven in [G(positive) 1996]:

If a manifold X° admits a map of degree d to such an X, then X° can't be decomposed into less than

$$N = const_k \cdot d \cdot |\chi(S_1)| \cdot |\chi(S_2) \cdot ... \cdot |\chi(S_k)|, \ const_k > 0,$$

cells.

unlike $\|X\|_\triangle$ defined with *manifolds*, rather than with *pseudomanifolds* mapped to X, this is unlikely to be true in general.)

Integral Strengthening of the Three Conjectures. The above conjectural inequalities 1,2,3, for the three topological invariants, call them here inv_i, $i = 1,2,3$, may, for all we know, hold (with no a priori assumption $Sc(X) \geq -n(n-1)$) in the following integral form,

$$inv_i \leq const_i \cdot \int_X |Sc_-(X,x)|^{\frac{n}{2}} dx,$$

where $Sc_-(x) = \min(Sc(x), 0)$, but no lower bound on this integral is anywhere in sight for $n \geq 5$. [255] (See section 3.16 for what is known for $n = 4$.)

3.14 Positive Scalar Curvature, Index Theorems and the Novikov Conjecture

Given a proper (infinity goes to infinity) smooth map between smooth oriented manifolds, $f : X \mapsto \underline{X}$ of dimensions $n = dim(X) = 4k + \underline{n}$ for $\underline{n} = dim(\underline{X})$, let $sign(f)$ denote the signature of the pullback $Y^{4k}_{\underline{x}} = f^{-1}(\underline{x})$ of a generic point $\underline{x} \in \underline{X}$, that is the signature of the (quadratic) intersection form on the homology $H_2(Y^{4k}_{\underline{x}}; \mathbb{R})$, where, observe, orientations of X and $\underline{X}$ define an orientation of $Y^{4k}_{\underline{x}}$ which is needed for the definition of the intersection index.

Since the f-pullbacks of generic (curved) segments $[\underline{x}_1, \underline{x}_2] \subset \underline{X}$ are manifolds with boundaries $Y^{4k}_{\underline{x}_1} - Y^{4k}_{\underline{x}_2}$, (the minus sign means the reversed orientation),

$$sign(Y^{4k}_{\underline{x}_1}) = sign(Y^{4k}_{\underline{x}_2}),$$

as it follows from the Poincaré duality for manifolds with boundary by a two-line argument. Similarly, one sees that $sign(f)$ depends only on the proper homotopy class $[f]_{hom}$ of f.

Thus, granted $\underline{X}$ and a proper homotopy class of maps f, the signature $sign[f]_{hom}$ serves as a *smooth invariant* denoted $sign_{[f]}(X)$, (which is actually equal to the value of some polynomial in Pontryagin classes of X at the homology class of $Y^{4k}_{\underline{x}_2}$ in the group $H_{4k}(X)$).

If X and $\underline{X}$ are closed manifolds, where $dim(X) > dim(\underline{X}) > 0$, and if $\underline{X}$, is *simply connected*, then, by the Browder-Novikov theory, as one varies the smooth structure of X in a given homotopy class $[X]_{hom}$ of X, the values of $sign_{[f]}(X)$ run through *all integers* $i = sign_{[f]}(X) \mod 100n!$ (we exaggerate for safety's sake), provided $dim(\underline{X}) > 0$ and $Y^{4k}_{\underline{x}} \subset X$ is non-homologous to zero.

However, according to the (illuminating special case of the) *Novikov conjecture*c

if $\underline{X}$ is a *closed aspherical* manifold[256] then this $sign_{[f]}(X)$ *depends only on the homotopy type of* X. [257]

[255] One doesn't even know if there are such bounds for $\|X\|_\triangle$ and/or $\|[X]_L\|$ in terms of the *full Riemannian curvature tensor* $R(X,x)$, namely the bounds

$$\|X\|_\triangle, \|[X]_L\| \leq const_n \int_X \|R(X,x)\|^{\frac{n}{2}} dx.$$

[256] *Aspherical* means that the universal cover of $\underline{X}$ is contractible

[257] Our topological formulation, which is motivated by the history of the Novikov conjecture, is deceptive: in truth, Novikov conjecture is 90% about infinite groups, 9% about geometry and only 1% about manifolds.

Originally, in 1966, Novikov proved this, by an an elaborated surgery argument, for the torus $\underline{X} = \mathbb{T}^{\underline{n}}$, where $X = Y \times \mathbb{T}^{\underline{n}}$ and f is the projection $Y \times \mathbb{T}^{\underline{n}} \to \mathbb{T}^{\underline{n}}$.

In 1972, Gheorghe Lusztig found a proof for general X and maps $f : X \to T^{\underline{n}}$ based on *the Atiyah-Singer index theorem for families of differential operators D_p parametrised by topological spaces P,* where the index takes values not in $\mathbb{Z}$ anymore but in the K-theory of P, namely, this index is defined as the K-class of the (virtual) vector bundle over P with the fibers $ker(D_p) - coker(D_p)$, $p \in P$, (Since the operators D_p are Fredholm, this makes sense despite possible non-constancy of the ranks of $ker(D_p)$ and $coker(D_p)$.)

The family P in Lusztig's proof in [Lusztig(Novikov) 1972] is composed of the *signature s* on X twisted with complex line bundles L_p, $p = P$, over X, where these L are induced by a map $f : X \to \mathbb{T}^{\underline{n}}$ from *flat* complex unitary line bundles $\underline{L}_p$ over $\mathbb{T}^{\underline{n}}$ parametrised by P (which is the $\underline{n}$-torus of homomorphism $\pi_1(\mathbb{T}^{\underline{n}}) = \mathbb{Z}^{\underline{n}} \to \mathbb{T}$).

Using the the Atiyah-Singer index formula, Lusztig computes the index of this , shows that it is equal to $sign(f)$ and deduce from this the homotopy invariance of $sign_{[f]}(X)$.

What is relevant for our purpose is that Lusztig's computation equally applies to the Dirac operator twisted with L_p and shows the following.

Let X be a closed orientable spin manifolds of even dimension $\underline{n}$ and $f : X \to \mathbb{T}^{\underline{n}}$ be continuous map of non-zero degree. Then

$$ind(\mathcal{D}_{\otimes\{L_p\}}) \neq 0.$$

Therefore, there exits a point $p \in P$, such that X carries a harmonic L_p-twisted spinor

But if $Sc(X) > 0$, this is incompatible with the the Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) formula which says for *flat* L_p that

$$\mathcal{D}_{\otimes L_p} = \nabla^2_{\otimes L_p} + \frac{1}{4} Sc(X).$$

Thus,

the existence of a map $f : X \to \mathbb{T}^{\underline{n}}$ with $deg(f) \neq 0$ implies that X carries no metric with $Sc > 0$.

Moreover, Lusztig's computation applies to manifolds X of all dimensions $n = \underline{n} + 4k$, shows that if a generic pullback manifold $Y^4_p = f^{-1}(p) \subset X$ (here f is smooth) has *non-vanishing $\hat{\alpha}$-invariant* defined in section 3.2 (that is the $\hat{A}$-genus for $4k$-dimensional manifolds), then the index $ind(\mathcal{D}_{\otimes\{L_p\}})$ doesn't vanish either and, assuming X is spin, it *can't carry metrics with $Sc > 0$.*

Remark on $X = (X, g_0) = \mathbb{T}^{\underline{n}}$. If (X, g_0) is isometric to the torus, then the only g_0-harmonic L_p-twisted spinors on X are parallel ones, which allows a direct computation of the index of $\mathcal{D}_{\otimes\{L_p\}}$. Then the result of this computation extends to all Riemannian metrics g on $\mathbb{T}^{\underline{n}}$ by the invariance of the index of $\mathcal{D}_{\otimes\{L_p\}}$ under deformations of $\mathcal{D}$, where the essential point is that, albeit the harmonic spinors of the (untwisted) $\mathcal{D}$ may (and typically do) disappear under a deformation $\mathcal{D}_{g_0} \rightsquigarrow \mathcal{D}_g$, they re-emerge as harmonic spinors of $\mathcal{D}_g$ twisted with a non-trivial flat bundle L_p.

The index theorem for families can be reformulated with P being replaced by the algebra $cont(P)$ of all continuous functions on P, where in Lusztig's case the algebra $cont(\mathbb{T}^{\underline{n}})$ is Fourier isomorphic to the algebra $C^*(\mathbb{Z}^{\underline{n}})$ of bounded linear operators on the Hilbert space space $l_2(\mathbb{Z}^{\underline{n}})$ of square-summarable functions on the group $\mathbb{Z}^{\underline{n}}$, which commute with the action of $\mathbb{Z}^{\underline{n}}$ on this space.

A remarkable fact is that a significant portion of Lusztig's argument generalizes to all discrete groups Π instead of $\mathbb{Z}^{\underline{n}}$, where the algebra $C^*(\Pi)$ of bounded operators on $l_2(\Pi)$ is regarded as the algebra of continuous functions on a "non-commutative space" dual to Π (that is the actual space, namely that of of homomorphisms $\Pi \to \mathbb{T}$ for commutative Π.)

This allows a formulation of what is called in [Rosenberg(C^*-algebras - positive scalar) 1984] the *strong Novikov Conjecture*, the relevant for us special case of which reads as follows.

$\mathcal{D}_{\otimes C^*}$-Conjecture. If a smooth closed orientable Riemannian spin n-manifold X for n even admits a continuous map F to the classifying space $B\Pi$ of a group Π, such that the homology homomorphism F_* sends the fundamental homology class $[X] \in H_n(X;\mathbb{R})$ to *non-zero* element $h \in H_n(B\Pi;\mathbb{R})$, then

the Dirac operator on X twisted with some flat unitary Hilbert bundle over X has non-zero kernel.

(Here "unitary" means that the monodromy action of $\pi_1(X)$ on the Hilbert fiber $\mathcal{H}$ of this bundle is unitary and where an essential structure in this $\mathcal{H}$ is the action of the algebra $C^*(\Pi)$, which commute with the action of $\pi_1(X)$.)

This, if true, would imply, according to the Schroedinger-Lichnerowicz-Weitzenboeck formula, the spin case of the conjecture stated in section 3.2. saying that

X admits no metric with $Sc > 0$.

Also "Strong Novikov" would imply, as it was proved by Rosenberg, the validity of the

Zero in the Dirac Spectrum Conjecture. Let $\tilde{X}$ be a complete *contractible* Riemannian manifold the quotient of which under the action of the isometry group $iso(\tilde{X})$ is *compact*.

Then the spectrum of the Dirac operator $\tilde{\mathcal{D}}$ on $\tilde{X}$ contains zero, that is, for all $\varepsilon > 0$, there exist L_2-spinors $\tilde{s}$ on $\tilde{X}$, such that

$$||\tilde{\mathcal{D}}(\tilde{s}|| \le \varepsilon||\tilde{s}||.$$

This, confronted with the Schroedinger-Lichnerowicz-Weitzenboeck formula, would show that $\tilde{X}$ can't have $Sc > 0$.

Are we to Believe in these Conjectures? A version of the *Strong Novikov conjecture* for a rather general class of groups, namely those which *admit discrete isometric actions on spaces with non-positive sectional curvatures*, was proven by Alexander Mishchenko in 1974.

Albeit this has been generalized since 1974 to many other classes of groups Π and/or representatives $h \in H_n(B\Pi;\mathbb{R})$, (most recent results and references can be found in [GWY (Novikov) 2019]) the sad truth is that one has a poor understanding of what these classes actually are, how much they overlap and what part of the world of groups they fairly represent.

At the moment, there is no basis for believing in this conjecture and there is no idea where to look for a counterexample either.[258]

The following is a more geometric version of the above conjecture.

Coarse $\mathcal{D}$-Spectrum Conjecture. Let $\hat{X}$ be a complete *uniformly contractible* Riemannian manifold, i.e. there exists a function $R(r) \geq r$, such that the ball $B_{\hat{x}}(r) \subset \hat{X}$, $x \in X$, of radius r is contractible in the concentric ball $B_{\hat{x}}(R(r))$ for all $\hat{x} \in \hat{X}$ and all radii $r > 0$.

Then the spectrum of the Dirac operator on $\hat{X}$ contains zero.

This conjecture, as it stands, must be, in view of [DRW(flexible) 2003], *false*, but finding a counterexample becomes harder if we require the bounds $vol(B_{\hat{x}}(r)) \leq \exp r$ for all $\hat{x} \in \hat{X}$ and $r > 0$.[259]

And although this conjecture remains unsettled for $n = dim(X) \geq 4$, its significant corollary –

non-existence of complete uniformly contractible Riemannian n-manifolds with positive scalar curvatures

was recently proved for n=4 and 5 by means of torical symmetrization of stable μ-bubbles,[260]

3.14.1 Almost Flat Bundles and $\otimes_\varepsilon$-Twist Principle

Let us recall Dirac operators twisted with *almost flat unitary bundles* and construction of such bundles over *profinitely hyperspherical* manifolds such as n-tori, for example.

Let X be a Riemannian manifold and $L = (L, \nabla)$ be a complex vector bundle L with unitary connection. If the curvature of L is ε-close to zero,

$$\|\mathcal{R}_L\| \leq \varepsilon,$$

then, locally, L looks, approximately as the flat bundle $X \times \mathbb{C}^r$, $r = rank_{\mathbb{C}}(L)$, and the Dirac twisted with L, denoted $\mathcal{D}_{\otimes L}$, that acts on the spinors with values in L, is locally approximately equal to the direct sum $\underbrace{\mathcal{D} \oplus ... \oplus \mathcal{D}}_{r}$.

It follows that if $Sc(X) \geq \sigma > 0$ and if ε is much smaller than σ, then by the (obvious) continuity of the Schroedinger-Lichnerowicz-Weitzenboeck formula, this twisted Dirac operator has trivial kernel, $ker(\mathcal{D}_{\otimes L}) = 0$ and, accordingly,

$$ind(\mathcal{D}^+_{\otimes L}) = 0,$$ [261]

where, by the Atiyah-Singer index theorem, this index is equal to a certain topological invariant

$$ind(\mathcal{D}^+_{\otimes L}) = \hat{\alpha}(X, L).$$

[258] Geometrically most complicated groups are those which represent one way or another universal Turing machines; a group, the k-dimensional homology (L-theory?) of which, say for $k = 3$, models such a "random" machine, would be a good candidate for a counterexample.

[259] See [F-W(zero-in-the-spectrum) 1999] for what is known about the similar conjecture by John Lott for the DeRham-Hodge .

[260] See [Chodosh-Li(bubbles) 2020] and [G(aspherical) 2020].

[261] Here we assume that $n = dim(X)$ is *even*, which makes $\mathcal{D}$ *split* as $\mathcal{D} = \mathcal{D}^+ \oplus \mathcal{D}^-$, such that $ind(\mathcal{D}^+) = -ind(\mathcal{D}^-)$, see section 4.

For instance, if X is an even dimensional topological torus, and if the top Chern class of L doesn't vanish, $c_m(L) \neq 0$ for $m = \frac{dim(X)}{2}$, then $\alpha(X, L) \neq 0$ as well.

On the other hand, given a Riemannian metric g on the torus $\mathbb{T}^n$, $n = 2m$, and $\varepsilon > 0$,

there exists a finite covering $\tilde{\mathbb{T}^n}$ of the torus, which admits an ε-flat vector bundle $\tilde{L} \to \tilde{\mathbb{T}^n}$ of $\mathbb{C}$-rank $r = m = \frac{n}{2}$ with $c_m(L) \neq 0$,

where the "flatness" of $\tilde{L}$, that is the norm of the curvature $\mathcal{R}_{\tilde{L}}$ regarded as a 2-form with the values in the Lie algebra of the unitary group $U(r)$, $r = rank_{\mathbb{C}}(\tilde{L})$, is measured with the lift $\tilde{g}$ of the metric g to $\tilde{\mathbb{T}^n}$.

Indeed, let $\hat{L} \to \mathbb{R}^n$, $n = 2m$, be a vector bundle with a unitary connection, such that $\hat{L}$ is isomorphic (together with it connection) at infinity to the trivial bundle and such that $c_m(\hat{L}) \neq 0$, where such an $\hat{L}$ may be induced by a map $\mathbb{R}^n \to S^n$, which is constant at infinity and has degree one, from a bundle $\underline{L} \to S^n$ with $c_m(\underline{L}) \neq 0$.

Let $\hat{L}_\varepsilon$ be the bundle induced from $\hat{L}$ by the scaling map $x \mapsto \varepsilon x$, $x \in \mathbb{R}^n$. Clearly, the curvature of $\hat{L}_\varepsilon$ tends to 0 as $\varepsilon \to 0$.

Since the finite coverings $\tilde{\mathbb{T}^n}$ of the torus converge to the universal covering $\mathbb{R}^n \to \mathbb{T}^n$ this $\hat{L}_\varepsilon$ can be transplanted to a bundle $\tilde{L}_\varepsilon \to \tilde{\mathbb{T}^n}$ over a sufficiently large finite covering $\tilde{\mathbb{T}^n}$ of the torus, where the top Chern number remains unchanged and where the curvature of $\tilde{L}$ with respect to the flat metric on $\tilde{\mathbb{T}^n}$ can be assumed as small as you wish, say $\leq \epsilon$.

But then this very curvature with respect to the lift $\tilde{g}$ of a given Riemannian metric g on $\mathbb{T}^n$ also will be small, namely $\leq const_g \epsilon$ and our claim follows.[262]

With this, we obtain

one of the (many) proofs of nonexistence of metrics g with $Sc(g) > 0$ on tori.

Seemingly Technical Conceptual Remark. The above rough qualitative argument admits a finer quantitative version, which depends on the twisted Schroedinger-Lichnerowicz-Weitzenboeck formula

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \frac{1}{4} Sc(X) + \mathcal{R}_{\otimes L},$$

where $\mathcal{R}_{\otimes L}$ is an operator on twisted spinors, i.e. on the bundle $\mathbb{S} \otimes L$, associated with the curvature of L and where an essential feature of $\mathcal{R}_{\otimes L}$ is a bound on its norm by the it norm $||\mathcal{R}_L||$ of the curvature $\mathcal{R}_L$ of L, with a constant *independent* of the rank of L.

Thus, for instance, the above proof of nonexistence of metrics g with $Sc(g) > 0$ on tori, that was performed with the twisted Dirac $\mathcal{D}_{\otimes \tilde{L}}$ over a *finite covering* $\tilde{X}$ of our torical X, can be brought *back to* X by pushing forward $\tilde{L}$ from the $\tilde{X}$ to X, where this push forward bundle $(\tilde{L})_* \to X$ has

$$rank(\tilde{L})_* = N \cdot rank(\tilde{L})$$

[262] Why do we need twelve lines to express, not even fully at that, so an obvious idea? Is it due to an imperfection of our mathematical language or it is something about our mind that makes instantaneous images of structurally protracted objects? Probably both, where the latter depends on the *parallel processing* in the human *subliminal* mind, which can't be well represented by any sequentially structured language that follows our *conscious* mind and where besides "*parallel*" there are many other properties of "*subliminal*" hidden from our conscious mind eye.

for N being the number of sheets of the covering.

(The lift of $(\tilde{L})_*$ to $\tilde{X}$ is the Whitney's sum of N-bundles obtained from $\tilde{L}$ by the deck transformations of $\tilde{L}$.)

This property of $\mathcal{R}_{\otimes L}$, in conjunction with the shape of the Atiyah-Singer index formula, for the Dirac operator twisted with Whitney's N-multiples

=

$$L \oplus ... \oplus L = \underbrace{L \oplus ... \oplus L}_{N},$$

which implies that in the relevant cases

$$ind(\mathcal{D}^+_{\otimes(L\oplus...\oplus L)}) = \alpha(X, L \oplus ... \oplus L) = N \cdot \hat{\alpha}(X, L) + O(1),$$

allows $N \to \infty$ and even $N = \infty$ in a suitable sense, e.g. in the context of infinite coverings and/or of C^*-algebras as was mentioned in the previous section.

What is also crucial, is that twisting with almost flat bundles is a *functorial* operation, where this functoriality yields the following.

$\otimes_\varepsilon$**-Twist Principle**. *All* (known) *arguments with Dirac operators for non-existence of metrics with* $Sc \geq \sigma > 0$ *under specific topological conditions on* X *can be* (more or less) *automatically transformed to inequalities between* σ *and certain geometric invariants of* X *defined via* ε*-flat bundles over* X.

$\otimes_\varepsilon$***-Problem.*** Can one turn $\otimes_\varepsilon$-Twist Principle to a $\otimes_\varepsilon$*-theorem?*

At the present moment, an application of the $\otimes_\varepsilon$-principle necessitates tracking *step by step*, let it be in a purely mechanical/algorithmic fashion, a particular Dirac theoretic argument, rather than a direct application of this principle to *the conclusion* of such an argument.

What, apparently, happens here is that the true outcomes of Dirac operator proofs are *not* the geometric theorems they assert, but certain linearized/hilbertized generalization(s) of these, possibly, in the spirit of Connes' non-commutative geometry.

To understand what goes on, one needs, for example, to reformulate (reprove?) Llarull's, Min-Oo's and Goette-Semmelmann's inequalities in such a "linearized" manner.[263]

Twists with non-Unitary Bundles. Available (rather limited) results concerning scalar curvature geometry of manifolds X, which support almost flat non-unitary bundles and of (global spaces of possibly) non-linear fibrations with almost flat connections over X, are discussed in section **??**.

Flat or Almost Flat? Lusztig's approach to the Novikov conjecture via the signature operators twisted with (families of) *finite dimensional non-unitary*

[263] A promising approach is suggested by the convept of *quantitative K-theory*, which was successfully used in [Guo-Xie-Yu(quantitative K-theory) 2020] for a new proof of the $\frac{pi}{n}$-bounds in the width of Riemannian bands with $Sc \geq n(n-1)$.

This theory encodes the geometric information on the underlying Riemannian manifold X in term of the *propagation radius* r of operators in the *Roe translation algebra* that correspond to (linear combinations) of r-*translations* of X that are self mappings $a : X \to X$ with $dist(a(x), x) \leq r$.

This faithfully reflects the *distance geometry* of X, but the quantitative K-theory, as it stands now, can't adequately capture the area geometry; conceivably this can be achieved by incorporation ideas from Cecchini's long neck paper into this theory.

flat bundles was superseded, starting with the work by Mishchenko and Kasparov, by more general index theorems, for *infinite dimensional flat unitary* bundles.

Then it was observed in [GL(spin) 1980] and proven in a general form in [Rosenberg(C^*-algebras - positive scalar) 1984]) that all these results can be transformed to the corresponding statements about Dirac operators on spin manifolds, thus providing obstructions to $Sc > 0$ essentially for the same kind of manifolds X, where the generalized signature theorems were established.

Besides following topology, the geometry of the scalar curvature suggested a quantitive version of these topological theorems by allowing twisted Dirac and signature operators with *non-flat vector bundles* with *controllably small curvatures*, thus providing geometric information on X with $Sc \geq \sigma > 0$, which complements the information on pure topology of X.

At the present moment, there are two groups of papers on twisted (sometimes untwisted) Dirac operators on manifolds with $Sc > \sigma$.

The first and a most abundant one goes along with the work on the Novikov conjecture, where it is framed into the KK-theoretic formalism.

A notable achievement of this is

Alain Connes' topological obstruction for leaf-wise metrics with $Sc > 0$ *on foliations*,

where

a geometric shortcut through the KK-formalism of Connes' proof is unavailable at the present moment.

Another direction is a geometrically oriented one, where we are not so much concerned with the K-theory of the C^*-algebras of fundamental groups $\pi_1(X)$, but with geometric constraints on X implied by the inequality $Sc(X) \geq \sigma$.

This goes close to what happens in the papers inspired by the general relativity, where one is concerned with specified (and rather special, e.g. asymptotically flat) geometries at infinity of complete Riemannian manifolds and where one plays, following Witten and Min-Oo, with Dirac operators, which are asymptotically adapted at infinity to such geometries. (In this context, the Schoen-Yau and the related methods relying of the *mean curvature flows* are also used.)

In the present paper, we are primarily concerned with geometry of manifolds, while topology is confined to *an auxiliary,* let it be irreplaceable, role.

3.14.2 Relative Index of Dirac Operators on Complete Manifolds

Most (probably, not all) bounds on the scalar curvature of *closed* Riemannian manifolds derived with twisted Dirac operators $\mathcal{D}_{\otimes L}$ have their counterparts for *complete* manifolds X, where one uses a relative version of the Atiyah-Singer theorem for *pairs of Dirac operators which agree at infinity* [264] the simplest and the most relevant case of this theorem applies to vector bundles $L \to X$ with unitary connections which are *flat trivial at infinity.*

In this case the pair in question is $(\mathcal{D}_{\otimes\mathcal{L}}, \mathcal{D}_{\otimes|\mathcal{L}|})$, where $|L|$ denotes the trivial flat bundle $X \times \mathbb{C}^k \to X$ for $k = rank_{\mathbb{C}}(L)$, which comes along with an isometric

[264] See [GL(complete) 1983], [Bunke(relative index) 1992], [Roe(coarse geometry) 1996]), and more recent papers [Zhang(Area Decreasing) 2020], [Cecchini(long neck) 2020], [Cecchini-Zeidler(generalized Callias) 2021], [Cecchini-Zeidler(Scalar&mean) 2021].

connection preserving isomorphism between L and $|L|$ outside a compact subset in X.

f^*-Example. Let $f : X \to S^n$ be a smooth map which is locally constant at infinity (i.e. outside a compact subset) and let $\underline{L} \to S^n$ be a bundle with a unitary connection on S^n.

Then the pullback bundle $f^*(\underline{L}) \to X$ is an instance of such an L.

The relative index theorem, similarly to its absolute counterpart implies that if the scalar curvature of X is *uniformly positive (i.e. $Sc \geq \sigma > 0$) at infinity* and if

a certain topological invariant, call it $\hat{\alpha}(X, L)$,[265] *doesn't vanish, then either X admits a non-zero (untwisted) harmonic L_2-spinor s on X, that is a solution of $\mathcal{D}(s) = 0$, or there is a non-zero L-twisted harmonic L_2-spinor on X.*[266]

f^*-Sub-Example. Let $L = f^*(\underline{L})$ be as in the f^*-example, let where $n = dim(X)$ is even, and let the bundle $\underline{L} \to S^n$ has non-zero top Chern class (e.g. $\underline{L}$ is the bundle of spinors on the sphere, $\underline{L} = \mathbb{S}_+(S^n)$). If the map $f : X \to S^n$ has non-zero degree, then $\hat{\alpha}(X, L) \neq 0$.

Finally, since the twisted Schroedinger-Lichnerowicz-Weitzenboeck formula (obviously) applies to L_2-spinors, one obtains, for example, as an application of the $\otimes_\varepsilon$-Twist Principle the following relative version of the Lichnerowicz' theorem for k-dimensional manifolds from section 3.2, that, let us remind it, says that

$$\hat{A}[X] \neq 0 \Rightarrow Sc(X) \not> 0 \text{ for closed spin manifolds } X.$$

If a complete Riemannian orientable spin manifolds X (of dimension $n+4k$) admits a proper λ -Lipschitz map $f : X \to \mathbb{R}^n$ for some $\lambda < \infty$, then the pullbacks of generic points $y \in \mathbb{R}^n$ satisfy $\hat{A}[f^{-1}(y)] = 0$.

This, in the case $dim(X) = n$, shows that

the existence of proper Lipschitz map $X \to \mathbb{R}^n$ implies that $\inf_x Sc(X, x) \leq 0$.[267]

Moreover,

it follows from Zhang's theorem stated below, that, in fact, $\inf_x Sc(X, x) < 0$.

The relative index theorem combined with the linear-algebraic analysis of the L-curvature term $\mathcal{R}_{\otimes L}$ in the twisted Schroedinger-Lichnerowicz-Weitzenboeck formula due to Llarull , Min-Oo, Goette-Semmelmann and Listing allows an extension of their inequalities from compact manifolds) to non-compact complete manifolds.

For instance,

★ *If a complete Riemannian orientable spin manifolds X (of dimension $n + 4k$) with $Sc(X) > n(n-1)$ admis a locally constant at infinity 1 -Lipschitz map $f : X \to S^n$, then the pullbacks of generic points $y \in \mathbb{S}^n$ satisfy $\hat{A}[f^{-1}(y)] = 0$.*[268],

[265] See section 3.2 for the definition of this invariant.

[266] If we don't assume that $Sc(X)$ is uniformly positive at infinity, then one can only claim the existence of either non-zero untwisted or non-zero twisted *almost harmonic* L_2-spinors, i.e. satisfying $\int_X \mathcal{D}^2(s)dx \leq \varepsilon \int_X ||s(x)||^2$ or $\int_X \mathcal{D}^2_{\otimes L}(s)dx \leq \varepsilon \int_X ||s(x)||^2$, for arbitrarily small $\varepsilon > 0$.

[267] This has a variety of generalizations and applications, (see e.g. [GL(spin) 1980], [GL(complete) 1983], [Roe(partial vanishing) 2012] and references therein), such as non-existence of metrics with $Sc > 0$ on tori.

[268] See [Llarull(sharp estimates) 1998] and also sections 3.4.1

4.1.5, 4.2.

Zhang's Extension of the Relative Index Theorem with Applications to maps $X \to S^n$. The above stated relative index theorem needs *uniform* positivity of the scalar curvature of X at infinity, i.e. the bound $Sc(X) \geq \sigma > 0$.

This uniformity condition was removed in [Zhang(Area Decreasing) 2020] by using a small zero order perturbation of the relevant twisted Dirac at infinity making the resulting positive at infinity and thus, proving the following theorem.

Let a complete orientable spin n-manifold X of non-negative scalar curvature, $Sc(X) \geq 0$ and let X admit a smooth *area decreasing* locally constant at infinity (i.e. outside a compact subset) map $f : X \to S^n$ of *non-zero degree.*

Then

★★ *the scalar curvature of X on the support of the differential of f* (where $df \neq 0$) *satisfies:*

$$\inf_{x \in supp(df)} Sc(X, x) \leq n(n-1),$$ [269]

and if n is even, then

$$\inf_{x \in supp(df)} Sc(X, x) < n(n-1),$$

unless X is compact and f is an isometry.

Remark It remains unclear, even for compact X, if the spin condition is essential, but the completeness condition can be significantly relaxed as we shall explain in the next section.

3.14.3 Roe's Translation Algebra, Dirac Operators on Complete Manifolds with Boundaries and Cecchini's Long Neck Theorem for Non-Complete manifolds

C^*-algebras bring forth the following interesting perspective on *coarse geometry* of non-compact spaces proposed by John Roe following Alain Connes' idea of non-commutative geometry of foliations.

Given a metric space Ξ, e.g. a discrete group with a word metric, let $\mathcal{T} = Tra(\Xi)$ be the semigroup of translations of M that are maps $\tau : \Xi \to \Xi$, such that

$$\sup_{\xi \in \Xi} dist(\xi, \tau(\xi)) < \infty.$$

The (reduced) Roe C^*-algebra $R^*(\Xi)$ is a certain completion of the semigroup algebra $\mathbb{C}[\mathcal{T}]$. For instance if Ξ is a group with a word metric for which, say the left action of Ξ on itself is isometric, then the right actions lie in $\mathcal{T}$ and $R^*(\Xi)$ is equal to the (reduced) algebra $C^*(\Xi)$.[270]

Using this algebra, Roe proves in [Roe(coarse geometry) 1996], (also see [Higson(cobordism invariance) 1991], [Roe(partial vanishing) 2012]) a *partitioned index theorem*, which implies, for example, that.

✏ *the toric half cylinder manifold $X = \mathbb{T}^{n-1} \times \mathbb{R}_+$ admits no complete Riemannian metric with $Sc \geq \sigma > 0$.*[271]

[269] This also follows from Cecchini's long neck theorem stated in the next section.

[270] "Reduced" refers to a minor technicality not relevant at the moment. A more serious problem – this is not joke – is impossibility of definition of "right" and "left" without an appeal to violation of mirror symmetry by weak interactions.

[271] I must admit I haven't fully understood Roe's argument.

Nowadays ✏ can be proved with the techniques of minimal hypersurfaces and of stable μ-bubbles, (sections 3.6, 3.6.1) as well as withDirac theoretic techniques with potentials developed by Zeidler and by Cecchini and by the Guo-Xie-Yu in the framework of the quantitative K-theory, (see below) where these techniques yield not only the bound $\inf_x Sc(X,x) \le 0$ but a *quadratic decay* of the scalar curvature on $\mathbb{T}^1 \times \mathbb{R}_+$.

Also notice in this regard that if X is sufficiently "thick at infinity", then ✏ follows by a simple argument with twisted Dirac operators and the standard bound on the number of small eigenvalues in the spectrum of the Laplace (or directly of the Dirac) operator in vicinity of ∂X, which applies to all manifolds with boundaries and which yields, in particular, (see section 4.6.3) the following.

✐ Let X be a complete oriented Riemannian spin n-manifold *with compact boundary*, such that

there exists a sequence of smooth area decreasing maps $f_i : X \to S^n$, which are constant in a (fixed) neighbourhood $V \subset X$ of the boundary ∂X as well as away from compact subsets $W_i \subset V$, and such that

$$deg(f_i) \underset{i\to\infty}{\to} \infty.$$

Then the scalar curvature of X satisfies

$$\inf_{x\in X} Sc(X,x) \le n(n-1).$$

Quantitative K-theory and Long Neck Principle. It seems that most (all?) results for *complete* Riemannian manifolds with $Sc \ge \sigma$ have their counterparts for manifolds X with boundaries insofar as this concerns the part of X that lies far from the boundary ∂X.

Definite results in this regard were recently obtained by Hao Guo, Zhizhang Xie and Guoliang Yu who, if I understand this correctly, developed a quantitative version of Roe's theory, and also by Rudolf Zeidler and Simone Cecchini who obtained index theorems for Dirac operators with potentials on manifolds with boundaries.[272]

Here is an instance of some of new results.

Cecchini's Bound on Hyperspherical Radii of Long Neck manifolds.[273] Let X be a compact n-dimensional orientable *spin* Riemannian manifolds with a boundary, let $Sc(X) \ge \sigma_0 > 0$ and let $f : X \to S^n(R)$ be a smooth area decreasing map, which is locally constant in a neighbourhood of the boundary $\partial X \subset X$ and which have *nonzero degree.*

Let the scalar curvature of the support of the differential of f be bounded from below by σ (where typically but not necessarily $\sigma \ge \sigma_0$),

$$Sc(X,x) \ge \sigma,\ \ x \in supp(df).$$

[272] See [Cecchini(long neck) 2020], [Guo-Xie-Yu(quantitative K-theory) 2020], [Zeidler(bands) 2019], [Zeidler(width) 2020], [Cecchini-Zeidler(generalized Callias) 2021], [Cecchini-Zeidler(scalar&mean) 2021].

[273] This was a response to a question from an earlier version of this manuscript.

If f satisfies the following "long neck condition",

$$dist(supp(df),\partial X)\geq\pi\sqrt{\frac{n-1}{n\sigma_0}},$$

then the radius of the sphere $S^n(R)$ is bounded, similarly to the case of complete X, by

$$R\leq\sqrt{\frac{n(n-1)}{\sigma}},$$

where in the case of *odd* n one additionally assumes (this is, probably, redundant) that f is *constant* (not just locally constant) in the neighbourhood of $\partial X\subset X$.

Question. What are (preferably sharp) long neck counterparts of the *spin-area convex* and *spherical trace area* extremality theorems from section 3.4.1

3.15 Foliations With Positive Scalar Curvature

According to the philosophy (supported by a score of theorems) of Alain Connes much of the geometry and topology of manifolds with discrete group actions, notably, those concerned with index theorems for Galois actions of fundamental groups on universal coverings of compact manifolds, can be extended to foliations.

In particular, Connes shows in [Connes(cyclic cohomology-foliation) 1986] that compact manifolds X which carry foliations $\mathscr{L}$ with leaf-wise Riemannian metrics with positive scalar curvatures behave in many respects as manifold which themselves admit such metrics.

For instance,

★ *if $\mathscr{L}$ is spin, i.e. the tangent (sub)bundle $T(\mathscr{L})\subset T(X)$ of such an $\mathscr{L}$ is spin then,*

by Connes' theorem, $\hat{A}[X]=0$.

This generalises Lichnerowicz' theorem from section **??** for oriented spin manifolds of dimensions $n=4k$, where, recall, $\hat{A}[X]$ is the value of a certain rational polynomial $\hat{A}(p_i)$ in the Pontryagin classes $p_i\in H^{4i}(X:\mathbb{Z})$ (see section 4) on the fundamental homology class $[X]\in H_n(X)$.[274]

In fact, the full Connes' theorem implies among other things

vanishing of the $\smile$-products of the $\hat{A}$-genus $\hat{A}(p_i)$, $j=0,1,...,k=\frac{n}{4}$, with all polynomials in the Pontryagin classes of the "normal" bundle $T^\perp(\mathscr{L})=T(X)/T(\mathscr{L})$, in the case where $\mathscr{L}$ is spin.

Connes' argument, which relies on Connes-Scandalis *longitudinal index theorem for foliations*), delivers a non-zero almost harmonic spinor on some leaf of $\mathscr{L}$ and an alternative and simpler proof of the existence of such spinors under suitable conditions was given in [Bern-Heit(enlargeability-foliations) 2018], where $\mathscr{L}$, besides being spin, is required to have *Hausdorff homotopy groupoid.*[275]

Another simplified proof of (a part of) Connes' theorem was also suggested in [Zhang(foliations) 2016], where the *manifold X itself*, rather than the tangent

[274] By definition, the values of the p_i-monomials $P_d=\smile_j p_{i_j}\in H^d(X)$, $4\sum_i i_j=d$, on $[X]$ equals zero for all $d\neq n$.

[275] One finds a helpful explanation of the meaning this condition in [Con(foliation) 1983] and in the lectures [Meinrenken(lectures) 2017].

bundle $T(\mathscr{L})$ is assumed *spin* [276] and where the existence of almost harmonic spinor is proven on some auxiliary manifolds associated with X.

One can get more mileage from the index theoretic arguments in these papers by applying the $\otimes_\varepsilon$-Twisting Principle from section 3.14.1, but this needs honest checking all steps in the proofs in there. This was (partly) done in [Bern-Heit(enlargeability-foliations) 2018], in [Zhang(foliations:enlargeability) 2018], [Su(foliations) 2018] and [Su-Wang-Zhang(area decreasing foliations) 2021] in the context of the index theorems used by the authors in their papers. [277] ,

Here is a geometric conjecture in this regard.[278]

Long Neck Foliated Conjecture. Let X be a compact oriented n-dimensional Riemannian manifold with a boundary, let $\mathscr{L}$ be a smooth m-dimensional, $2 \le m \le n$, foliation on X, such that the induced Riemannian metrics on the leaves of $\mathscr{L}$ have positive scalar curvatures,

$$Sc(\mathscr{L}) > \sigma_0,$$

let $f : X \to S^n$ be a smooth map, which is locally constant in a neighbourhood of the boundary $\partial X \subset X$ and which have *nonzero degree*, let the scalar curvature of L on the support of the differential of f be bounded from below by the trace norm of the second exterior power of the differential of f on the tangent bundle of $\mathcal{L}$ as follows

$$Sc(X, x \in supp(f)) \ge 2trace(\wedge^2 df_{\mathcal{L}})(x)).$$

Then the leaf-wise distance $D = dist_{\mathcal{L}}(supp(df), \partial X)$[279] *is bounded by some universal function of σ_0,*

$$d \le \theta(\sigma_0).$$[280]

Non-Integrable Question. Is there a "good" bound on the scalar curvature of non-integrable subbundles $\mathcal{T} \subset T(X)$ of rank m (instead of the tangent subbundles $T(\mathcal{L})$ of foliations $\mathcal{L}$)?

Here, $Sc(\mathcal{T}, x)$ is defined as the sum of the sectional curvatures of X in an orthonormal frame of bi-vectors in the space $\mathcal{T}_x$, and where, besides the scalar curvature, such an inequality must contain a non-integrability correction term.

[276] In the ambience of Connes' arguments [Connes(cyclic cohomology-foliation) 1986], these two spin conditions reduce one to another.

[277] I recall going through Connes' paper long time ago and observing in §9$\frac{2}{3}$ in [G(positive) 1996]). that Connes' argument yields the following.

Complete manifolds X with infinite K-cowaist$_2$ (called "K-area" in [G(positive) 1996]) , *e.g.* $\mathbb{R}^n$*, carry no spin foliations, where the induced Riemannian metrics in the leaves satisfy* $Sc \ge \sigma > 0$,

but my memory is uncertain at this point.

[278] This may follow by what is done in The techniques (results?) from [Zhang(foliations:enlargeability) 2018], [Cecchini(long neck) 2020], [GWY (Novikov) 2019] may be useful for settling this question.

[279] This D, that is the infimum of the length of curves in the leaves between the intersections of these leaves with $supp(df) \subset X$ and with $\partial X \subset X$, isIn general, greater than the distance $d = dist_X((supp(df), \partial X)$ For instance, if no leaf intersects both subsets, $supp(df) \subset X$ and $\partial X \subset X$, than $D = \infty$.

[280] Combined arguments from [Su-Wang-Zhang(area decreasing foliations) 2021]and [Cecchini(long neck) 2020] may lead to the proof if either X or $\mathcal{L}$ is spin and $\theta(\sigma_0) = \pi\sqrt{\frac{n-1}{n\sigma_0}}$.

If this correction term is sufficiently small in the C^1-topology, then the above conjecture could apply to families of approximate integral manifolds of $\mathcal{T}$; however, the resulting bound on $Sc(\mathcal{T}, x)$ seems very rough.

But what we look for is a sharp or a nearly sharp inequality approaching model examples, such as the standard codimension one (contact) subbundles on the odd dimensional spheres and codimension three subbundles on the $(4k-1)$-spheres.

Next, we want to work out a concept of scalar curvature of *sub-Riemannian* (it Carnot-Caratheodory) manifolds and show, for instance, that (self-similar) nilpotent Lie groups admit no such metrics quasi-isometric to the standard (self-similar) ones.

Stable Complementation Question [$\star$?]. Let (X, g) be a (possibly non-complete) Riemannian n-manifold with a smooth foliation, such that scalar curvature of the induced metric on the leaves satisfies $Sc \geq \sigma > 0$.

Does the product of X by a Euclidean space, $X \times \mathbb{R}^N$, admit an $\mathbb{R}^N$*-invariant* Riemannian metric $\tilde{g}$, such that $Sc(\tilde{g}) \geq \sigma$ and the quotient map $(X \times \mathbb{R}^N, \tilde{g})/\mathbb{R}^N \to (X, g)$ is 1-Lipschitz, or, at least, $const_n$-Lipschitz?

(See §$1\frac{7}{8}$ in [G(positive) 1996] and section 6.5, 6.5.2, 6.5.4 for partial results in this direction based on the geometry of *Connes' fibrations*.)

Notice that even the complete (positive) resolution of [$\star$?] wouldn't yield the entire Connes' vanishing theorem from [Connes(cyclic cohomology-foliation) 1986], nor would this fully reveal the geometry of foliated Riemannian manifolds X with scalar curvatures of the leaves bounded from below, e.g. an answer to the following questions.

1. Do compact Riemannian n-manifolds with constant curvature -1 admit k-dimensional foliations, $2 \leq k \leq n-1$, such that the scalar curvatures of the induced Riemannian metrics in the leaves are bounded from below by $-\varepsilon$ for a given $\varepsilon > 0$?

2. What would be a foliated version of the Ono-Davaux Spectral Inequality?

3.16 Scalar Curvature in Dimension 4 via the Seiberg Witten Equation

The simplest examples of 4-manifolds where non-existence of metrics with $Sc >$ follows from non-vanishing of Seiberg-Witten invariants are complex algebraic surfaces X in $\mathbb{C}P^3$ of degrees $d \geq 3$. (If d is even and these X are spin, this also follows from Lichnerowicz' theorem from section **??**.)

In fact, it was shown by LeBrun (see [Salamon(lectures) 1999] and references therein) that

no minimal (no lines with self-intersections one) *Kähler surface X admits a Riemannian metric with $Sc > 0$, unless X is diffeomorphic to $\mathbb{C}P^2$ or to a ruled surface* .

Furthermore, LeBrun following Witten shows in [LeBrun(Yamabe) 1999] that

if such an X has *Kodaira dimension* 2, which is the case, for instance, for the algebraic surfaces $X \subset \mathbb{C}P^3$ of degree $d \geq 5$, then

the total squared scalar curvature is bounded by the first Chern number of

X,

$$\int_X Sc(X,x)^2 dx \geq 32\pi^2 c_2(X),$$

where, moreover this inequality is sharp.

Although one doesn't expect anything comparable to the Seiberg-Witten equations for $n = dim(x) > 4$, one wonders if some coupling between the twisted Dirac $\mathcal{D}_{\otimes L}$ and an energy like functional in the space of connections in L may be instrumental in the study of the scalar curvature of X and lead to bounds on $\int_X Sc(X,x)^{\frac{n}{2}} dx$ for a manifolds X of dimension $n > 4$ and, even better on $\int_X |Sc_-(X,x)|^{\frac{n}{2}} dx$ for $Sc_-(X,x) = \min(Sc(X,x),0)$.

For instance,

Let a closed orientable Riemannian n-manifold X admits a map of non-zero degree to a closed locally symmetric manifold $\underline{X}$ with negative Ricci curvature, e.g. with constant negative curvature.

Does then the scale invariant integral of the negative part of the scalar curvature is bounded from below as follows:

$$\int_X |Sc_-(g,x)|^{\frac{n}{2}} dx \geq \int_{\underline{X}} |Sc(\underline{X},\underline{x})|^{\frac{n}{2}} d\underline{x}?$$

(Three conjectures related to this one are formulated in section 3.13.)

Question. What is the Seiberg-Witten 4D-version of geometric inequalities on manifolds with boundaries and manifolds with corners?

3.17 Topology and Geometry of Spaces of Metics with $Sc \geq \sigma$.

Non-connectedness of the space of metrics with $Sc > 0$ starts with the following observation.

Let a closed n-manifold X be decomposed as $X_- \cup X_+$ where X_- and X_+ are smooth domains (n-submanifolds) in X with a common boundary $Y = \partial X_- = \partial X_+$ and where $X_\mp$ are equal to regular neighbourhoods of disjoint polyhedral subsets $P_\mp \subset X$ of dimensions $n_\mp$ such that $n_- + n_+ = n-1$.

If $n_\mp \leq n-2$, then, by an easy elementary argument, both manifolds X_- and X_+ admit Riemannian metrics, say $g_\mp$, such that

the restrictions of these $g_\mp$ to Y, call them $h_\mp$, both have *positive* scalar curvatures.

And if X admits no metric with positive scalar curvature, e.g. if X is homeomorphic to the n-torus or to product of two Kummer surfaces, then h_- and h_+ can't be joined by a homotopy of metrics with positive scalar curvatures.

Indeed, such a homotopy, h_t, $t \in [-1,+1]$ could be easily transformed to a metric on the cylinder $Y \times [-1,+1]$ with positive scalar curvature and with relatively flat boundaries isometric to (Y, h_-) and (Y, h_+), which would then lead in obvious way to a metric on $X = X_- \cup Y \times [-1,+1] \cup X_+$ with $Sc > 0$ as well.

The first case of disconnectedness of spaces of metrics with $Sc > 0$ goes back to Hitchin's paper [Hitchin(spinors)1974] , where it is shown, among many other things, that *the sphere S^n, $n = 8k, 8k+1$, admits a diffeomorphism $\phi : S^n \to S^n$, such that the pullback $g_1 = \phi^*(g_0)$ of the standard metric g_0 can't be joined with g_0 by a homotopy g_t with $Sc(g_t) > 0$,*

where appropriate ϕ are those for which

the exotic spheres obtained by gluing pairs of $(n+1)$-balls across their boundaries according to ϕ have non-vanishing $\hat{\alpha}$-invariants (see section **??**) and where the proof relies on the index theorem for families of Dirac operators. Similarly, Hitchin finds non-contractible loops in the spaces of metrics g on S^n with $Sc(g) > 0$ for $n = 8k-1, 8k$.

This kind of argument combined with thin surgery with $Sc > 0$ and empowered by "higher" index theoretic invariants of families of diffeomorphisms, leads to the following results.

[HaSchSt 2014]. *If m is much greater than k then the kth homotopy group of the space $\mathcal{G}_{Sc>0}(S^{4m-k-1})$ of Riemannian metrics with $Sc > 0$ on the sphere S^{4m-k-1} is infinite.*

[EbR-W 2017]. *There exists a compact Spin 6-manifold X such that the space $\mathcal{G}_{Sc>0}(X)$ has each rational homotopy group infinite dimensional.* [281]

However, there is *no closed manifold* of dimension $n \geq 4$, which admits a metric with $Sc > 0$ and where the (rational) homotopy type, or even the set of connected components, of *the space of such metrics is fully determined.* [282]

Let us formulate two specific questions motivated by the following vague one:

What is the "*topology of the geometric shape*" of the (sub)space of metrics with $Sc \geq \sigma$?

Question 1. Given a Riemannian manifold $\underline{X}$, numbers $\lambda, \sigma > 0$ and an integer $d \neq 0$, let $G(X; \underline{X}, \lambda, \sigma, d)$ be the space of pairs (g, f) where g is a Riemannian metrics on a X with $Sc(g) \geq \sigma$ and $f : X \to \underline{X}$ is a λ-Lipschitz map of degree d.

What is the topology and geometry of this space and of the natural embeddings

$$G(X; \underline{X}, \lambda_1, \sigma_1) \hookrightarrow G(X; \underline{X}, \lambda_2, \sigma_2)$$

for $\lambda_2 \geq \lambda_1$ and $\sigma_2 \leq \sigma_1$.

More specifically,

what is the the supremum $\sigma_+ = \sigma_+(\lambda) = \sigma_+(\underline{X}, \lambda, d)$ of σ, such that the manifold $\underline{X}$ receives a λ-Lipschitz map $f : X \to \underline{X}$ of degree $d(\neq 0)$, where $Sc(X) \geq \sigma$?

Notice, for instance, that if $\underline{X}$ is the unit sphere, $\underline{X} = S^n$, then $\sigma_+(\lambda) = n(n-1)/\lambda^2$ by Llarull's inequality. (Here as ever we need to assume X is spin.)

Question 2. Let D be some natural distance function on the space G of smooth Riemannian metrics g on a closed manifold X. For instance $\mathsf{D}(g_1, g_2)$

---

[281] It seems, judging by the references in [Ebert-Williams(infinite loop spaces) 2017] and [Ebert-Williams(cobordism category) 2019] , that all published results in this direction depend on the Dirac operator techniques which do not cover the above example, if we take a *Schoen-Yau-Schick manifold* for X.

[282] Connectedness $\mathcal{G}_{Sc>0}(X^3)$ is proven in [Marques(deforming Sc > 0)2012] by means of the Ricci flow. Conceivably, a similar argument may reduce the study of the homotopy structure of $\mathcal{G}_{Sc>0}(X^3)$ to the space of "standard metrics" with $Sc > 0$ on X^3.

may be defined as log of the infimum of $\lambda > 0$, such that

$$\lambda^{-1} g_1 \le g_2 \le \lambda g_1.$$

Let $\mathsf{D}_\sigma(\mathsf{g})$ denote the D-distance from $g \in G$ to the subspace of metrics with $Sc \ge \sigma$ and $\tilde{\mathsf{D}}_\sigma(g)$ be the D-distance from the $diff(X)$-orbit of g to this subspace.

What are topologies, e.g. homologies, of the *a-sublevels*, $a \ge 0$, of the functions $D_\sigma : G \to [0, \infty)$ and $\tilde{\mathsf{D}}_\sigma : G \to [0, \infty)$ and of the inclusions

$$D_\sigma^{-1}(0, a] \hookrightarrow D_\sigma^{-1}(0, b], \text{ and } \tilde{\mathsf{D}}_\sigma^{-1}(0, a] \hookrightarrow \tilde{\mathsf{D}}_\sigma^{-1}(0, b] \text{ for } b > a?$$

Observe that the counterpart of the above σ_+ call it $\sigma_+^+(\lambda) = \sigma_+^+(X, \lambda)$, satisfies

$$\lim_{\lambda \to 1} \sigma_+^+(X, \lambda) = \inf_{x \in X} Sc(X, x)$$

by the C^0-closure theorem from section 3.1.3, and it is plausible that the function $\sigma_+^+(\lambda)$ is *Hölder continuous in* λ.

3.18 Domination, Extremality and Rigidity of Manifolds with Corners

Recall that a *corner structure* on an n-manifolds X is defined by (a coherent sets of) diffeomorphisms of small neighbourhoods of all points in X to neighbourhoods of points in convex polyhedra P in $\mathbb{R}^n$.

A corner structure is called *simple* and/or *cosimpicial* if these P are intersections of $m \le n$ half-spaces in $\mathbb{R}^n$ in general position, i.e. such that the dimension of the intersection of their boundaries is equal to $n - m$.

Most (all?) theorems concerning closed manifolds X with $Sc \ge \sigma$ and, more visibly, manifolds with smooth boundaries $Y = \partial X$, have (some proven, some conjectural) counterparts for Riemannian manifolds X with *corners* on the boundary, where the mean curvature $mean.curv(\partial X)$ for the smooth part of ∂X plays the role of *singular/distributional* scalar curvature supported on ∂X and where the *dihedral angles* $\angle$ along the corners, or rather the complementary angles $\pi - \angle$, can be regarded as *singular/distributional* mean curvature supported on the corners.

We bring several examples in tis section illustrating this idea starting with the following definitions.

Domination with Corners. A proper continuous map between manifolds with corners, $f : X \to \underline{X}$ called *corner proper* if the codimension 1 faces $F_i \subset \partial X$ are equal to the pullbacks $f^{-1}(\underline{F}_i)$ of the codimension 1 faces of $\underline{F}_i \subset \partial \underline{X}$.

Such a map f between *equidimensional* manifolds is called *proper domination* if both manifolds are orientable and f has non-zero degree. [283]

Extremality. Given a class $\mathcal{F}$ of manifolds X along with dominating maps f from X to a Riemannin manifold $\underline{X}$ with corners, call $\underline{X}$ *extremal* with respect to $\mathcal{F}$ if no map $f \in \mathcal{F}$ can be simultaneously

[283] This definition can be generalized by allowing proper maps that sends some of the ends of X to points and also maps from spin manifolds with with non-zero $\hat{A}$-degrees, i.e. with non-zero $\hat{A}$-genera of pullbacks of generic points, but we don't do it this here, since we want to emphasize the corner aspect of the story.

(a) "geometrically contracting" and

(b) *"scalar and mean curvatures decreasing"* at all points,

where an appropriate (but not the only one) specific meaning of these (a) and (b) is expressed by the following three pointwise inequalities, call them **three "≤"**, concerning

the scalar curvatures versus area contraction in the interiors of the manifolds,

the mean curvatures of their boundaries in (the interiors of) the $(n-1)$-faces $F_i \subset \partial X$,

the dihedral angles along the $(n-2)$-faces $F_{ij} \subset F_i \cap F_j$.

$$[codim = 0] \qquad Sc(f(x)) \geq \| \wedge^2 df(x)\| \cdot Sc(x), \ x \in X,$$

$$[codim = 1] \qquad mean.curv(\underline{F}_i, f(y)) \leq \|df\| \cdot mean.curv(F_i, y), \ y \in F_i \subset \partial X,$$

$$[codim = 2] \qquad \pi - \angle(\underline{F}_{ij}, f(z)) \leq \pi = \angle(F_{ij}, z), \ z \in F_{i,j} \subset F_i \cap F_j \subset \partial X.$$

Observe that the fist inequality [codim=0] is satisfied by all maps f, whenever $\underline{X}$ is scalar flat ($Sc(\underline{X} = 0)$ and $Sc(X) \geq 0$. No condition on the norms of the differentials df or on the exterior powers $\wedge^2 df$ is needed here. However, we (usually) require in this case that $Sc(X, x) \geq 0$ even at the points $x \in X$ where $df(x) = 0$.

Similarly the second inequality is automatic, if the faces $\underline{F}_i$ are minimal (mean.curv=0) and the faces the boundary is mean convex, $mean.curv F_i \geq 0$, e.g. where $\underline{X}$ is a convex polyhedron in $\mathbb{R}^n$. In this case, however we (usually) require that $Sc(X) \geq 0$ and the boundary of X is mean convex.

Now, an orientable n-manifold $\underline{X}$ with corners is called *extremal* with respect to a class $\mathcal{F}$ of domination maps f from Riemannian manifolds&maps from $\mathcal{F}$ if none of these inequalities **three "≤"** for $(X, f) \in \mathcal{F}$ can be strict at any point, i.e. **three "≤"** imply that

$$Sc(f(x)) = \| \wedge^2 df(x)\| \leq Sc(x), \ x \in X,$$

$$mean.curv(\underline{F}_i)(X) f(y)) = mean.curv(F_i, y), \ y \in F_i \subset \partial X,$$

$$\angle(\underline{F}_{ij}, f(z)) = \angle(F_{ij}, z), \ z \in F_{i,j} \subset F_i \cap F_j \subset \partial X.$$

Exercises. (a) Show that the set of *extremal* Riemannian metrics $\underline{g}$ on a smooth manifold with corners $\underline{X}$ (extremality of $\underline{g}$ means that for the manifold $(\underline{X}, \underline{g})$) is closed in the C^2-topology in the space of Riemannin metrics on $\underline{X}$, provided this extremality is understood for a class $\mathcal{F}$ in which manifolds X have $Sc(X) \geq 0$ and $mean.curv(F_i) \geq 0$.

Hint. Adapt the redistribution of curvature arguments from section 11.2 in [G(inequalities) 2018].

(b) Let g_0 be a smooth Riemannin metric on a manifold X with corners and let $x_0 \in X$ be a point in X.

Show that there exists a smooth deformation g_t, $t \geq 0$, of g_0 supported in a given arbitrarily small neighbourhood $U_0 \subset X$ of x_0 and such that

$\bullet_0$ if x_0 lies in the interior of $\subset X$ then the Scalar curvature $Sc(X, x_0)$ is strictly decreasing;

$\bullet_1$ if x_0 lies in the interior of a codimension 1 face $F_i \subset X$ then the mean curvature of $mean.curv_{g_t}(F_i, x_0)$ is strictly decreasing, while the scalar curvature of X is nowhere decreasing;

$\bullet_2$ if x_0 is in the interior of a codimension 2 face $F_{ij} \cap F_j \subset F_j$, then the dihedral angle at this point $\angle_{g_t}(x_0)$ is strictly increasing while the scalar curvature of X and the mean curvatures of the faces are nowhere decreasing.

(i) the curvatures of g are constant, $Sc(X) = \sigma$;

(ii) the faces of all edges F_i are also constant, $mean.curv_g(F_i) = M_i$,

and such that g these g are *locally extremal* :

if a deformation of g doesn't decrease the scalar curvature of X, of the mean curvatures of F_i, and of the complementary angles between the edges $\pi - \angle_{i,j}$,

Rigidity. A Riemannian manifold X with corners is called *rigid* in $\mathcal{F}$ if **three "≤"** imply that small neighbourhoods $U_x \subset X$ of all points x are *isometric* to some neighbourhoods $\underline{U}_{\underline{x}}$ (depending on of the the image points $\underline{x} = f(x) \in \underline{X}$.

Recall that in the scalar flat case of complete manifolds rigidity often (but not always) follows from extremality via the Bourguignon-Kazdan -Warner perturbation theorem. Below is a possible generalization of this theorem to manifolds with corners, that however has limited applications.

Perturbation Conjecture. Let $\underline{X} = (\underline{X}, \underline{g}_0)$ be a complete Riemannin manifold with corners, such that $Sc(\underline{g}_0) = 0$ and such that all codimension 1 faces are minimal, $mean.curv(F_i) = 0$.

Then either $Ricci(\underline{X}) = 0$ and all faces F_i are totally geodesic, or the Riemannian metric g admits a bounded deformation g_t, which increases the scalar curvature and the mean curvatures of the faces

$$Sc(\underline{g}_t) > 0 \text{ and } mean.curv_{\underline{g}_t}(F_i) > 0, \text{ for } t > 0,$$

and also decreasing the dihedral angles, $\angle_{ij}(\underline{g}_t) = \angle_{g_t} F_i, F_j) < \angle_{ij}(\underline{g}_0)$.

Remarks/Questions. (a) It is unclear what is a similar perturbation property (if any) for *non-scalar flat* (potentially extremal) manifolds with corners.

(b) It is easy to see that extremal surfaces ($\underline{X}$) with corners are rigid: these have constant curvatures and in the case of $Sc \geq 0$, they have geodesic edges.

(c) Quadrilaterals $\underline{X}$ in the hyperbolic plane, such that

(i) all angles $\frac{\pi}{2}$;

(ii) two opposite geodesic edges ($mean.curv = 0$) of equal length, and the two other segments are concentric horospherical (with $mean.curv = \pm 1$),

are rigid.

Let an X dominate $\underline{X}$, Then, if and $Sc(X) \geq -2$, if $mean.curv F_i \geq mean.curv(\underline{F}_i$ and if the angles between adjacent edges in X are all $\leq \frac{\pi}{2}$, then X is *isometric to a hyperbolic* (i)&(ii)-*quadrilateral.*

It seems, there are no similarly rigid hyperbolic k-gons besides these quadrilaterals.

Extremality/Rigidity Problem. Identify/classify extremal and rigid Riemannian manifolds $\underline{X}$ with corners for various classes $\mathcal{F}$ of manifolds X and dominating maps $f : X \to \underline{X}$.

Two motivating examples, where this problems was solved, is the *rigidity of flat metrics on closed manifolds*[284] and

the Goette-Semmelmann theorem, extended by Lott to compact Riemannian manifolds $\underline{X}$ with *smooth* that claims that the following three conditions are *sufficient for extremality of an orientable $\underline{X}$ in the class of spin manifolds X that dominate $\underline{X}$ and have $Sc(X) \geq 0$ and mean convex boundaries.*

(1) The curvature operator of $\underline{X}$ is *non-negative*.

(2) The boundary of $\underline{X}$ is *convex*.

(3) The dimension n of $\underline{X}$ is *even* and the Euler characteristic of $\underline{X}$ is non-zero.

Moreover, such an $\underline{X}$ =X=(X, g) is rigid in certain cases, e.g. if

$$0 < Ricci(\underline{g}) < \frac{1}{2}Sc(\underline{g}) \cdot \underline{g}.$$

Conjecturally, this holds for *all compact Riemannian manifolds with corners*, which satisfy (1) and (2) and with no extra topological assumptions, i.e. possibly *non-spin and with* $\chi(\underline{X}) = 0$.

This may be too strong to be true even for Riemannin flat manifolds, where this reads as follows.

Flat Corner Domination Conjecture. Let $\underline{X}$ be a compact orientable Riemannin flat n-manifold with corners, such that all codimension 1 faces $\underline{F}_i$ are flat, e.g. X is a convex polyhedron in the Euclidean space $\mathbb{R}^n$.

Then $\underline{X}$ is *rigid*:

if a proper corner map f of *non-zero degree* from a compact Riemannian manifold X with $Sc(X) \geq 0$, with mean convex faces F_i and with the dihedral angles between these faces at all points bounded by the corresponding angles $\angle(\underline{F}_i, \underline{F}_j)$, then

X is also Riemannin flat, the faces F_i are flat, the dihedral angles between F_i and F_j are equal to $\angle(\underline{F}_i, \underline{F}_j)$; moreover, at all points $x \in X$ the manifold X is locally isometric to $\underline{X}$ at $f(x) \in \underline{X}$.

Although this remains problematic even in the category of *convex polyhedra*, where rigidity is known only for infinitesimal deformations, see section 3.1.1 and IV below, the following results are available.

I. $\times\blacktriangle^i$*-Inequality.* Let $X_0 \subset \mathbb{R}^n$. Let $\underline{X}$ be a compact orientable Riemannian flat n-manifold with corners, where all $(n-1)$-faces $\underline{F}_i$ are flat.

If all dihedral angles $\angle_{i,j} = \angle(\underline{F}_i, \underline{F}_j)$ in $\underline{X}$ are $\leq \frac{\pi}{2}$ then $\underline{X}$ is spin extremal:

if an orientable *spin* manifold X, which *dominates* $\underline{X}$, i.e. comes with a proper corner map $f : X \to \underline{X}$ with *non-zero degree* and such that

$\bullet_0$ $Sc(X) \geq 0$

$\bullet_1$ $mean.curv(F_i) \geq 0$

$\bullet_2$ $\angle(F_i, F_j) \leq \angle(\underline{F}_i, \underline{F}_j)$,

then

$$Sc(X) = 0,\ mean.curv(F_i) = 0,\ \angle(F_i, F_j) = \angle(\underline{F}_i, \underline{F}_j).$$

Remark/Example. (a) If $\underline{X}$ simply connected, thus, is isometric to a convex polyhedron in $\mathbb{R}^n$ then the condition $\angle_{i,j} \leq \frac{\pi}{2}$ implies (by an elementary argument)

[284]This, recall, in the case of non-spin manifolds X of dimensions $n \geq 10$, needs Lohkamp's or Schoen-Yau's desingularizations theorems.

that $\underline{X}$ is the product of simplices with dihedral angles $\leq \frac{\pi}{2}$, such as the n-cube, fo instance.

About the Proof. The condition $\angle(\underline{F}_i, \underline{F}_j) \leq \frac{\pi}{2}$, shows (see section 4.4) that a suitable smoothing of the boundaries of $\underline{X}$ and X reduces the problem to the rigidity in the smooth case. For instance if $\underline{X}$ is a convex polyhedron one may use the mean curvature spin extremality theorem $[Y_{spin} \to \bigcirc]$ from section 3.5.(If n is even, is follows from the above Goette-Semmelmann-Lott theorem.)

Exercise. Directly prove the $\times\blacktriangle^i$- Inequality in the case, where the faces $F_i \subset X$ are it convex, rather than only mean convex.

$\blacktriangle$ – *Remark.* If both $\underline{X}$ and X are affine n-simplices, then the implication

$$\angle_{ij}(X) \leq \angle_{ij}(\underline{X}) \Rightarrow \angle_{ij}(X) = \angle_{ij}(\underline{X})$$

follows from the Kirszbraun theorem with no need for the condition $\angle_{ji} \leq \pi/2$.

But there is no no direct elementary proof of this (unless I am missing something obvious) if X has *convex*, rather than flat, faces

Question. Are there "good" local boundary conditions for Dirac operators on manifolds with corners suitable for proving this kind of theorems similar to what is done by John Lott in [Lott(boundary) 2020] and by Christian Bär with Bernhard Hanke in [Bär]-Hanke(boundary) 2021] for manifolds with smooth boundaries?

(Such conditions seem plausible for orbifold like corners, especially for good orbifolds[285] but the general case is not so clear.)

II. *Reflection Orbifolds.* Let $\hat{X}$ be a smooth manifold acted upon by a (reflection) group Γ generated by reflections in cooriented hypersurfaces $\hat{F}_i \subset X$ and let $X \subset \hat{X}$ be the fundamental domain for this action that is the intersection of the "half-spaces" $\hat{X}_i \subset \hat{X}_i$ bounded by $\hat{F}_i \subset \hat{X}_i$ in X.

This $X = \hat{X}/\Gamma$ comes with a natural corner structure and if the action of Γ is isometric for a Riemannian metric $\hat{g}$ on $\hat{X}$, then all codimension 2-faces $F_{ij} \in X$ are endowed with angles of the form $\alpha_{ij}(\Gamma) = \frac{\pi}{2l}$, $l = 1, 2, \ldots$.

We have already explained in section 3.1.1 that

if $\hat{X}$ admits no Γ-invariant metric with $Sc > 0$ then X satisfies the following

No $Sc > 0$ Property. Let g be a Riemannian metric g on X, such that

$\bullet_{Sc}$ *the scalar curvature of g is non-negative: $Sc(g) \geq 0$;*

$\bullet_{mean}$ *all $(n-1)$-faces F_i of X are mean convex: $mean.curv_g(F_i) \geq 0$;*

$\bullet_{\angle}$ *The dihedral angles $\angle_{ij}$ of X at all points of all $(n-2)$-faces $F_{ij} = F_i \cap F_j \subset X$ are bounded by the canonical ones, $\angle_{ij} \leq \alpha_{ij}(\Gamma)$.*

Then

$$Sc(g) = 0, \quad mean.curv_g(Y_{reg}) = 0, \text{ and } \angle_{ij} \leq \alpha_{ij}(\Gamma)$$

About the Proof. This is shown by reflecting X around its $(n-1)$-faces, smoothing around the edges and applying the corresponding result for closed manifolds as it was done in [G(billiard] 2014][286] for cubical X, and where the

[285] Compare with what is done in [Bunke(orbifolds) 2007] and in related paper cited in there.

[286] When writing this paper I overlooked the paper by Brendle, Marques and Neves [[Bre-Mar-Nev(hemisphere) 2011], where an essential step of smoothing codimension one corners appears as theorem 5.

general case needs an intervention of arguments from [G(inequalities) 2018], where the (non-spin) case $n \geq 9$ relies on [SY(singularities) 2017].

An immediate application of of this to manifolds X which dominate Euclidean reflection domains $\underline{X}$ is the following

Extremality of Euclidean Reflection Domains. *If $\hat{X} = \mathbb{R}^n$ and the reflections in Γ are isometric then the orbifold $\underline{X} = \mathbb{R}^n/\Gamma$ is extremal.*

Remark. Since the reflection domains have their dihedral angles $\angle_{ij} \leq \frac{pi}{2}$ their *spin* extremality follows from the above $\times\blacktriangle^i$-Inequality.

▲-Rigidity. This says that, in fact, *X is Riemannin flat and all faces F_{ij} are also flat.*

Proof . The quickest proof of the rigidity is technical, namely it relies on the *regularization theorem* proven in [Burkhart-Guim(regularizing Ricci flow) 2019]:

If a continuous metric g_0 on a Riemannin manifold can be C^0-approximated by smooth metrics g_ε, $\varepsilon > 0$, with $Sc(g_\varepsilon \geq \sigma_0 - \varepsilon$ for $\varepsilon \to 0$, then it can be approximated by smooth metrics with $Sc \geq \sigma_0$.

We apply this to the γ-invariant metric $\hat{g}_0$ on $\hat{X}$ that extend the, a priori *non-Riemannian*, metric g_0 on $X \subset \hat{X}$, but but because of the equalities $\angle_{ij} \leq \alpha_{ij}(\Gamma)$ guarantied by the weak rigidity, this g_0 Riemannian and the *regularization theorem* does apply and then an easy argument shows that the metric g_0 itself is Riemannian flat and the faces F_i are flat as well.

Remarks. (a) The rigidity for cubical X of dimension ≤ 7 was originally proven by Ciao Li in [Li(rigidity) 2019] and then extended in the second version of his paper to manifolds with the corner structures combinatorially isomorphic to that in the product of the cube $\square^{n-2}$ by an acute angled triangle $\triangle$, where an essential novel point in this paper is the proof of a sufficient regularity of minimal surfaces at the corners that allows one to argue as in the proof of the rigidity of flat tori. [287]

(The products of cubes by general triangles considered by Li are not, in general reflection orbihedra. On the other hand, the above argument with reflections+Ricci flow, implies, for instance, *rigidity of products* of several *regular* triangles, where no present day minimal hypersurface argument applies.)

(b) (*Recapturing Rigidity while Smoothing the Corners .*

III. *Pyramids and Quasi-Prisms.* A counterpart $\times\blacktriangle^i$-*inequality* is known to hold for certain polytopes P with dihedral angles $> \frac{\pi}{2}$, which, much as the above products of simplices, are *extremal* in the sense that

can't make the dihedral angles smaller, while keeping the faces mean convex and the scalar curvature ≥ 0

The simples such extremal P are (convex) k-gonal prisms, where for $k \geq 4$ some dihedral angles are always $> \frac{\pi}{2}$. This is shown in [G(billiards) 2014] by looking at minimal surfaces with free boundary on the side-part of the boundary of P.

More generally Chao Li [Li(comparison) 2017] proved a similar property for convex *pyramids* and *quasi-prisms* P where the latter are convex polyhedra in $\mathbb{R}^3$, where all vertices are contained in a pair of parallel planes and where

[287] I haven't read Li's paper carefully and I am not certain on how actually he does it, but, granted regularity, the μ-bubble perturbation argument as in section 5.7 applies in the case considered by Li.

the proof follows by a construction and analysis of suitable μ-bubble (capillary surfaces) pinched between these planes.

(A technical limitation on deformations of the flat geometry in P, a mild lower bound on the dihedral angles between side faces allowed by Li's argument, was removed in his later paper.)

IV. *Polyhedral Extremality Problem.* The above kind of extremality, even the local one, remains problematic in general even for *simple* n-polytopes, where at most n faces of dimension $n-1$ may meet at the vertices:

it is unknown which pairs of combinatorially equivalent polytopes P and P' (convex polyhedra) may have their corresponding dihedral angles satisfying $\angle_{ij} \geq \angle'_{ij}$ without all corresponding angles being mutually equal.[288]

laer V. *Extremality and Rigidity of Hyperbolic Manifolds with Corners.* It is unclear, in general, what kind of extremality/rigidity one can expect from manifolds which may have have negative scalar curvature at some points, some non-mean convex faces and/or some dihedral angles $> \pi$.

But the extremality of flat manifolds $\underline{X} = (\underline{X}, \underline{g} = \underline{g}(x))$ with corners passes to the hyperbolic cylinders

$$\underline{X}^{\rtimes}_d(-1) = (\underline{X} \times [0,d], g^{\rtimes}_{\exp} = e^{2t}\underline{g}(x) + dt^2), 0 \leq t \leq d,$$

with constant sectional curvatures $\kappa(g^{\rtimes}_{\exp}) = -1$. Namely, these $\underline{X}^{\rtimes}_d(-1)$ can't be dominated with manifolds with strict increase of the scalar curvature, increase of the mean curvatures of the faces and decrease of the dihedral angles, in the case of extremal X.

In fact, the above reflection, doublings and smoothing arguments apply to these $\underline{X}^{\rtimes}_d(-1)$ in conjunction with the existence and basic properties of stable μ-bubles Y in the cylinders $\underline{X}^{\rtimes}_d(-1)$, which separate the "bottom" $\underline{X} \times \{0\} \subset \underline{X}^{\rtimes}_d(-1)$ from "top" $\underline{X} \times \{d\} \subset \underline{X}^{\rtimes}_d(-1)$, which have constant mean curvatures $n-1$ and such that some warped products $Y\mathbb{T}^1$ have non-negative scalar curvatures. see section 5.6,

However, there are two technical caveats to this reasoning.

(1_{reg}) If $n+1 = dim(\underline{X}^{\rtimes}_d(-1)) \geq 8$ the bubles Y may, a priori, have stable singularities where the present day state of desingularization art of Lohkamp-Schoen-Yau is not, at least not immediately, applicable to all cases of interest.

(2_{reg}) Even for $n \leq 7$, the bubbles $Y \subset \underline{X} \times \{d\}$ are not fully smooth, at the corners, where the dihedral angles $\angle_i j(x) \neq \frac{pi}{2k}$, and where, the unconditional implication

$$\text{X is extremal} \Rightarrow \underline{X}^{\rtimes}_d(-1) \text{ is extremal}$$

and even more so

$$\text{X is rigid} \Rightarrow \underline{X}^{\rtimes}_d(-1) \text{ is rigid}$$

needs a bit of technical reasoning.

Motivations for Corners. Besides opening avenues for generalisations of what is known for smooth manifolds, Riemannian manifolds with corners and $Sc \geq \sigma$ may do good to the following.

1. Suggesting new techniques, (calculus of variations, Dirac operator) for the study of Euclidean polyhedra.

[288] As we mentioned in 3.1.1, Karim Adiprasito told me that *Schläfli formula* (see [Souam (Schläfli) 2004]) implies that *no convex polytope admits an infinitesimal deformability on simultaneously decreasing all its dihedral angles.*

2. Organising the totality of manifolds with $Sc \geq 0$ (or, more generally with $Sc \geq \sigma$) into a nice category (A_∞-category?) $\mathcal{P}^\square$, that would include, as objects manifolds Y with Riemannian metrics h and functions M on them and where *morphisms* are *(co)bordisms* (*h-cobordisms*?) (X, g), $\partial X = Y_0 \cup Y_1$, where g is a Riemannian metric on X with $Sc \geq 0$, which restricts to h_0 and to h_1 on Y_0 and Y_1 and where the the mean curvature of Y_0 with inward coorientation is equal to $-M_0$ while the mean curvature of Y_1 with the outward coorientation is equal to M_1.

Conceivably, the variational techniques for "flags" of hypersurfaces from [SY(singularities) 2017] or its generalisation(s), may have a meaningful interpretation in $\mathcal{P}^\square$, while a suitably adapted Dirac operator method may serve as a quantisation of $\mathcal{P}^\square$.

Comprehensive $Sc \geq \sigma$ Existence Problem for Manifolds with Corners.

Let X be a smooth compact manifold with corners and let X_i, $i \in I$, [289] be tote the faces here X_i he set of faces of X of all (co)dimensions, where, we agree that $X_0 = X$ and let $\sigma_i : X_i \to [-\infty, \infty)$, $\mu_{i,k} : X_i \to [-\infty, \infty)$ and $\alpha_{ijk} : X_i \cap X_j \to (0, 2\pi)$ be continuous functions, where

μ_{ik} are defined for all i and those k for which X_i serve as codimension one faces in X_k;

α_{ijk} are defined for the pairs of codimension one sub-faces $X_i, X_j \subset X_k$, such that $dim(X_i \cap X_j) = dim(X_i) - 1 = dim(X_j) - 1 = dim(X_k) - 2$.

When does there exist a smooth Riemannian metric g on X, such that

$\bullet_{Sc}$ the scalar curvatures of g restricted to X_i are bounded from below by σ_i, that is

$$Sc(g_{|X_i}, x) \geq \sigma_i(x),\ x \in X_i;$$

$\bullet_{mean}$ the mean curvatures of $X_i \subset X_k$ with respect to g, for all X_i and X_k, where $dim(X_i) = dim(X_k) - 1$, are bounded from below by μ_{ik};

$\bullet_\angle$ the dihedral angles between X_i and X_j in X_k satisfy

$$\angle_g(X_i, X_j) \leq \alpha_{ijk}.$$

More generally, one wants to understand the topology (e.g. the homotopy type) of the spaces $G(\sigma_i, \mu_{ik}, \alpha_{ijk})$ of metrics g on X, which satisfy $\bullet_{Sc}$, $\bullet_{mean}$ and $\bullet_\angle$, as well as of the inclusions

$$G(\sigma_i, \mu_{ik}, \alpha_{ijk}) \hookrightarrow G(\sigma'_i, \mu'_{ik}, \alpha'_{ijk})$$

for $\sigma'_i \leq \sigma_i$, $\mu'_{ik} \leq \mu_{ik}$ and $\alpha'_{ijk} \geq \alpha'_{ijk}$ and restriction maps from these spaces to the corresponding ones on submanifolds $Y \subset X$ compatible with the corner structures. .

Exercise. Let X be a smooth n-manifold with cornered boundary $Y = \partial X$, and prescribed mean curvatures of the top dimensional faces X_i and the dihedral angles between them, that is, in the above notation:

$\sigma_i = -\infty$, unless $dim(X_i) = n - 1$,

[289]We switched the notation from F_i to X_i to place all faces, includung X itself, on equal footing.

$\mu_{i,k} = -\infty$, unless $k = 0$, i.e. $dim(X_i) = dim(X_k) - 1 = n - 1$ and
$\alpha_{ijk} = 2\pi$, unless $dim(X_i) = din(X_j) = dim(X_k) - 1 = n - 1$.

Show that X admits a smooth metric g, such that

the scalar curvature of g is positive in a (small) neighbourhood of $Y \subset X$, and such that g satisfies the above conditions $\bullet_{mean}$ and $\bullet_{\angle}$.

Hint. Construct g in a small neighbourhood of the union of the i-dimensional faces by induction on $i = 1, 2, ..., n-1$.

Measure Valued Curvature. The mean curvature of the boundary ∂X and the complementary dihedral angles $\pi - \angle_{ij}$ can be regarded as measures with continuous densities on the faces which represent singular scalar curvature, where this becomes especially clear if you think in terms of the double $\mathbb{D}X$.

With this in mind, the above problem can reformulated as follows:

given a triangulated n-dimensional manifold X(pseudomanifold?) and numbers $\sigma = \sigma(\Delta)$ assigned to all simplices Δ of codimensions 0, 1 and 2.

When does there exist a continuous piecewise smooth Riemannian metric on X, such that its scalar curvature, understood as a measure, is bounded from below by $\sigma(\Delta)$ on all of the above Δ,

where the inequality $Sc(X|\Delta) \geq \sigma(\Delta)$ is understood as earlier, namely,

(i) if $dim(\Delta) = n$ this is the usual $Sc \geq \sigma$;

(ii) if $dim(\Delta) = n-1$ this is the σ-bound on the sum of the mean curvatures of this Δ in the two adjacent n-simplices;

(iii) if $codim(\Delta) = n-2$ this is 2π minus the sum of the dihedral angles of the n-simplices adjacent to Δ.

Remark on Higher Codimension Singularity. Strictly speaking the above applies not to triangulated but to stratified manifolds X, where

- there are only strata of codimensions 0, 1 and 2,
- the codimension 2 strata are smooth submanifolds in X,
- the codimension 1 strata Σ_{-1} are submanifolds with boundaries with all components of these boundaries being codimension 2 strata Σ_{-2}, where different Σ_{-1} with common components Σ_{-2} of their boundaries meet transversally at these Σ_{-2},
- the Riemannian metics in question are piecewise smooth with respect to this stratification.

One may also to allow singularities of codimensions ≥ 3, but this is a different matter (compare with (c) in 5.4.1).

3.18.1 Corners, Plateauhedra and Bubble Spaces

Central geometric examples of manifolds with corners are convex polyhedra in the Euclidean spaces and, more generally domains in spaces with constant sectional curvatures κ that are intersections on of half spaces bounded by by umbilic hypersurfaces, that are spheres, hyperplanes and, for $\kappa < 0$, horospheres and equidistances of hyperplanes.

What are Riemannin Counterparts of these?

Below are candidates for answers.

$\bullet_b$ *Bubblehedra.* A bubblehedron Q in a Riemannian n-manifold X is the boundary of a domain $Q_> \subset X$ with corners

$$Q = \partial Q_> = \bigcup_{i=1,2,...} Q_i$$

where all $(n-1)$-faces $Q_i \subset \partial X$ have *constant mean curvatures* M_i, where all *dihedral angles* $\angle_{ij}$ *are constant* along the $(n-2)$-faces $Q_i \cap Q_j$ and where one may require these angles to be $\leq \pi$.

A special case of these where all $M_i = 0$ are called *plateuhedra.*

Remark. The common description of minimal varieties in terms of currents doesn't seem appropriate for such Q, and even less so for similar arrangements of minimal subvarieties $Q_i \subset X$ of codimensions >1.

Example 1: Normal Plateauhedra and Bubblehedra. An attractive instance of these is where all dihedral angles $\angle_{ij} = \frac{\pi}{2}$ and where, moreover, each face Q_i is (n-1)-volume minimizing with free boundary in the union of the remaining edges,

$$\partial Q_i \subset \bigcup_{j \neq i} Q_j.$$

A more general similar case is where $M_i > 0$ and each Q_i with free boundary in $\bigcup_{j\neq i} Q_j$.. minimizes $vol_{n-1}(Q_i) - M_i \cdot vol_n(Q_>)$.

Example 2: Normal Plateau Webs. Let X be a compact Riemannin manifold and let $Y_1 \subset X$ be a a closed locally minimizing minimal hypersurface in X. Next, let Y_2 be a locally minimizing minimal hypersurface in the complement of Y_1 in X with free boundary in ∂Y_1, i.e.

$$Y_2 \setminus \partial Y_2 \subset X \setminus Y_1 \text{ and } \partial Y_2 \subset Y_1.$$

Then continue with minimal $Y_3, ..., Y_i...$, where $Y_i \subset X$ lies in the complement of all $Y_j, j < i$ and has free boundary in the union of these Y_j,

$$Y_i \setminus \partial Y_i \subset X \setminus \bigcup_{j<i} Y_j, \ \partial Y_i \subset \bigcup_{j<i} Y_j.$$

Thus we divide X into mean convex domains with 90° dihedral angles.

Questions. What do combinatorics of such webs $\{Y_i\}$ tell you about the topology and geometry of X?

How much does positivity of the scalar curvature X restrict combinatorics of such a $\{Y_i\}$?

If X is complete non-compact, where "plateau" may be too restrictive, and asks:

what geometric/topological condition(s) on X would guarantee the existence of normal *bubblehedra* $Q \subset X$ of given combinatorial types?

Conjectural Example 3: Dodecahedral and Similar Exhaustions of Large Manifolds. If $n = 3$ and $\tilde{X}$ is a the universal covering of a compact Riemannian manifold X, where this X admits a hyperbolic Riemannin metric (probbaly, non-zero simplicial volume will do), then it seems not hard to show that it can be exhausted by (compact domains $Q_> \subset \tilde{X}$ bounded by) such normal bubblehedra $Q \subset \tilde{X}$ of dodecahedral combinatorial types, i.e. admitting proper corner maps of degree 1 to (the boundary of) the dodecahedron).

Also "hyperbolically looking" manifolds $\tilde{X}$ of dimensions ≥ 4, e.g. the universal coverings of compact manifolds X with *non-zero simplicial volumes*, can probably be exhausted by similar Q.

For instance, if X is the product of surfaces of genera ≥ 2, then such an exhaustion is expected by normal bubblehedra Q of combinatorial types of products of k-gons.

Example 4; Local Riemannian Realization of Euclidean P. Let P be a convex polyhedron in a tangent space $T_{x_0}(X) = \mathbb{R}^n$, let us scale P by a small $\varepsilon > 0$ and let $P'_\varepsilon \subset X$ be the image of this $\varepsilon P \subset T_x(X)$ under the exponential map $\exp : \varepsilon P \subset T_x(X) \to X$.

This $P'_\varepsilon \subset X$, which is not a true but only an *ε-approximate plateauhedron*, already may carry some information about the scalar curvature $Sc(X,x)$, in terms of the mean curvatures of its $(n-1)$-faces and the dihedral angles along its $(n-2)$-faces similar to the ⊛ representation of the inequality $Sc(X,x_0) < Sc(X',x'_0)$ in section 1.1 by comparison the integral mean curvatures of the ε-spheres around the points x_0 and x_0 in two manifolds.

Next, to make this $P'_\varepsilon \subset X$ look prettier, one can slightly perturb it and thus turn it into a true bubblehedron $Q \subset X$ by solving the Plateau soap bubble problems with free boundaries for all $(n-1)$-faces one after another[290] where, depending of what one wants, one can either make all its *faces with zero mean curvatures*, or all its *dihedral angles equal those of the original P*. (This seems easy but I didn't try to carefully check it.[291])

We know, however that if $Sc(X,x_0) > 0$, there are some constraints on possible values of the mean curvatures $M_i(Q) = mean.curv(Q_i)$ and the dihedral angles $\angle_{ij}(Q)$ of such a Q, e.g. if $\angle_{ij} \leq \frac{\pi}{2}$, then, for small $\varepsilon \to 0$, one can't have all $M_i(Q) \geq M_i(P)$ and $\angle_{ij}(Q) \leq \angle_{ij}(Q)$, where conjecturally this is true for all P.

Despite this, in general, if $n \geq 3$, the space $\mathcal{Q}$ of all bubblehedra (or plateuhedra) Q in a small neighbourhood of P'_ε is typically *infinite dimensional*.

Example 5: Too Many Q. Let a plateuhedron Q in a Riemannin manifold X contains only two $(n-1)$-faces Q_1 and Q_2, which are compact smooth hypersurfaces in X the common boundary of which makes the only $(n-2)$-face of Q,

$$Q_{12} = Q_1 \cap Q_2 = \partial Q_1 = \partial Q_2$$

Imagine that Q_1 extends in $X \supset Q_1$ beyond its boundary to a minimal $Q_{1+} \supset Q_1$, such that

$\bullet_{1+}$ the extended face $Q_{1+} \subset X$ is *a strictly locally minimizing hypersurface*[292] with respect to its boundary $Z = \partial Q_{1+}$;

[290] One can't, a priori, guarantee the full (not even C^2) regularity of the $(n-k)$ faces for $k \geq 2$, but in view of high non-uniqueness of these Q explained below, such regularity seems non-impossible in many cases.

[291] To fully include ⊛ in this picture, one had to start with a P, which has *spherical* as well as planar faces. But then perturbing P'_ε into a bubblehedron Q becomes a more delicate matter. For instance, if, as it is in ⊛, our $P \subset T_x(X)$ is a ball bounded by a single spherical "face", then the corresponding $Q \subset X$ (bounded by a single hypersurface of constant mean curvature) may (and usually will) drift away from the point x.

[292] "*Strict* local minimum" usually means "*isolated* local minimum" – this is sufficient for most our present geometric purposes. But if following an analytic vein of thinking, "strict" should be understood as *strict positivity* of the second variation operator.

$\bullet_2$ the face Q_2 is *strictly locally minimising with free boundary* in Q_{1+}.

Then small deformations Z' of the (smooth closed) $(n-2)$-submanifold $Z \subset X$ are (by an elementary elliptic perturbation argument) accompanied by *unique minimal* deformations Q'_{1+} of Q_{1+} i.e. submanifolds Q'_{1+} are minimal) followed by *minimal* Q'_2 that are small deformations of Q_2, such that the boundary of such a of Q'_2 is *contained in* Q'_{1+} and where Q'_2 is *normal* to Q'_{1+} along $\partial Q'_2$.

Thus the local moduli space of these Q contains, as subspace, the *full space of small functions on* Z corresponding to small deformations of Z *normal* to Q_{1+}.[293]

Example 6: Too few Q. Let *both faces* Q_1 and Q_2 in the above example extend to minimal hypersurfaces beyond their boundaries, say to $Q_{1+} \supset Q_1$ and $Q_{2+} \supset Q_2$, and let both be *a strictly locally minimizing hypersurface* with respect to their boundaries $Z_1 = \partial Q_{1+}$ and $Z_2 = \partial Q_{2+}$.

Let Z'_1 and Z'_2 be small perturbations of these $\partial Q'_{1+}$ and $\partial Q'_{2+}$, let Q'_{1+}, and Q'_{2+} be the corresponding minimal perturbations of Q_{1+}, and Q_{2+}, let

$$Q'_{12} = Q'_{1+} \cap Q'_{2+}$$

and let $\angle'_{12}$ be the dihedral angle between Q'_{1+}, Q'_{2+} regarded as a function on the perturbed intersection Q'_{12} of the two $(n-1)$-faces of Q,

$$\angle'_{12} = \angle'_{12}(q'),\ q' \in Q'_{12}.$$

Here, the situation is opposite to that in the previous example:

the operator (map)

$$(Z'_1, Z'_2) \mapsto \angle'_{12}$$

from the space of small deformations of the boundaries of Q_{1+} *and* Q_{2+} *to the space of functions on* Q_{12}[294] *is (by elliptic regularity) compact.*

Hence, only

a *minority of functions on* Q_{12} is realizable by dihedral angles of (not quite) plateuhedra with *minimally extendable faces.*

Ouroboros Example 7: Biting its Own Tail. Let us describe a class of hypersurfaces, where the two opposite phenomena from the above examples strike a balance and make the Plateau problem "well posed", in particular, allowing its *a Fredholm representation.*[295]

Let $Q \subset X$ be the image of a compact $(n-1)$-manifold,[296] $n = dim(X)$, with boundary, say $\hat{Q}$, immersed to X,

$$h : \hat{Q} \to X,\ h(\hat{Q}) = Q,$$

such that

$\bullet_{int}$ the immersion h is one-to one on the interior of Q,

$$h : \hat{Q} \setminus \partial\hat{Q} \hookrightarrow X,$$

[293]Deformations of Z within Q_{1+} don't affect Q.

[294]Normally project perturbed intersections Q'_{12} to Q_{12} and thus identify the spaces of functions on all Q'_{12} with the space f functions on the unperturbed Q_{12}.

[295]The concept of "Fredholm" strikes as artificial in the present geometric picture and begs for something more adequate. Perhaps, I am missing something in the literature.

[296]Much of what follows makes sense for Q of codimension >1.

where the images of the connected components of $\hat{Q}$ serve as the $(n-1)$-faces Q_i of Q;

$\bullet_\partial$ the immersion h is one-to one on the boundary of $\hat{Q}$;

$$h : \hat{Q} \setminus \partial\hat{Q} \hookrightarrow X;$$

;

$\bullet_\subset$ the image of the boundary of $\hat{Q}$ is contained in the image of its interior,

$$h(\partial\hat{Q}) \subset h(\hat{Q} \setminus \partial\hat{Q}),$$

where the images of the connected components of $\partial\hat{Q}$ serve as the $(n-2)$-faces or corners of Q;

$\bullet_{min}$ *Q is minimal*: it has zero mean curvature and it is normal to itself along the corners.

Sub-Example 8. The most transparent instance of this is where Q is the union of two faces that are smooth submanifolds in X with boundaries,

$$Q = Q_1 \cup Q_2 \subset X,$$

such that the boundary of one is contained in the interior of another,

$$\partial Q_1 \subset int(Q_2) \text{ and } \partial Q_2 \subset int(Q_1).$$

Thus, the the corner of Q is equal the intersection of the two faces of Q,

$$Q_{12} = Q_1 \cap Q_2,$$

and where one may think of Q_1 as the solution of the Plateau problem with free boundary in Q_2 and, similarly, Q_2 is minimal with boundary in Q_1.

Proposition/Example 9: Finite dimensionality of Deformations and Codeformations. It seems obvious, (I didn't check this carefully) that, by the standard elliptic estimates, the space of the above compact minimal Q in a small C^∞-neighbourhood of a given minimal Q_0 is finite dimensional.

It is slightly less obvious that, given an above minimal $Q \subset X = (X, g)$, there exists a *finite dimensional*[297] linear space Δ of C^∞-smooth quadratic forms δ on X, such that, for all Riemannin metrics g' sufficiently C^∞-close to g

there exit a small $\delta \in \Delta$ and C^∞-small perturbation Q' of Q such that

Q' is *minimal with respect to the Riemannin metric* $g' + \delta$.

Let us explain this in he simplest case where $Q = Q_1 \cup Q_2 \subset X = (X, g)$ as in the above sub-example, where we assume that both Q_1 and Q_2 are *strictly locally minimizing* with the free boundary conditions $\partial Q_1 \subset Q_2$ and $\partial Q_2 \subset Q_1$.

Slightly C^∞-perturb the Riemannin metric in X, say $g \rightsquigarrow g'$, and show that Q can be accordingly deformed to $Q' \subset X$, which is strictly $(n-1)$-volume minimizing with respect to g' with similar free boundary conditions . [298]

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[297] This dimension can be bounded by the index of the second variation operator for Q.

[298] In the classical case, where $Q \subset X$ is a smooth closed strictly locally minimizing submanifold (no boundaries), it is not hard to show that it is stable under C^0-*small* perturbations of g; probbaly the same applies to Q with smooth edge(s) Q_{12} and, possibly to general *semi-regular* Q presented later in this section.

The simplest way to do it is by consecutively minimizing g'-volumes of Q_1 with free boundary $\partial Q_1 \subset Q_2$, then of the volume of Q_2 with boundary in the new g'-minimal Q_1, etc

Then, for sufficiently small $g - g'$, the strict minimality of P implies the convergence of this process to Q' which lies C^∞-close to Q (for the obvious C^∞-topology in the space of our $Q \subset X$) and g'-volume minimizing with free boundary positioned on the non-singular part of Q. [299]

Example 10: Higher Order Corners. Let us generalize the above sub-example by allowing piecewise smooth

$$Q = \cup Q_i \subset X$$

where all $Q_i \subset X$, $i = 1, 2, ..., k$, are submanifolds with corners, such that the boundary of each of them is contained in the union of others,

$$\partial Q_i \subset \bigcup_{j \neq i} Q_j.$$

More general Q of this kind is where such a decomposition exits only locally at all points in Q:

given a point $q \in Q \subset X$, there exists a neighbourhood of this point in X, say $U(x) \in X$, such that the intersection $Q \cap U(q)$ admits the above kind of decomposition

$$Q \cap U(q) = \bigcup_i Q_i(q), \text{ where } \partial Q_i(q) \subset \bigcup_{j \neq i} Q_j(q).$$

It still make sense here to speak of *minimal* Q, i.e. with all $mean.curv(Q_i(q)) = 0$ and where, minimality with free boundary in $\bigcup Q_j(q)$ is also well defined for all $Q_i(q)$.

Question. What is the most general assumption(s) on local topology of such Q that would imply the above kind Deformations and Codeformations finite dimensionality properties?

Example 11: Semi-regular P and Q. Recall that a *simple cone* in $\mathbb{R}^{n-1}$ is the intersection of at most $n - 1$ half spaces, with mutually transversal boundary hyperplanes.

Now, call a piecewise linear linear cone $P \subset \mathbb{R}^n$ *semi-regular* if it is equal to the union of $k \leq n$ mutually transversal simple cones P_i in some hyperplanes in $\mathbb{R}^n$,

$$P = \bigcup_i P_i,$$

such that the boundary of each P_i is contained in the union of the remaining ones,

$$\partial P_i \subset \bigcup_{j \neq i} P_j,$$

and, moreover, such that the interior of each $(n-2)$-face in P_i, for all i, is contained in the interior of some Q_j.

[299] This Q' can be *defined* as a fixed point of a self mapping in the space of P behind this iteration process, where the strictness of minimality makes this self-mapping (which, by the way, is compact) *contracting.*

Example 12: Cones of rank k=1, 2 and 3. The cones of rank 1 are just hyperplanes in $\mathbb{R}^n$.

A cones of rank 2 is a union of a hyperplane $P_1 \subset \mathbb{R}^n$ and a half-hyperplane $P_2 \subset \mathbb{R}^n$ with its boundary (an $(n-2)$-subspace) in P_1.

If $k = 3$, then there are two possibilities for the position of the third face $P_3 \subset P$: This can be either a half-hyperplane positioned in the halfspace bounded by P_1 on the other side from P_2 or be an $(n-1)$-cone with two $(n-2)$-faces which is positioned in one of the two convex cones bounded by P_1 and P_2.

Semi-Regular $Q \subset X$. A piece wise smooth $Q \subset X$ is called *semi-regular* if, locally, at each point it is diffeomorphic to a semi-regular cone.

Conjecture. *Minimal Semi-regular $Q \subset X = (X, g)$* enjoy the deformations and codeformations finite dimensionality properties.

Remark (a) This conjecture, is probbaly, not hard to prove but the semi-regularity condition is too restrictive and much of it seem unneeded, such as the transversality condition between the half-hyperplanes P_2 and P_3 attached along their boundaries on two different side to a hyperplane $P_1 \subset \mathbb{R}^n$ in the above rank 3 example.

More seriously, semi-regularity excludes singular minimal Q in dimensions $N \geq 8$.

(b) It remains unclear if our minimal Q have any global geometric significance.

(c) The full transversality condition, albeit, probbaly, redundant, implies the following convenient (irrelevant?) simple property.

Let $Q \subset X$ be compact semi-regular and let $Q' \subset X$ be C^∞-close to Q, which means all, locally defined, $(n-1)$-faces of Q' are close to the corresponding faces of Q. (This is formulated more carefully below.) Then there is a diffeomorphism $\Phi' : X \to X$ that moves Q to Q'; moreover, there is a C^∞-continuous map $\Psi' : Q' \to \Phi'$ from the space of $Q' \subset X$ to $Diff$ such that $\Psi'(Q')(Q) = Q'$ and $\Psi'(Q) = Id$.

Bubble-Spaces $Q \subset X$ with Variable Mean Curvatures. The basic properties, including deformations and codeformations finite dimensionality properties for semi-regular minimal Plateau spaces Q extend verbatim to to bubble spaces with constant mean curvatures M, including stability of strictly minimizing ones under variations of M keeping M constant.

But one needs be more careful with variable mean curvature of Q, since it is not and is *not supposed to be continous* as function on Q with the topology induced by the the *embedding* $Q \subset X$.

Another problem is comparing the mean curvatures of two different spaces, Q and Q' in X, let them even be very close one to another.

To handle this, we recall that Q is the image of a smooth manifold with boundary under a smooth immersion,

$$h : \hat{Q} \to X.$$

Accordingly, the mean curvature is required to be continuous, smooth if you wish, as a function on this $\hat{Q}$.

Also the C^r-distance for all $r < \infty$ between Q and Q' is defined as the infimum of the numbers d_r, such that there exists a diffeomorphism $\phi : \hat{Q} \to \hat{Q}'$ for which the C^r distance between the immersions $h : \hat{Q} \to \mathbb{R}^n$ and $\phi \circ h' : \hat{Q} \to \mathbb{R}^n$ for $h' : h : \hat{Q}' \to \mathbb{R}^n$ is $\leq d_r$.

Finally, the measures μ behind (the existence theorems for) μ-bubbles are not defined on $X \supset Q$ or on a neighbourhood $U = U_Q \supset Q$ in X but on an n-manifold $\hat{U} = \hat{U}_Q$, which comes with an *immersion* $\alpha : \hat{U} \to X$ and an *embedding* $\beta : \hat{Q} \to \hat{U}$ such that

$$\alpha \circ \beta = h : \hat{Q} \to Q \subset X.$$

Granted that one sees the same picture of *small* deformation and codeformation of bubble-spaces as for constant mean curvature, where codeformations refer here to finite dimensional families of functions (or measures) on $\hat{U}$. But understanding global properties of such μ-bubbles remains even more limited than that for constant M.

But can one prove (or just conjecture) nevertheless something non-trivial about these Q in relation to the scalar curvature problems?

Wouldn't it be, perhaps, more sensible to switch to a reasonably regular class of *minimal varifolds*, e. g. $V^{n-1} \in X$, where the singular locus of such a V^{n-1} is a smooth $F^{n-2} \subset V^{n-1}$, where there are three local branches of V^{n-1} meeting along this F^{n-2} and where the dihedral angles between these branches are $\frac{2\pi}{3}$?

Can, one, alternatively, "rigidify" bubblehedra by minimizing a single functional, a weighted combination of volumes of faces of different dimensions and /or involving also dihedral angles and (directions of vectors of) means curvatures of low dimensional faces?

3.19 Stability of Geometric Inequalities, Metrics and Topologies in Spaces of Manifolds, Limits and Singular Spaces with Scalar Curvatures bounded from Below

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Inequalities relating geometric quantities $\mathcal{A}$ and $\mathcal{B}$ of geometric objects Ob progress along the following lines.

1. *Rough Inequalities.* This says that $\mathcal{A}(Ob)$ is bounded by *some* function of $\mathcal{B}(Ob)$.

For instance, volumes of Euclidean domains $V \subset \mathbb{R}^n$ are bounded by the (n-1)-volumes of their boundaries.

≀ (There is about a dozen of "direct elementary" proofs, of this which generalize to a variety of situations, e.g. to Riemannian manifolds with certain restrictions to their curvatures.)

2. *Sharp Inequalities.* These specify the maximal values of $\mathcal{A}(Ob)$ among all Ob with a given bound on $\mathcal{B}(Ob)$, say, in the form $\mathcal{A}(Ob) \le E_{sharp}(\mathcal{B}(Ob))$.

For instance, the Euclidean domains satisfy the sharp *isoperimetric inequality* $vol(V) \le \gamma_n \cdot (vol_{n-1}\partial V)^{\frac{n-1}{n}}$, where γ_n is equal to the volume of the n-ball with unit $(n-1)$-volume of the boundary.

≀ (There is no direct elementary proof of this, except for $n = 2$ and 4, and the present day "non-elementary" proofs don't generalize to the expected cases, such as complete manifolds with non-positive sectional curvatures.)

3. *Rigidity.* This is a description of all *extremal* Ob that maximize $\mathcal{A}(Ob)$ with a given bound on $\mathcal{B}(Ob)$, that is where $\mathcal{A}(Ob) = E_{sharp}(\mathcal{B}(Ob))$.

≀

For instance, the balls in $\mathbb{R}^n$ are "isoperimetrically extremal": they are the *only* Euclidean domains, where the isoperimetric inequality becomes equality, $vol(V) = \gamma_n(vol_{n-1}\partial V)^{\frac{n-1}{n}}$.

4. *Stability.* An extremal object Ob_{extr} is *stable* if convergence $\mathcal{A}(Ob_\varepsilon) \to \mathcal{A}(O_{extr})$ and $\mathcal{B}(Ob_\varepsilon) \to \mathcal{B}(Ob_{extr}))$ implies that $Ob_\varepsilon \to Ob_{extr}$ in a "suitable sense", where determination of this "sense" is the main problem here.

For instance, the the balls B in $\mathbb{R}^n$ are *isoperimetrically stable (modulo translations) with respect to the flat topology,* : if $vol(V_\varepsilon) \to vol(B_0)$ and $vol_{n-1}(\partial V_\varepsilon) \to vol_{n-1}(\partial B_0)$, then translates V'_ε of V_ε converge to B_0 in the *flat topology.* This means in the present case that that

• $vol(B_0 \cap V'_\varepsilon) \to vol(B_0)$
• $vol(B_0 \setminus V'_\varepsilon) \to 0$
• the δ-neighbourhoods of B_0 for $\delta \to_{\varepsilon\to 0} 0$ contain almost all of the boundary of V'_ε, that is $Vol_{n-1}(U_\delta(B_0)) \cap \partial V'_\varepsilon)) \to Vol_{n-1}(\partial V'_\varepsilon)$.

Turning to scalar curvature, observe, that poofs of sharp inequalities, be they Dirac theoretic or relying on the μ-bubble, are easily adaptable in most known cases, at least for compact manifolds, for identification of rigid objects, such as

(i) Riemannian *flat metrics* $g_{extr} = g_{fl}$ for the inequality $inf Sc(g) \geq 0$ on the torus,

(ii) metrics $g_{extr} = g_{sph}$ with constant curvature one on the n-sphere S^n for the inequality $\inf Sc(g) \geq n(n-1)$ for metrics $g \geq g_{sph}$ on S^n.

However, the following two questions remain unsettled.

Problem 1. Fully describe in the case (1) metric g_ε, $\varepsilon > 0$, on the n-torus with $Sc(g_\varepsilon) \geq -\varepsilon \to 0$, and, in the case (ii), metrics $g \geq g_{sph}$ on S^n with $Sc(g_\varepsilon) \geq n(n-1) - \varepsilon$.

Problem 2. Find a minimal set of reasonable additional conditions on ε, such that the metrics g_ε would converge to g_{extr}

The following example indicates what can be expected in regard to problem 1.

Bubble-Convergence. Let X be a Riemannian n-manifold, $n \geq 3$, and let $X_i = X_{N_i,\varepsilon_i}$, $\varepsilon_i > 0$, be the connected sum of X with closed Riemannian manifolds $X_{i,j}$, $j = 1, 2, ,, ,.N_i$, where the connected sum is realized by ε_i-thin surgery localised at N_i disjoint ε_i-balls $B_{i,j} = B_{ij}(\varepsilon_i) \subset X$, $j = 1, ...N_i$.

If $\varepsilon_i \to 0$, then, this is geometrically clear, that

X "emerges" from the sequence X_i in the limit for $i \to \infty$,

where "emerges" becomes "*Hausdorff converges*" if $diam(X_i) \to 0$ and "*converges to X in the intrinsic flat topology*",[300] if

$$\sum_{i=1}^{N_i} vol(X_i) \to 0.$$

We explained in section 1.3 that if $Sc(X) \geq \sigma$ and $Sc(X_{i,j}) \geq \sigma$, then the manifolds X_i "naturally" carry metrics with $Sc(X_i) \geq \sigma - \epsilon_i$, where $\epsilon_i \to 0$.

[300] The definition of this metric, introduced by Christina Sormani and Stefan Wenger in [Sormani-Wenger(intrinsic flat) 2011], is given in later on in this section.

More interestingly, the argument indicated in section 3.1.3 can be used to show[301] that

if the set of the centers $x_{i,j} \in X$ of all these balls is dense in X, i.e. all open sets $U \subset X$ contain some balls $B_{i,j}$,

if the distances between the balls are much larger than their radii,

$$dist(B_{i,j_1}, B_{i,j_2})/\varepsilon_i \to \infty \text{ for } i \to \infty$$

and if the scalar curvatures of the manifolds $X_{i,j}$ are bounded from below, $Sc(X_{i,j}) \geq \sigma$,

then $Sc(X) \geq \sigma$.

(One doesn't need here any bounds on the diameters and/or volumes of X_i, and, probably, the lower bound on the distances between $B_{i,j}$ is redundant.)

Intrinsic Flat Distance. Given two compact oriented n-dimensional pseudo-manifolds with peace-wise Riemannian metrics X_1 and X_2 define $dist_{if}(X_1, X_2)$ as the infimum of the numbers $D \geq 0$ such that there exists an oriented $(n+1)$-dimensional piecewise Riemannian pseudomanifold W with a boundary, such that

• the oriented boundary of W is $\partial W = X_1 \sqcup -X_2$, where the imbeddings $X_1, X_2 \hookrightarrow W$ are isometric with respect to the *distance functions* associated to the Riemannian structures in these spaces;

• $vol_{n+1}(W) \leq d$.

Remark. If X_1 and X_2 are Riemannian *manifolds*, then one can also take a Riemannian manifold for W, but now with a larger boundary
$\partial W = X_1 \sqcup -X_2 \sqcup X_3$ and with the condition $vol_{n+1}(W) + vol_n(X_3) \leq d$.

The following conjecture, in agreement with the Penrose inequality, gives an idea of how wild metrics with $Sc \geq -\varepsilon$ can/can't be.

Sormani Conjecture. Let X_i be a sequence of Riemannian manifolds homeomorphic to the torus $\mathbb{T}^3$, such that

$$Sc(X_i) \geq -\varepsilon_i \underset{i\to\infty}{\to} 0.$$

If the volumes *and the* diameters *of all X_i are bounded by a constant and the* areas of all closed minimal surfaces *in X_i are bounded from below by a* positive *constant,*

then a subsequence of X_i converges to a flat torus with respect to the intrinsic flat distance in the space of Riemannian 3-manifolds.[302]

Exercise. Show that the above condition $\sum_{i=1}^{N_i} vol(X_i) \to 0$ does imply the intrinsic flat convergence $X_i \to X$ as it is claimed in the above example.

Hint. Use the filling volume inequality:[303]

Given a compact Riemannian n-manifold $X = (X, g)$, there exists a Riemannian metric $g_\circ$ on the cylinder $W_\circ = X \times (0,1]$, such that:

[301] If $dim(X) \geq 9$ or if some among manifolds $X_{i.j}$ are non-spin, then one needs new not formally published results by Lohkamp and/or by Schoen and Yau on "desingularization" of minimal hypersurfaces.

[302] See [Sormani(scalar curvature-convergence) 2016], [AH-VPPW (almost non-negative) 2019], [Sormani(conjectures on convergence) 2021], [Allen(conformal to tori) 2020], [Pa-Ke-Pe(graphical tori) 2020].

[303] See [G(filling) 1983], [Wenger(filling) 2007], [Katz(systolic geometry) 2017].

(i) the metric $g_\circ$ is conical near 0,

$$g_\circ(x,t) = t^2 dx^2 + dt^2, \text{ for } t \le \varepsilon = \varepsilon_X > 0;$$

(ii) the distance function $dist_{g_\circ}$ on $X = X \times \{1\} \subset W_\circ$ is equal to $dist_g$;
(iii) the volume of $W_\circ$ is universally bounded by that of X

$$vol_{n+1}(W_\circ) \le const_n \cdot vol_n(X)^{\frac{n+1}{n}}.$$

.

Other kinds of convergence. Besides the intrinsic flat there are other distances in the spaces of Riemannian manifolds (more or less adapted to scalar curvature such as the *directed Lipschitz metric* in section 10 in [G(Hilbert) 2012] and the $d_{p,g}$-distance introduced in in [Lee-Naber-Neumayer](convergence) 2019] which well goes along with $Sc \ge -\sigma$ under a lower bound on Perelman's ν-functional.

Once you have a metric in the space $\mathcal{X}$ of Riemannian manifolds, you are inclined to complete this space and study the resulting singular spaces X from this completion.

Then you isolate the essential properties of these X and define more general " *singular spaces X with $Sc(X) \ge \sigma$* "

Then you dream of an abstract category of "objects" with $Sc \ge \sigma$ that carry the essence of what we know (and don't know) about the scalar curvature.

3.20 Who are you, Scalar Curvature?

There are two issues here.

1. What are most general geometric objects that display features similar to these of manifolds with positive and more generally, bounded from below, scalar curvatures?

2. Is there a direct link between Dirac operators and minimal varieties or their joint appearance in the ambience of scalar curvature is purely accidental?

Notice in this regards that there are two divergent branches of the growing tree of scalar curvature.

A. The first one is concerned with the effects of $Sc > 0$ on the *differential structure* of spin (or spin$^{\mathbb{C}}$) manifolds X, such as their $\hat{\alpha}$ and Seiberg-Witten invariants.

B. The second aspect is about coarse geometry and topology of X with $Sc(X) \ge \sigma$, the (known) properties of which are derived by means of minimal varieties and twisted Dirac operators; here the spin condition, even when it is present, must be redundant.

To better visualise separation between A and to B, think of possible *singular spaces X with $Sc(X) \ge 0$* corresponding to A and to B – these must be grossly different.

For instance, if X is an *Alexandrov space* with (generalised) sectional curvature $\ge \kappa > -\infty$ then the inequality $Sc \ge 0$ makes perfect sense and, probably most (all?) of B can be transplanted to these spaces. [304]

[304] It seems, much of the geometric measure theory extends to Alexandrov spaces but it is unclear what would correspond to twisted Dirac operators on these spaces.

But nothing of spin related results makes sense for singular Alexandrov spaces.

And if you start from the position of 2 you better go away from conventional spaces and start dreaming of geometric magic glass ball with ghosts of harmonic spinors and of minimal varieties dancing within. (See section 6.9 for continuation of this discussion.)

In concrete terms one formulates two problems.

A. What is the largest class of spaces (singular, infinite dimensional ...) which display the basic features of manifolds with $Sc \geq 0$ and/or with $Sc \geq \sigma > -\infty$ and, more generally, of spaces X, where the properly understood $-\Delta + \frac{1}{2}Sc(X)$ is positive or, at least not too negative?

For instance, which (isolated) conical singularities and which singular volume minimising hypersurfaces belong to this class?

B. Is there a partial differential equation, or something more general, the solutions of which would mediate between twisted harmonic spinors and minimal hypersurfaces (flags of hypersurfaces?) and which would be non-trivially linked to scalar curvature?

Could one non-trivially couple the twisted Dirac $\mathcal{D}_{\otimes L}$ with some equation $\mathcal{E}_{\mathcal{L}}$ on the connections in the bundle L the Dirac operator in the spirit of the Seiberg-Witten equation?[305]

4 Dirac Operator Bounds on the Size and Shape of Manifolds X with $Sc(X) \geq \sigma$

4.1 Spinors, Twisted Dirac Operators, and Area Decreasing maps

The Dirac $\mathcal{D}$ on a Riemannian manifold X tells you by itself preciously little about the geometry of X, but the same $\mathcal{D}$ twisted with vector bundles L over X carries the following message:

manifolds with scalar curvature $Sc \geq \sigma > 0$
can't be too large area-wise.

Albeit the best possible result of this kind (due to Marques and Neves, see B in section 3.10, which is known for X homeomorphic to S^3 and which says that

if $Sc(X) \geq 6 = Sc(S^3)$, then X can be "swept over" by 2-spheres of areas $\leq 4\pi$,

was proven by means of minimal surfaces, all known bounds on "areas" of Riemannian manifolds of dimensions ≥ 4 depend on Dirac operators $\mathcal{D}$ twisted (or "non-linearly coupled" for n=4) with complex vector bundles L over X with unitary connections in L, where, don't forget it, the very definition of $\mathcal{D}$ needs X to be spin.[306]

[305] Natural candidates for $\mathcal{E}_{\mathcal{L}}$ are equations for critical points of energy-like functional on spaces of connections, where, observe, L-twisted harmonic spinors $s : X \to \mathbb{S} \otimes L$ themselves minimize $s \mapsto \int_X \langle \mathcal{D}_{\otimes L}(s(x))\mathcal{D}_{\otimes L}(s(x))\rangle\, dx$.

[306] Recently, ¯Jintian Zhu [Zhu(rigidity) 2019] and Thomas Richard [Richard(2-systoles) 2020] established new kind of bounds on areas of surfaces applicable to higher dimensional non-spin manifolds by using geometric calculus of variations, but these bounds depend on

Recall that the *twisted* Dirac operator, denoted

$$\mathcal{D}_{\otimes L} : C^\infty(\mathbb{S}\otimes L) \to C^\infty(\mathbb{S}\otimes L),$$

acts on the tensor product of the spinor bundle $\mathbb{S} \to X$ [307] with $L \to X$, where it is related to the (a priori positive Bochner Laplace operator) $\nabla^2_{\otimes L} = \nabla^2_{\otimes L} = \nabla^*_{\otimes L}\nabla_{\otimes L}$ in the bundle $\mathbb{S}\otimes L$, by the *twisted* Schroedinger-Lichnerowicz-Weitzenboeck formula

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \tfrac{1}{4}Sc(X) + \mathcal{R}_{\otimes L},$$

where $\nabla_{\otimes L}$ denotes the covariant derivative in $\mathbb{S}\otimes L$ and $\mathcal{R}_{\otimes L}$ is a certain (zero order) which acts in the fibers of the twisted spin bundle $\mathbb{S}\otimes L$ and which is derived from the curvature of the connection in L.

If we are not concerned with the sharpness of constants, all we have to know is that $\mathcal{R}_{\otimes L}$ is controlled by

$$||\mathcal{R}_{\otimes L}|| \le const \cdot ||curv(L)||$$

for $const = const(n, rank(L))$, where a little thought (no computation is needed) shows that, in fact, this constant depends only on $n = dim(X)$. (The actual formula for $\mathcal{R}_{\otimes L}$ is written down in the next section, also see [L-M(spin geometry) 1989] and [MarMin(global riemannian) 2012] for further details and references.)

We regard a closed orientable *even dimensional* Riemannian manifold X *area wise large*, if it carries a *homologically substantial* or *essential bundle* L over it with *small curvature*, where "homologically substantial" signifies that some Chern number of L doesn't vanish. It is easy in this case[308] that there exists an associated bundle $L^\wedge$, such that

$$|curv|(L^\wedge) \le const_n |curv|(L)$$

and such that the Chern character in the index formula guaranties non-vanishing of the cup product $\hat{A}(X) \smile Ch(L^\wedge)$ evaluated at $[X]$,

$$(\hat{A}(X) \smile Ch(L^\wedge))[X] \ne 0$$

and, thus, by Atiyah-Singer theorem, the presence of *non-zero harmonic twisted spinors*: sections s of the bundle $\mathbb{S}\otimes L^\wedge$ for which $\mathcal{D}_{\otimes L^\wedge}(s) = 0$.

If the dimension n of X is odd, the above applies to $X \times S^1$ for a sufficiently long circle S^1.

For instance, n-manifolds, which admit area decreasing non-contractible maps to spheres $S^n(R)$ of large radii R are area-wise large, where the relevant bundles L are induced from non trivial bundles over the spheres. (One may take $L^\wedge = L$ for these L.)

---

lower distance bounds (that may be hidden in the topological assumptions, such as in the Zhu paper) and are not sufficient, for instance, to show that the unit sphere S^n for $n \ge 4$ admits no metric g with $Sc(g) \ge n(n-1)$ and such that the g-areas of all surfaces Σ in S^n satisfy $area_g(\Sigma) \ge C \cdot area_{S^n}(\Sigma)$ for arbitrary large C.

[307] All you have to know about $\mathbb{S}(X)$ is that it is a vector bundle associated with the tangent bundle $T(X)$, which can be defined for spin manifolds X, where "spin" is needed, since the structure group of $\mathbb{S}(X)$ is the double cover of the orthogonal group $O(n)$ rather than $O(n)$ itself.

[308] See $(L^\wedge)$ in section 4.1.3 and references therein.

But if the scalar curvature of X is $\geq \sigma$ for a *large* $\sigma > 0$, where this *"large"* properly matches the above *"small"*, then by the Schroedinger-Lichnerowicz-Weitzenboeck formula the $\mathcal{D}_{\otimes L^\wedge}$ is positive and no such harmonic twisted spinor exists; therefore, a suitably defined "area"(X) must be bounded by $\frac{const}{\sigma}$. (See the sections 3.3.4, 4.1.4 for a definition of this "area" called K-*area* and K-*cowaist.*)

Next, recall that the $\hat{A}$-genus,

$$\hat{A}(X) = 1 - \frac{1}{24}p_1 + \frac{1}{5760}(-4p_2 + 7p_1^2) + ... \in H^*(X)$$

is a certain polynomial in Pontryagin classes $p_i \in H^{4i}(X)$ of X and

$$Ch(L) = rank_{\mathbb{C}}(L) + c_1(L) + \frac{1}{2}(c_1(L)^2 - 2c_2(L)) + ... \in H^*(X)$$

is a polynomial in Chern classes $c_i(L) \in H^{2i}(X)$ of L, while $[X] \in H_n(X)$ denotes the fundamental class of X.

If $n = dim(X)$ is even, the spin bundle $\mathbb{S}$ naturally splits, $\mathbb{S} = \mathbb{S}^+ \oplus \mathbb{S}^-$, the $\mathcal{D}_{\otimes L}$ also splits: $\mathcal{D}_{\otimes L} = \mathcal{D}^+_{\otimes L} \oplus \mathcal{D}^-_{\otimes L}$, for

$$\mathcal{D}^{\pm}_{\otimes L} : C^\infty(\mathbb{S}^\pm \otimes L) \to C^\infty(\mathbb{S}^\mp \otimes L)$$

and the index formula reads:

$$ind(\mathcal{D}^{\pm}_{\otimes L}) = \pm(\hat{A}(X) \smile Ch(L))[X].)$$

Relative Index Theorem on Complete Manifolds. Let X be a *complete* Riemannian manifold the scalar curvature of which is *uniformly positive at infinity.*[309] Then the Schroedinger-Lichnerowicz-Weitzenboeck formula implies that

the Dirac operator is positive at infinity, i.e. outside some compact subset $V \subset X$:

$$\int_X \langle \mathcal{D}^2 s(x), s(x)\rangle dx \geq \varepsilon \int_X \|s(x)\|^2 dx$$

for some $\varepsilon = \varepsilon(X) > 0$ and all L_2-spinors s supported outside V. This (easily) implies, in turn, that the operators $\mathcal{D}^\pm$ are Fredholm but the indices of these operators depend on delicate information on geometry of X at infinity and no simple formula for $ind(\mathcal{D}^\pm)$ is available.

However if there are two operators $\mathcal{D}_1$ and $\mathcal{D}_2$, which are *equal at infinity*, e.g. $\mathcal{D}_1 = \mathcal{D}^+_{\otimes L}$, and $\mathcal{D}_2 = \mathcal{D}^+_{\otimes |L|}$, where $L \to X$ is a bundle with a unitary connection, where $|L|$ is the trivial bundle of rank $k = rank_{\mathbb{C}} L$ over X and where L comes with an isometric connection preserving isomorphism with $|L|$ at infinity, as in section 3.14.2, then the difference of their indices – both are Fredholm for the same reason as $\mathcal{D}^\pm$– satisfy the Atiyah-Singer formula:

$$ind(\mathcal{D}^+_{\otimes L}) - ind(\mathcal{D}^+_{\otimes |L|}) = (\hat{A}(X) \smile (Ch(L) - Ch|L|))[X].$$

where,

$$Ch(L) - Ch|L| = c_1(L) + \frac{1}{2}(c_1(L)^2 - 2c_2(L)) + ...$$

309 It is shown in [Zhang(Area Decreasing) 2020] that "uniformly"can be dropped – "positive at infinity" suffices.

is understood as a cohomology class with compact supports and $[X]$ is the fundamental homology class with infinite supports.

More generally, if $\mathcal{D}_i = \mathcal{D}_{\otimes L_i}$, $i = 1,2$, where L_1 is equated with L_2 at infinity, then

$$ind(\mathcal{D}_1^+) - ind(\mathcal{D}_2^+) = (\hat{A}(X) \smile (Ch(L_1) - Ch(L_2))[X],$$

where one needs the operators $\mathcal{D}_i$ be positive at infinity.

The proof of this can be obtained by adapting any version of the *local proof* of the compact Atiyah-Singer theorem (see (see [GL(complete) 1983], [Bunke(relative index) 1992], [Roe(coarse geometry) 1996]).

Namely, the index is represented by the difference of the traces of families of auxiliary operators $K^+_{1,t} - K^+_{2,t}$ and $K^-_{1,t} - K^-_{2,t}$, $t > 0$, where

(i) these $K_{...,t}$-s are given by continuous kernels $K_{...,t}(x,y)$ which are supported in the *t-neighbourhood of the diagonal* in $X \times X$, i.e. where $dist(x,y) \leq t$;

(ii) $K^{\pm}_{1,t}(x,y) = K^{\pm}_{2,t}(x,y)$ for x and y in the *complement of a compact subset* $V_t \subset X$, where $V_{t_1} \subset V_{t_2}$ for $t_2 > t_1$ and where $\bigcup_t V_t = X$;

(iii) $trace(K^+_{1,t} - K^+_{2,t}) - trace(K^1_{1,t} - K^1_{2,t}) = (\hat{A}(X) \smile (Ch(L_1) - Ch(L_2))[X]$;

for all $t > 0$;

(iv) the operators $K^{\pm}_{i,t}$, $i = 1,2$, *weakly converge*[310] for $t \to \infty$ to the projection operators on the kernels of $\mathcal{D}\pm_i$.

The quickest way to get such $K_{...,t}$ is by taking suitable functions ψ_t of the corresponding Dirac operators, where the Fourier transforms of ψ_t have *compact supports*, and where (as in all arguments of this kind) the essential issue is the proof of *uniform bounds* on the traces of the operators $K^{\pm}_{1,t} - K^{\pm}_{2,t}$ for $t \to \infty$.

Specific bounds for particular $K_{...,t}$ are crucial for an (approximate) extension of the index theory to non-complete manifolds, but these bounds are often buried in the K-theoretic formalism of the recent papers. Also, I must admit, this point was not explained (overlooked?) in the exposition of Roe's argument in my paper [G(positive) 1996].

4.1.1 Negative Sectional Curvature against Positive Scalar Curvature

A characteristic topological corollary of the above is as follows.

$[\kappa \leq 0] \rightsquigarrow [Sc \ngtr 0]$: *If a closed orientable spin n-manifold X admits a map to a complete Riemannian manifold $\underline{X}$ with $sect.curv(\underline{X}) \leq 0$,*

$$f : X \to \underline{X},$$

such that the homology image $f_*[X] \in H_n(\underline{X}; \mathbb{Q})$ doesn't vanish, then X admits no metric with $Sc(X) > 0$.

Two Words about the Proof. All we need of $sect.curv \leq 0$ is the existence of distance decreasing maps from the universal covering of $\underline{X}$ to (large) spheres,

$$F_{\underline{x}} : \underline{X} \to S^{\underline{n}}(R),\ \underline{n} = dim(\underline{X}),\ \underline{x} \in \underline{X},$$

[310]The corresponding functions $K_{...,t}(x,y)$ uniformly converge on compact subsets in $X \times X$.

which can be (trivially) obtained with a use of the inverse exponential maps

$$\exp_{\underline{x}}^{-1} : \tilde{\underline{X}} \to T_{\underline{x}}(\underline{X}),\ \underline{x} \in \underline{X}.$$

To make the idea clear, let $\underline{X}$ be compact, the fundamental group of $\underline{X}$ be residually finite, (e.g. $\underline{X}$ having constant sectional curvature or, more generally being a locally symmetric space) and X be embedded to $\underline{X}$.

Let $X^\perp \subset \underline{X}$ be a closed oriented submanifold of dimension $m = \underline{n} - n$ for $\underline{n} = dim(\underline{X})$, which has *non-zero intersection index* with $X \subset \underline{X}$.

Also assume that the restriction of the tangent bundle of $\underline{X}$ to $\underline{X}^\perp \subset \underline{X}$ is trivial.

Then – this is rather obvious – there exist finite covers $\tilde{\underline{X}}_i \to \underline{X}$, such that the products of the lifts (i.e. pull-backs) of X and of $X^\perp$ to $\tilde{\underline{X}}_i$, denoted $\tilde{X}_i \times \tilde{X}_i^\perp$, admit smooth maps to the spheres of radii R_i,

$$F_i : \tilde{X}_i \times X_i^\perp \to S^{\underline{n}}(R_i),$$

where

•$_1$ $R_i \to \infty$,

•$_2$ $deg(F_i) \neq 0$,

• the maps F_i are *distance decreasing on the fibers* $\tilde{X}_i \times x^\perp$ for all $x^\perp \in X_i^\perp$ for the Riemannian metric in these fibers induced by the embedding $\tilde{X}_i \times x^\perp = \tilde{X}_i \subset \tilde{\underline{X}}_i$.

It follows that for *arbitrary* Riemannian metrics g and $g^\perp$ on X and on $X^\perp$ there exists (large) constants λ and C independent of i, such that

the maps F_i are C-*Lipschitz* with respect to the sum of the lift of the metric g to $\tilde{X}_i$ and the lift of $\lambda \cdot g^\perp$ to $\tilde{X}_i^\perp$ that is the metric

$$\tilde{g}_i \oplus \lambda \cdot \tilde{g}_i^\perp \text{ on } \tilde{X}_i \times \tilde{X}_i^\perp.$$

If $Sc(g) \geq \sigma > 0$, then also $Sc(\tilde{g}_i \oplus \lambda \cdot \tilde{g}_i^\perp) \geq \sigma' > 0$ for all sufficiently large λ, which, for large R_i, rules out non-zero harmonic spinors on $\tilde{X}_i \times \tilde{X}_i^\perp$ twisted with the bundle $L^* = F_i^*(L)$ induced from any given bundle L on $S^{\underline{n}}$.

But if $\underline{n} = 2k$ and the Chern class $c_k(L)$ is non-zero, then non-vanishing of $deg(F_i)$ implies non-vanishing of of $ind(\mathcal{D}_{\otimes L})$ via the index formula and the resulting contradiction delivers the proof for even $\underline{n}$ and the odd case follows with $\underline{X} \times S^1$.

Remarks. This argument, which is rooted in Mishchenko's proof of Novikov conjecture for the fundamental group of the above $\underline{X}$, which was adapted to scalar curvature in [GL(complete) 1983] and further generalized/formalised in [Rosenberg(C^*-algebras - positive scalar) 1984], and [CGM(Lipschitz control) 1993], doesn't really need compactness of $\underline{X}$, residual finiteness of $\pi_1(\underline{X})$ and triviality of $T(\underline{X})|X^\perp$. Beside, the spin condition for X can be relaxed to that for the universal cover of X.

Moreover, since the bound on the size of $\tilde{X}_i \times \mathbb{T}^{\underline{n}-n}$ by $\frac{const}{\sqrt{\sigma}}$ can be obtained with the use of minimal hypersurfaces (see §12 in [GL(complete) 1983]), [G(inequalities) 2018] and section 5.4) the spin condition can be dropped altogether.

Question. Are there other topological non-spin obstructions to $Sc > 0$?

For instance, is the following true?

Conjecture. Let X be a closed orientable Riemannian n-manifold, such that no closed orientable n-manifold X' which admits a map $X' \to X$ with non-zero degree carries a metric with $Sc > 0$. Then there exists an integer m and a sequence of maps

$$F_i : \tilde{X}_i \times \mathbb{R}^m \to S^{n+m}(R_i),$$

where $\tilde{X}_i$ are (possibly infinite) coverings of X, such that

- the maps F_i are constant at infinity and they have non-zero degrees,
- $R_i \to \infty$,
- the maps F_i are distance decreasing on the fibers $\tilde{X} + i \times x^\perp$ for all $x^\perp \in \mathbb{R}^m$.[311]

Apparently, there is no instance of a *specific* homotopy class $\mathcal{X}$ of closed manifolds X of dimension $n \geq 5$, where a Dirac theoretic proof of non existence of metrics with $Sc > 0$ on all $X \in \mathcal{X}$ couldn't be replaced by a proof via minimal hypersurfaces.

(This seems to disagree with what was said concerning the "quasisymplectic theorem" $\otimes_{\wedge\omega}$ in section 2.7.

In fact the *general* condition for $Sc \ngtr 0$ in $\otimes_{\wedge\omega}$, can't be treated, not as it stands, with minimal hypersurfaces, but this may be possible in all *specific examples*, where this condition was *proven to be* fulfilled.)

And it is conceivable when it comes to the Novikov conjecture, that its validity in all proven specific examples, can be derived by an elementary argument from the invariance of rational Pontryagin classes under ε-homeomorphisms.[312])

But even though the relevance of twisted Dirac theoretic methods is questionable as far as *topological* non-existence theorems are concerned, these methods seem irreplaceable when it comes to *geometry* of $Sc \geq \sigma$.

4.1.2 Global Negativity of the Sectional Curvature, Singular Spaces with $\kappa \leq 0$, and Bruhat-Tits Buildings

The essential feature of complete spaces $\kappa \leq 0$ (these often come under heading of CAT(0)-spaces) needed for $[Sc \ngtr 0]$ is as follows.

$[\circlearrowleft_\varepsilon]$ *Self-contraction Property.* $\underline{X}$ *admits a family of proper* ε*-Lipschitz selfmaps* $\phi_\varepsilon : \underline{X} \to \underline{X}$*, for all* $\varepsilon > 0$*, where these maps are properly homotopic to the identity map id.* [313]

If $\underline{X}$ is a topological n-manifold, than this property implies the existence of proper Lipschitz maps $\underline{X} \to \mathbb{R}^n$ of degree one, but unlike the latter it makes sense for singular spaces that are not topological manifolds or pseudomanifolds.

On the other hand, if a possibly singular, say finite dimensional polyhedral space X satisfies $\circlearrowleft_\varepsilon$, then there exists a manifold $\underline{X}^+ \supset \underline{X}$, which also satisfies

[311] See [Dranishnikov(asymptotic) 2000], [Dranishnikov(macroscopic) 2010], [DFW)flexible) 2003], [Dranishnikov(hypereuclidean) 2006] [BD(totally non-spin) 2015] and references therein for relations between various largeness conditions (e.g. of universal covering of compact manifolds) and their roles in the proofs of the Novikov conjecture and of non-existence of metrics with $Sc > 0$.

[312] The original proof of topological invariance of Pontryagin classes by Novikov, as well as simplified versions and modifications of his proof in [G(positive) 1996) automatically apply to ε-homeomorphisms and, sometimes, to homotopy equivalences

[313] See [G(large) 1986] for more about such manifolds.

$\circlearrowleft_\varepsilon$, where the most transparent case is that of spaces $\underline{X}$ which come with free isometric actions by discrete groups Γ with compact quotients $\underline{X}$.

To derive $\underline{X}^+$ from $\underline{X}$ in this case, embed $\underline{X}/\Gamma \hookrightarrow \mathbb{R}^N$, take a small regular neighbourhood $U \subset \mathbb{R}^N$ of $\underline{X}^+/\Gamma \subset \mathbb{R}^N$ and let $\tilde{U} \to U$. be the universal covering of U.

Then this $\tilde{U}$ with a *suitably blown-up metric* serves for $\underline{X}^+$, where the simplest such blow up is achieved by multiplying the (locally Euclidean) metric in $\tilde{U}$ by the function $\frac{1}{dist(\tilde{u}, \partial U)}$.

In fact, what is truly needed for the non-existence argument, and what is satisfied by complete simply connected spaces $\underline{X}$ with $\kappa < 0$ is the following parametric version of $[\circlearrowleft_\varepsilon]$.

$[\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon]$. There exist a continuous map $\Phi_\varepsilon : \underline{X} \times \underline{X} \to \underline{X}$ with the following properties.

$\bullet_\varepsilon$ *the maps* $\phi_{\varepsilon,\underline{x}_0} = \Phi_\varepsilon : \underline{X} = \underline{x}_0 \times \underline{X} \to \underline{X}$ *are proper* ε*-Lipschitz for all* $\underline{x}_0 \in \underline{X}$ *and all* $\varepsilon > 0$*;*

$\bullet_n$ *the restrictions of these maps* $\phi_{\varepsilon,\underline{x}_0} : \underline{X} \to \underline{X}$ *to the* n*-skeleton* $\underline{X}^{(n)} \subset \underline{X}$ *are proper homotopic to the inclusions* $\underline{X}^{(n)} \subset \underline{X}$*;*[314]

$\bullet_\Gamma$ *the family* $\phi_{\varepsilon,\underline{x}_0}$ *is equivariant under the isometry group of* $\underline{X}$*:*

if $\gamma : \underline{X} \to \underline{X}$ *is an isometry, then*

$$\phi_{\varepsilon,\gamma(\underline{x}_0)} = \gamma \circ \phi_{\varepsilon,\underline{x}_0}.$$

The above argument combined with that in the previous section yields the following generalization of the non-existence theorem $[\kappa \le 0] \rightsquigarrow [Sc \ngtr 0]$.

$[\kappa \le 0]_{global} \rightsquigarrow [Sc \ngtr 0]$: *If a complete Riemannian spin manifold* $\tilde{X}$ *of dimension* n *with a discrete (not necessarily free) co-compact isometric action of a group* Γ *admits a proper* Γ*-equivariant map to an* $\underline{X}$ *which satisfies* $\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon$*, then* $\inf_x(Sc(X,x) \le 0$.

Corollary. Let Γ *be a finitely generated subgroup in the linear group* $GL_N(\mathbb{C})$*,*[315] *let* X *be a compact oriented Riemannian spin* n*-manifold with* $Sc(X) > 0$ *and let* $f : X \to \mathsf{B}(\Gamma)$ *be a continuous map, where* $\mathsf{B}(\Gamma)$ *denotes the classifying (Eilenberg MacLane) space of* Γ*.*

Then the image

$$f_*[X]_\mathbb{Q} \in H_n(\mathsf{B}(\Gamma);\mathbb{Q})$$

of the rational fundamental class

$$[X]_\mathbb{Q} \in H_n(X;\mathbb{Q}) \text{ for } f_* : H_*(X;\mathbb{Q}) \to H_*(\mathsf{B}(\Gamma);\mathbb{Q})$$

is zero.[316]

Proof. A finite index subgroup in Γ freely,[317] discretely and isometrically acts on the product $\underline{X}$ of Riemannian symmetric spaces and *Bruhat-Tits buildings*, where such products, according to Bruhat-Tits are

[314] Here we assume that $\underline{X}$ is triangulated and n denotes the dimension of a manifold X we are going to map to $\underline{X}$;

[315] One may place here any field instead of $\mathbb{C}$.

[316] A more sophisticated theoretic version of this in the context of the Novikov conjecture appears in [Kasp-Scan (Novikov) 1991].

[317] Finite index was needed for his "freely"

complete simply connected polyhedral space with $\kappa(X) \leq 0$.

Since $\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon$ apply to such spaces, the proof of the corollary follows.

Historical Remark. Around 1950, A.D. Alexandrov, H. Pedersen and Busemann who suggested (two somewhat different) definitions of $\kappa \leq 0$ applicable to *singular* metric spaces, and their followers focused on essentially local geometric properties of these spaces X, and tried to *alleviate effects of singularities* by adding extra assumptions on X. [318]

The theory of $\kappa \leq 0$ has acquired a global mathematical status in early seventies with the discoveries of Bruhat-Tits buildings (1972)[319] and the link of $\kappa \leq 0$ with the index theory and the Novikov conjecture by Mishchenko(1974).

This has eventually led to the modern perspective on CAT(0)-spaces, i.e. those with $\kappa \leq 0$, the main interest in which is due to a multitude of significant examples of singular CAT(0)-spaces with interesting fundamental groups inspired by the ideas behind the construction(s) and applications of the Bruhat-Tits buildings.

Hyperbolic Remark. "ε-Lipschitz" in the theorem $[\kappa \leq 0]_{global} \rightsquigarrow [Sc \not> 0]$ is only needed on the *large scale*, that is expressed by the inequality

$$dist(f_{\varepsilon,x_0}(x_1), f_{\varepsilon,x_0}(x_2)) \leq \varepsilon dist(x_1, x_2) + const.$$

Thus, for instance,

the non-existence conclusion for metrics with $Sc > 0$ on X applies, where $\underline{X}$ is the *Vietoris-Rips complex* of a *hyperbolic group*.

It follows, that the conclusion of the above corollary holds for hyperbolic groups Γ:

Let X be a closed orientable Riemannian spin manifold with $Sc(X) > 0$ and let Γ be a hyperbolic group. Then the class $f_*[X]_{\mathbb{Q}} \in H_n(\mathsf{B}(\Gamma);\mathbb{Q})$ vanishes for all continuous maps $f : X \to \mathsf{B}(\Gamma)$.

"ε-Area" Remark. Instead of "ε-Lipschitz" one may require "*ε-area contracting*" or some large scale counterpart to this condition.

This may be significant, because the ε-area version of $[\kappa \leq 0]_{global} \rightsquigarrow [Sc \not> 0]$ is not-approachable with the (known) techniques of minimal hypersurfaces and/or of stable μ-bubbles, while the above "ε-Lipschitz" $[\kappa \leq 0]_{global} \rightsquigarrow [Sc \not> 0]$ can be proved in many, probably in all, cases with these techniques having an advantage of not requiring manifolds X to be spin.

On the other hand, for all I know, there is no example of an $\underline{X}$, say with a cocompact action of an isometry group Γ, which satisfies a version of $\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon$ with the ε-contracting area property but not with the ε-Lipschitz one.[320]

4.1.3 Curvatures of Unitary Bundles, Virtual Bundles and Fredholm Bundles

Let us try to formalise the concept of

[318] A brief overview of this circle of ideas is given in section 2.3 of [G(hyperbolic)2016] and contributions by the Alexandrov's school are presented in [AKP(Alexandrov spaces) 2017].

[319] Bruhat and Tits independently developed the local and global theory of their spaces being unaware of definitions of $\kappa \leq 0$ suggested by differential geometers.

[320] Neither, it seems, there are examples of $\underline{X}$ with compact quotients $\underline{X}/\Gamma$, which satisfy $\circlearrowleft_\varepsilon$ but not $\circlearrowleft_\varepsilon \circlearrowleft_\varepsilon$.

"area", of a Riemannian manifold X, where this "area" is associated with curvatures of vector bundles over X and which has the property of being bounded by $const \cdot \frac{1}{\sigma}$, for $\sigma = \inf_x Sc(X, x) > 0$.

$||curv(L)||$. Given a vector bundle (L, ∇) with an orthogonal (unitary in the complex case) connection, over a Riemannian manifold X, let

$$||curv(L)||(x) = ||curv(\nabla)||(x) = ||curv(L, \nabla)||(x)$$

denote

the infimum of positive functions $C(x)$, such that the maximal rotation angles $\alpha \in [-\pi, \pi]$ of the parallel transports along the boundaries of smooth discs D in X satisfy

$$|\alpha| = |\alpha_D| \leq \int_D C(d).$$ [321]

(The holonomy splits into the direct sum of rotations $z \mapsto \alpha_i z$, $z \in \mathbb{C}$, $\alpha_i \in \mathbb{T} \subset \mathbb{C}$, $i = 1, 2, ..., rank(L)$, and our $\alpha = \max_i \alpha_i$.)

For instance, if D is a geodesic digon in S^2 with the angles $\beta\pi$, $\beta \leq 1$, then the holonomy of the tangent vectors around the boundary of D satisfies:

$$|\alpha_D| = 2\beta\pi = area(D),$$

which agrees with the equality $|curv|(T(S^2)) = 1$.

It follows that the curvature of the tangent bundle (complexified if you wish) of the product of spheres, satisfies

$$\left|\left|curv\left(T\left(\bigtimes_i S^{n_j}(R_j)\right)\right)\right|\right| = \frac{1}{\min_j R_j^2}.$$

What is more amusing is that the even dimensional spheres S^n, $n = 2m$, support unitary bundles L with with twice smaller curvatures and *non-zero* top Chern classes,

$$|curv|(L) = \tfrac{1}{2} \text{ and } c_m(L) \neq 0.$$

For instance, if $n = 2$, then the Hopf bundle, that is the square root of the tangent bundle, has these properties and in general, the positive $\mathbb{C}$-spin bundle $\mathbb{S}^+$ can be taken for such an L.

This is the smallest curvature a non-trivial bundle over S^n may have:

Unitary vector bundles over S^n with $|curv| < \frac{1}{2}$ are trivial.

Proof. Follow the parallel transport of tangent vectors from the north to the south pole.

More generally

there are bundles L on the products of even dimensional spheres $\bigtimes_i S^{n_j}(R_j)$, which are induced by λ-Lipschitz maps to S^n, $n = \sum n_j$, $\lambda = \frac{1}{\min_j R_j^2}$, such that $|curv| \leq \frac{1}{2\min_j R_j^2}$ and such that *some Chern numbers of these L are non-zero*, and this is the best one can do.

In fact,

[321] This definition is adapted to vector bundles over rather general metric spaces, e.g. polyhedra with piecewise smooth metrics.

If a unitary vector bundle $L = (L, \nabla)$ over a product manifold $S^n \times Y$ has $|curv|(L) < \frac{1}{2}$, then all Chern numbers of L vanish. (see §13 in [G(101) 2017]).

The role of the Chern numbers here is motivated by the following observation (see [GL (spin) 1980, [G(positive) 1996]).

Let X be a closed orientable spin manifold of dimension $n = 2m$ and $L = (L, \nabla)$ a unitary vector bundle, such that *some Chern number of L doesn't vanish.* Then

$(L^\wedge)$ *there exists an associated bundle $L^\wedge$, which is a polynomial in the exteriors powers of L, such that*

$$ind(\mathcal{D}_{\otimes L^\wedge}) \neq 0$$

.

Since (it is easy to see) the degree and the coefficients of such a polynomial must be bounded by a constant depending only on n, the curvature of $L^\wedge$ satisfies

$$|curv|(L^\wedge) \leq const_n \| curv|(L);$$

Therefore,

• *if the scalar curvature of a closed orientable $2m$-dimensional spin manifold satisfies $Sc(X) \geq \sigma > 0$, then* – this is explained in the previous section – *non-vanishing $c_m(L) \neq 0$, implies the following lower bound on the curvature of the bundle L:*

$$|curv|(L) \geq \epsilon \cdot \sigma, \ \ \epsilon = \epsilon(n) > 0.$$

Open Problem. Prove • without the spin condition.

The above suggest the definition of "area"(X) of a Riemannian manifold X as the supremum of $\frac{1}{|curv|(L)}$ over all unitary vector bundles $(L = L, \nabla)$ with non-zero Chern numbers.

However, the "area" terminology we introduced in [G(positive) 1996], despite several natural/functorial properties of this "area" (see [G(positive) 1996] and [G(101 2017]), seems inappropriate, since this "area" is *by no means additive.* A more adequate word , which we prefer to use from now on is *K-cowaist.*

Virtual Hilbert and Fredholm. To define this, we represent the *(Grothendieck) classes* **h** of vector bundles over X, which are also called *virtual (Fredholm) bundles*, by *Fredholm homomorphisms* between Hilbert bundles with unitary connections $\mathcal{L}_i = (\mathcal{L}_i, \nabla_i)$, $i = 1, 2$,

$$h : \mathcal{L}_1 \to \mathcal{L}_2,$$

where these h must *almost commute*, i.e. *commute modulo compact s*, with the parallel transports in in $\mathcal{L}_1$ and $\mathcal{L}_2$ along smooth paths in X.

(This idea for flat bundles goes back to [Atiyah(global) 1969], [Kasparov(index) 1973], [Kasparov(elliptic) 1975], [Mishchenko(infinite-dimensional) 1974] and where non-flat generalizations and applications are discussed in §$9\frac{1}{6}$ of [G(positive) 1996].)

(Such an **h** represents the finite dimensional virtual (not quite) bundle $ker(h) - coker(h)$.)

Define

$$|curv|(h) = \max(|curv|\|(\mathcal{L}_1), |curv|(\mathcal{L}_2))$$

and let

$$|curv|(\mathbf{h}) = \inf |curv|(h)$$

where the infimum is taken over all h in the class $\mathbf{h}$.

Why Hilbert? If one limits the choice of representatives of $\mathbf{h}$ to virtual *finite dimensional* bundles $L \to X$, then the resulting curvature function on $K^0(X)$ may only increase:

$$|curv|(\mathbf{h})_{fin.dim} \geq |curv|(\mathbf{h}).$$

Apparently, this must be standard, the Hilbert spaces in the definition of Fredholm bundles can be approximated by finite dimensional Euclidean ones, [322] that implies that

$$|curv|(\mathbf{h})_{fin.dim} = |curv|(\mathbf{h}),$$

but even so "Hilbert" allows greater flexibility of certain constructions, example of which we shall see below.

Naive (Strong Novikov) Conjecture. Let Y be a compact *aspherical*[323] Riemannian manifold, possibly with a boundary. Then
all (classes of complex vector bundles) $\mathbf{h} \in K^0(Y)$ *satisfy:*

$$\inf_N |curv|(N \cdot \mathbf{h}) = 0, \;\; N = 1, 2, 3, ..., \;.$$

Exercises. (a) Show that the equalities $|curv|(\mathbf{h}) = 0$ and $\inf_N |curv|(N \cdot \mathbf{h}) = 0$ are *homotopy invariants* of Y.

(b) Show that if Y satisfies this naive conjecture and X is a closed Riemannian orientable spin n-manifold with $Sc(X) > 0$, then all continuous maps $f : X \to Y$ send the fundamental rational homology class $[X]_{\mathbb{Q}} \in H_n(X, \mathbb{Q})$ to zero in $H_n(Y, \mathbb{Q})$.

4.1.4 Area, Curvature and K-Cowaist

K-cowaist$_2$. Given a Riemannian manifold Y (or a more general space, e.g. a polyhedral one with a piecewise smooth metric), define the *K-cowaist* on the homology classes $h_* \in H_*(Y)$, denoted K-*cowaist*$_2(h)$ [324] as the infimum of $|curv|(\mathbf{h})$ over all $\mathbf{h} \in K^0(Y)$, such that $\mathbf{h}(h_*) \neq 0$, where this equality serves as an abbreviation for the value of the Chern character of $\mathbf{h}$ on h_*,

$$\mathbf{h}(h_*) =_{def} Ch(\mathbf{h})(h_*).$$

In these terms the above $\bullet$ can be reformulated as follows.

K-cowaist Inequality for Closed Manifolds. The K-cowaists of (the fundamental classes of) closed orientable $2m$-dimensional spin manifolds X with $Sc(X) \geq \sigma > 0$ satisfy:

$\bullet_{wst}$ $$K\text{-}cowaist_2[X] \leq \frac{const_m}{\sigma}.$$

[322] This is an exercise that the author delegates to the reader.
[323] The universal covering of X is contractible.
[324] Subindex 2 is to remind that curvature of bundle L over Y is seen on restrictions of L to surfaces in Y.

Notice, that *conjecturally,* a similar inequality also holds for the *ordinary 2-waist*, (see [Guth(waist) 2014] for an exposition of this "waist") where it is confirmed for 3-manifold by the Marques-Neves theorem (see section 3.10)

Exercises. Show that the K-cowaist is bounded by the hyperspherical radius defined in section 3.10.1 as follows,

$$\text{K-}cowaist_2[X] \leq 4\pi Rad^2_{S^{2m}}(X)$$

(b) Show that K-$cowaist_2(S^n) = 4\pi$.

Almost Flat Bundles Over Open Manifolds. If X is a non-compact manifold, then we deal with the K-theory with compact support that is represented by Fredholm homomorphisms

$$h : \mathcal{L}_1 \to \mathcal{L}_2$$

which are *isometric and connection preserving isomorphisms at infinity*, i.e. away from compact subsets in X where the corresponding K-group is denoted $K^0(X/\infty)$. (If X is compact then $K^0(X/\infty) = K^0(X)$

Here the Hilbertian nature of "Fredholm" allows a painless (and obvious by deciphering terminology) definition of the *pushforward homomorphism* for possibly *infinitely* sheeted covering maps $F : X_1 \to X_2$,

$$F_\star : K^0(X_1/\infty) \to K^0(X_2/\infty),$$

where, clearly,

$$|curv|(F_\star(\mathbf{h})) \leq |curv|(\mathbf{h})$$

for all $\mathbf{h} \in K^0(X_1/\infty)$.

It follows that

$$K\text{-}cowaist_2[X_1] \leq K\text{-}cowaist_2[X_2]$$

for coverings $X_1 \to X_2$ between orientable Riemannian manifolds.

On K-cowast Contravariance. The *compact support* property of (virtual) bundles $L \to X_2$ is preserved under pullbacks by *proper* maps $F : X_1 \to X_2$, e.g. by finite coverings, but it fails, for instance, for *infinitely sheeted* coverings $F : X_1 \to X_2$.

This makes the inequality

$$K\text{-}cowaist_2[X_1] \geq K\text{-}cowaist_2[X_2]$$

(that is obvious for *finitely sheeted* coverings) *problematic* for infinite covering maps $F : X_1 \to X_2$.

This should be compared with the *covariance problem* for *max-scalar curvature* which is defined in section 5.4.1 and which obviously lifts under covering maps,

$$Sc^{\max}_{prop}[X_1] \geq Sc^{\max}_{prop}[X_2],$$

while the opposite inequality causes a problem (see section 5.4.1).

Question. Can one match the covariance of $Sc^{\max}$ by a somehow generalized K-$cowaist_2$ that would be invariant under (finite and infinite) covering maps $F : X_1 \to X_2$?

Specifically, one looks for *almost flat* (virtual) *infinite dimensional* Hilbert bundles in a suitable K-theory, which would be compatible with the index theory and with the Schroedinger-Lichnerowicz-Weitzenboeck formula in the spirit of Roe's C^*-algebras. c

Amenable Cutoff Subquestion. Let X_2 be a closed orientable Riemannian manifold of dimension $n = 2k$ and let $L \to X_2$ be a vector bundle induced by an ε-Lipschitz map $f : X_2 \to S^n$ from the positive spinor bundle $L = \mathbf{S}^+ = \mathbf{S}^+(S^n) \to S^n$. c Suppose that the fundamental group $\pi_1(X_2)$ is *amenable*, let $X_1 = \tilde{X}_2 \to X_2$ be the universal covering map and let

$$\tilde{L} = F^*(L) \to X_1$$

be the pullback of L.

When do there exist unitary bundles $\tilde{L}_i \to X_1$, $i = 1, 2, ...$, with unitary connections, such that

$\bullet_\infty$ the bundles $\tilde{L}_i$ are flat trivial at infinity;

$\bullet_{|\tilde{L}}$ there is an exhaustion of X_1 by compact *Følner subsets*

$$V_1 \subset ... \subset V_i \subset ... \subset X_1,$$

such that the restrictions of $\tilde{L}_i$ to V_i are equal to the restrictions of $\tilde{L}$,

$$(\tilde{L}_i)_{|V_i} = \tilde{L}_{|V_i};$$

$\bullet_\int$ the integrals of the k-th powers of the curvatures of L_i are dominated by such integrals for $\tilde{L}$ over V_i,

$$\frac{\int_{X_1} |curv|^k(\tilde{L}_I)dx_1}{\int_{V_i} |curv|^k(\tilde{L})dx_1} \underset{i\to\infty}{\to} 0;$$

$\bullet_\epsilon$ the curvatures of all $\tilde{L}_i$ are bounded by

$$|curv|(\tilde{L}_i) \le \epsilon,$$

where $\epsilon = \epsilon_n(\varepsilon) \to 0$ for $\varepsilon \to 0$.

(The Federer Fleming isoperimetric/filling inequality in the rendition of [MW(mapping classes) 2018] may be useful here.)

Non-Amenable Cutoff Example. Let $X(= X_2)$ be a closed orientable Riemann surface of genus ≥ 0 and $L \to X$ a complex line bundle with a unitary connection, e.g. L is the tangent bundle $T(X)$, the Chern number of which $c_1(T(X))[X] = \chi(X)$ doesn't vanish for $genus(X) > 0$.

Let $\tilde{L} \to \tilde{X}$ be the lift (pullback) of L to the universal covering $\tilde{X}(= X_1)$ of X and observe that there exit disks $\tilde{D}^2(R) \subset \tilde{X}$, such that the parallel translates over the boundary circles $\tilde{S}^1(R) = \partial\tilde{D}^2(R)$ are a multiples of 2π and where the radii R of such disks can be arbitrary large.

Then the restriction of $\tilde{L} \to \tilde{X}$ to such a disk $\tilde{D}^2(R) \subset \tilde{X}$ *extends to a bundle*, call it $\tilde{L}_R \to \tilde{X}$, which is *trivial outside* $\tilde{D}^2(R)$ and such that

$$c_1(\tilde{L}_R/\tilde{S}^1(R)) \sim area(\tilde{D}^2(R)) \underset{R\to\infty}{\to} \infty,$$

provided the curvature of L (that is a closed 2-form on X) *doesn't vanish.*

Problem for $n > 2$. The main difficulty in similarly trivializing at infinity bundles over n-dimensional Riemannian manifolds X for $n = dim(X) \geq 3$ seems to be associated with the following *questions.*

Let $\mathcal{U}_b(k) = \mathcal{U}_b(k, X)$, $b \geq 0$, be the space of the unitary connections ∇ on a trivial bundle $L \to X$ of rank k, such that $|curv|(\nabla) \leq b$.

(a) *For which values b_1 and $b_2 > b_1$ are the connections from $\mathcal{U}_{b_1}(k)$ homotopic in $\mathcal{U}_{b_2}(k) \supset \mathcal{U}_{b_1}(k)$?*

(b) When do the homomorphisms of the homotopy groups

$$\pi_i(\mathcal{U}_{b_1}(k)) \to \pi_i(\mathcal{U}_{b_2}(k)),\ i \geq 1,$$

induced by the inclusions $\mathcal{U}_{b_1}(k) \hookrightarrow \mathcal{U}_{b_2}(k)$ vanish?

(c) How do the Whitney sum homomorphisms

$$\mathcal{U}_b(k_1) \times \mathcal{U}_b(k_2) \to \mathcal{U}_b(k_1 + k_2)$$

behave in this respect?

In particular, what happens to the homomorphisms $\pi_i(\mathcal{U}_{b_1}(k)) \to \pi_i(\mathcal{U}_{b_2}(k))$ under stabilization

$$\underbrace{\mathcal{U}_b(k) \times \ldots \times \mathcal{U}_b(k)}_{N} \rightsquigarrow (\mathcal{U}_b(Nk))$$

for $N \to \infty$?

Exercise. Let X be a complete orientable even dimensional Riemannian manifold with nonpositive sectional curvature. Show that there exists a K-class $\mathbf{h} \in K^0(X/\infty)$, such that

$$|curv|(\mathbf{h}) = 0 \text{ and } \mathbf{h}[X] \neq 0,$$

where $[X]$ denotes the fundamental homology class of X with *infinite supports.*

4.1.5 Sharp Algebraic Inequalities for the L-Curvature in the Twisted SLW(B) Formula

Normalization of Curvature. In so far as the scalar curvature is concerned we are interested not in the curvature $|curv|(L)$ per se but rather in the norm of the endomorphism()

$$\mathcal{R}_{\otimes L} : \mathbb{S} \otimes L \to \mathbb{S} \otimes L$$

in the Schroedinger-Lichnerowicz-Weitzenboeck formula for the twisted Dirac operator,

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \tfrac{1}{4} Sc(X) + \mathcal{R}_{\otimes L},$$

(see the previous section) where this $\mathcal{R}_{\otimes L}$ is as following linear/tensorial combination of the values of the curvature of L on the tangent bivectors in the manifold X, (see [GL(spin) 1980],[Lawson&Michelsohn(spin geometry) 1989] and section 3.3.3)

$$\mathcal{R}_{\otimes L}(s \otimes l) = \frac{1}{2} \sum_{i,j} (e_i \circ e_j \circ s) \otimes R^L_{e_i \wedge e_j}(l),$$

where

$e_i \in T_x(X)$, $i = 1,...n = dim(X)$ is an orthonormal frame of tangent vectors at a point $x \in X$,

$s \in \mathbb{S}$, are spinors,

$l \in L$ vectors in the bundle L,

$R^L(e_i \wedge e_j) : L \to L$ is the curvature of L (written down as the valued 2-form on X)

and

"$\circ$" denotes the Clifford multiplication.

This suggest the definition of

$$\lambda_{\min}[curv]_{\otimes\mathbb{S}}(L)$$

as the smallest (usually negative) eigenvalue of the $||\mathcal{R}_{\otimes L}||$.

$\ast$ Example: Llarull's algebraic inequality. [Llarull(sharp estimates) 1998] *Let $f : X \to S^n$ be a smooth 1-Lipschitz, or more generally, an area non-increasing map and let $L \to X$ be the pullback the spinor bundle $\mathbb{S}(S^n)$. Then this minimal eigenvalue of the $\mathcal{R}_{\otimes L}$ satisfies:*

$$\lambda_{\min}[curv]_{\otimes\mathbb{S}}(L) = -\frac{1}{4}(n(n-1) = -\frac{1}{4}Sc(S^n).$$

(We return to this in correctedthe next section,)

Using this $\lambda_{min}[curv]$ instead of the $|curv|$ one defines

$$\lambda_{min}[curv]_{\otimes\mathbb{S}}(\mathbf{h}),\ \mathbf{h} \in K^0(X),$$

as the *supremum* of $\lambda_{min}[curv]_{\otimes\mathbb{S}(L)}$ for all (virtual) bundles L in the class of $\mathbf{h}$,

Accordingly one modifies the above K-*cowaist*$_2(\mathbf{h})$ and define the corresponding *K-cowaist coupled with spinors*, denoted K-*cowaist*$_{\otimes\mathbb{S},2}(h_*)$, $h_* \in H_*(X)$, as the supremum of $\lambda_{min}[curv]_{\otimes\mathbb{S}}(\mathbf{h})$ over over all $\mathbf{h} \in K^0(Y)$, such that $\mathbf{h}(h_*) \neq 0$.

Then, for instance, the above $\bullet_{wst}$ for spin manifolds X takes more elegant form:

$$\text{K-}waist_{\otimes\mathbb{S},2}[X] \leq \frac{4}{\sigma} \text{ for } \sigma = \inf_x Sc(X,x) > 0.$$

Notice that this inequality, combined with the above $\ast$, implies Llarull's *geometric inequality* $Rad_{S^n}(X) \leq \sqrt{\frac{n(n-1)}{\sigma}}$, which we discuss at length in the next section.

Also this may give better formulae for K-cowaists of product of manifolds.

(See section 5.4.1 and also [G(positive) 1996] and [G(101) 2017] for other known and conjectural properties of $|curv|(\mathbf{h})$ formulated in these papers in the language of the K-area.)

4.2 Llarull's and Goette-Semmelmann's Sc-Normalised Estimates for Maps to Convex Hypersurfaces in Symmetric Spaces.

Let us now look closer at the above

$$\mathcal{R}_{\otimes L}(s \otimes l) = \frac{1}{2}\sum_{i,j}(e_i \circ e_j \circ s) \otimes R_{ij}(l),$$

that is the endomorphism of (on) the bundle $\mathbb{S} \otimes L \to X$,

which appears in the zero order term in the twisted Dirac

$$\mathcal{D}^2_{\otimes L} = \nabla^2_{\otimes L} + \frac{1}{4}Sc(X) + \mathcal{R}_{\otimes L},$$

for

$$\mathcal{D}_{\otimes L} : C^\infty(\mathbb{S} \otimes L) \to C^\infty(\mathbb{S} \otimes L).$$

Example of $L = \mathbb{S}$ *on* S^n. Since the norm of the curvature of (the Levi-Civita connection on) the tangent bundle is one, the norm of the curvature operators $R_{ij} : \mathbb{S} \to \mathbb{S}$ are at most (in fact, are to) $\frac{1}{2}$,

$$||R_{ij}(s)|| \le \frac{1}{2},$$

since the spin bundle $\mathbb{S}(X)$ serves as the "square root" of the tangent bundle $T(X)$, where this is literally true for $n = dim(X) = 2$, that formally implies the inequality $||R_{ij}(s)|| \le \frac{1}{2}$ for all $n \ge 2$.

And since the Clifford multiplication operators $s \mapsto e_i \cdot e_j \cdot s$ are unitary,

$$||\mathcal{R}_{\otimes L}(s \otimes l)|| \le \frac{1}{4}n(n-1) = \frac{1}{4}Sc(S^n)$$

This doesn't, a priori, imply this inequality for all (non-pure) vectors v on the tensor product $\mathbb{S} \otimes L$ for $L = \mathbb{S}$, but, by diagonalising the Clifford multiplication operators in a suitable basis and by employing the *essential constancy*[325] of the curvature R_{ij} of S^n, [Llarull(sharp estimates) 1998] shows that

$$||\langle \mathcal{R}_{\otimes L}(\underline{\theta}), \underline{\theta} \rangle|| \ge -\frac{1}{4}n(n-1)$$

for all unit vectors $\underline{\theta} \in \mathbb{S}(S^n) \otimes \mathbb{S}(S^n)$.

This inequality for twisted spinors on S^n trivially yields the corresponding inequality on all manifolds X mapped to S^n, where the bundle $L \to X$ is the induced from the spin bundle $\mathbb{S}(S^n)$.

Namely, let $X = (X, g)$ be an n-dimensional Riemannian manifold, $f : X \to S^n$ be a smooth map, $L = f^*(\mathbb{S}(S^n))$, let $df : T(X) \to T(S^n)$ be the differential of f and

$$\wedge^2 df : \wedge^2 T(X) \to \wedge^2 T(S^n)$$

be the exterior square of df.[326]

Then the

$$\mathcal{R}_{\otimes L} : \mathbb{S}(X) \otimes L \to \mathbb{S}(X) \otimes L$$

satisfies

$$||\langle \mathcal{R}_{\otimes L}(\theta), \theta \rangle|| \ge -|| \wedge^2 df|| \frac{n(n-1)}{4}, \ L = f^*(\mathbb{S}(S^n)),$$

for all unit vectors $\theta \in \mathbb{S}(X) \otimes f^*(\mathbb{S}(S^n))$.

Moreover, – this is formula (4.6) in [Llarull(sharp estimates) 1998] –

[325] Some eigenvalues of this are ± 1 and some zero.

[326] Recall that the norm $|| \wedge^2 df||$ measures by how f contracts/expands surfaces in X. For instance the inequality $|| \wedge^2 df|| 1$ signifies that f decreases the areas of the surfaces in X.

$$||\langle \mathcal{R}_{\otimes L}(\theta), \theta\rangle|| \geq -\frac{1}{4}|trace \wedge^2 df|,$$

where $trace \wedge^2 df$ at a point $x \in X$ stands for

$$\sum_{i\neq j} \lambda_i \lambda_j,$$

for the differential $df : T_x(X) \to T_{f(x)}(S^n)$ diagonalised to the orthogonal sum of multiplications by λ_i.

This inequality, restricted to $L^+ = f^*(\mathbb{S}^+(S^n))$ together with the index formula, which says for this L_+ that

$$ind(\mathcal{D}_{\otimes L^+}) = \frac{|deg(f)|}{2}\chi(S^n),$$

provided X is a closed oriented spin manifold.

Thus we arrive at the proof of Llarull's theorem in the Sc-normalized trace form suggested by Mario Listing in [Listing(symmetric spaces) 2010].

★ *trace* $\wedge^2$ *df-Extremality of* S^n.[327] Let X be a closed orientable Riemannian spin n-manifold and $f : X \to S^n$ a smooth map of nonzero degree.

If

$$Sc(X, x) \geq \frac{1}{4}|trace \wedge^2 df(x)|$$

at all points $x \in X n$ then, in fact, $Sc(X) = \frac{1}{4}|trace \wedge^2 df|$ everywhere on X.

In fact, if n is even and $\chi(S^n) = 2 \neq 0$, this follows from the above. And if n is odd, there are (at lest) three different reductions to the even dimensional case (see [Llarull(sharp estimates) 1998], [Listing(symmetric spaces) 2010], [G(inequalities) 2018]), but these are artificial and conceptually unsatisfactory.

Also see see [Llarull(sharp estimates) 1998] and [Listing(symmetric spaces) 2010] for characterisation of maps f, where $Sc(X) = \frac{1}{4}|trace \wedge^2 df|$.

Llarull's estimate for the bottom of the spectrum of the curvature operator in spin bundle $\mathbb{S}(S^n)$, was generalized by Goette and Semmelmann [Goette-Semmelmann(symmetric) 2002] to the other Riemannian manifolds $\underline{X}$ with *non-negative curvature operators*, and (in the Sc-normalized form suggested Listing) resulted in the following.

★★ $\wedge^2 df$*-Extremality Theorem.*[328] Let $\underline{X} = (\underline{X}, \underline{g})$ and $X = (X.g)$ be a closed orientable Riemannian *spin* n-manifolds. where $\underline{X}$ has *non-negative curvature operator* and let $f : X \to \underline{X}$ be a smooth map of *non-zero* degree.

If $\underline{X}$ has non-zero Euler characteristics, then this map can't be strictly area decreasing with respect to the Sc-normalised metrics $g^\circ = Sc(g) \cdot g$ and $\underline{g}^\circ = Sc(\underline{g}) \cdot \underline{g}$.

This means that

if of the exterior square of the differential of f with respect to the original metrics g and $\underline{g}$ is related to the scalar curvatures of the two manifolds by the *inequality*

$$Sc(g, x) \geq || \wedge^2 df(x)||Sc(\underline{g}, f(x))$$

[327] This is *Spherical Trace Area Extremality Theorem* from section 3.4.1.

[328] This generalizes *Spin-Area Convex Extremality Theorem* from section 3.4.1.

at all $x \in X$, where $\| \wedge^2 df(x)\|$ stands fo the sup-norm with respect to the metrics g and $\underline{g}$,

then the equality holds:

$$Sc(X, x) = \| \wedge^2 df\| Sc(\underline{g}, f(x)).$$

Examples, Remarks, Conjectures. (a) All *compact symmetric* spaces have non-negative curvature operators.

Also

(a_1) *the induced metrics in convex hypersurfaces in these spaces also have the curvature operator non-negative,*[329]

and

(a_2) *Riemannian products of manifolds with curv.oper* ≥ 0 *have curv.oper* ≥ 0.

(a_3) By a theorem of Alan Weinstein [Weinstein(Positively curved) 1970], *submanifolds* $\underline{X}^n \subset \mathbb{R}^{n+2}$ *with non-negative sectional curvatures have non-negative curvature operators.*

(b) Llarull, Goette-Semmelmann, and Listing also analyzed the equality cases in their papers and proved the corresponding *rigidity theorems.*

(c) Goette and Semmelmann also state in their paper an extremality/rigidity result for odd dimensional $\underline{X}$, that was scrutinized and generalized in [Goette(alternating torsion)2007].

(d) Besides symmetric spaces, Goette and Semmelmann proved $\wedge^2 df$-extremality for was proven for Kähler manifolds with positive Ricci curvature.[330]

(e) Conjecturally, neither *spin* nor $\chi(\underline{X}) \neq 0$-condition are necessary for the $\wedge^2 df$-extremality.

In fact, Goette and Semmelmann (as well as Min-Oo) prove their theorems not only for spin manifolds but also for certain *spin^c-manifolds* and also for for *spin maps* $f : X \to \underline{X}$ between non-spin manifolds, i.e. where f pulls back the Stiefel-Whitney class $w_2(\underline{X})$ to $w_2(X)$.

(f) The above extremality theorems were generalized in the original papers to maps $f : X \to \underline{X}$, where $dim(X) = n = \underline{n} + 4k$, $\underline{n} = dim(\underline{X})$, and where f has non-zero $\hat{A}$-degree, i.e. where the pullback $f^{-1}(\underline{x}) \subset X$ of a generic point $\underline{x} \in \underline{X}$ has *non-zero* $\hat{A}$*-genus.*

$\mathbb{T}^{\rtimes}$*-Stabilization of Extremality Theorems and Generalizations.* The above (f) suggests the following.c

Conjecture. If a Riemannin manifold $\underline{X}$ is $\wedge^2 df$-extremal, then, for all X and all smooth maps $f : X \to \underline{X}$, such that

$$Sc(g, x) > \| \wedge^2 df(x)\| Sc(\underline{g}, f(x)),$$

the generic pullback $f^{-1}(\underline{x}) \subset X$ is homologous (even bordant) in X to a submanifold Y, which *supports a metric with positive scalar curvature.*

As it stands, this seems not very realistic.

[329]This was explained to me by Anton Petrunin, who introduced a class of metrics inherited by convex hypersurfaces, see [Petrunin(convex) 2003].

[330]See [Goette-Semmelmann(Hermitian) 1999] and the earlier "symmetric" paper [Min-Oo(Hermitian) 1998].

However, if the extremality of $\underline{X}$ follows by the above kind of argument relying on a *sharp* SLW(B)-*inequality* for the Dirac operator on X twisted with the pullback $L^* = f^*(\underline{L})$ of some bundle $\underline{L} \to \underline{X}$, with a unitary connection, then, as we shall explain below,

★ ★ ★ *the inequality* $Sc(g,x) > \| \wedge^2 df(x)\| Sc(\underline{g}, f(x))$ *implies vanishing not only of* $\hat{A}(f^{-1}(\underline{x}))$ *but of more general (all?) Dirac theoretic obstructions for* $Sc > 0$ *on* $(n - \underline{n})$*-dimensional manifolds.*[331]

The basic (and fairly general) instance of this is where X supports ε-flat bundles $L_\varepsilon \to X$ for all $\varepsilon > 0$ (i.e. L_ε are endowed with unitary connections the curvatures of which are bounded in norm by ε), such that that the indices of the Dirac operator $\mathcal{D}$ on X twisted with $L^* \otimes L_\varepsilon$, as expressed by the index formula, don't vanish for $\varepsilon \to 0$.

Since the norm of the connection curvature term in SLW(B)-formula for the operator $\mathcal{D}_{\otimes(L^*\otimes L_\varepsilon)}$ converges, for $\varepsilon \to 0$, to that for $\mathcal{D}_{\otimes L^*}$, the inequality

$$Sc(g,x) - \| \wedge^2 df(x)\| Sc(\underline{g}, f(x)) \ge \delta > 0$$

implies vanishing of of the index of $\mathcal{D}_{\otimes(L^*\otimes L_\varepsilon)}$ for $\varepsilon << \delta$ and the proof follows.

Remarks and Examples. (a) The bundles L_ε can be understood in a fairly general way, e.g. is virtual Fredholm bundles, as families of such bundles or, more generally as moduli over the (reduced) C^*-algebra of a quotient group of the fundamental group of X.

(b) If $\underline{X} = \underline{X}_0 \times \underline{X}_1$, where X_1 is a compact orientable Riemannin spin manifold with $curv.oper(X_0) \ge 0$ as in ★★, if X is orientable spin, and if $f : X \to \underline{X}$ is a map of non-zero degree, such that $Sc(g,x) > \| \wedge^2 df(x)\| Sc(\underline{g}, f(x))$, then, probably, *the rational Rosenberg index*(see [Zeidler(width) 2020])

$$\alpha(\underline{X}_1) \in (KO_{\underline{n}_1}(C^*\pi_1(\underline{X}_1))) \otimes \mathbb{Q}$$

vanishes.

I feel shaky in these matters (this must be obvious to the readers well versed in the K-theory of C^*-algebras) but the proof of this is transparent in many cases.

For instance, this is so

if the universal covering $\tilde{\underline{X}}_1$ *of* $\underline{X}_1$ *is* $\wedge^2$*-hyper-Euclidean* i.e. there exists a smooth proper area non-decreasing map $\tilde{\underline{X}}_1 \to \mathbb{R}^{n_1}$, $n_1 = dim(X_1)$,

In fact, the above considerations and the relative index theorem yield the following more general proposition.

★ **Non-compact Extremality Theorem.** Let X and $\underline{X}_0$ be connected orientable Riemannin *spin* manifolds of dimensions n and $\underline{n}_0$, where X is *complete* and $\underline{X}_0$ is *compact*, and let *the curvature operator of* $\underline{X}_0$ *be non-negative.*

Let

$$f = (f_0, f_1) : X \to \underline{X}_0 \times \mathbb{R}^m, \;\; m = n - n_0,$$

be a smooth proper map with non-zero degree.

[331] The simplest instance of this, where $\underline{X} = S^{\underline{n}}$ and where X is a warped extension $X_0 \rtimes_\varphi \mathbb{T}^1$, was observed in §5$\frac{4}{9}$ in [G(positive) 1996], and used for the proof of a special case of C^0-*closure theorem* from section 3.1.3.

Let $\underline{X}_0$ be simply connected, let the Euler characteristics of $\underline{X}_0$ don't vanish, $\chi(\underline{X}_0) \neq 0$, and let the map $f_1 : X \to \mathbb{R}^m$ be area non-increasing. Then

$$\inf_{x \in X} \left(Sc(g,x) - \| \wedge^2 df(x) \| \cdot Sc(\underline{g}, f(x)\right) \leq 0.$$

Proof. Here, the relevant bundle $L^* \to X$ is the f_0-pull back of the positive spin bundle $\mathbb{S}^+(\underline{X}_0)$ (as in ★ and in ★★) from $\underline{X}_0$ to X, while the bundles $L_\varepsilon \to X$ are the f_1 pullbacks of the complex ε- *flat bundles of ranks* $l = \frac{m}{2}$ (if m is odd, multiply X and $\underline{X}_0$ by $\mathbb{R}^1$) on $\mathbb{R}^m$ with compact supports (e.g. flat split at infinity) and such that *they have their relative Chern numbers* $c_l \to \infty$ *for* $\varepsilon \to 0$.

This and non-vanishing of $\chi(\underline{X}_0$ imply (half line computation) the non-vanishing of the relative index of $\mathcal{D}_{\otimes(L^* \otimes L_\varepsilon)}$ by the relative index theorem. and the proof concluded is with the (ε-perturbed) SLW(B)-formula for X_0 as in the Goette-Semmelmann theorem ★★.

Remarks/Corollaries (a) Probbaly, it is is not hard to prove rigidity of $\underline{X}_0$ in this case.

(b) Instead of "*simply connected and* $\chi(\underline{X}_0) \neq 0$" one could requirer that *the universal covering* $\tilde{\underline{X}}_0$ *has non-zero Euler characteristics.*

Indeed, by the Gromoll-Meyer theorem, $\tilde{\underline{X}}_0$ isometrically splits,

$$\underline{X}_0 = \underline{X}'_0 \times \mathbb{R}^k,$$

where $\underline{X}'_0$ is compact simply connected and the theorem apples to $\underline{X}'_0 \times \mathbb{R}^{k+m}$.

(c) The above proof, similarly to these of ★ and ★★, easily generalizes to maps f with *non-vanishing* $\hat{A}$*-degrees.*

Question. Can one approach the above conjecture from the opposite angle by actually *constructing* $(n-m)$*-submanifolds in* X *with positive scalar curvatures* in the homology class of $f^{-1}(\underline{x})$?

(Application of μ-bubbles, as we know, allows such constructions, but these *fail to deliver sharp inequalities* of this kind).

4.3 Bounds on Mean Convex Hypersurfaces

Recall that the *spherical radius* $Rad_{S^{n-1}}(Y)$ of a connected orientable Riemannian manifold of dimension $(n-1)$ is the supremum of the radii R of the spheres $S^{n-1}(R)$, such that X admits a distance decreasing map $f : Y \to S^{n-1}(R)$ of non-zero degree, where this f for non-compact Y this map is supposed to be constant at infinity.[332]

We already indicated in section 3.5 also see [G(boundary) 2019] that Goette-Semmlenann's theorem (above ★★), applied to smoothed doubles $\mathbb{D}X$ and $\mathbb{D}\underline{X}$ yields the following corollary.

$\bigcirc^{n-1}$*Let* X *be a compact orientable Riemannian manifold with boundary* $Y = \partial X$.

If $Sc(X) \geq 0$ *and the mean curvature of* Y *is bounded from below by* $mean.curv(Y) \geq \mu > 0$, *then the hyperspherical radius of* Y *for the induced Riemannian metric*

[332] Alternatively, one might require f to be *locally* constant at infinity, or more generally, to have the limit set of codimension≥ 2 in $S^{n-1}(R)$.

is bounded by

$$Rad_{S^{n-1}}(Y) \leq \frac{n-1}{\mu}.$$

In fact, the proof of this indicated in section 3.5 (also see [G(boundary) 2019]) together with the above ★ yields the following more general theorem.

★$_{mean}$ **Non-Compact Mean Curvature Inequality**. Let X and $\underline{X}_0$ be connected orientable Riemannin *spin* manifolds of dimensions n and $\underline{n}_0$ with boundaries, where X is *complete* and $\underline{X}_0$ is *compact*, and let *the curvature operator of* $\underline{X}_0$ *be non-negative.*

Let

$$f = (f_0, f_1) : X \to \underline{X}_0 \times \mathbb{R}^m, \ m = n - \underline{n}_0,$$

be a smooth proper map, which sends $\partial X \to \partial \underline{X}_0 \times \mathbb{R}^m$ and which has *non-zero* degree.

Let $\underline{X}_0$ *be simply connected, let the Euler characteristics of* $\underline{X}_0$ *don't vanish,* $\chi(\underline{X}_0) \neq 0$, *let the map* $f_1 : X \to \mathbb{R}^m$ *be area non-increasing and let the restriction of this map to the boundary of* X, *be distance non-increasing, i.e.*

$$||df_1(x)|| \leq 1 \ for \ x \in \partial X.$$

If

scal$_{\geq}$ $$Sc(g,x) - || \wedge^2 df(x)|| \cdot Sc(\underline{g}, f(x)) \geq 0$$

then

mean$_{\leq}$ $$\inf_{x \in \partial X} (mean.curv(\partial X, x) - ||df(x)|| \cdot mean.curv(\partial \underline{X}_0 \times \mathbb{R}^m, f(x))) \leq 0.$$

Remarks. (a) If $Sc(\underline{X}_0) = 0$, (hence, $\underline{X}_0$ is Riemannian flat) then the condition scal$_{\geq}$ reduces to $Sc(X) \geq 0$.

(b) The inequality mean$_{\leq}$ also yields some information for manifolds X with *negative scalar curvatures bounded from below.*

For instance, if X is compact and $Sc(X) \geq -2$), then mean$_{\leq}$, this is achieved by applying ★$_{mean}$ to maps from $X \times S^2$ to the unit balls $B^{n+2} \subset \mathbb{R}^{n+2}$ (see [G(boundary) 2019]).

However, the sharp inequalities for $Sc(X) < 0$, such, for instance, as *optimality* of the hyperspherical radii of the boundary spheres of balls $B^n(R)$ in the hyperbolic spaces $\mathbb{H}^n_{-1}$, remain *conjectural*.[333]

(c) It is unknown if the spin condition on X is necessary, but it can be relaxed by requiring the universal cover of X, rather than X itself is spin.(This done with the L_2-version of the Goette-Semmelman theoremGoette-Semmelmann

And if one is content with a non-sharp bound

$$Rad_{S^{n-1}}(Y) \leq \frac{const_n}{\inf mean.curv(Y)},$$

[333]This "optimality" means that if $Sc(X) \geq -n(n-1)$ and $mean.curv(\partial X) \geq mean.curv(\partial B^n(R))$ than $Rad_{S^{n-1}}(\partial X) \leq Rad_{S^{n-1}}(\partial B^n(R))$.

then one and can prove this without the spin assumption by the by a capillary version of the (iterated) warped product argument for manifolds with boundaries 5.6. 5.8.1.

(d) Unavoidable *approximation error terms* in the smoothing of the corners in the doubles $\mathbb{D}X$ and $\mathbb{D}\underline{X}_0$ make our proof of $\bigstar_{mean}$ poorly adjusted for *establishing rigidity* of $\underline{X}_0 \times \mathbb{R}^m$.

For this purpose, it would be better to use Lott's index theorem for manifolds with boundary.

In fact, Lott himself proves in [Lott(boundary)2020] a *non-normalized* rigidity theorem for *compact* manifolds $\underline{X}_0$ of *even* dimension n.

Apparently, Lott's argument extends to the Sc- and mean.curv- normalized case and non-compactness of $\underline{X}_0$ also causes no serious problem. But it is unclear how handle the case of odd n without an approximation argument.

The simplest case, where this difficulty arises is for maps from compact manifolds X to *odd dimensional* balls $B^n \subset \mathbb{R}^n$ and to products of such balls by tori, $\underline{X}_0 = B^{2k+1} \times \mathbb{T}^{n-2k-1}$, where $\bigstar_{mean}$ applies to the universal coverings of these manifolds.

Possibly on can resolve the problem with a generalized *Bourguignon-Kazdan -Warner perturbation theorem* or with (also generalized) *Burkhart-Guim's regularized Ricci flow argument.*

4.4 Lower Bounds on the Dihedral Angles of Curved Polyhedral Domains

We want to generalise the above $\bigstar_{mean}$ to manifolds X with non-smooth boundaries with suitably defined mean curvatures bounded from below, where we limit ourself to manifolds with rather simple singularities at their boundaries.

Namely, let X and $\underline{X}$ be Riemannian n-manifolds *with corners*, which means that their boundaries $Y = \partial X$ and $\underline{Y} = \partial \underline{X}$ are decomposed into $(n-1)$-faces F_i and $\underline{F}_i$ correspondingly, where, locally, at all points $y \in Y$, and $\underline{y} \in \underline{Y}$ these decompositions are is diffeomorphic to such decomposition of the boundary of a convex n-dimensional polyhedron (polytope) in $\mathbb{R}^n$.

Let $f : X \to \underline{X}$ be a smooth map, which is compatible with the corner structures in X and $\underline{X}$:

f sends the $(n-1)$-faces F_i of X to faces $\underline{F}_i$ of $\underline{X}$.

Assume as earlier that

$$\text{scal}_{\geq} \qquad Sc(X,x) \geq \| \wedge^2 df \| \cdot Sc(\underline{X}, f(x)) \text{ for all } x \in X$$

and replace mean$_{\leq}$ by the opposite inequalities applied to *for all faces* $F_i \subset Y$ individually,

$$\text{mean}_{\geq} \qquad mean.curv(F_i, y) \geq \|df\| \cdot mean.curv(\underline{F}_i, f(y)) \text{ for all } y \in F_i \ .$$

Let $\angle_{i,j}(y)$ be the dihedral angle between the faces F_i and F_j at $y \in F_i \cap F_i$ and let us impose our main inequality between these $\angle_{i,j}(y)$ for all F_i and $F)j$ and the dihedral angles between the corresponding faces faces $\underline{F}_i$ and $\underline{F}_j$ at the points $f(y) \in \underline{F}_i \cap \underline{F}_j$:

$$[\leq]^{\angle ij} \qquad \angle_{i,j}(y) \leq \angle_{i,j}(f(y)) \text{ for all } F_i, F_j \text{ and } y \in F_i \cap F_j.$$

Besides the above, we need to add the following condition the relevance of which remains unclear.

Call a point $y \in Y = \partial X$ *suspicious* if one of the following two conditions is satisfied

(i) the corner structure of X at y is *non-simple* (not cosimplicial), where simple means that a neighbourhood of y is diffeomorphic to a neighbourhood of a point in the n-cube, which is equivalent to transversality of the intersection of the $(n-1)$-faces which meet at $\underline{y}$;

(ii) there are two $(n-1)$-faces in X which contain y, say $F_i \ni y$ and $F_j \ni y$, such that the dihedral angle $\angle_{ij} = \angle(F_i, F_j$ is $> \frac{\pi}{2}$;

Then out final condition says that

$[\equiv]^{\angle ij}$
$$\angle_{i,j}(y) = \angle_{i,j}(f(y)).$$

for all suspicious points y.

◆$_{\angle ij}$ **Compact Dihedral Extremality Theorem**. . Let X and $\underline{X}$ be compact connected orientable Riemannin *spin* manifolds of dimension n with corners, where *the curvature operator of* $\underline{X}$ *be non-negative*, all faces $\underline{F}_i \subset \partial \underline{X}$ are mean convex. Let $f : X \to \underline{X}$ be a smooth *proper corner* map, (it respects the corner structure) of *non-zero degree* and let f satisfy the four conditions $\mathsf{scal}_{\geq}$ and $\mathsf{mean}_{\geq}$, $[\leq]^{\angle ij}$ and $[\equiv]^{\angle ij}$

If the universal covering of $\underline{X}$ *has non-zero the Euler characteristics* $\chi(\tilde{\underline{X}}) \neq 0$, *then* f *satisfies the equalities corresponding to the inequalities* $\mathsf{scal}_{\geq}$, $\mathsf{scal}_{\geq}$, $[\leq]^{\angle ij}$:

$$Sc(X, x) = \| \wedge^2 df \| \cdot Sc(\underline{X}, f(x)) \text{ for all } x \in X,$$

$$mean.curv(F_i, y) = \|df\| \cdot mean.curv(\underline{F}_i, f(y)) \text{ for all } y \in F_i,$$

$$\angle_{i,j}(y) = \angle_{i,j}(f(y)) \text{ for all } F_i, F_j \text{ and } y \in F_i \cap F_j.$$

About the Proof. This is shown by smoothing the boundaries of X and of $\underline{X}$ and applying ★$_{mean}$ from the previous section, to the universal covering of $\underline{X}$ and the corresponding (induced) covering of X [334] where an essential feature of non-suspicious points follows from the following

Elementary Lemma. Let $\Delta \subset S^n$ be a spherical simplex with all edges of length $\geq l \geq \frac{\pi}{2}$. Then there exists a continuous family of simplices $\Delta_t \subset S^n$, $t \in [0,1]$ with the following properties.

- *$\Delta_0 = \Delta$ and Δ_1 is a regular simplex with the edge length l;*
- *all Δ_t have the edges of length $\geq l$;*
- *$\Delta_{t_2} \subset \Delta_{t_1}$ for $t_2 \geq t_1$;*
- *for each $t < 1$ there exists an $\varepsilon > 0$, such that n (out of $n+1$) vertices of $\Delta_{t+\varepsilon}$ coincide with those of Δ_t.*[335]

[334] If, instead of "X is spin" we only assume "the universal covering of X is spin", then we pass to this universal covering of X and use there the L_2-index theorem.

[335] This Lemma explains the role of the condition $[=]^{\angle ij}$ in our proof. The conclusion of the Lemma fails, in general to be true for obtuse angles (it seems OK if there is a single obtuse angle at each vertex , e.g, as it is for products of convex polygons) but it remains unclear if this condition is needed for the validity of the theorem itself.

The proof of the lemma is a high school exercise while *construction of adequate smoothing of* X with the help of this lemma, which is straightforward and boring, will be given elsewhere.

Notice that the $\times\blacktriangle^i$-*Inequality* from section 3.18, which says that

convex polyhedra $\underline{X} \subset \mathbb{R}^n$ with the dihedral angles $\leq \frac{\pi}{2}$ admit no deformations which would *decrease their dihedral angles* and simultaneously *increase the mean curvatures of their faces,*

is an immediate corollary of $\blacklozenge_{\angle_{ij}}$.

Two Problems. 1. There is little doubt that the above extremal manifolds with corners $\underline{X}$ are *rigid*, but our argument, as we explained this in the previous section is, technically, not good enough for proving it, and no index theorem theorem for general manifolds with corners is available, at least not at the present moment.

2. It remains unclear what is the *full class* of extremal polyhedra and manifolds with corners in general, but the following generalization of $\blacklozenge_{\angle_{ij}}$ is easily available.

Fundamental Domains of Reflection Groups. What underlies the double $\mathbb{D}$-construction, $X \rightsquigarrow \mathbb{D}X$ in the proof of the $\blacklozenge_{\angle_{ij}}$ theorem is the doubling $S^n = \mathbb{D}S^n_+$, which is associated with the reflection of S^n with respect to the equatorial subsphere.

With this in mind, one can generalise everything from this section to general reflection groups, including spherical, Euclidean and hyperbolic ones, (such as we met in section 3.1.1) and also to products of these.

Example of Corollary. Let X be a compact manifold with corners, where the (combinatorial) corner structure is isomorphic to that of the product of an $(n-m)$-simplex $\blacktriangle$ with the rectangular fundamental domain $\blacksquare$ (orbifold) of a cocompact reflection group in an aspherical m-manifold.[336]

If X *is spin, than it admits* no Riemannian metric g, *such that* $Sc(g) \geq 0$, *where all faces have* $mean.curv_g \geq 0$ *and where the dihedral angles are smaller than the corresponding angles in the product of the regular Euclidean simplex* $\blacktriangle$ *by* $\blacksquare$ *with* $\frac{\pi}{2}$ *dihedral angles.*

Problem with Rigidity. If not for

4.5 Stability of Geometric Inequalities with $Sc \geq \sigma$ and Spectra of Twisted Dirac Operators.

Sharp geometric inequalities, as we explained in section 3.19, beg for a company of their nearest neighbours.

For instance, the Euclidean isoperimetric inequality for bounded domains $X \subset \mathbb{R}^n$, which says that

$$vol_n(X) \leq \gamma_n vol_{n-1}(\partial X)^{\frac{n}{n-1}} \text{ for } \gamma_n = \frac{vol(B_n)}{vol_{n-1}(S^{n-1})^{\frac{n}{n-1}}},$$

goes along with the following.

A. *Rigidity. If* $vol_n(X) = \gamma_n vol_{n-1}(\partial X)^{\frac{n}{n-1}}$, *then* X *is a ball.*

[336] These exist for all $m \geq 4$ by Michael Davis 1983 theorem, see his lectures [Dav(orbifolds) 2008] and references therein.

B. *Isoperimetric Stability.* Let $X \subset \mathbb{R}^n$ be a bounded domain with $vol_n(X) = vol_n(B^n)$ and $vol(\partial X) \le vol_{n-1}(S^n) + \varepsilon$.

Then there exists a ball $B = B^n_x(1+\delta) \subset \mathbb{R}^n$ of radius δ with center $x \in X$, where $\delta \underset{\varepsilon\to 0}{\to} 0$, such that the volume of the difference satisfies

$$vol_n(X \smallsetminus B) \le \delta_1,$$

and, moreover,

$$vol_{n-1}(\partial B \cap X) \le \delta_2, \text{ and } vol_{n-2}(\partial B \cap \partial X) \le \delta_3,$$

where

$$\delta_1, \delta_2, \delta_3 \underset{\varepsilon\to 0}{\to} 0.$$

(Unless $n = 2$ and X is connected, there is no bound on the diameter of X, but the constants $\delta, \delta_1, \delta_2, \delta_3$ can be explicitly evaluated even for moderately large ε.)

In the case of sharp scalar curvature inequalities, their poofs by Dirac theoretic methods[337](more or less) automatically deliver rigidity. For instance,

★ *if a manifold $\underline{X}$ homeomorphic to S^n, besides having $curv.oper(\underline{X}) \ge 0$ has $Ricci(\underline{X}) > 0$ and if X is a closed orientable spin Riemannian manifold with $Sc(X) \ge n(n-1)$ then, all* smooth 1-Lipschitz *maps $X \to \underline{X}$ of non-zero degrees are* isometries. [338]

What we want to understand next is what happens if the inequality $Sc(X) \ge n(n-1)$ is relaxed to $Sc(X) \ge n(n-1) - \varepsilon$ for a small $\varepsilon > 0$, where an application of thin surgery1.3 delivers the following.

Example.[339] Let $\Sigma \subset S^n$ be a compact smooth submanifold of dimension $\le n-3$. Then there exists an arbitrary small ε-neighbourhood $U_\varepsilon = U_\varepsilon(\Sigma) \subset S^n$ with a smooth boundary $\partial_\varepsilon = \partial U_\varepsilon$ and a family of smooth metrics $g_{\varepsilon,\epsilon}$ on the double

$$\mathbb{D}(S^n \smallsetminus U_\varepsilon) = (S^n \smallsetminus U_\varepsilon) \cup_{\partial_\varepsilon} (S^n \smallsetminus U_\varepsilon),$$

where $Sc(g_{\varepsilon,\epsilon}) \ge n(n-1) - \varepsilon - \epsilon$ and which, for $\epsilon \to 0$, *uniformly converge* to the natural continuous Riemannian metric on $\mathbb{D}(S^n \smallsetminus U_\varepsilon(\Sigma)$.

Moreover, if $\Sigma \subset S^n$ is contained in a hemisphere, then – this follows from the *spherical Kirszbraun theorem* – the (double) manifolds $\mathbb{D}(S^n \smallsetminus U_\varepsilon, g_{\varepsilon,\epsilon})$ admit 1-Lipschitz maps to the sphere S^n with degrees one, for all sufficiently small $\varepsilon > 0$ and , $\epsilon = \epsilon(\varepsilon) \underset{\varepsilon\to 0}{\to} 0$.

For instance, if $n \ge 3$ and Σ consists of a single point, then $\mathbb{D}(S^n \smallsetminus U_\varepsilon)$, that is the connected sum $S^n \# S^n = S^n \#_{S^{n-1}(\varepsilon)} S^n$ of the sphere S^n with itself

[337]See [Llarull(sharp estimates) 1998], [Min-Oo(Hermitian) 1998], [Goette-Semmelmann(symmetric) 2002], [Listing(symmetric spaces) 2010], [Zeidler(width) 2020], [Zhang(area decreasing) 2020], [Lott(boundary) 2020], [Guo-Xie-Yu(quantitative K-theory) 2020].

[338]Even if Ricci vanishes somewhere, one still may have a satisfactory description of the extremal cases. For instance, if $\underline{X} = (S^{n-m} \times \mathbb{R}^m)/\mathbb{Z}^m$, e.g. $\underline{X} = S^{n-m} \times \mathbb{T}^m$, then all (orientable spin) X with $Sc(X) \ge Sc(\underline{X}) = (n-m)(n-m-1)$, which admit maps $f : X \to \underline{X}$ with $deg(f) \ne 0$, are *locally isometric* to $\underline{X}$ (albeit the map f itself doesn't have to be a local isometry.

[339]Compare with [GL(classification) 1980], [BaDoSo(sewing Riemannian manifolds) 2018] and section 2 in [G(101) 2017].

(where the ε-sphere $S^{n-1}(\varepsilon)$ serves as ∂_ε and $S^n\# S^n$ is homeomorphic to S^n), admits, for small ε, a 1-Lipschitz map to S^n with degree 2.

Furthermore, iteration of the connected sum construction, delivers manifolds (topologically spheres)

$$(S^n)^{k\#_\varepsilon} = \underbrace{S^n\#_{S^{n-1}(\varepsilon)}S^n\#...\#_{S^{n-1}(\varepsilon)}S^n m}_{k}$$

which carry metrics with $Sc(S^n)^{k\#_\varepsilon} \geq n(n-1) - \varepsilon - \epsilon$ and, at the same time, admit maps to S^n of degree k, where these maps are 1-Lipschitz everywhere and which are locally isometric away from $\sqrt{\varepsilon}$-neighbourhoods of $k-1$ ε-spherical "necks" in $(S^n)^{k\#_\varepsilon}$.

(For general Σ and even k one has such maps f with $deg(f) = k/2$.

Conjecturally, this example faithfully represents possible geometries of closed Riemannian n-manifolds X with $Sc(X) \geq n(n-1) - \varepsilon$, which admit 1-Lipschitz maps to the unit sphere S^n, but only the following two, rather superficial, results of this kind are available.

1. Let $X = (X, g)$ be a closed oriented Riemannian spin n-manifold with $Sc(X) \geq n(n-1) - \varepsilon$ and let $f : X \to \underline{X} = S^n$ be a smooth 1-Lipshitz map of degree $d \neq 0$ and let $J_f(x) = \wedge^n df$ denote the Jacobian of f.

Let $X_{\leq\lambda} \subset X$ denotes the subset, where $|J_f(x)| \leq \lambda$, for some $\lambda < 1$.

Then the signed f-volume of $X_{\leq\lambda}$ satisfies

$$[|X_{\leq\lambda}| \leq] \qquad |vol_f(X_{\leq\lambda})| =_{def} \left|\int_{X_{\leq\lambda}} J_f(x)dx\right| \leq c_{\lambda,n,\tilde{V}}(\varepsilon) \underset{\varepsilon\to 0}{\to} 0.$$ [340]

(Observe that since f is 1-Lipschitz, $|J_f| \leq 1$ and $1 - |J_f(x)|^{-1}$ measures the the distance from the differential $df(x) : T_x(X) \to T_{f(x)}(\underline{X})$ to being isometry.)

Sketch of the Proof. Since the twisted Dirac $D_\otimes$ in Llarull's rigidity argument from [Llarull(sharp estimates) 1998] has non-zero kernel, its square $D^2_\otimes$ is non-positive (we assume here that $n = dim(X) = dim(\underline{X})$ is even), and, by the Bochner-Schrödinger-Lichnerowicz-Weitzenböck formula (that is above $[D^2_\otimes]_f$), this implies non-positivity of

$$\nabla^2 + \frac{1}{4}Sc(X) + \mathcal{R}_\otimes.$$

Consequently, $-\Delta_g - \frac{1}{4}(\varepsilon + (1 - \underline{l}(x)))$, where Δ_g is an ordinary Laplace on $X = (X, g)$, also non-positive, since the coarse (Bochner) Laplacian ∇^2 is "more positive" than the (positive) Laplace(-Beltrami) $-\Delta$ as it follows from the *Kac-Feynman* formula and/or from the *Kato inequality.*

(In general, this applies in the context of the above rigidity theorem ★ and yields non-positivity of $-\Delta_g - \frac{1}{4}(\varepsilon + \underline{C}(1 - \underline{l}_f(x))$ with $\underline{C}$ depending on the smallest eigenvalue of $Ricci(\underline{X})$.)

In order to extract required geometric information concerning the metric $\tilde{g}$ from this property of the metric g, we observe that the essential part of X, that

[340] This was incorrectly stated in an earlier version of this text for *non-signed* volume of f, that is $\int |J_f(x)|dx$; the error was pointed out to me by Bernhard Hanke.

is the one, where we need to bound from below the L_2-norms of the g-gradients of functions $\phi(x)$ (to which the above Δ_g applies) is where

$$\lambda \geq \underline{l}_f(x) \geq \lambda_{\tilde{V}} > 0$$

for some $\lambda_{\tilde{V}} > 0$, and where the geometries of g and of $\tilde{g}$ are mutually $(\lambda_{\tilde{V}})^{-1}$-close.

Thus, the relevant lower g-gradient estimate for $\phi(x)$ comes from the isoperimetric inequality for $\tilde{g}$ which, in turn, follow from such an inequality in $\underline{X}$, that is the sphere in the present case. (Filling in the details is left to the reader.)

Remark. (a) The above example shows that the g-volume of $X_{\leq\lambda} \subset X$ can be large and that the bound on $\tilde{V}$ concerns not only the subset $X_{\leq\lambda}$ but its complement $X \setminus X_{\leq\lambda}$ as well.

Corollary + Question. (a) Let X be a closed orientable Riemannian spin n-manifold with $Sc(X) \geq n(n-1)$ and let $f: X \to S^n$ a (possibly non-smooth!) 1-Lipshitz map of degree $\neq 0$.

If the map Y is a homeomorphism, then it is an isometry.

(b) Is this remain true for *all* 1-Lipshitz maps?

The inequality $[|X_{\leq\lambda}| \leq]$ doesn't take advantage of $deg(f)$ when this is large, but the following proposition does just that.

2. Let X be a compact oriented Riemannian spin n-manifold with a boundary $Y = \partial X$, such that $Sc(X) \geq n(n-1) + \varepsilon$, $\varepsilon > 0$.

Let $f: X \to S^n$ be a smooth map, which is constant on Y, which is *area contracting away from the a neighbourhood $U \subset X$ of $Y = \partial X \subset X$,*

$$\| \wedge^2 df(x)\| \leq 1 \textit{ for all } x \in X \setminus U,$$

and where

$$\| \wedge^2 df(x)\| \leq C_o \text{ for all } x \in X \setminus U \text{ and some constant } C_o > 0.$$

Then the degree of f is bounded by a constant d depending only on U and on C_o,

$$|deg(f)| \leq d = const_{U,C_o}.$$

Sketch of the Proof. (Compare with §§$5\frac{1}{2}$ and 6 in [G(positive) 1996].) Let $s(x)$ be the (Borel) function on X which equals to ε away from U and is equal to $E = -C_n \times C_o$ on U for some universal $C_n \approx n^n$.

Then arguing (essentially) as in the first part of the above proof, we conclude that the spectrum of the $-\Delta + s(x)$ on the (smoothed) double $\mathbb{D}(X)$ contains at least $d = deg(f)$ negative eigenvalues.

This an easy argument would deliver d eigenvalues λ_i of the $-\Delta$ on $\mathbb{D}(U)$, where the corresponding eigenfunctions vanish on the two copies of the boundary of U in X (but not, necessarily on Y), and such that $\lambda_i \lessapprox E$.

This would yield the required bound on d. (Here again, the details are left to the reader.)

Remark + Example + Two Problems. (a) If the boundary of $Y = \partial X$ admits an orientation reversing involution, then the constancy of f on Y can be relaxed to $dim(f(Y)) \leq n-2$, where the constant d will have to depend on the geometry of this involution and of the map $Y \to S^n$.

(It is unclear if the existence of such an involution is truly necessary.)

(b) This (a) apply, for instance, to coverings $X = \Sigma^2_{d,\delta}$ of the 2-sphere minus two δ-discs as well as to the products of these $\Sigma^2_{d,\delta}$ with the Euclidean ball $B^{n-1}(R)$ of radius $R > \pi$.

(c) What are the sharp and/or comprehensive versions of these **1** and **2**?

(d) Let Y be a homotopy sphere of dimension $4k-1$, which bounds a Riemannian manifold X with $Sc \geq \varepsilon > 0$. Give an *effective* bound on the $\hat{A}$-genus of X in terms of the geometry of Y and its second fundamental form $h = \mathrm{II}(Y \subset X)$ and study the resulting invariant

$$Inv_\varepsilon(Y,h) = \sup_X |\hat{A}(X)|, \text{ where } \partial X = Y,\ Sc(X) \geq \varepsilon,\ \mathrm{II}(Y \subset X) = h.$$

4.6 Dirac Operators on Manifolds with Boundaries

When I was delivering these lectures in the Spring 2019, all known relevant for us index theorems for (twisted) Dirac operators $\mathcal{D}$ directly applied only to *complete* Riemannian manifolds. [341] But then Cecchini, Guo-Xie-Yu and Zeidler [342] have developed

an index theory for manifold with boundary including the solution of the long neck problem for spin manifolds by Cecchini (see section **??**).

Even though, much(all?) what is presented in this and the following sections 4.6.1-4.6.5 may follow from the recent results of these authors, we keep it as it was originally written, since this suggests an additional perspective on the role of the Dirac operator in the geometry of scalar curvature.

As far as the scalar curvature is concerned, all the index theorems are needed for is delivering *non-zero harmonic* or *approximately harmonic* (often twisted) spinors on Riemannian manifolds X under certain certain geometric/topological conditions on X, which, a priori, have nothing to do with the scalar curvature but which are eventually used to obtain upper bounds on $Sc(X)$ via the (usually twisted) Bochner-Schrödinger-Lichnerowicz-Weitzenböck formula.

The index theorems for Dirac operators on closed manifolds can yield a non-trivial information on existence of approximately harmonic spinors on non-complete manifolds as well as on manifolds with boundaries, where the main issue, say for manifolds with boundaries, can be formulated as follows.

Spectral $\mathcal{D}^2$-Problem. Let X be a compact Riemannian spin manifold with a boundary and $L \to X$ be a (possibly infinite dimensional Hilbert) vector bundle with a unitary connection.

Under which geometric/topological conditions does the first eigenvalue of the twisted Dirac $\mathcal{D}_{\otimes L}$ on X with the zero boundary condition is $\leq \lambda > 0$?

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[341] This is not quite true: Roe partitioned index theorem and its generalization do allow boundaries, see [Roe(partial vanishing) 2012], [Higson(cobordism invariance) 1991], [Schick-Zadeh(multi-partitioned) 2015], [Karami-Zadeh-Sadegh(relative-partitioned) 2018] and section 3.14.3.

[342] [Cecchini(long neck) 2020], [Guo-Xie-Yu(quantitative K-theory) 2020], [Zeidler(bands) 2019], [Zeidler(width) 2020], [Cecchini-Zeidler(generalized Callias) 2021], [Cecchini-Zeidler(scalar&mean) 2021].

In other words, when does X support a smooth *non-zero* twisted spinor $s : X \to \mathbb{S}(X) \otimes L$, which *vanishes on the boundary* of X and such that

✄$_{\lambda}$ $$\int_X \langle \mathcal{D}^2_{\otimes L}(s(x)), s(x)\rangle dx \leq \lambda^2 \int_X \|s(x)\|^2 dx$$

for a given constant $\lambda \geq 0$?[343]

Motivating Example. If X is obtained from a complete manifold $X_+ \supset X$ by cutting away $X_+ \setminus X$, and if X_+ carries a non-vanishing (twisted) L_2-spinor s_+ delivered by applying the relative index theorem, then the cut-off spinor $s = \phi \cdot s_+$, for a "*slowly* decaying" positive function ϕ with supports in X satisfies ✄$_{\lambda}$ with "*rather small* " λ.

Potential Corollary. Since

$$\mathcal{D}^2_{\otimes L}(s) \geq \nabla^2_{\otimes L}(s) + \frac{1}{4} Sc(X)(s) - const'_n |curv|(L)$$

by the Bochner-Schrödinger-Lichnerowicz-Weitzenböck formula and since

$$\int \langle \nabla^2_{\otimes L}(s), s\rangle = \int_X \langle \nabla_{\otimes L}(s), \nabla_{\otimes L}(s)\rangle \geq 0$$

for $s_{|\partial X} = 0$, the inequality ✄$_{\lambda}$ implies

✄$_{Sc}$ $$\inf_x Sc(X, x) \leq \frac{4const_n}{\rho^2} + 4const'_n |curv|(\nabla).$$

for some universal positive constants $const_n$ and $const'_n$.

From a geometric perspective, the role of above is to advance the solution of the following.

Long Neck Problem. Let X be an orientable (spin?) Riemannian n-manifold with a boundary and $f : X \to S^n$ be a smooth area decreasing map.

What kind of a *lower bound on* $Sc(X, x)$ and a *lower bound on* the "length of the neck" of (X, f), that is

the distance between the support of the differential of f *and the boundary of* X, would make $deg(f) = 0$?

An instance of a desired result[344] would be

$$[Sc(X) \geq n(n-1)] \& [dist(supp(df), \partial X) \geq const_n] \Rightarrow deg(f) = 0,$$

but it is more realistic to expect a weaker implication

$$[Sc(X) \geq n(n-1)] \& [dist(supp(df), \partial X) \geq const_n \cdot \sup_{x \in X} |||df(x)|||] \Rightarrow deg(f) = 0.$$

[343]Recall that the first eigenvalue of the Dirichlet problem is the infimum of $\int_X \|\mathcal{D}_{\otimes L}(s(x))\|^2 dx$ taken over all L-twisted spinors $s(x)$, such that $s|\partial X = 0$ and $\int_X \|s\|^2 dx = 1$.

[344]This is settled for spin manifolds in [Cecchini(long neck) 2020].

In fact, Roe's proof of the partitioned index theorem as well as the proof of the relative index theorem, e.g. via the *finite propagation speed* argument, combined with Vaffa-Witten kind spectral estimates (see $6\frac{1}{2}$ in [G(positive) 1996]) suggest that

if a compact orientable Riemannian spin manifold of even dimension n with boundary admits a a smooth map $f : X \to S^n$, which is locally constant on the boundary of X and which has *non-zero degree*, then there exists a *non-zero spinor* s, twisted with the pullback bundle $L = f^*(\mathbb{S}(S^n))$ such that *s vanishes on the boundary ∂X* and which satisfies ✂$_{\lambda}$,

$$\int_X \langle \mathcal{D}^2_{\otimes L}(s), s\rangle \le \lambda^2 \int_X \|s\|^2 dx,$$

where

$$\lambda \le const_n \frac{\sup_{x\in X} \|df(x)\|}{dist(supp(f), \partial X)}.$$

This still remains problematic, but we prove in the sections below some inequalities in this regard for manifolds X with certain restrictions on their local geometries.[345]

4.6.1 Bounds on Geometry and Riemannian Limits

Some properties of manifolds X with boundaries trivially follow by a limit argument from the corresponding properties of complete manifolds as follows.

A sequence of manifolds X_i marked with distinguished points $\underline{x}_i \in X_i$ is said to *Lipschitz converge* to a marked Riemannian manifold $(X_\infty, \underline{x}_\infty)$, if

there exist $(1+\varepsilon_i)$-*bi-Lipschitz* maps [346] from the balls $B_{\underline{x}_i}(R_i) \subset X_j$ to the balls $B_{\underline{x}_\infty}(R_i) \subset X_\infty$, say

$$\alpha_i : B_{\underline{x}_i}(R_i) \to B_{\underline{x}_\infty}(R_i + 1),$$

which send $\underline{x}_i \to \underline{x}_\infty$ and where

$$\varepsilon_i \to 0 \text{ for } i \to \infty.$$

Observe that if

$$dist(\underline{x}_i, \partial X_i) \to \infty \text{ for } i \to \infty,$$

then *the limit manifold X_∞ is complete.*

★ *Cheeger Convergence Theorem.* If the (local) C^k-geometries of Riemannian manifolds X_i at the points $x_i \in B_{\underline{x}_i}(R_i)$ for $R_i \to \infty$ are bounded (as defined below) by $c(dist(x_i, \underline{x}_i))$ for some continuous function $b(d)$, $d \ge 0$ independent of i, then some subsequence of X_i converges to a C^{k-1}-smooth Riemannian manifold X_∞.

See [Boileau(lectures) 2005] for the proof and further references.

Definition of Bounded Geometry. The C^k-geometry of a smooth Riemannian n-manifold X is bounded by a constant $\underline{g}eq0$ at a point $x \in X$, if the ρ-ball $B_x(\rho) \subset X$ for $\rho = \frac{1}{b}$ admits a smooth $(1+b)^2$-bi-Lipschitz map $\beta : B_x(\rho) \to \mathbb{R}^n$,

[345] An influence of the metric geometry of a Riemannian manifold X on the spectra of twisted Dirac operators on X is briefly duscussed in §6 of [G(positive) 1996].

[346] Here and below "λ-bi-Lipschitz" is understood as the λ-bound on the *norms of the differentials* of our maps and their inverse.

such that the norms of the kth covariant derivatives of β in $B_x(\rho)$ are bounded by b.

Notice that the *traditionally defined bound* on geometry in terms of the curvature and the injectivity radius of X, implies the above one:

if the norms of the curvature tensor of X and its kth-covariant derivatives are bounded by β^2 and there is no geodesic loop in X based at x of length $\leq \frac{1}{\beta}$, then (the proof is very easy) the C^{k+1}-geometry of X at x is bounded by $b(\beta)$ for some universal continuous function $b(\beta) = b_{n,k}(\beta)$.

✪ *Application of* ★ *to Scalar Curvature.* Let $b = b(d) \geq 0$, $d > 0$, be a continuous function and let $(X, \underline{x} \in X)$ be a marked compact Riemannian n-manifold with a boundary, such that the local geometry of X at $x \in X$ is bounded by $b(dist(x, \underline{x}))$ and let

$$R = dist(\underline{x}, \partial X).$$

Let d_0 be a positive number and let $f : X \to S^n$ be a smooth *area decreasing* map which is constant within distance $\geq d_0$ from $\underline{x} \in X$ and which has *non-zero* degree.

A. *If X is spin and $n = dim(X)$ is even, then there exists a spinor s on X twisted with the induced spinor bundle $L = f^*(\mathbb{S}(S^n)) \to X$, such that s vanishes on the boundary ∂X of X and such that*

$$\int_X \langle \mathcal{D}^2_{\otimes L}(s), s\rangle \leq \lambda(R)^2 \int_X ||s||^2 dx$$

where $\lambda = \lambda_{n,b,d_0}(R)$ is a certain universal function in R, which asymptotically vanishes at infinity,

$$\lambda(R) \underset{R\to\infty}{\to} 0.$$

B. *The scalar curvature of X is bounded by*

$$\inf_{x\in X} \leq n(n-1) + \lambda'_{n,b,d_0}(R),$$

where, similarly to the above λ, this $\lambda'(R) \to 0$ for $R \to \infty$.(One can actually arrange $\lambda' = \lambda$.)

Proof. According to Cheeger's theorem, if $R = dist(\underline{x}, \partial X)$ is sufficiently large , then X can be well approximated by a complete manifold X_∞, where such an X_∞ supports a non-zero L-twisted harmonic spinor s_∞ by the relative index theorem.

Then this s can be truncated to s_i by multiplying it with a slowly decaying function on X with compact support and then transporting it to the required spinor on X.

This takes care of A and B follows by Llarull's inequality.

Remarks. (a) The major drawback of ✪ is an excessive presence and non-effectiveness of the bounded geometry condition.

We *don't know what the true dependence of λ on the geometry of X is*, but we shall prove several inequalities in the following sections that suggest what one may expect in this regard.

(b) If the "area decreasing" property of the above map $f : X \to S^n$ is strengthened to "1-Lipschitz", then a version of B follows from the double puncture theorem (see sections 3.9 and 5.5), which needs neither spin nor the bounded geometry conditions.

4.6.2 Construction of Mean Convex Hypersurfaces and Applications to $Sc > 0$

Since doubling of manifolds with mean convex boundaries preserves positivity of the scalar curvature (see section 1.4), some problems concerning $Sc > 0$ for manifolds X with boundaries can be reduced to the corresponding ones for closed manifolds by doubling *mean convex* domains $X_{\bigcirc} \subset X$ across their boundaries $\partial X_{\bigcirc}$.

To make use of this, we shall present below some a simple criterion for the existence of such $X_{\bigcirc}$ and apply this for establishing effective versions of the above B.

Let X be a compact n-dimensional *Riemannian band* (capacitor), that is the boundary of X is divided into two disjoint subsets, that are certain unions of boundary components of X,

$$\partial X = \partial_- \cup \partial_+$$

and let us give a condition for the existence of a domain $X_{\bigcirc} \subset X$ which *contains* ∂_- and the boundary of which is smooth and has *positive* mean curvature.

Lemma. Let the boundaries of all domains $U \subset X$, which *contain the d_0-neighbourhood of ∂X_-* for a given $d_0 < dist(\partial_-, \partial_+)$, satisfy

[$*_1$] $$vol_{n-1}(\partial U) > vol_{n-1}(\partial_-)$$

and let all *minimal*[347] *hypersurfaces $Y \subset X$*, the boundaries of which are contained in ∂_+ and which themselves contain points $y \in Y$ far away from ∂_+, namely, such that

$$dist(y, \partial_+) \geq dist(\partial_-, \partial_+) - d_0,$$

satisfy

[$*_2$] $$vol_{n-1}(Y) > vol_{n-1}(\partial_-).$$

Then there exists a domain $X_{\bigcirc} \subset X$ which contains ∂_- and such that the boundary of which is smooth with positive mean curvature.

Proof. Let $X_0 \subset X$ minimises $vol_{n-1}(\partial X_0)$ among all domains in X which contain ∂_- and observe that, because of [$*_1$], the boundary of X_0 contains a point $y \in \partial X_0$ with $dist(y, \partial_+) \geq dist(\partial_-, \partial_+) - d_0$ and, because of [$*_2$], this X_0 doesn't intersect ∂_+.

Then, by an elementary argument (see [G(Plateau-Stein) 2014]) the hypersurface ∂X_0 can be smoothed and its mean curvature made everywhere positive.

[$\star\star$] *Two Words about* [$*_2$]. There are several well known cases of manifolds where the lower bound on the volumes of minimal hypersurfaces $Y \subset X$, where $\partial Y \subset partial X$ and where $dist(y, \partial) X \geq R$ for some $y \in Y$, are available.

[347] Here "minimal" means "volume minimizing" with a given boundary.

For instance if X is λ-bi-Lipschitz to the R-ball in the simply connected space X^n_κ with constant curvature κ, then the volume of Y is bounded from below in terms of the volume of the R-ball $B^{n-1}_0(R) \subset X^{n-1}_\kappa$ as follows.

Let $g = dr^2 + \phi^2(r)ds^2$, $r \in [0, R]$, be the metric in the ball $B(R) = B^{n-1}_0(R) \subset X^{n-1}_\kappa$ in the polar coordinates where ds^2 is the metric on the unit sphere S^{n-1} and let $g_\lambda = dr^2 + \phi^2_\lambda(r)ds^2$ be the metric (which is typically singular at $R = 0$), such that the volumes of the concentric balls and of their boundaries satisfy

[⋆] $$\frac{vol_{g_\lambda,n-1}B(r)}{vol_{g_\lambda,n-2}(\partial B(r))} = \Psi_\lambda(r) = \lambda^{2n-3}\frac{vol_{g,n-1}B(r)}{vol_{g,n-2}(\partial B(r))}.$$

Then the standard relation between $vol(Y)$ and the filling volume bound in X says that,

the volume of the above Y is bounded by $vol_{g_\lambda,n-1}(B(R))$.[348]

Notice that [⋆] uniquely and rather explicitly defines the function ϕ_λ.

In fact, since

$$vol_{g_\lambda,n-2}(\partial B(r)) = \phi^{n-2}_\lambda \sigma_{n-2}$$

for $\sigma_{n-2} = vol(S^{n-2})$, and since

$$\frac{dvol_{g_\lambda,n-1}(B(r)}{dr} = vol_{g_\lambda,n-2}(\partial B(r))$$

this [⋆] can be written as the following differential equation on ϕ_λ

$$\phi^{n-2}_\lambda = \frac{d(\phi^{n-2}_\lambda \Psi_\lambda)}{dr},$$

where our ϕ_λ satisfies $\phi_\lambda(0) = 0$. .

Examples of Corollaries.

A. Let X be a complete Riemannian n-manifold with *infinite* $(n-1)$*-volume at infinity*, which means that the boundaries of compact domains which exhaust X,

$$U_1 \subset U_2 \subset \ldots \subset U_i \subset \ldots \subset X,$$

have $vol_{n-1}(U_i) \to \infty$.

If X contains no complete non-compact minimal hypersurface with finite $(n-1)$-volume, then X can be exhausted by compact smooth domains the boundaries of which have positive mean curvatures.

Notice that according to [⋆⋆],

no such minimal hypersurface exists in manifolds with uniformly bounded, or even, slowly growing, local geometries.

Also notice that

infinite non-virtually cyclic coverings $\tilde{X}$ of *compact* Riemannian manifolds X, besides having *uniformly bounded* local geometries, also have *infinite* $(n-1)$*-volumes at infinity*; hence they can be exhausted by compact smooth mean convex domains.

[348] The quickest way to show this is with a use of *Almgren's sharp isoperimetric inequality.* But since this still remains unproved for $\kappa < 0$, one needs a slightly indirect argument in this case, which, possibly – I didn't check it carefully – gives a slightly weaker inequality, namely $Vol(Y) \geq c_n \cdot vol_{g_\lambda,n-1}(B(R))$ for some $c_n > 0$.

And even the *virtually cyclic coverings* $\tilde{X}$ admit such exhaustions unless they are isometric cylinders $Y \times \mathbb{R}$.

Also notice that if $\tilde{X}$ is a Galois (e.g. universal) covering with *non-amenable* deck transformation (Galois) group, then it can be exhausted by U_i with $mean.curv(\partial U_i) \geq \varepsilon > 0$. (See 1.5(C) in [G(Plateu-Stein) 2014].)

Exercises. (a) Show that if a complete connected non-compact Riemannian n-manifold X has uniformly bounded local geometry, then $X \times \mathbb{R}$ has infinite n-volume at infinity.

(b) Show that if X has $Ricci(X) > -(n-1)$, then $X \times \mathbf{H}^2_{-1}$ has infinite $(n+1)$-volume at infinity and that it can be exhausted by compact smooth mean convex domains.

B. Let A be λ-bi-Lipschitz to the annulus $\underline{A} = \underline{A}(r, r+R)$ between two concentric spheres of radii r and $r+R$ in the Euclidean space $\mathbb{R}^n$. [349]

If $R \geq 100\lambda r$, then A contains a hypersurface Y which separates the two boundary components of A and such that

$$mean.curv(Y) \geq \frac{100}{r}.$$

C. Let $\underline{X}$ be a complete simply connected n-dimensional manifold with non-positive sectional curvature and such that $Ricci(X) \leq -(n-1)$, e.g. an irreducible symmetric space with $Sc(X) = -n(n-1)$.

Let A be a compact Riemannian manifold which is λ-bi-Lipschitz to the annulus between two concentric balls $B(r)$ and $B(r+R)$ in $\underline{X}$.

There exists a (large) constant $const_n > 0$, such that if $R \geq const_n \cdot \log\lambda$, then there exists a smooth closed hypersurface $Y \subset A$, which separates the two boundary components in A and such that

$$mean.curv(Y) \geq \frac{n-1}{\lambda + const_n(\lambda - 1)}.$$ [350]

About the Proof. If $\kappa(X) \leq -1$ this follows from [∗∗], while the general case needs a minor generalization of this.

First Application to Scalar Curvature. Since

$$Rad_{S^{n-1}}(Y) \geq \lambda^{-1} Rad_{S^{n-1}}(\partial B(r)) \gtrapprox \exp r,$$

the above inequality together with Remark (b) after $\bigcirc^{n-1}$ from section 4.3. yields the following.

If a Riemannian manifold X is λ-bi-Lipschitz to the ball $B(R) \subset \underline{X}$, where $R \geq const_n \log\lambda$, then the scalar curvature of X is bounded by:

$$\inf_{x \in X} Sc(X, x) \leq -\frac{1}{const_n \cdot \lambda^2}.$$

[349] This means the existence of a λ-Lipschitz homeomorphism from $\underline{A}$ onto A, the inverse of which $A \to \underline{A}$ is also λ-Lipschitz.

[350] The sign convention for the mean curvature is such that the *mean convex* part of V bounded by Y is the one which contains the boundary component *corresponding to the sphere* $\partial B(r)$ in $\underline{X}$.

Second Application to Scalar Curvature. It may happen that a manifold X with $Sc(X) > 0$ itself contains no mean convex domain, but it may acquire such domains after a modification of its metric that doesn't change the sign of the scalar curvature. Below is an instance of this.

Let $X = (X, g)$ be a compact n-dimensional Riemannian band, as in the above ***Lemma***, where the boundary of a compact Riemannian manifold $X = (X, g)$ with $Sc(X) \geq 0$ is decomposed as earlier, $\partial X = \partial_- \cup \partial_+$.

Let $Sc(X) > 0$ and let us indicate possible modifications of the Riemannian metric g, that would enforce the conditions $[*_1]$ and $[*_2]$ in the ***Lemma***, while keeping the scalar curvature positive.

We will show below that this can be achieved in some cases by multiplying g by a positive function $e = e(x)$, which is equal one near $\partial_- \subset X$ and which is as large far from ∂_- as is needed for $[*_1]$ and where we also need the Laplacian of $e(x)$ to be bounded from above by $\varepsilon_n Sc(X, x)$ in order to keep $Sc > 0$ in agreement with the Kazdan-Warner conformal change formula from section 2.6.

The simplest case, where there is no need for any particular formula, is where the sectional curvatures of X are pinched between $\mp b^2$, no geodesic loop in X of length$< \frac{1}{b}$ exists, while the scalar curvature of X is bounded from below by $\sigma > 0$.

In this case, let

$$e_0(x) = c\frac{\sqrt{\sigma}}{b+1} dist_g(x, \partial_- 0)$$

and observe that if $c = c_n > 0$ is sufficiently small, then $e_0(x)$ has a *small* (generalized) gradient $\nabla(e_0)$ and, because the the geometry of X is suitably bounded, the function e_0 can be approximated by a smooth function $e(x)$ with second derivatives significantly smaller than σ,

thus, *ensuring the inequality* $Sc(eg) > 0$.

On the other hand, if

$$dist(\partial_-, \partial_+) \geq C(b+1)\|\nabla(e)\|^{-1} vol(\partial_-)^{\frac{1}{n-1}},$$

for a large $C = C_n$,

then

the *condition* $[*_1]$ *is satisfied*, say with $d_0 = \frac{1}{2} dist(\partial_-, \partial_+)$,

and, due to the bound on the geometry of X,

the *condition* $[*_2]$ *is satisfied* as well.

Now let us look closer at what kind $e(x)$ we need and observe the following

[1] The bound on the geometry of X is needed only, where the gradient of e doesn't vanish.

Thus, it suffices to have the geometry of X

bounded only in the $\frac{1}{b}$*-neighbourhoods of the boundaries of domains* U_i,

$$\partial_- \subset U_1 \subset \ldots \subset U_i \subset \ldots \subset U_k \subset X,$$

where $dist(U_i, \partial U_{i+1}) \geq \frac{1}{b}$ and where $\frac{k}{b}$ is sufficiently large.

[2] Since, the by the standard comparison theorem(s),

Laplacians of the distance-like functions are bounded from above in terms of the Ricci

curvature,

the b-bound on the full local geometry can be replaced by $Ricci(X,x) \geq -b^2 g$.

Summing up, this yields the following refinement of B in ✪ from the previous section.

Let $X = (X, g)$ be a, *possibly non-complete* Riemannian n-manifold, such that

$$Sc(X) \geq 0,$$

and let

$$f : X \to S^n$$

be an *area non-increasing map*, such that the support of the differential of f is compact and the scalar curvature of X in this support is bounded from below by that of S^n,

$$\inf_{x \in supp(df)} Sc(X,x) \geq n(n-1).$$

Let A_i be disjoint "bands" in X, that are *a_i-neighbourhoods of the boundaries* of compact domains U_i, such that

$$supp(df) \subset U_1 \subset ... \subset U_i ... \subset U_k \subset X.$$

Let us give an effective *criterion for vanishing of the degree of the map f* in terms of the geometries of A_i.

Proposition. Let the scalar and the Ricci curvatures of X in A_i for $i = 2, ...k-1$ be bounded from below by

$$Sc(A_i) \geq \sigma_i \text{ and } Ricci(A_i) \geq -b^2 g,\ 2 \leq i \leq k-1,$$

and set

$$\beta_i = \frac{\sqrt{\sigma}_i}{b_i}.$$

Let the *sectional curvatures* of U_k outside U_{k-1} be bounded from above by

$$\kappa(U_k \smallsetminus U_{k-1}) \leq c^2,\ c > 0,$$

and let the complement $U_k \smallsetminus U_{k-1}$ contains *no geodesic loop of length* $\leq \frac{1}{c}$.

If the following weighted sum of a_i (that are half-widths of the bands A_i) *is sufficiently large,*

$$\sum_{1<i<k} \beta_i a_i \geq const_n \frac{(vol_{n-1}(\partial U_1))^{\frac{1}{n-1}}}{\frac{a_k}{c}},$$

and if X is orientable spin, then

$$deg(f) = 0.$$

Proof. Arguing as above, one finds a smooth function $e(x)$, the differential of which is supported in the union of A_i, $1 < i < k$, such that $Sc(e \cdot g)$ remains nonnegative (and even can be easily made everywhere positive) and such that U_k satisfy the assumptions $[*_1]$ and $[*_2]$ of the above ***Lemma***, that yields a subdomain

$$X_{\bigcirc} \subset U_k,$$

which is mean convex with respect to the metric eg and to a smoothed double of which compact Llarull's theorem applies.

Remarks. (a) Even in the case of *complete* manifolds X, this doesn't (seem to) directly follow from Llarull's theorem, since the latter, unlike the former, needs *uniformly positive* scalar curvature at infinity.

(b) The above proposition, as well construction of mean-convex hypersurfaces in general, doesn't advance, at least not directly, the solution of the *spectral $\mathcal{D}^2$-problem* formulated in section 4.6.

Let $X = (X, g)$ be a complete Riemannian n-manifold, let $f : X \to S^n$ be a smooth *area contracting* map the differential df of which has *compact* support.

Let

$$|d| = \sup_{x \in X} ||df(x)||$$

and

$$r = r(x) = dist(x, supp(df)).$$

Let the Ricci curvature of X outside $supp(df)$ be bounded from below by

$$Ricci(x) \geq -b(r(x))^2 g(x)$$

for some continuous function $b(r)$, $r \geq 0$.

If the function $b(r)$ grows sufficiently slowly for $r \to \infty$, e.g. $\sigma(r) \leq \sqrt[3]{r}$ for large r, then there is an effective lower bound

$$Sc(X, x) \geq \sigma(r(x)),$$

which implies that

the map f has zero degree,

where $\sigma(r)$, $r \geq 0$, is a certain "universal" function, which is "small negative" at infinity.

More precisely, there exists a universal effectively computable family of functions in r,

$$\sigma(r) = \sigma_{b,|d|,N,}(r),\ r \geq 0,\ N = 1, 2, ...,$$

with the following five properties

(i) the functions $\sigma(r)$ are *monotone decreasing* in $r \geq 0$,

(ii) $\sigma_{b,|d|,N,}(r)$ is *monotone decreasing* in N,

(iii) $\sigma_{b,|d|,N,}(r)$ *is monotone increasing in b and in $|d|$,*

(iv) $$\sigma(0) = N(N-1), \text{ while } \sigma(r) \underset{r\to\infty}{\to} -\infty$$

(v) $$\sigma_N(r) = \sigma_{b,|d|,N,}(r) \underset{N\to\infty}{\to} -\infty \text{ for fixed } b, |d| \text{ and } r > 0,$$

such that

$[\times \bigcirc^{N-n}]$ *if $Sc(X, x) \geq \sigma_{b,|d|,N,}(r(x))$ for all $x \in X$ and some $N \geq n + 2$, then, assuming X is orientable and spin, the degree of f is zero.*[351]

[351]Compare with "inflating balloon" used in 7.36 of [GL(complete) 1983].

Proof. The bound on $\Delta\varphi(x)$ for $Ricci \geq -b^2$ (compare with [2] from the previous section) shows that there exists $\sigma_{b,|d|,N,}(r)$ with the above properties (i)-(v) and a positive function $\varphi(x)$ on X, such that

(a) φ is equal to $|d|$ on the support $supp(df) \subset X$

and such that

$$(b)\ \sigma(r(x))+\frac{m(m-1)}{\varphi(x)^2}-\frac{m(m-1)}{\varphi^2(x)}||\nabla\varphi(x)||^2-\frac{2m}{\varphi(x)}\Delta\varphi(x) \geq \varepsilon > 0 \text{ for } r(x) > 0.$$

Therefore, by the formula ($\star\star$) from section 2.4.1 for the scalar curvature of the warped product metrics $g_\varphi = g + \varphi^2 ds^2$ on $X \times S^m$, $m = N - n$,

$$Sc(g_\varphi)(x,s) = Sc(g)(x) + \frac{m(m-1)}{\varphi(x)^2} - \frac{m(m-1)}{\varphi^2(x)}||\nabla\varphi(x)||^2 - \frac{2m}{\varphi(x)}\Delta\varphi(x),$$

the metric g_φ has uniformly positive scalar curvature and because of (a) the map $f : X \to S^n$ suspends to an area decreasing map $(X \times S^m, g_\varphi) \to S^{n+m}$ of the same degree as f. Then Llarull's theorem applies and the proof follows.

On Manifolds with Boundaries. If X is a compact manifold with a boundary, the above can be applies to the smoothed double $X \cup_{\partial X} X$, where the scalar curvature of such a double near the smoothed boundary can be bounded from below by the geometry of X near the boundary and the (mean) curvature of the boundary $\partial X \subset X$.

Thus, the above yields a condition for $deg(f) = 0$ in terms of the lower bound on $Sc(X,x)$ and on $dist(x, supp(df))$, which is similar to, yet is different from such a condition from the previous section.

Dirac operators with Potentials. The recent

relative index theorem for the Dirac operators with potentials

by Weiping Zhang[352],

which applies to complete manifolds X with non-negative scalar curvatures at infinity and which is more efficient in many (all?) cases than multiplication of X by spheres, makes most (all?) of the above redundant.

4.6.3 Amenable Boundaries

If the *volume of the boundary* of a compact manifold X is *significantly smaller than the volume of* X and if it is additionally supposed that the manifold is *not very much curved near the boundary*, then we shall see in this section that

the index theorem applied to the double of such an X with a smoothed metric, yield geometric bounds on the area-wise size of X in terms of the lower bound on the scalar curvature of X.

Elliptic Preliminaries. Let V be a (possibly non-compact) Riemannian manifold with a boundary, and let l be a section of a bundle $L \to V$ with a unitary connection ∇, such that l satisfy the following (elliptic) *Gårding* $(\delta_\circ, C_\circ)$-*inequality*: the C^1-norm of l at $v \in V$ is bounded at by the L_2-norm of l in the

[352]See [Zhang(area decreasing) 2020], [Zhang(deformed Dirac) 2021].

$\delta_\circ$-ball $B = B_v(\delta_\circ) \subset V$ as follows

$$||l(v)|| + ||\nabla l(v)|| \le C_\circ \sqrt{\int_B ||(l)||^2 dv}$$

for all points $v \in V$, where
$$dist(v, \partial V) \ge \delta_\circ.$$

Let
$$\rho(v) = dist(v, \partial V) \text{ and } \beta = \sup_{v \in V} vol(B_v(\delta_\circ))$$

Lemma. If l vanishes on an ε-net $Z \subset V$, then

$$||l(v)|| + ||\nabla l(v)|| \le (10 C_\circ \varepsilon \beta)^{\rho(x) - 2\delta_\circ} \sqrt{\int_V l^2(v) dv}$$

Moreover, if V can be covered by $2\delta_\circ$-balls with the multiplicity of the covering at most m, then the L_2-norms of l and ∇l on the subset $V_{-\rho} \subset V$ of the points ρ-far from the boundary, that is

$$V_{-\rho} = V \smallsetminus U_\rho(\partial V) = \{v \in V\}_{dist(v, \partial V) \ge \rho},$$

satisfies

[⊕○] $$\sqrt{\int_{V_{-\rho}} ||l||^2(v) dv} \le \epsilon \sqrt{\int_V ||l||^2(v) dv}$$

for $\epsilon = m \left(10 C_\circ \varepsilon \beta\right)^{\rho(x) - 2\delta_\circ}$.

Proof. Combine Gårding's inequality with the following obvious one:

$$||l|| \le \varepsilon ||\nabla|| l$$

and iterate the resulting inequality i times insofar as $\rho - i\delta_\circ$ remains positive.

Remark. A single round of iterations suffices for our immediate applications.

Corollary. Let X be a complete orientable Riemannian manifold of dimension n with compact boundary (e.g. X is compact or homeomorphic to $X_0 \times \mathbb{R}_+$, where X_0 is a closed manifold), and let, y *for some $\rho > 0$ and $0 < \delta_\circ < \frac{1}{4}\rho$,*

the ρ-neighbourhood of the boundary of X, denoted $U = U_\rho(\partial X) \subset X$, has (local) geometry bounded by $\frac{1}{\delta_\circ}$,

where we succumb to tradition and define this bound on geometry as follows:

the sectional curvatures κ of U are pinched between $-\frac{1}{\delta_\circ^2}$ and $\frac{1}{\delta_\circ^2}$ and the injectivity radii are bounded from below by $\delta_\circ$ at all points $x \in U$, for which $dist(x, \partial X) \ge \delta_\circ$, that is, in formulas,

$$|\kappa(X, x)| \le \frac{1}{\delta_\circ^2} \text{ for } dist(x, \partial X) \le \rho \text{ and } injrad(X, x) \ge \delta_\circ \text{ for } \delta_\circ \le dist(x, \partial X) \le \rho.$$

Let the scalar curvature of X be non-negative $\frac{1}{2}\rho$-away from the boundary,

$$Sc(X, x) \ge 0 \text{ for } dist(x, \partial X) \ge \frac{1}{2}\rho.$$

Let $f : X \to S^n(R)$, where $S^n(R)$ is the sphere of radius R, be a smooth *area decreasing* map , which is constant on U_ρ, and, if X is non-compact, also locally constant at infinity.

Let the degree of this map be *bounded from below by the volume of* $U_\rho = U_\rho(\partial X)$ as follows.

$$d > Cvol(U_\rho) \text{ for some } C \geq 0.$$

If $\delta_\circ$, ρ and C are *sufficiently large*, then, provided X is *spin,* the scalar curvature of the complement

$$X_{-\rho} = X \smallsetminus U_\rho = \{x \in X\}_{dist(x,\partial X > \rho}$$

can't be everywhere much greater than $Sc(S^n(R)) = \frac{n(n-1)}{R^2}$. Namely

[◆•] $$\inf_{x \in X_{-\rho}} Sc(X,x) \leq \sigma_+ \frac{n(n-1)}{R^2} + \sigma,$$

where $\sigma = \sigma_n(\delta_\circ, \rho, C)$ is a positive function, which may be infinite for small $\delta_\circ$ and/or ρ and/or C and which has the following properties.

- *the function* σ *is monotone decreasing in* $\delta_\circ$, ρ *and* C*;*
- $\sigma_n(\delta_\circ, \rho, C) \to 0$ *for* $C \to \infty$ *and arbitrarily fixed* $\delta_\circ > 0$ *and* $\rho > \delta_\circ$.

Proof. Let $2X = \mathbb{D}X$ be a smoothed double of X and $L \to 2X$ the vector bundle induced from $\mathbb{S}^+(S^n)$ by f applied to a copy (both copies, if you wish) of $X \subset 2X$.

Assume $n = dim(X)$ is even, apply the index theorem and conclude that the dimension of the space of L-twisted harmonic spinors on $2X$ is $\geq d$.

Therefore, there exists such a non-zero spinor l that vanishes at given $d-1$ points in $2X$.

Let such points make a ε-net on the subset $2U_{\rho_\circ} = \mathbb{D}U_{\rho_\circ} \subset 2X$ with a minimal possible ε.

If d is much larger then $vol(2U_\rho) \approx 2vol(U_\rho)$, then this ε becomes small and, consequently, ϵ in the above inequality [◆○] also becomes small. Then, the inequality [◆○] applied to the domain $2U_\rho \subset 2X$, shows that the integral

$$\int_{2U_\rho} ||l||^2(x)dx$$

is much smaller then the integral of $||l||^2$ over the complement $2X_0 = 2X \smallsetminus 2U_\rho$.

Therefore, if σ_+ is large then the sign of the full integral

$$\int_{2X} Sc(X,x)||l||^2(x)dx = \int_{2X_\rho} Sc(X,x)||l||^2(x)dx + \int_{U_\rho} Sc(X,x)||l||^2(x)dx$$

is equal to the sign of $\int_{2X_\rho} Sc(X,x)||l||^2(x)dx$, which contradicts the Schroedinger-Lichnerowicz-Weitzenboeck formula for harmonic l.

Thus, modulo simple verifications and evaluations of constants left to the reader, the proof is completed.

Example 1. Let a complete non-compact orientable spin Riemannian n-manifold X with *compact boundary* admits smooth *area decreasing* maps $f_i : X \to S^n$ of *non-zero degrees*,[353] such that the "supports" of f_i, i.e. the subsets where these maps are *non-constant*, may lie arbitrarily far from the boundary of X,

$$\text{dist}\ (\text{"supp"} f_i, \partial X) \to \infty \text{ for } i \to \infty.$$

Then the scalar curvature of X can't be uniformly positive at infinity:

$$\liminf_{x\to\infty} Sc(X,x) \leq 0.$$

Moreover, the same conclusion holds, if

there exist i-sheeted coverings $\tilde{X}_i \to X$, which admit smooth area decreasing maps $f_i : \tilde{X}_i \to S^n$, such that

$$\frac{deg(f_i)}{i} \to \infty \text{ for } i \to \infty.$$

Example 2. Let Y_k be a k-sheeted covering of the unit 2-sphere $S^2 = S^2(1)$ minus two opposite balls of radii $\frac{1}{k^m}$, for some $m \geq 1$.

Then the product manifold $X_0 = Y_k \times S^{n-2}(k)$ admits an area decreasing map $f : X_0 \to S^n(R)$ constant on the boundary and such that

$$deg(f) \geq \frac{k}{10d}$$

and it follows from the above corollary that the Riemannian metric on X_0 can't be extended to a larger manifold $X \supset X_0$, with bounded geometry and $Sc \geq 0$ without adding much volume to X_0, say in the case $m = n-1$, although $vol_{n-1}(\partial X_0)$ remains bounded for $R \to \infty$.

Melancholic Remarks. Rather than indicating the richness of the field, the diversity of the results in the above sections 4.6.1- 4.6.4 is due to our inability to formulate and to prove the true general theorem(s).

4.6.4 Almost Harmonic Spinors on Locally Homogeneous and and Quasi-homogeneous Manifolds with Boundaries

Let X be a complete Riemannian manifold with a transitive isometric action of a group G, let $L \to X$ be a vector bundle with a unitary connection ∇ and let the action of G equivariantly lifts to an action on (L, ∇).

Let the L_2-index of the twisted Dirac operator $\mathcal{D}_{\otimes L}$ (see [Atiyah(L_2) and [Connes-Moscovici($L_2 - index$ for homogeneous) 1982], be non zero. For instance, if X admits a free discrete isometry group $\Gamma \subset G$ with compact quotient, then this is equivalent to this index to be non-zero on X/Γ.

The main class of examples of such X are *symmetric spaces with non-vanishing "local Euler characteristics* (compare with [AtiyahSch(discrete series) 1977]) i.e. where the corresponding (G-Invariant) n-forms, $n = dim(X)$ don't vanish.

The simplest instances of these are hyperbolic spaces $\mathbf{H}^{2m}_{-1}$, where the indices of the Dirac operators twisted with the positive spinor bundles don't vanish. In

[353] Here as everywhere in this paper, when you you speak of $deg(f)$ the map f is supposed to be locally constant at infinity as well as on the boundary of X.

fact, such an index for a compact quotient manifold $\mathbf{H}^{2m}_{-1}/\Gamma$ is equal to ±one half of the Euler characteristics of this manifold by the Atiyah-Singer formula (compare [Min(K-Area) 2002]).

Let (X, L) be an above homogeneous pair with $ind(\mathcal{D}_{\otimes L}) \neq 0$ and let $X_R \subset X$ be a ball of radius R. Then the restrictions of L_2-spinors on X (delivered by the L_2-index theorem) to X_R can be perturbed (by taking products with slowly decaying cut-off functions) to *ε-harmonic* spinors that *vanish on the boundary* of X_R, where $\varepsilon \to 0$ for $R \to \infty$ and where "ε-harmonic" means that

$$\int_{X_R} \langle \mathcal{D}^2_{\otimes L}(s), s\rangle \leq \varepsilon^2 \int_{X_R} ||s||^2 dx$$

as in c ✄$_\lambda$ in section 4.6.

In fact, it follows from the local proof of the L_2-index theorem in [Atiyah(L_2) 1976] or, even better, from its later version(s) relying on the finite propagation speed, that these ε-harmonic spinors can be constructed internally in X_R with no reference to the ambient $X \supset X_R$.

Moreover, a trivial perturbation (continuity) argument shows that

similar spinors exist on manifolds X'_R with these metrics close to these on X_R.

but it is unclear "how close" they should be. Here is a specific problem of this kind.

Let X_R be a compact Riemannian spin manifold with a boundary, such that

$$\sup_{x \in X} dist(x, \partial X_R) \geq R$$

and let the sectional curvatures of X are everywhere pinched between -1 and $-1-\delta$.

(A) *Under what conditions on R, δ and ε does X_R support a non-vanishing ε-harmonic spinor twisted with the spin bundle $\mathbb{S}(X_R)$?*

Besides, one wishes to have

(B) similar spinors on manifolds $\overline{X}$ mapped to X_R with *non-zero* degrees and with

controlled metric distorsions

in order to get *bounds on the scalar curvatures* of such $\overline{X}$

(See section 6.4.3)for continuation of this discussion to *fibrations* with quasi-homogeneous fibers.)

4.7 Topological Obstructions to Complete Metrics with Positive Scalar Curvatures Issuing from the Index Theorems for Dirac Operators

Obstruction on *homotopy types* of compact manifolds X implied by the existence of metrics of positive scalar curvature on X, obtained by Dirac theoretic methods usually (always?) generalize to *non-compact complete* manifolds, where "homotopy" means "proper homotopy", i.e. the maps being "homotopies" as well as the maps establishing homotopies must be proper: infinity-to-infinity.

Moreover, such obstructions not only rule out metrics with positive scalar curvatures on n-manifolds X' which are homotopy equivalent to X, but also on n-manifolds $\hat{X}$ that *dominate* (the fundamental homology class of) X, i.e. admits maps $f : \hat{X} \to X$ with $deg(f) = \pm 1$ to X in the orientable cases, and often, even with any $deg(f) \neq 0$.

Dimension+m-Domination. The above also applies to smooth proper maps of $(n+m)$-dimensional manifols to n-dimensional X, say $f : \hat{X}^{+m} \to X$, such that the pullbacks of generic points under f and by all smooth maps $X^{+m} \to X$ *homotopic to* f – these pullbacks (but not necessarily all m-manifolds homotopy equivalent to these pullbacks) admit no metrics with $Sc > 0$.

Example 1: *Maps of non-zero $\hat{A}$-degree to Enlargeable*[354] *Manifolds and Similar Maps.* If a compact *spin* $(n+m)$-manifolds $\hat{X}^{+m}$ admits a smooth map f to compact *enlargeable* n-manifolds X, (see section 3.10.1 e.g. to the torus $\mathbb{T}^n$, or, more generally, to a Riemannian manifold with non-positive sectional curvature, such that the pullback $f^{-1}(x) \subset \hat{X}^{+m}$ of a generic point $x \in X$ has *non-zero* $\hat{\alpha}$-invariants, e.g. $\hat{A}(f^{-1}(x)) \neq 0$ in the case $m = 4k$, then $\hat{X}^{+m}$ can't carry a metric with $Sc > 0$.

About Relevance of Spin. Probably, the same non-existence conclusion holds if only the pullback "$\hat{X}^{+m}$ is spin, for instance, where $\hat{X}^{+m}$ is diffeomorphic to $X \times X^m$, where X^m (but not necessarily X) is spin.

In fact, if $m+n \leq 8$ this follows from the the μ-bubble separation theorem in section 3.7, and if $m_n \geq 9$, this might follow from Lohkamp's desingularization results. (Schoen-Yau's 2017 theorem is non-sufficient for this purpose.)

On the other hand, the Dirac theoretic method has an advantage of being applicable to $\wedge^2$-*enlargeable* manifolds X defined in example 4 below.

Also Dirac operators serve well if the underlying X is a quasisymplectic $\otimes_{\wedge^k \tilde{\omega}}$-manifolds as in section 2.7, e.g. a closed aspherical 4-manifolds X with $H^2(X;\mathbb{Q}) \neq 0$.[355]

"Positive" versus "Uniformly Positive". If X is non-compact, one has to distinguish "just (strict or not) *positivity*" of the scalar curvature, $Sc(X) > 0$ along with $Sc(X) \geq 0$ – the existence of the former implies the existence of the latter except for a few exceptional "rigid" examples, such as Riemannian flat manifolds and Ricci flat Kähler (Calabi-Yau) manifolds, from "*uniform uniform positivity*", where $Sc(X) \geq \sigma > 0$.

Example 2: *Metrics with Positive Curvatures in the Plane and their High Dimensional Warped Descendants.* The products of tori $\mathbb{T}^{n-2}$ by the plane $\mathbb{R}^2$ (obviously) admit metrics with $Sc > 0$, but *no metric with* $Sc \geq \sigma > 0$, where the latter follows from Roe's partitioned index theorem.

Also one can do it with Zeidler's-Cecchini's Dirac theoretic $\frac{2\pi}{n}$-inequality for Riemannian spin bands, while our non-Dirac theoretic proof needs Lohkamp-Schoen-Yau desingularization theorem(s) for $n \geq 9$.

More generally, by the same token the product manifolds $X = X_0 \times \mathbb{R}^2$, support complete metrics with $Sc > 0$, but if X_0 admits *no domination* by a

[354] A compact Riemannian n-manifold X is enlargeable if it admits (finite or infinite) coverings $\tilde{X}$ with *arbitrarily large* hyperspherical radii, i.e. for all $R > 0$, there exists a covering $\tilde{X}$, which admits a locally constant at infinity distance decreasing map $\tilde{X} \to S^n$ with non-zero degree.

Notice that this condition doesn't depend on the Riemannian metric in X, moreover it is a homotopy (even domination) invariant.

[355] One should note, however, that no example is known of a *compact non-enlargeable* manifold that is $\wedge^2$-enlargeable or quasisymplectic $\otimes_{\wedge^k \tilde{\omega}}$ or a manifold with infinite K-area.

manifold with a complete metric with *positive* scalar curvature, then X admits *no domination* by a manifold with a complete metric with with *uniformly positive* scalar curvature.[356]

Exercise.[357] Show that products $X_1 \times X_2$ of *non-compact* manifolds X_1 and X_2 admit complete metrics with $Sc > 0$, while such triple products, $X_1 \times X_2 \times X_3$ admit complete metrics with $Sc \geq \sigma > 0$.

Example 3: *Simply Connected Manifold Dominated by* $Sc > 0$. There only instance of a Dirac theoretic obstruction for $Sc > 0$ on topology of compact *simply connected* manifolds, which is (this is an accident) a homotopy theoretic one, is Lichnerowicz' $\hat{A}[X] \neq 0$ for $n = dim(X) = 4$. (If $n \geq 5$ there is no constraints on rational Pontryagin classes of X except for the signature and none of higher $\hat{\alpha}$-invariants used in Hitchin's theorem is homotopy invariant either.)

But even this obstruction is not "domination invariant": connected sums $X_{\#\pm} = X\# - X$ have $\hat{A}[X\# - X] = 0$ for all X and, by Milnor' homotopy classification theorem, these $X_{\#\pm}$ are homotopy equivalent to manifolds which admits metrics with $Sc > 0$, namely to connected sums of CP^2 and $S^2 \times S^2$ by Milnor's 1958 theorem and by adding more copies of $S^2 \times S^2$ these become *diffeomorphic* to connected sums of CP^2 and $S^2 \times S^2$ by Wall's 1964 theorem.

All (known) Dirac theoretic non-domination results of compact n-manifolds X by compact $\hat{X}$ with $Sc(\hat{X}) > 0$ apply only to spin manifolds $\hat{X}$[358] and rely on existence of flat or almost flat (generalized, e.g. virtual Fredholm) unitary vector bundles over X (or over $X \times \mathbb{T}^1$) with non-zero Chern numbers.

In fact, the limit of applicability of such results would be (essentially) reached if one could resolve the following.

Problem A. Let B be a Riemannian manifold, let $X \subset B$ be a compact *relatively aspherical* submanifold, i.e. the inclusion homomorphisms of the higher homotopy groups, $\pi_i(X) \to \pi_i(B)$, vanish for all $i \geq 2$.

Prove (or disprove) that *for all complex vector bundles $L \to B$ and all $\varepsilon > 0$ there exist vector bundles $L_\varepsilon \to X$ with unitary connections, such that*

(i) the bundles L_ε are isomorphic to multiples of L restricted to X

$$L_\varepsilon = k \cdot L_{|X} \text{ for } k \cdot L = \underbrace{L \otimes L \otimes ... \otimes L}_{k};$$

(ii) The curvature operators R_{L_ε} of L_ε satisfy

$$||R_{L_\varepsilon}|| \leq \varepsilon.$$

In fact, as we know, that if X is spin, then the index theorem applied to the twisted Dirac operators $\mathcal{D}_{\otimes L_\varepsilon}$ (that act on spinors with values in the bundles L_ε) shows that the (untwisted) Dirac operators $\mathcal{D}$ on certain covering manifolds $\tilde{X}_\varepsilon$ contain zero in their spectra; thus $Sc(X) \ngtr 0$ by the Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) formula.

[356]This follows from the μ-bubble separation theorem (section 3.7) that relies on Lohkamp-Schoen-Yau desingularization for $n \geq 9$, but I am not certain how much of this can be proven for by Dirac theoretic methods in the case of spin manifolds.
[357]I haven't solved this exercise.
[358]In all known examples it suffices that the universal covering of $\hat{X}$ is spin.

In this in mind, one asks another question.

B. Suppose, an even dimensional compact spin submanifold X in an aspherical space B represents a *non-torsion* homology class in B. [359]

Does then the spectrum of the Dirac operator on some covering of X contain zero in the spectrum?

Now, let us look more systematically at what of the above generalizes to complete manifolds with $Sc > 0$ and with $Sc \geq \sigma > 0$.

Originally, the results for $Sc(X) > 0$ were derived from these, where $Sc \geq \sigma > 0$, namely applied to $X \times S^2(R)$ for suitably large R.

Nowadays, one has at one's disposal index theorems for *Dirac operators with potentials* proved in [Cecchini(Callias) 2018], [Cecchini(long neck) 2020] and in [Zhang(area decreasing) 2020].

Example 4: *$\wedge^2$-Enlargeability against $Sc > 0$.* A Riemannian metric g on a manifold X is $\wedge^2$-enlargeable if, for all $R > 0$, there exists coverings $\tilde{X}$, which admit locally constant at infinity g-area non-increasing maps with non-zero degrees to the R-spheres $S^n(R)$, where, a priori, such a covering may depend on R.

A smooth manifold ? is $\wedge^2$-enlargeable if all Riemannian metrics on it are enlargeable.

For instance,

metrics with infinite areas on connected surface are $\wedge^2$-enlargeable,

while

connected surfaces are enlargeable if they have infinite fundamental groups.

Exercises. (4a) Show that the products $X = X_1 \times X_2$, where both X_1 and X_2 are connected non-compact are not $\wedge^2$-enlargeable.

(4b) Show that the products of enlargeable manifolds by $\wedge^2$-enlargeable are $\wedge^2$-enlargeable.

(4c) Show that the product $X = X_0 \times \mathbb{R}$, where X_0 is enlargeable, is $\wedge^2$-enlargeable.

Probably the converse is also true: if $X_0 \times \mathbb{R}$ is $\wedge^2$-enlargeable, then X_0 is enlargeable. (This is close in spirit to stabilization conjecture in section 7.3)

Also it is *not impossible* that (b) also admits a converse: if the product $X_1 \times X_2$ is $\wedge^2$-enlargeable, then one of the two manifolds is $\wedge^2$-enlargeable and another one is enlargeable.

(4d) Show that if X dominates (a multiple of the fundamental class of) a $\wedge^2$-enlargeable manifold $\underline{X}$, i.e. if there is a quasi-proper map $f : X \to \underline{X}$ of non-zero degree,[360] then X $\wedge^2$-enlargeable.

For instance, complements to Cantor (closed zero-dimensional) subsets in enlargeable manifolds X and connected sums of X with arbitrary manifolds are $\wedge^2$-enlargeable.

Theorem 4e. *$\wedge^2$-Enlargeable manifolds X, the universal coverings $\tilde{X}$ of which are spin, admit no metrics with $Sc > 0$.*

This is proven in §6 in [GL(complete)1983] for spin manifolds X with a use of the relative index theorem applied to $X \rtimes S^2(R)$, where in the case of $\tilde{X}$ spin, one does it with relativized Atiyah's L_2-index theorem.

[359] One has little idea of what to expect for non-zero torsion classes.

[360] A map f is quasi-proper if it extends to a continuous map between the compactified spaces, from $X^{+ends} \supset X$ to $\underline{X}^{+ends} \supset \underline{X}$.

Example 5: *Obstruction on $Sc > 0$ of Complete Metrics for Manifolds with Infinite relative K-areas and for Quasisymplectic $\otimes_{\wedge^k\tilde{\omega}}$-Manifolds.* Let us formulate two special cases of general non-existence theorems for complete metrics with $Sc > 0$ from [Cecchini-Zeidler(generalized Callias) 2021]and from [Zhang(deformed Dirac) 2021] proved with a use of Dirac operators with potentials.[361]

Theorem 5a. Let X be an orientable manifold of even dimension n and let $X_0 \subset X$ be a compact subset, such that X has *infinite K-area relative to the complement $X \smallsetminus X_0$.*

This means that for some, hence for every, Riemannian metric g_0 on X the following holds.

For all $\varepsilon > 0$, there exist complex vector bundles $L_1, L_2 \to X$ with unitary connections, such that:

• the norms of the curvature operators of these connections with respect to g_0 are everywhere $\leq \varepsilon$;

• these norms vanish outside X_0, i.e. the connections are flat over $X \smallsetminus X_0$;

• There exists a parallel, i.e, connections preserving, isomorphism between the bundles L_1 and L_2 over $X \smallsetminus X_0$.

• some (relative) Chern number $c_I[X]$, $c_I \in H^n(X, X \smallsetminus X_0)$, of the virtual bundle $L_1 - L_2$ doesn't vanish.

If the universal covering of X is spin, then X admits no complete Riemannian metric g with $Sc(g) > 0$.

Theorem 5b. Let X be an orientable manifold of dimension $n = 2k$ and let $X_0 \subset X$ be a compact subset.

Let $h \in H^2(X, X \subset X_0)$ be a relative cohomology class, such that $h^k \neq 0$, while the lift of h to the universal covering of X, say $\tilde{h} \in H^2(\tilde{X}; \widetilde{X \smallsetminus X_0})$, *vanishes.*

If the universal covering $\tilde{X}$ of X is spin, then X admits no complete metric with $Sc > 0$.

Example 6. *Topology at Infinity of Complete Manifolds with Uniformly Positive Scalar Curvatures.* If instead of $Sc > 0$ we want to rule out complete metrics $Sc \geq \sigma > 0$, we need the above topological conditions on X satisfied only at infinity. Below is a specific formulation of this.

Theorem/Conjecture 6a. Let X be an orientable manifold of even dimension n, let $X^\circ \subset X$ be an open subset with a *compact* complement in X and let $X_i^\circ \subset X^\circ$, $i = 1, 2, \ldots$, be a sequence of compact subsets that tend to infinity in X, i.e. every compact subset in X intersects only finitely many X_i°.

Let one of the following two conditions be satisfied.

$\bullet_{area}$ The relative K-areas of X° with respect to $X^\circ \smallsetminus X_i^\circ$ are infinite for all i.

$\bullet_{sympl}$ There exists cohomology classes $h_i \in H^2(X^\circ, X^\circ \smallsetminus X_i^\circ)$, such that $h_i^k \neq 0$, while the lifts $\tilde{h}_i \in H^2(\tilde{X}^\circ; \widetilde{X^\circ \smallsetminus X_0^\circ})$, where $\tilde{X}^\circ$ denotes the universal covering of X°, vanish.

If the universal covering $\tilde{X}$ of X is spin, then X admits no complete Riemannian metric with $Sc \geq \sigma > 0$.

The proof of this must follow from a suitable version of Cecchini's long neck principle and from [Guo-Xie-Yu(quantitative K-theory) 2020] but I haven't carefully checked this. Nor am I certain that that the same conclusion holds under

[361] I want to thank Simone Cecchini and Weiping Zhang for explaining their results to me.

more general condition(s), where the subset X° is not fixed but dependent on (decreasing with) $i = 1, 2, \ldots$.

But we do know for sure that Roe's partitioned index theorem shows (in agreement with what follows from the μ-bubble separation theorem) that if a spin manifold X is *enlargeable at infinity*, i.e.

if there exists an exhaustion of X by compact domains $X_i \subset X$ with smooth boundaries $Y_i \subset \partial X_i$, such that the complements of all X_i admit sequences of coverings, say $\tilde{X}_{ij}^{\perp} \to X \setminus X_i$, where the hyperspherical radii of the corresponding coverings $\tilde{Y}_{ij} = \partial \tilde{X}_{ij}^{\perp}$ of Y_i tend to infinity for $j \to \infty$,

then X admits no complete metric with $Sc \geq \sigma > 0$.

REMARKS, PROBLEMS, CONJECTURES.

Question 7. Let X be an open aspherical n-manifold. Does *non-contractibility of X to the $(n-1)$-dimensional skeleton* $X^{[n-1]} \subset X$ imply that X is $\wedge^2$-enlargeable?

(It is not even clear, in the case where X admits a complete metric with non-positive sectional curvature, whether X admits a metric with positive scalar curvature.)

Question 8. Are there *"topological conditions at infinity"*, which prevent complete metrics with $Sc > 0$?

Or, conversely, given an open n-manifold X, there exists a n-manifold X', such that

(i) X' "contains X at infinity", i.e. a complement in X to a *relatively compact* open subset, admits a *proper* imbedding $X \setminus U \hookrightarrow X'$;

(ii) X' admits a complete metric with $Sc > 0$, or, at least, can be dominated by a complete manifold with $Sc > 0$.

Probably, the minimal surface argument from [Wang(Contractible) 2019] shows that

3-manifolds X', which "contains ends" of contractible non-simply connected at (their single ends at) infinity manifolds X, can't be dominated by manifolds with positive scalar curvatures.

But no such result is in sight for manifolds of dimensions $n \geq 4$.

(Over)Optimistic Existence Conjecture 9☹. *All open simply connected manifolds of dimensions $n \geq 4$ admit complete metrics with $Sc > 0$.*

Questionable Case. If X^{n-1} is a simply connected manifold, which admits no metric with $Sc > 0$, e.g, where $n = 4k + 1$ and $\hat{A}[X^{4k}] \neq 0$ or where X^{n-1} is Hitchin's sphere, the results by Cecchini, Zeidler and Zhang may imply that $X = X^n = X^{n-1} \times \mathbb{R}^1$, admits no complete metric with $Sc > 0$. (Unquestionably, these X admit *no metrics* with $Sc > \sigma > 0$,)

In view of this, it is safer to reformulate 9☹ as follows.

More Realistic Conjecture 9☹. *All open simply connected manifolds X of dimensions $n \geq 4$ with $H_{n-1}(X) = 0$ (which is equivalent to "connected at infinity" for $\pi_1(X) = 0$) admit complete metrics with $Sc > 0$.*

Example. The products $X = Y \times \mathbb{R}^2$, as we know do admit complete metrics with $Sc > 0$ for all Y and these can be made simply connected by thin surgery for $dim(X) \geq 4$.

*Non-Example **10☹**. .* There is no instance of a compact *contractible* manifold $\bar{X}$ with *aspherical* boundary, where we know whether the interior X of $\bar{X}$ admits a complete metric with $Sc > 0$.

Codimension one Optimistic Reduction Conjecture 10$_☹$. *Let X be a complete orientable n-manifold with $Sc(X) > 0$. If X is orientable, then all $(n-1)$-dimensional homology classes in X are realizable by by smooth closed oriented hypersurfaces $Y \subset X$, which support metrics with $Sc > 0$.*

But this *contradicts* to 4B$_☺$ in the questionable case. Maybe, it would be better to stick to a weaker conjecture, e.g. as follows.

Codimension one more Realistic Conjecture 10$_☹$. *All $(n-1)$-dimensional homology classes in X are realizable by the images of the fundamental homology classes of smooth closed $n-1$-manifolds Y under continuous maps $Y \to X$, where these Y support metrics with $Sc > 0$.*

If X is compact, one knows that deg ± 1 dominants of SYS-manifolds X_{SYS} and manifolds $X_{\kappa \leq 0}$ with non-positive sectional curvatures, as well as their products $X_{SYS} \times X_{\kappa \leq 0}$ have this $Sc \not> 0$ property: they have no dominants with $Sc > 0$;[362] we shall prove in section obstructions5 a similar property for open manifolds, thus confirming the following conjecture in special cases.

Non-compact Domination Conjecture 11$_☹$ *If a compact orientable n-manifold (or pseudomanifold) X_0 can't be dominated (with maps of degree 1) by compact manifolds with $Sc > 0$, then it can't be dominated by complete manifolds with $Sc > 0$.*

Despite the validity of this is known in a variety of *specific cases*, including complete manifolds X_0, where non-domination by complete X with $Sc(X) > 0$ via *proper* maps implies this property with *quasi-proper* ones(see section 1.5), one can't even rule out in general domination by complete manifolds with $Sc \geq \sigma > 0$.

5 Variation, Stabilization and Application of μ-Bubbles

Given a a Borel measure μ on an n-dimensional Riemannian manifold X, *μ-bubbles* are critical points of the following functional on a topologically defined class of domains $U \subset X$ with boundaries called $Y = \partial U$:

$$(U, Y) \mapsto vol_{n-1}(Y) - \mu(U).$$

Observe that in our examples, $\mu(U) = \int_U \mu(x)dx$ for (not necessarily positive) continuous functions μ on X and that $\mu(U)$ can be regarded as a *closed 1-form* on the space of cooriented hypersurfaces $Y \subset X$. Then $vol_{n-1}(Y) - \mu(U)$ also comes as such an 1-form which we denote $vol^{[-\mu]}_{n-1}(Y)(+const)$.

5.1 Second Variation Formula and Pointwise Scalar Curvature Estimates for $\mathbb{T}^{\rtimes}$-Stabilized Bubbles

The first and the second variations of $vol^{[-\mu]}_{n-1}(Y)(+const)$ are the sums of these for $Vol_{-1}(Y)$ and of $vol(U)$ where the former were already computed in section 2.5.

[362] I am inclined to think that products of SYS-manifolds may, in general, carry metrics with $Sc > 0$, but I am not certain about it.

And turning to the latter, it is obvious that the first derivative/variation of $\mu(U)$ under $\psi\nu$, where ν is the outward looking unit normal normal field to Y and $\psi(y)$ is a function on Y, is

$$\partial_{\psi\nu}\int_U \mu(x)dx = \int_Y \mu(y)\psi(y)dy$$

and the second derivative/variation is

$$\partial^2_{\psi\nu}\int_U \mu(x)dx = \partial_{\psi\nu}\int_Y \mu(y)\psi(y)dy = \int_Y (\partial_\nu\mu(y) + M(y)\mu(y))\psi^2(y)dy,$$

where the field ν is extended along normal geodesics to Y, (compare section 2.5) and where $M(y)$ denotes the mean curvature of Y in the direction of ν.

It follows that μ-bubbles Y, (critical points of $vol^{[-\mu]}_{n-1}(Y) = vol_{n-1}(Y) - \mu(U)$) have

$$mean.curv(Y) = \mu(y)$$

and that

second variation of *locally minimal bubbles* $Y \subset X$,

$$\partial_{\psi\nu}(vol^{[-\mu]}_{n-1}(Y)) = \partial_{\psi\nu}\left(vol_{n-1}(Y) - \int_U \mu(x)dx)\right),$$

is non-positive.

Then we recall, the formula [○○] from section 2.5

$$\partial^2_{\psi\nu}vol_{n-1}(Y) = \int_Y \|d\psi(y)\|^2 dy + R_-(y)\psi^2(y)dy$$

for

$$R_-(y) = -\frac{1}{2}\left(Sc(Y,y) - Sc(X,y) + M^2(y) - \sum_{i=1}^{n-1}\alpha_i(y)^2\right),$$

where $\alpha_i(y)$ are the principal curvatures of Y at y, and where $\sum \alpha_i^2$ is related to the mean curvature $M = \alpha_1 + ... + \alpha_{n-1}$, by the inequality

$$\sum \alpha_i^2 \geq \frac{M^2}{n-1}.$$

Thus, summing up all of the above, observing that

$$\partial_\nu\mu(x) \geq -\|d\mu(x)\|$$

and letting

[$R_+ =$] $$R_+(x) = \frac{n\mu(x)^2}{n-1} - 2\|d\mu(x)\| + Sc(X,x),$$

we conclude that

if Y *locally minimises* $vol^{[-\mu]}_{n-1}(Y)(= vol_{n-1}(Y) - \mu(U))$, *then*

$$\int \|d\psi\|^2 dy + \left(\frac{1}{2}Sc(Y) - \frac{1}{2}R_+(y)\right)\psi^2(Y)dy \geq \partial_{\psi\nu}vol^{[-\mu]}_{n-1}(Y) \geq 0$$

for all functions ψ *on* Y.

Hence,

✤$_{\geq 0}$ *the* $-\Delta + \frac{1}{2}Sc(Y,y) - \frac{1}{2}R_+(y)$, *for* $\Delta = \sum_i \partial^2_{ii}$ *is positive on* Y.

Examples. (a) Let $X = \mathbb{R}^n$ and $\mu(x) = \frac{n-1}{r}$, that is the mean curvature of the sphere of radius r. Then

$$R_+(x) = \frac{n(n-1)}{R} - 2\frac{n-1}{r^2} + 0 = \frac{(n-1)(n-2)}{r^2} = Sc(S^{n-1}(r)).$$

(b) Let $X = \mathbb{R}^{n-1}\times\mathbb{R}$ be the hyperbolic space with the metric $g_{hyp} = e^{2r}g_{Eucl} + dr^2$ and let $\mu(x) = n-1$. Then

$$R_+(x) = n(n-1) - 0 + (-n(n-1)) = 0 = Sc(\mathbb{R}^n).$$

(c) Let $X = Y \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ with the metric $\varphi^2 h + dt^2$, where the metric h is a metric on Y and where

$$\varphi(t) = \exp\int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}.$$

Then a simple computation shows that

$$R_+(x) = \frac{n(n-1)}{R} - 2\frac{n-1}{r^2} + 0 = \frac{(n-1)(n-2)}{r^2} = Sc(S^{n-1}(r)).$$

$$\frac{n\mu(x)^2}{n-1} - 2\|d\mu(x)\| + n(n-1) = 0.$$

Furthermore, if $Sc(h) = 0$, than $Sc(X(= n(n-1)$ and $R_+ = 0$.

Two relevant corollaries to ✤$_{\geq 0}$ are as follows.

Let X be a Riemannian manifold of dimension n, let $\mu(x)$ be a continuous function and Y be a smooth minimal μ-bubble in X.

✤$_{conf}$ *If*

$$R_+(x) = \frac{n\mu(x)^2}{n-1} - 2\|d\mu(x)\| + Sc(X,x) > 0,$$

then by Kazdan-Warner conformal change theorem (see section 2.6) *Y admits a metric with $Sc > 0$.*

✤$_{warp}$ *There exists a metric $\hat{g}$ on the product $Y \times \mathbb{R}$ of the form $g_Y + \phi^2 dr^2$ for the metric g_Y on Y induced from X, such that*

$$Sc_{\hat{g}}(y,r) \geq R_+(y).$$

which implies that $Sc_{\hat{g}}(y,r) \geq R_+(y)$, since $\lambda \geq 0$. QED.

5.2 On Existence and Regularity of Minimal Bubbles

Let X be a compact connected Riemannian manifold of dimension n with boundary ∂X and let $\partial_- \subset \partial X$ and $\partial_+ \subset \partial X$ be disjoint compact domains in ∂X.

Example. Cylinders $Y \times [-1,1]$ naturally come with such a $\partial_\mp$-pair for $\partial_- = Y \times \{-1\}$ and $\partial_+ = Y \times \{1\}$, where, observe, $\partial_- \cup \partial_+ = \partial(Y \times [-1,1])$ if and only if Y is a manifold without boundary.

Let us agree that the mean curvature of ∂_- is evaluated with the incoming normal field and $mean.curv(\partial_+)$ is evaluated with the outbound field.

For instance, if the boundary of X is *concave*, as for instance for X equal to the sphere minus two small disjoint balls, t then $mean.curv(\partial_-) \geq 0$ and $mean.curv(\partial_+) \leq 0$.

Barrier $[\gtrless \mp mean]$-*Condition.* A continuous function $\mu(x)$ on X is said to satisfy $[\gtrless \mp mean]$-condition if

$[\gtrless \mp mean]$ $$\mu(x) \geq mean.curv(\partial_-, v) \text{ and } \mu(x) \leq mean.curv(\partial_+, x)$$

for all $x \in \partial_- \cup \partial_+$.

It follows by the maximum principle in the geometric measure theory that

★ *the* $[\gtrless \mp mean]$-*condition ensures the existence of a minimal μ-bubble $Y_{min} \subset X$. which separates ∂_- from ∂_-+.*

If this condition is *strict*, i.e. if $\mu(x) > mean.curv(\partial_-)$ and $\mu(x) < mean.curv(\partial_+)$ and if X has no boundary apart from $\partial_\mp$, then $Y_{min} \subset X$ doesn't intersect $\partial\mp$; in general, the intersections $Y_{min} \cap \partial_\mp$ are contained in the *side boundary* of X that is the closure of the complement $\partial X \setminus (\partial_- \cup \partial_-)$. (This, slightly reformulated, remains true for non-strict $[\gtrless \mp mean]$.)

If $dim(X) = n \leq 7$, then, (this well known and easy to see) Federer's regularity theorem(see section 2.7) applies to minimal bubbles as well as to minimal subvarieties and the same can be said about Nathan Smale's theorem on non-stability of singularities for $n = 8$. Thus, in what follows we may assume our minimal bubbles smooth for $n \leq 8$.

Then, by the stability of Y_{min} (see section 5.1 above),

$\bullet_{\varphi_\circ}$: *there exits a function $\phi_\circ = \phi_\circ(y) > 0$ defined in the interior $^\circ Y$ of Y, i.e. on $Y \setminus \partial X$, such that the metric*

$$g_{\varphi_\circ} = \varphi_\circ^2 g_Y + dt^2 \text{ on the cylinder } {}^\circ Y \times \mathbb{R},$$

where g_Y is the Riemannian metric on Y induced from X, satisfies

$\bigcirc$ $$Sc_{g_{\varphi_\circ}}(y,t) \geq Sc(X,y) + \frac{n\mu(y)^2}{n-1} - 2\|d\mu(y)\|$$

for all $y \in {}^\circ Y$.[363]

What if $n \geq 9$?.

The overall logic of the proof indicated in [Lohkamp(smoothing) 2018] leads one to believe that, assuming strict $[\gtrless \mp mean]$, there always exists a smooth $Y_o \subset X$, which separates $\partial_\mp$ and and which admits a function $\phi_\circ$ with the property $\bigcirc$.

The proof of this, probably, is automatic, granted a full understanding Lohkamp's arguments. But since I have not seriously studied these arguments, everything which follows in sections 5.3-5.8 should be regarded as *conjectural* for $n \geq 9$.[364]

[363]Since the metric $g_{\varphi_\circ}$ is $\mathbb{R}$-*invariant* its scalar curvature is *constant* in $t \in \mathbb{R}$.

[364]In some cases, a generalization of Schoen- Yau's theorem 4.6 from [SY(singularities) 2017] can be used instead of Lohkamp's theory; namely, this is possible in those applications, which don't depend on the Dirac operators on these bubbles, but can be obtained by relying only on the geometric measure theory.

Barrier $[\gtrless mean = \mp\infty]$*-Condition.* Let X be a non-compact, possibly non-complete, Riemannian manifold X and let the set of the ends of X is subdivided to $(\partial_\infty)_- = (\partial_\infty)_-(X)$ and $(\partial_\infty)_+ = (\partial_\infty)_+(X)$, where this can be accomplished, for instance, with a proper map from X to an open (finite or infinite) interval (a_-, a_+) where "convergence" $x_i \to (\partial_\infty)_\mp$, $x_i \in X$, is defined as $e(x_i) \to a_\mp$.

For example, if X is the open cylinder, $X = Y \times (a, b)$, where Y is a compact manifold, possibly with a boundary, this is done with the projection $Y \times (a_-, a_+) \to (a_-, a_+)$.

Obvious Useful Observation. If a function $\mu(x)$ satisfies

$$\mu(x_i) \to \pm\infty \text{ for } x_i \to (\partial_\infty)_\mp$$

then X can be exhausted by compact manifolds X_i with distinguished domains $(\partial_\mp)_i \subset \partial X_i$, such that

- these $(\partial_\mp)_i$ separate $(\partial_\infty)_-$ from $(\partial_\infty)_-$ for all i and

$$(\partial_\mp)_i \to (\partial_\infty)_\mp;$$

- restrictions of μ to $(X_i, (\partial_\mp)_i)$ satisfy the barrier $[\gtrless \mp mean]$-condition.

This ensures the existence of locally minimising μ-bubbles in X which separate $(\partial_\infty)_-$ from $(\partial_\infty)_+$.

5.3 Bounds on Widths of Riemannian Bands and on Topology of Complete Manifolds with $Sc > 0$

Let us prove the following version of the $\frac{2\pi}{n}$-inequality from section 3.6.

$\frac{2\pi}{n}$*-Inequality*.* Let X be an open, possibly non-complete Riemannian manifold of dimension n and let

$$f : X \to (-l, l)$$

be a proper (i.e. infinity → infinity) smooth distance non-increasing map, such that the pullback $f^{-1}(t_o) \subset X$ of a generic point t_o the interval $(-l, l)$ is non-homologous to zero in X.

If $Sc(X) \geq n(n-1) = Sc(S^n)$ and if the following condition $\nparallel_{Sc\not>0}$ is satisfied, then

$$l \leq \frac{\pi}{n}.$$

$\nparallel_{Sc\not>0}$ No smooth closed cooriented hypersurface in X homologous to $f^{-1}(t_o)$ admits a metric with $Sc > 0$.

Proof. Assume $l > \frac{\pi}{n}$. and let $\underline{\mu}(t)$ denote the mean curvature of the hypersurface $\underline{Y} \times \{t\}$ in the warped product metric $\varphi^2 h + dt^2$. on $\underline{Y} \times \left(-\frac{\pi}{n}, \frac{\pi}{n}\right)$ for

$$\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan \frac{nt}{2} dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}$$

as in example (c) from the previous section.

Since $\underline{\mu}(t) \to \pm\infty$ for $t \to \mp\frac{\pi}{n}$, the barrier $[\gtrless mean = \mp\infty]$-condition from the section 5.2 guaranties the existence of a locally minimizing μ-bubble in X for μ being a slightly modified f-pullback of $\underline{\mu}$ to X.

Let us spell it out in detail.

Assume without loss of generality that the pullbacks $Y_{\mp} = f^{-1}\left(\mp\frac{\pi}{n}\right) \subset X$ are smooth, and let $\mu(x)$ be a smooth function on X with the following properties.

$\bullet_1$ $\mu(x)$ is constant on X on the complement of $f^{-1}\left(-\frac{\pi}{n},\frac{\pi}{n}\right)$ for $\left(-\frac{\pi}{n},\frac{\pi}{n}\right) \subset (-i,i)$;

$\bullet_2$ $\mu(x)$ is equal to $\underline{\mu} \circ f$ in the interval $\left(-\frac{\pi}{n}+\varepsilon,\frac{\pi}{n}-\varepsilon\right)$ for a given (small) $\varepsilon > 0$;

$\bullet_3$ the absolute values of the mean curvatures of the hypersurfaces $Y_{\mp}$ are everywhere smaller than the absolute values of μ;

$\bullet_4$ $\frac{n\mu(x)^2}{n-1} - 2\|d\mu(x)\| + n(n-1) \geq 0$ at all points $x \in X$.

In fact, achieving $\bullet_3$ is possible, since $\underline{\mu}(t)$ is infinite at $\mp\frac{\pi}{n}$, while the mean curvatures of the hypersurfaces $Y_{\mp}$ and what is needed for $\bullet_4$ are the inequality $\|df\| \leq 1$ and the equality

$$\frac{n\underline{\mu}(t)^2}{n-1} - \left|\frac{d\underline{\mu}(t)}{dt}\right| + n(n-1) = 0$$

indicated in example (c) from section 5.1).

Because of $\bullet_3$, the submanifolds $Y_{\mp}$ serve as barriers for μ-bubbles (see the previous section) between them; this implies the existence of a minimal μ-bubble Y_{min} in the subset $f^{-1}\left(-\frac{\pi}{n},\frac{\pi}{n}\right) \subset X$ homologous to Y_o. by ★ in section 5.2.

Due to $\bullet_4$, the $\Delta + \frac{1}{2}Sc(Y)$ is positive by ✤$_{\geq 0}$ from the section 5.1.

Hence, by ✤$_{conf}$ the manifold Y_{min} admits a metric with $Sc > 0$ and the inequality $l \leq \frac{\pi}{n}$ follows.

On Rigidity. A a close look at minimal μ-bubbles (see section 5.7) shows that

if $l = \frac{\pi}{n}$, then X is isometric to a warped product , $X = Y \times \left(-\frac{\pi}{n},\frac{\pi}{n}\right)$ with the metric $\varphi^2 h + dt^2$, where the metric h on Y has $Sc(h) = 0$ and where

$$\varphi(t) = \exp\int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \quad -\frac{\pi}{n} < t < \frac{\pi}{n}.$$

Exercises. (a) Let X be an open manifolds with two ends, Show that if no closed hypersurface in X that separates the ends admits a metric with positive scalar curvature then X admits no metric with $Sc > 0$ either.[365]

(b) Let X be a complete Riemannian manifold, and let

$$S(R) = \min_{B(R)} Sc(X)$$

denote the minimum of the scalar curvature (function) of X on the ball $B(R) = B_{x_0}(R) \subset X$ for some centre point $x_0 \in X$. Show that

if X is homeomorphic to $\mathbb{T}^{n-2} \times \mathbb{R}^2$, then there exists a constant $R_0 = R_0(X, x_0)$, such that

$[\asymp \frac{4\pi^2}{R^2}]$ $$S(R) \leq \frac{4\pi^2}{(R-R_0)^2} \text{ for all } R \geq R_0.$$[366]

[365] This, for a class of spin manifolds X, was shown in [GL(complete) 1983] by applying a relative index theorem for suitably twisted Dirac operators on $X \times S^2(R)$.

[366] A rough version of this for a class of spin manifolds X can be proved by Dirac operator methods.

Hint. Since the bands between the concentric spheres of radii r and $r+R$, call them $X(r,r+R) = B(r+R) \smallsetminus B(r)$, are, for large r, quite similar to the cylinders $\mathbf{T}^{N-1} \times [0,R]$, the $\frac{2\pi}{n}$*-Inequality** applies to them and says that their scalar curvatures satisfy

$$S(R) = \inf Sc_x(X(r,r+R),x) \leq \frac{4(n-1)\pi^2}{nR^2}.$$

Question. What are topological obstructions, if any, for the existence of a complete Riemannian metric g on an open manifold X (possibly with a boundary), such that $Sc(g) > 0$ and/or $Sc(g) \geq 0$ at infinity, i.e. outside a given compact subset in X. mple

We describe obstructions on topology implied by positivity of the scalar curvature in sections 4.7 and 5.10; [367] here we make a couple of preparatory remarks concerning this issue.

(a) If the sectional curvature of a complete Riemannian manifold X is non-negative at infinity, then, by the standard argument, X admits a proper continuous function $f : X \to [0,\infty)$, where the levels $f^{-1}(r) \subset X$ are *convex hypersurfaces for all* $r \in [1,\infty)$. Thus,

X is topologically cylindrical at infinity.

(b) Similarly to (a), if $Ricci(X) \geq 0$ at infinity, then X admits a proper continuous function $f : X \to [0,\infty)$, where the levels $f^{-1}(r) \subset X$ have *non-negative (generalized) mean curvatures for all* $r \in [1,\infty)$. Thus,

X has finitely many ends.

(c) Let $X = Y_0 \times \mathbb{T}^{n-2}$, where Y_0 is connected surface of infinite topological type, e.g. with infinitely many ends.

Then "the first "-Torical symmetrization from section 3.6.1, at least if $n \leq 8$, brings us to a complete surface $Y \subset X$ of infinite topological type, such that a (generalized) warped product g° metric on $Y \times \mathbb{T}^{n-2}$ has $Sc > 0$ at the points $x \in Y$, where $Sc(X,x) > 0$.

Now, to prove that g° can't have $Sc(g^\circ) > 0$ at infinity, one needs to find a complete $\mathbb{T}^{n-2}$-invariant volume minimizing hypersurface that doesn't intersect a given compact subset $K \subset Y \times \mathbb{T}^{n-2}$, where such a hypersurface can be seen as a minimizing geodesic in the surface Y with the metric, which is conformally equivalent to the metric dy^2 induced from X, and where the conformal factor is equal to $(vol(\mathbb{T}^{n-2}_y)^2$.

In general, the volumes of the tori $\mathbb{T}^{n-2}_y = \mathbb{T}^{n-2} \times \{y\}$ may grow very fast for $y \to \infty$, such that all minimal hypersurfaces intersect the subset K, where $Sc \leq 0$, but there is a limit to such a growth due to the differential inequality satisfied by the conformal factor (see section 2.4.1. Besides, a significant growth of this factor, may allow stable μ-bubbles away from K.

But it is unclear if this can be made rigorous and

non-existence of complete metrics with $Sc > 0$ on the above $Y_0 \times \mathbb{T}^{n-2}$ remains conjectural.

[367] Also see [GL(complete) 1983], [Cecchini(Callias) 2018], [Wang(Contractible) 2019]) for the existence of complete metrics with *everywhere* positive scalar curvatures, where the techniques from [Cecchini(long neck) 2020] and/or from [Zhang(Area Decreasing) 2020] may be(?) applicable to $Sc > 0$ *at infinity.*

On the other hand, this kind of reasoning rules out complete metrics with $Sc > 0$ everywhere on many manifolds with sufficiently complicated topologies, which may include manifolds considered in [Cecchini(Callias) 2018] and/or in [Wang(Contractible) 2019].[368]

5.4 Equivariant Separation and Bounds on Distances Between Opposite Faces of Cubical Manifolds with $Sc > 0$

Recall the following general purpose proposition from section **??**.

||| $_\circlearrowleft$ **Equivariant Separation Theorem**. Let X be an n-dimensional, Riemannian band, possibly non-compact and non-complete.

Let

$$Sc(X,x) \geq \sigma(x) + \sigma_1, \ ,$$

for a continuous function $\sigma = \sigma(x) \geq 0$ on X and a constant $\sigma_1 > 0$, where σ_1 is related to $d = width(X) = dist_X(\partial_-, \partial_+)$ by the inequality

$$\sigma_1 d^2 > \frac{4(n-1)\pi^2}{n}.$$

(If scaled to $\sigma_1 = n(n-1)$, this becomes $d > \frac{2\pi}{n}$.)

Then there exists a smooth hypersurface $Y \subset X$, which separates ∂_- from ∂_+, and a smooth positive function ϕ on Y, such that the scalar curvature of the metric $g_\phi = g_{Y_{-1}} + \phi^2 dt^2$ on $Y \times \mathbb{R}$ is bounded from below by

$$Sc(g_\phi, x) \geq \sigma(x).$$

Furthermore

if X is isometrically acted upon by a compact connected group G, then the separating hypersurface $Y \subset X$ and the function ϕ on Y can be chosen invariant under this action.

Proof. The general case of this reduces to that of $\sigma = n(n-1)$ by on obvious scaling/rescaling argument and when $\sigma = n(n-1)$ we use the same μ as above associated with $\varphi(t) = \exp \int_{-\pi/n}^{t} -\tan\frac{nt}{2}dt, \ -\frac{\pi}{n} < t < \frac{\pi}{n}$. Then, as earlier, since

$$Sc_{g_{\varphi_\circ}}(y,t) \geq Sc(X,y) + \frac{n\mu(y)^2}{n-1} - 2||d\mu(y)||$$

by ◯ from the previous section, the above equality $\frac{n\mu(t)^2}{n-1} - |\frac{d\mu(t)}{dt}| + n(n-1) = 0$ implies the requited bound $Sc(g_o) \geq \sigma_1$. QED.

Example of Corollary Let X be an *orientable spin* manifold, let $\partial_- \cup \partial_+ = \partial X$ and let $f : X \to S^{n-1} \times [-l, l]$ be a smooth map, such that $\partial_\mp \to S^{n-1} \times \{\mp l\}$.

Let the following conditions be satisfied.

- $deg(f) \neq 0$,
- the map $X \to S^{n-1}$, that is the composition of f with the projection $S^{n-1} \times [-l, l] \to S^{n-1}$, is *area decreasing*;

[368] Cecchini's proof, which applies to spin manifolds of all dimensions, depends on the index theory for Dirac-type operators, while Wang's argument, which relies on specifically 2-dimensional properties of minimal surfaces, shows that certain contractible 3-manifolds admit no metrics with $Sc > 0$.

• $Sc(X) \geq (n-1)(n-2) + \sigma_1$ for some $\sigma_1 \geq 0$.
Then in conjunction with the (stabilised) Llarull theorem shows that

$$dist(\partial_-, \partial_+) \leq \frac{2\pi}{n} \frac{n(n-1)}{\sqrt{\sigma_1}} = \frac{2\pi(n-1)}{\sqrt{\sigma_1}}.$$

Remark. This inequality if it looks sharp, then only for $\sigma_1 \to 0$, while sharp(er) inequality of this kind need different functions μ.

$\square^{n-m}$-***Theorem.*** Let X be a compact connected orientable Riemannian manifold of dimension n with a boundary and let $\underline{X}_\bullet$ is a closed orientable manifold of dimension $n-m$, e.g. a single point $\bullet$ if $n = m$.

Let

$$f : X \to [-1,1]^m \times \underline{X}_\bullet$$

be a continuous map, which sends the boundary of X to the boundary of $[-1,1]^m \times \underline{X}_\bullet$ and which has *non-zero degree*.

Let $\partial_{i\pm} \subset X$, $i = 1, ..., m$, be the pullbacks of the pairs of the opposite faces of the cube $[-1,1]^m$ under the composition of f with the projection $[-1,1]^m \times \underline{X}_\bullet \to [-1,1]^m$.

Let X satisfy the following condition:

$\pitchfork^m_{Sc \not> 0}$ No transversal intersection $Y_{-m\pitchfork} \subset X$ of m-hypersurfaces $Y_i \in X$ which separates ∂_{i-} from ∂_{i+}, admits a metric with $Sc > 0$; moreover, the products $Y_{-m\pitchfork} \times \mathbb{T}^m$ admit no metrics with $Sc > 0$ either.[369]

If $Sc(X) \geq n(n-1)$, *then the distances* $d_i = dist(\partial_{i-}, \partial_{i+})$ *satisfy the following inequality* (which generalise $\square^n$-**inequality** from section 3.8).

$$\square_\Sigma \qquad \sum_{i=1}^m \frac{1}{d_i^2} \geq \frac{n^2}{4\pi^2}$$

Consequently

$$\square_{\min} \qquad min_i dist(\partial_{i-}, \partial_{i+}) \leq \sqrt{m}\frac{2\pi}{n}.$$

Proof. Let

$$\sigma_i' = \left(\frac{2\pi}{n}\right)^2 \frac{n(n-1)}{d^2} = \frac{4\pi^2(n-1)}{nd^2}$$

and rewrite $\square_\Sigma$ as

$$\sum_i \sigma_i' \geq n(n-1).$$

Assume $\sum_i \sigma_i' < n(n-1)$ and let $\sigma_i > \sigma_i'$ be such that $\sum_i \sigma_i < n(n-1)$.

Then, by induction on $i = 1, 2, ..., m$ and using $\mathbb{R}^{i-1}$-invariant $\square$-Lemma on the ith step, construct manifolds $X_{-i} = Y_{-i} \times \mathbb{R}^i$ with $\mathbb{R}^i$-invariant metrics g_{-i}, such that

$$Sc(X_{-i}) > n(n-1) - \sigma_1 - .. - \sigma_i.$$

[369] This "moreover" is unnecessary, since the relevant for us case of stability of the $Sc \not> 0$ condition under multiplication by tori is more or less automatic. (The general case needs some effort.)

The proof is concluded by observing that this for $i = m$ would contradict to $\|^m_{Sc \not> 0}$.

Remarks. (a) As we mentioned earlier, this inequality is non-sharp starting from $m = 2$, where the sharp inequality

$$\square^2_{\min} \qquad min_{i=1,2} dist(\partial_{i-}, \partial_{i+}) \leq \pi.$$

for squares with Riemannian metrics on them with $Sc \geq 2$ follows by an elementary argument.

(b) One can show for all n that

$$min_i dist(\partial_{i-}, \partial_{i+}) \leq \sqrt{m}\frac{2\pi}{n} - \varepsilon_{m,n},$$

where $\varepsilon_{m,n} > 0$ for $m \geq 2$.

(c) A possible way for sharpening $\square_\Sigma$, say for the case $m = n$, is by using $n-2$ inductive steps instead of n and then generalizing the elementary proof of $\square^2_{\min}$ to $\mathbb{T}^{n-2}$-invariant metrics on $[-1,1]^2 \times \mathbb{T}^{n-2}$.

In fact, all theorems for surfaces X with positive (in general, bounded from below) sectional curvatures beg for their generalisations to $\mathbb{T}^{m-2}$-invariant metrics on $X \times \mathbb{T}^{m-2}$ with positive (and/or bounded from below) scalar curvatures.

5.4.1 Max-Scalar Curvature with and without Spin

It remains a big open problem of making sense of the inequality $Sc(X) \geq \sigma$, e.g. for $\sigma = 0$, for *non-Riemannian* metric spaces, e.g. for piecewise smooth polyhedral spaces P.

But lower bounds on Lipschitz constants of homologically substantial maps $X \to P$ entailed by the inequality $Sc(X) \geq \sigma > 0$, that, for a fixed P, tell you something about the geometry of X, can be used the other way around for the definition of scalar curvature-like invariants of general metric spaces P as follows.

Given a metric space P [370] and a homology class $h \in H_n(P)$ define $Sc^{\max}(h)$ as the supremum of the numbers $\sigma \geq 0$, such that H can be *dominated with* $Sc \geq \sigma$. Here (slightly unlike how it is in 1.5) this means that

there exists a closed orientable Riemannian n-manifold X and a 1-Lipschitz map $f : X \to P$, such that the fundamental homology class $[X]$ goes to h,

$$f_*[X] = h.$$

Similarly, one defines $Sc^{\max}_{sp}(h)$ by allowing only *spin* manifolds X, where, for instance, the discussion in section 4.1.1 shows that

$$Sc^{\max}_{sp}(h) \leq const_n \cdot \text{K-}waist_2(h).$$

Below are a few observations concerning these definitions.

$\bullet_1$ $Sc^{\max}[X] \geq \inf_x Sc(X,x)$ for all closed *Riemannian* manifolds X, where the equality $Sc^{\max}[X] = Sc(X,x)$, $x \in X$, holds for what we call *extremal* manifolds X.

[370] To be specific we assume that P is locally compact and locally contractible, e.g. it is locally triangulable space

•$_2$ More generally, the product homology class $h \otimes [X] \in H^{n+m}(P \times X)$, $m = dim(X)$, where $P \times X$ is endowed with the Pythagorean product metric, satisfies

$$Sc^{\max}(h \otimes [X]) \geq Sc^{\max}(h) + \inf_x Sc(X, x).$$

•$_3$ *Possibly,*

$$Sc^{\max}(h \otimes [S^m]) = Sc^{\max}(h) + m(m-1),$$

but even the rough inequality

$$Sc^{\max}(h \otimes [S^m]) \leq Sc^{\max}(h) + const_m.$$

remains beyond splitting techniques from section 5.3. [371]

•$_4$ If $F : X_1 \to X_2$ is a finitely sheeted covering between closed orientable Riemannian manifolds, then

$$Sc^{\max}_{sp}[X_1] \geq Sc^{\max}_{sp}[X_2] \text{ as well as } Sc^{\max}_{sp}[X_1] \geq Sc^{\max}_{sp}[X_2],$$

but the equality may fail to be true, e.g. for SYS-manifolds X_2 defined in section 2.7

(It is less clear when/why this happens to *infinitely sheeted* coverings, where the problem can be related to possible failure of contravariance of K-$waist_2$, see section 4.1.4)

Non-Compact Spaces and $Sc^{\max}_{prop}$. The above definitions naturally extends to homology with infinite supports in non-compact spaces , e.g. to the fundamental classes $[P]$ of open manifolds and pseudomanifolds P, where the Riemannian manifolds X mapped to these spaces are now non-compact and not even complete.

Also we use the notation $Sc^{\max}_{prop}$ for fundamental classes of (psedo)manifolds P *with boundaries*, where *proper maps* $X \to P$ are those sending $\partial X \to \partial P$.

Stabilized max-Scalar Curvatures. These for a space P are defined as

$$stabSc^{max}_{...}(P) = Sc^{max}_{...}(P \times \mathbb{T}^N)$$

where $\mathbb{T}^N$ is flat torus that may be assumed arbitrarily large (this proves immaterial at the end of day), where N is also large and where the implied metric in the product is the Pythagorean one:

$$dist((p_1, t_1), (p_2, t_2)) = \sqrt{dist(p_1, p_2)^2 + dist(t_1, t_2)^2}.$$

Examples. (a) Llarull's and Goette-Semmelmann's inequalities from section 4.2 can be regarded as sharp bounds on $Sc^{\max}_{sp}$ for (the fundamental homology classes of) spheres and convex hypersurfaces.

(b) The $\square$-inequalities from the previous section provide similar bounds on stabilised $Sc^{\max}_{prop}(P)$ for the fundamental homology classes of the rectangular solids $P = \times_{i=1}^{n}[0, a_i]$.

[371] These techniques deliver such an inequality for the stabilized max-scalar curvature: $Sc^{\mathsf{maxstab}}(h) = \lim_{m\to\infty}(h \otimes [\mathbb{T}^m])$, where one may additionally require the manifolds X mapped to $P \times \mathbb{T}^m$ to be isometrically acted upon by the m-tori

(It seems, there are interesting examples in the spirit of SYS-spaces from section 2.7 where one needs to allow $f_*[X]_{\mathbb{Z}/l\mathbb{Z}} \neq 0$, at least for for odd l.

Also one may ask in this regard if $Sc^{\max}_{prop}$ of the universal covering of a closed orientable manifold X with a residually finite fundamental group is equal to the limit of $Sc^{\max}_{prop}$ of the finite coverings of X.)

(c) *Spaces with S-Conical Singularities and $Sc \geq \sigma$.* Let us define classes $\mathscr{S}^n_{\geq\sigma}$, $n = 2, 3, ...$ of piecewise Riemannian spaces with $Sc \geq \sigma > 0$ by induction on dimension $n \geq 2$ as follows.

Let $Y = Y^{n-1}$ from $\mathscr{S}^{n-1}_{\geq\sigma}$ be isometrically realized by a piecewise smooth $(n-1)$-dimensional subvariety in a $(N-1)$-dimensional sphere, $N >> n$, that serves as the boundary of the N-dimensional hemisphere,

$$Y \subset S^{N-1}(R) = \partial S^N_+(R),$$

where the radius of the sphere satisfies,

$$R \geq \sqrt{\frac{(n-1)(n-2)}{\sigma}}$$

and where "isometrically" means preservation of the lengths of piecewise smooth curves in Y.

Then the *spherical cone* of Y, that is the union of the geodesic segments which the center of the spherical n-ball $S^N_+ \subset S^N$ to all $y \in Y$ is, by definition, belongs to $\mathscr{S}^n_{\geq\sigma'}$ for

$$\sigma' = \sigma\frac{n}{n-2}$$

and, more generally, a piecewise smooth Y is in $\mathscr{S}^n_{\geq\sigma'}$ if its scalar curvature at all non-singular points is $\geq \sigma'$ and near singularities Y is isometric to a spherical cone over a space from $\mathscr{S}^{n-1}_{\geq\sigma}$.

To conclude the definition, we agree to start the induction with $n-1 = 1$, where our admissible spaces are circles of length $\leq 2\pi$ and, if we allow boundaries, segments of any length.

$Y \subset S^{N-1}$ be a closed submanifold of dimension $n-1 \geq 2$, and let $S(Y) \subset S^N \supset S^{N-1}$ be the *spherical suspension* of Y, that is the union of the geodesic segments which go from the north and the south poles of S^N to Y.

Notice that this $S(Y)$ with the induced Riemannian metric is smooth away from the poles, where it is singular unless the induced Riemannian metric in Y has constant sectional curvature $+1$ and Y is simply connected (hence, isometric to S^{n-1}).

Let Y be a space from $\mathscr{S}^n_{\geq\sigma}$ with k isolated singular points $y_i \in Y$ where X is locally isometric to S-cones over $(n-1)$-manifolds, call them V_i, $i = 1, ...k$ such that every such V_i bounds a Riemannian manifold W_i, where $Sc(W_i) > 0$ and the mean curvature of $V_i = \partial W_i$ is positive. Then

$$Sc^{\max}_{prop}(Y) \geq \sigma.$$

Sketch of the Proof. Arguing as in [GL(classification) 1980], one can, for all $\varepsilon > 0$, deform the metric in X near singularities keeping $Sc \geq \sigma - \varepsilon$, such that the resulting metric on Y minus the singular points y_i becomes complete, where its

k ends are isometric to the cylinders $\varepsilon V_i \times [\infty)$, where εV stands for an V with its Riemannian metric multiplied by ε^2.

This complete manifold, call it Y_ε, admits a locally constant at infinity 1-Lipschitz map $Y_\varepsilon \to Y$ of degree 1, and then the closed manifold $\bar{Y}_\varepsilon$, obtained from Y_ε by attaching εW_i to $\varepsilon V_i \times \{t_i\}$, for large $t_i \in [0, \infty]$ admits a required 1-Lipschitz map to Y as well. QED

Remark. Instead of filling V_i by W_i *individually* it is sufficient to fill in their (correctly oriented!) disjoint union $V = \bigsqcup_i V_i$ by W. For instance, if there are only two singular points, where V_1 and V_2 are isometric and admit orientation reversing isometries then $V_1 \sqcup -V_2$ bounds the cylinder W between them.

This kinds of "desingularization by surgery" also applies to Y, where the singular loci $\Sigma \subset Y$ have dimensions $dim(\Sigma) \geq 1$, similarly to how it is done to manifolds with corners (see section 1.1 in [G(billiard0 2014]) but the filling condition becomes less manageable.

In fact even if $dim(\Sigma) = 0$, it is unclear how essential our filling truly is, especially for evaluation $Sc^{\max}$ of a *multiple* of the fundamental class of an Y; yet, the spaces $Y \in \mathscr{S}^n_{\geq\sigma}$ with isolated singularities seem to enjoy the same metric properties as smooth manifolds with $Sc \geq \sigma$ filling or no filling.

For instance, if the non-singular locus of such an Y is spin then the hyperspherical radius Y is bounded in the same way as it is for smooth manifolds:

$$Rad_{S^n}(Y) \leq \sqrt{\frac{n(n-1)}{\sigma}},$$

as it follows from Llarull's theorem for complete manifolds.

In fact, the construction from [GL(classification) 1980] for connected sums of manifolds with $Sc > 0$, when applied to $Y \setminus \Sigma$, achieves a blow-up of the metric g of Y on $Y \setminus \Sigma$ to a complete one, say g_+, such that $g_+ \geq g$ and $\inf_x Sc(g_+, x) \geq \inf_x Sc(g, x) - \varepsilon$ for an arbitrarily small $\varepsilon > 0$.

Also mean convex cubical domains U in Y with none of the singular $y_i \in Y$ lying on the boundary ∂U satisfy the constraints on the dihedral angles similar to those for smooth Riemannian manifolds with $Sc \geq \sigma$

But the picture becomes less transparent for $dim(\Sigma) > 0$, as it is exemplified by the following.

Question. Does the inequality $Rad^2_{S^n}(Y) \leq const_n \frac{n(n-1)}{\sigma}$ hold true for all $Y \in \mathscr{S}^n_{\geq\sigma}$?

Perspective. In view of [Cheeger(singular) 1983], [GSh(Riemann-Roch) 1993] and[AlbGell(Dirac operator on pseudomanifolds) 2017], it is tempting to use the Dirac operator on the non-singular locus $Y \setminus \Sigma$ with a controlled behavior for $y \to \Sigma$, but it remains unclear if one can actually make this work for $dim(\Sigma) > 0$.

The only realistic approach at the present moment is offered by the method of minimal hypersurfaces (and/or of stable μ-bubbles), which may be additionally aided by surgery desingularization, such as multi-doubling similar to that described in [G(billiards) 2014] for manifolds with corners.

Max-Scalar Curvature Defined via Sc*-Normalized Manifolds* . Given a Riemannian manifold $X = (X, g)$ with positive scalar curvature, let $g_\sim = Sc(g) \cdot g$, consider Lipschitz maps f of closed oriented Riemannian manifolds $X = (X, g)$ with $Sc(X) > 0$ to P, such that $f_*[X] = h$, for a given $h \in H_n(P)$, let $\lambda^{min}_\sim$ be the infimum of the Lipschitz constants of these maps with respect to the metrics

$g_{\sim}$ and let

$$Sc^{\max}_{\sim}(h) = \frac{1}{(\lambda^{min}_{\sim})^2}.$$

And if P is a *a piecewise smooth polyhedral space* (e.g. a Riemannian manifold), define $Sc^{\max}_{\wedge^2_{\sim}}(h)$ by taking the infimum $\inf_f \sup_{x\in X} \| \wedge^2 df(x)\|$ instead of the $\lambda^{min}_{\sim}$ (as in $\wedge^2$-inequality from section 4.2[372]):

$$Sc^{\max}_{\wedge^2_{\sim}}(h) = \frac{1}{\inf_f \sup_{x\in X} \| \wedge^2 df(x)\|}.$$

Clearly,

$$Sc^{\max} \le Sc^{\max}_{\sim} \le Sc^{\max}_{\wedge^2_{\sim}}.$$

(Similar inequalities are satisfied by the spin and by proper versions of $Sc^{\max}$), where *most bounds on* $Sc^{\max}$ we prove and/or conjecture below *can be* more or less automatically *sharpened to their* $Sc^{\max}_{\sim}$ *and* $Sc^{\max}_{\wedge^2_{\sim}}$ (as well as to their spin and proper) *counterparts.*)

Problem. Evaluate $Sc^{\max}_{prop}$ of (the fundamental classes of) "simple" metric space, such as products of m_i-dimensional balls of radii a_i where $\sum_i m_i = n$ and the product distance is l_p, i.e. $dist_{l_p}((x_i),(y_i)) = \sqrt[p]{\sum_i dist(x_i,y_i)^p}$, e.g. for $p = 2$.

This is related to the problem of a general nature of evaluating $Sc^{\max}(h_1 \otimes h_2)$ of $h_1 \otimes h_2 \in H_{n_1+n_2}(P_1 \times P_2)$ in terms of $Sc^{\max}(h_1) \in H_{n_1}(P_1)$ and $Sc^{\max}(h_2) \in H_{n_2}(P_2)$.

It follows from the additivity of the scalar curvature (see section 1) that

$$Sc^{\max}(h_1 \otimes h_2) \ge Sc^{\max}(h_1) + Sc^{\max}(h_2),$$

but it is unrealistic (?) to expect that, in general

$$Sc^{\max}(h_1 \otimes h_2) \le const_{n_1+n_2} \cdot (Sc^{\max}(h_1) + Sc^{\max}(h_2)),$$

albeit the geometric method from the section 5.4 does deliver non-trivial bounds on $Sc^{\max}_{prop}$ of *product spaces* whenever lower bounds on the *hyperspherical radii of the factors* are available.[373]

5.5 Extremality and Rigidity of log-Concave Warped products

The inequalities proven in section 5.3 say, in effect, that the metric

$$g_\phi = \phi^2 g_{flat} + dt^2 \text{ on } \mathbb{T}^{n-1} \times \mathbb{R} \text{ for } \phi(t) = \exp \int_{-\pi/n}^{t} -\tan \tfrac{nt}{2} dt$$

is *extremal*: *one can't increase g_ϕ without decreasing its scalar curvature,*[374]

[372]The definition of $\| \wedge^2 df(x)\|$ makes sense for Lipschitz maps (at almost all x) but the arguments with Dirac operators need smoothness of the maps. But it may be interesting to go beyond smooth manifolds and maps to general continues maps with *bounded area dilations*, where, probably, the most adequate definition of "area" in non-smooth metric spaces P is the Hilbertian one in the sense of [G(Hilbert) 2012].

[373]One may define $Rad_{S^n}(h)$, $h \in H^n(P)$, as the suprema of the radii R of the n-spheres, for which P admits a 1-Lipschitz map $f : P \to S^n(R)$, such that $f_*(h) \ne 0$.

where the essential feature of ϕ

(implicitly) used for this purpose was log-*concavity of* ϕ:

$$\frac{d^2 \log\phi(t)}{dt^2} < 0.$$

We show in this section that the same kind of extremality (accompanied by rigidity) holds for other log-concave functions, notably for $\varphi(t) = t^2$, $\varphi(t) = \sin t$ and $\varphi(t) = \sinh t$ which results in

rigidity of punctured Euclidean, spherical and hyperbolic spaces.

More generally, let $X = Y \times \mathbb{R}$ comes with the warped product metric $g_\phi = \phi^2 dg_y + dt^2$. Then the mean curvatures of the hypersurfaces $Y_t = Y \times \{t\}$, $t \in \mathbb{R}$, satisfy (see 2.4)

$$mean.curv(Y_t) = \mu(t) = (n-1)\frac{d\log\phi(t)}{dt} = \frac{\phi'(t)}{\phi(t)},$$

and, obviously, are these $Y_t \subset X$ are locally (non-strictly) minimizing μ-bubbles.[375]

Now, clearly, ϕ *is log-concave, if and only if*

$$\frac{d\mu}{dt} = -\left|\frac{d\mu}{dt}\right|.$$

Thus, R_+ defined (see section 5) as

$$R_+(x) = \frac{n\mu(x)^2}{n-1} - 2\|d\mu(x)\| + Sc(X,x)$$

is equal in the present case to

$$\frac{n\mu(t)^2}{n-1} + 2\mu'(t) + Sc(g_\phi(t)) = \frac{2(n-1)\phi''(t)}{\phi(t)} + (n-1)(n-2)\left(\frac{\phi'}{\phi}\right)^2 + Sc(g_\phi(t))$$

which implies (see section 5) that

$$(R_+)_{Y_t} = \frac{1}{\phi^2}Sc(g_{Y_t}) = Sc(g_{Y_t}) \text{ for } g_{Y_t} = \phi^2 g_Y.$$

Here our $-\Delta_{Y_t} + \frac{1}{2}Sc(g_{Y_t}) - (R_+)_{Y_t}$ from section 5.1) is equal to $-\Delta_{Y_t}$, the lowest eigenvalue which is *zero* with *constant* corresponding eigenfunctions and the corresponding ($\mathbb{T}^1$-invariant warped product) metrics on $Y_t \times \mathbb{T}^1$ are (non-warped) $g_{Y_t} + dt^2$ for $Y_t = Y \times \{t\} \subset X = Y \times \mathbb{R}$ and all $t \in \mathbb{R}$.

This computation together with ✤$_{warp}$ in section 5.1 yield the following.

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[374] To be precise, one should say that

one can't modify the metric, such that the *scalar curvature increases* but the metric itself *doesn't decrease.*

The relevance of this formulation is seen in the example of $X = S^n \times S^1$, where one can stretch the obvious product metric g in the S^1-direction without changing the scalar curvature, but one *can't increase* the scalar curvature by deformations that increase g.

[375] If Y is non-compact, the minimization is understood here for variations with compact supports.

Comparison Lemma. Let $\underline{X} = \underline{Y} \times [\underline{a}, \underline{b}]$ be an $\underline{n}$-dimensional warped product manifold with the metric

$$g_{\underline{X}} = g_{\underline{\phi}} = \underline{\phi}^2 g_{\underline{Y}} + dt^2, \ \underline{t} \in [\underline{a}, \underline{b}],$$

where $\underline{\phi}(\underline{t})$ is a smooth positive log-*concave* function on the segment $[\underline{a}, \underline{b}]$.

Let X be an n-dimensional Riemannian manifold, with a smooth function $\mu(x)$ on it and let $Y = Y_\mu \subset X$ be a stable, e.g. locally minimising μ-bubble in X.

Let $g^{\rtimes} = g_{\phi_\rtimes} = \phi_\rtimes^2 g_Y + dt^2$ be the metric on $Y \times \mathbb{T}^1$ where g_Y is the metric on Y induced from X, and where $\phi_\rtimes$ is the first eigenfunction of the

$$-\Delta + \tfrac{1}{2} Sc(g_Y, y) - R_+(y) \text{ for } R_+(x) = \tfrac{n\mu(x)^2}{n-1} - 2||d\mu(x)|| + Sc(X, x)$$

($\phi_\rtimes$ is not assumed positive at this point).

Let $f : X \to \underline{X}$ be a smooth map let $f_{\underline{Y}} : X \to \underline{Y}$ denote the $\underline{Y}$-component of f, that is the composition of f with the projection $\underline{X} = \underline{Y} \times [\underline{a}, \underline{b}] \to \underline{Y}$.

Let

$$f_{[\underline{a},\underline{b}]} : X \to [\underline{a}, \underline{b}]$$

be the $[\underline{a}, \underline{b}]$-component of f, let

$$\underline{\mu}^*(x) = \underline{\mu} \circ f_{[\underline{a},\underline{b}]}(x) \text{ for } \underline{\mu}(\underline{t}) = (\underline{n} - 1)\frac{d \log \underline{\phi}(\underline{t})}{d\underline{t}} = mean.curv(\underline{Y}_{\underline{t}}), \ \underline{t} = f_{[\underline{a},\underline{b}]}(x)$$

and let

$$\underline{\mu}'^* = \underline{\mu}' \circ f_{[\underline{a},\underline{b}]}(x) \text{ where } \underline{\mu}' = \underline{\mu}'(\underline{t}) = \frac{d\underline{\mu}(\underline{t})}{d\underline{t}}.$$

Let

$$\underline{R}_+^*(x) = \frac{n\underline{\mu}^*(x)^2}{\underline{n} - 1} - 2||d\underline{\mu}^*(x)|| + Sc(\underline{X}, f(x))$$

If

$$R_+(x) \geq \underline{R}_+^*(x),$$

then the function $\phi_\rtimes$ *is positive and the scalar curvature of the metric* $g^{\rtimes} = g_{\phi_\rtimes}$ *on* $Y \times \mathbb{T}^1$ *satisfies*

$$Sc_{g^{\rtimes}}(y, t) \geq \frac{1}{||df_{[\underline{a},\underline{b}]}(y)||^2} Sc(\underline{Y}, f_{\underline{Y}}(y)) = Sc(\underline{Y}_{\underline{t}}, f(y)) \text{ for } \underline{Y}_{\underline{t}} \ni f(y).$$

The main case of this lemma, which we use below, is where

($\bullet_{df_{[\underline{a},\underline{b}]}}$) *the function* $f_{[\underline{a},\underline{b}]} : X \to [\underline{a}, \underline{b}]$ *is 1-Lipschitz, i.e.* $||df_{[\underline{a},\underline{b}]}|| \leq 1$,

and

($\bullet_\mu$) $\mu(x) = \underline{\mu} \circ f_{[\underline{a},\underline{b}]}$, *that is* $\mu(x) = mean.curv(\underline{Y}_{\underline{t}}, f(x))$ *for* $\underline{Y}_{\underline{t}} \ni f(x)$ and where the conclusion reads:

$$[Sc \geq]. \qquad Sc_{g_\phi}(y, t) \geq \frac{1}{(f_{[\underline{a},\underline{b}]}(y))^2} Sc(\underline{Y}, f_{\underline{Y}}(y)) + Sc(X, y) - Sc(\underline{X}, f(y)).$$

Corollary. Let $X^{\rtimes}$ denote the above Riemannian (warped product) manifold $(Y \times \mathbb{T}^1, g^{\rtimes} = g_{\phi_\rtimes})$ and let $f_\rtimes : X^{\rtimes} \to \underline{Y}$ be defined by $(y, t) \mapsto f_{\underline{Y}}(y)$.

If besides $\bullet_{df_{[a,b]}}$ *and* $\bullet_\mu$,

$$\| \wedge^2 df\| \le 1, \ \ e.g. \ \|df\| \le 1$$

and if

$$Sc(X,y) \ge Sc(\underline{X}, f(y)),$$

then the map $f_\rtimes$ *satisfies*

$$Sc(X^\rtimes, x_\rtimes) \ge \|df_\rtimes\|^2 Sc(\underline{Y}, f_\rtimes(x_\rtimes)) \ge \| \wedge^2 df_\circ\| Sc(\underline{Y}, f_\rtimes(x_\rtimes)).$$

Now, the existence of minimal bubbles under the barrier $[\gtrless$ *mean* $= \mp\infty]$-condition (see section 5.2) and a combination of the above with the Llarull *trace* $\wedge^2$ *df*-inequality from section 4.2 yields the following.

$\odot_{S^n}$. *Extremality of Doubly Punctured Spheres.* Let X be an oriented Riemannian spin n-manifold, let $\underline{X}$ be the n-sphere with two opposite points removed and let $f : X \to \underline{X}$ be a *smooth 1-Lipschitz* map of *non-zero* degree.

If $Sc(X) \ge n(n-1) = Sc(\underline{X}) = Sc(S^n)$, *then*

(A) the scalar curvature of X *is constant* $= n(n-1)$*;*

(B) the map f *is an isometry.*

Proof. The spherical metric on $\underline{X} = S^n \smallsetminus \{s, -s\}$ is the warped product $S^{n-1} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ where the warping factor $\underline{\phi}(t) = \cos t$ which is logarithmically concave, where $\underline{\mu}(t) = \frac{d \log \underline{\phi}(t)}{dt} \to \pm\infty$ for $t \to \mp\frac{\pi}{2}$. [376]

This implies (A) while (B) needs a little extra (rigidity) argument indicated in section 5.7.

1-*Lipschitz Remark.* As it is clear from the proof, the 1-Lipshitz condition can be relaxed to the following one.

The radial component $f_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} : X \to \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ of f, which corresponds to the signed distance function from the equator in $S^n \smallsetminus \{s, -s\}$ is 1-Lipschitz and (the exterior square of) the differential of the S^{n-1} component $f_{S^{n-1}} : X \to S^{n-1}$ satisfies

$$\wedge^2 df_{S^{n-1}} \le \frac{1}{\left(\cos f_{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}(x)\right)^2}.$$

Non-Spin Remark. If $n = 4$, one can drop the spin condition, since μ-bubbles $Y \in X$, being 3-manifolds, are spin.

Similarly to $\odot_{S^n}$ one shows the following.

$\odot_{\mathbb{R}^n}$. Let Let X be as above, let $\underline{X}$ be $\mathbb{R}^n$ with a point removed and let $f : X \to \underline{X}$ be a *smooth 1-Lipschitz* map of *non-zero* degree.

If $Sc(X) \ge n(n-1) \ge 0$ *and if* X *is an isometry at infinity, then*

(A) $Sc(X) = 0$*;*

(B) the map f *is an isometry.*

[376] If a log-concave function ϕ on the segment $[-l, l]$ is positive for $-l < t < l$ and it vanishes at $-l$, then the logarithmic derivative of ϕ goes to ∞ for $t \to -l$; similarly,

$$\frac{\phi'}{\phi} \underset{t \to l}{\to} -\infty,$$

if ϕ vanishes at $t = l$.

$\odot_{\mathbf{H}^n}$. Let Let X be as above, let $\underline{X}$ be the hyperbolic space with a point removed and let $f: X \to \underline{X}$ be a *smooth 1-Lipschitz* map of *non-zero* degree.

If $Sc(X) \geq -n(n-1)$ and if X is an isometry at infinity, then

(A) $Sc(X) = -n(n-1)$;

(B) the map f is an isometry.

Question. Let $d_0(\underline{x}) = dist(\underline{x}, \underline{x}_0)$ be the distance function in $\underline{X}$ (used in $\odot_{\mathbb{R}^n}$ and/or in $\odot_{\mathbf{H}^n}$) to the point $\underline{x}_0$, which was removed from $\mathbb{R}^n$ or from $\mathbf{H}^n$, and let $d_f(x) = d_0(f(x))$.

Can one relax the 1-Lipschitz condition in the propositions $\odot_{\mathbb{R}^n}$ and in $\odot_{\mathbf{H}^n}$ by requiring that not f but only the function $d_f(x)$ is 1-Lipschitz?

We conclude this section with the following proposition, that is proven (in a different form) in [Richard(2-systoles) 2020] (compare [Zhu(rigidity) 2019]) and which provides a useful geometric information on manifolds with scalar curvature $\geq \sigma > 0$ on the scale $\sim \frac{1}{\sqrt{\sigma}}$.

Richard's Lemma. Let X be an oriented m-dimensional Riemannian manifold (possibly non-compact and non-complete) with compact boundary and $X_0 \subset X$ be an open subset with smooth boundary such that the complement $X \setminus X_0$ is compact. Let $h \in H_{m-2}(\partial X)$ and $h_0 \in H_{m-2}(X_0)$ be homology classes, which have *equal images* under the homomorphisms induced by the inclusions $\partial X \hookrightarrow X \hookleftarrow X_0$, that are

$$h \in H_{m-2}(\partial X) \to H_{m-2}(\partial X) \leftarrow H_{m-2}(X_0) \ni h_0.$$

Let

$$Sc(X) \geq \sigma > 0,$$

and

$$dist^2(X_0, \partial X) \geq \frac{m(m-1)\pi^2}{\sigma}.$$

Then (we can vouch 100% here, as everywhere in this text, only for $n \leq 8$)

the image of the homology class h in $H_{m-2}(X)$ can be realized by a closed smooth $(m-2)$-dimensional submanifold $Y \subset X$, on which there exists a smooth positive function $\phi(y)$, such that the metric $g_* = dy^2 + \phi(y)^2 dt^2$ on the product manifold $Y \times \mathbb{R}^2$ satisfies

$$Sc(g_*) \geq \frac{m-2}{m}\sigma,$$

where dy^2 denotes the Riemannian metric on Y induced from $X \supset Y$ and dt^2 is the Euclidean metric in the plane $\mathbb{R}^2$.

Proof. Use the codimension 2 argument as in the proof of the *quadratic decay theorem* in section 1 in [G(inequalities) 2019] (see also section 7 in [GL(complete) 1983] and §$9\frac{3}{11}$ in [G(positive) 1996]) together with the above comparison lemma combined with the equivariant separation theorem from section 5.4.

(A version of Richard's lemma is also established in [Chodosh-Li(bubbles) 2020] in the course of their proof of non-existence of metric with $Sc > 0$ on aspherical 4- and 5-manifolds; also, this lemma is used for a similar purpose in [G(aspherical) 2020].)

5.6 On Extremality of Warped Products of Manifolds with Boundaries and with Corners

We explained in section 4.4 how reflection+ smoothing allows an extension of the Llarull and Goette-Semmelmann theorems from section 4.2 to manifolds with smooth boundaries and to a class of manifolds with corners. This, combined with the above, enlarges the class of manifolds with corners to which the conclusion of the extremality $\blacklozenge_{\angle_{ij}}$ *theorem* applies. example.

Let $\triangle^{n-1}\subset S^{n-1}$ be the regular spherical simplex with flat faces and the dihedral angles $\frac{\pi}{2}$ and let $\mathsf{S}^*_*\triangle^{n-1}\subset S^n\subset S^{n-1}$ be the spherical suspension of $\triangle^{n-1}$ and let $\underline{X}=\mathsf{S}^b_a(\triangle^{n-1})\subset\mathsf{S}^*_*\triangle^{n-1}$, $a,b\in(-\frac{\pi}{2},\frac{\pi}{2})$, be the region of $\mathsf{S}^*_*\triangle^{\mathsf{n}-1}$ between a pair of $(n-1)$-spheres concentric to our equatorial $S^{n-1}\subset S^n$.

Let X be an n-dimensional orientable Riemannian spin manifold with corners and let $f:X\to\underline{X}$ be a smooth 1-Lipschitz map which respects to the corner structure and which has non-zero degree.

Spherical $\mathsf{S}^b_a(\triangle)$-Inequality. If $Sc(X)\geq Sc(\underline{X})=n(n-1)$, if all $(n-1)$-faces $F_i\subset\partial X$ have their mean curvatures bounded from below by those of the corresponding faces in $\underline{X}$, [377]

$$mean.curv(F_i)\geq mean.curv(\underline{F}_i),$$

and if all dihedral angle of X are bounded by the corresponding ones of $\underline{X}$,

$$\angle_{ij}\leq\underline{\angle}_{ij}=\frac{\pi}{2},$$

then

$$Sc(X)=n(n-1),$$

$$mean.curv(F_i)=mean.curv(\underline{F}_i)$$

and

$$\angle_{ij}=\frac{\pi}{2}.$$

Exercise. (a) Recall $\blacksquare$-hyperbolic comparison theorem for cubical manifolds diffeomorphic to

$$\underline{V}=[0,1]\rtimes[0,1]^{n-1}\subset\mathbb{H}^n=(\mathbb{R}^1\times\mathbb{R}^{n-1},dt^2+e^{2t}dx^2)$$

from section 3.1 and generalize it to all compact cubical manifolds V (to be sure, of dimension $n\leq 8$).

(b) Formulate and prove (for $n\leq 8$) the Euclidean and hyperbolic versions of the $\mathsf{S}^b_a(\triangle)$-inequality for spin manifolds V with corners.[378]

Question. Do the counterparts to the $\mathsf{S}^b_a(\triangle)$-inequality hold for other simplices and polyhedra?

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[377] All these but two have zero mean curvatures.

[378] See [Li(parabolic) 2020) for further results in this direction.

5.7 On Rigidity of Extremal Warped Products

Let us explain, as a matter of example, that

doubly punctured sphere $\underline{X} = S^n \setminus \{\pm s\}$ is spin-rigid.

This means that

if an oriented Riemannian spin n-manifold X with $Sc(X) \geq n(n-1) = Sc(\underline{X} = Sc(S^n)$ admits a smooth proper 1-Lipschitz map $f : X \to \underline{X}$ such that $deg(f) \neq 0$, then, in fact, such an f is an isometry.

Proof. We know (see the the proof of $\odot_{S^n}$ in 5.5) that X contains a minimal μ-bubble Y, which separates the two (union of) ends of X, where $\mu(x)$ is the f-pullback of the mean curvature function of the concentric $(n-1)$-spheres in $\underline{X} = S^n \setminus \{\pm s\}$ between the two punctures and that this m-bubble must be umbilic, where we assume at this point that Y is non-singular, e.g. $n \leq 7$.

What we want to prove now is that these bubbles *foliate all of* X, namely they come in a continuous family of mutually disjoint minimal μ-bubbles Y_t, $t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, which together cover X.

Indeed, if the *maximal* such family Y_t wouldn't cover X, then the would exists a small perturbation $\mu'(x)$ of $\mu(x)$ in the gap between two Y_t in the maximal family, such that $|\mu'| > |\mu|$ in this gap, while $||d\mu'|| = ||d\mu||$ in there and such that there would exist a minimal μ'-bubble Y' in this gap.

But then, by calculation in section 5.5, the resulting warped product metric on $Y' \times S^1$ would have $Sc > n(n-1)$, thus proving "no gap property" by contradiction.

Therefore, X itself is the warped product, $X = Y \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ with the metric $dt^2 = (\sin t)^2 g_Y$, where $Sc(g_Y) = n(n-1)$ and which by Llarull's rigidity theorem, has constant sectional curvature. QED.

Remarks (a) On the positive side, this argument is quite robust, which makes it compatible with approximation of bubble and metrics. For instance it nicely works for $n = 8$ in conjunction with Smale's generic regularity theorem and, probably, for all n with Lohkamp's smoothing theorem.

But it is not quite clear how to make this work for non-smooth limits of smooth metrics.

For instance,

let g_i be a sequence of Riemannian metrics on the torus $\mathbb{T}^n$, such that

$$Sc(g_i) \geq -\varepsilon_i \underset{i\to\infty}{\to} 0$$

and such that g_i uniformly converge to a continuous metric g.

Is this g, say for $n \leq 7$, Riemannian flat?[379]

(The above argument shows that, given an indivisible $(n-1)$-homology class in $\mathbb{T}^n$, there exists a foliation of $\mathbb{T}^n$ by g-minimal submanifolds from this class. But it is not immediately clear how to show that these submanifolds are totally geodesic.)

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$$Sc(g_i) \geq -\varepsilon_i \underset{i\to\infty}{\to} 0$$

and such that g_i uniformly converge to a continuous metric g.

[379]Yes, according to [Burkhart-Guim(regularizing Ricci flow) 2019].

The above argument shows that, given an indivisible $(n-1)$-homology class in $\mathbb{T}^n$, there exists a foliation of $\mathbb{T}^n$ by g-minimal submanifolds from this class, but it is not clear how to show that these submanifolds are totally geodesic, that is needed for the proof of flatness of g,

Yet,

the Ricci flow argument from [Burkhart-Guim(regularizing Ricci flow) 2019] does show the metric g is flat.

5.8 Capillary Surfaces: μ-Bubbles with Measures μ_∂ Supported on Boundaries

In order to extend extremality and other results to more general manifolds with boundaries, such, e.g. as conical domains in $\mathbb{R}^n$, one shouldn't limit oneself to the definition of a μ bubble from section 5, where the admissible measures μ on X are of the form $\mu(x)dx$ for *continuous functions* $\mu(x)$.

In fact the definition of μ-bubbles makes sense for more general measures, where a geometrically interesting case is that of a manifold X with boundary, here denoted $S = \partial X$, and our measure is of the form $\mu_\bullet(x)dx + \mu_\partial(c)ds$, where $\mu_\bullet$ and μ_∂ are continuous (or measurable) functions on X and on $S = \partial X$, and where we let

$$|\mu_\partial| < 1$$

for a reason that becomes clear later on.

Let $\mathcal{Y}$ be the set of cooriented hypersurfaces $Y \subset X$ with boundaries contained in S,

$$Z = \partial Y \subset S = \partial X,$$

where the unit field normal to Y, which defines the coorientation is called the *upward* field and denoted $\nu = \nu_{Y\uparrow}$ and let

$$\mu = \mu_\bullet(x)dx + \mu_\partial(s)ds.$$

Then a hypersurface $Y \in \mathcal{Y}$ is called a *μ-bubble* (compare 5.1), if it is extremal or, at least, stationary for

$$Y \mapsto vol_{n-1}^{[-\mu]}(Y) =_{def} vol_{n-1}(Y) - \mu(X_<),$$

where $X_< \subset X$ is the region in X "below" $Y \subset X$, where

$$\mu(X_<) = \int_{X_<} \mu_\bullet(X)dx + \int_{S_<} \mu_\partial(s)ds$$

for

$$S_< = S \cap X_< \subset S = \partial X.$$

This kind of (2-dimensional) Y for constant functions $\mu_\bullet$ and μ_∂ are called *capillary surfaces.*

An essential for our geometric purposes feature of such surfaces and of $(\mu_\bullet+\mu_\partial)$-bubbles in general is a particular algebraic property of the second variation formula for stationary Y, similar to that for μ-bubles with continuous $\mu(x)$ on manifolds without boundary that is proved and used at the beginning of section 5; the derivation of this formula for capillary surfaces was given in [Ros-Souam(capillary) 1997] and used in [Li(comparison) 2017] for the proof of extremality of certain polyhedra. [380]

In what follows we present a geometrically transparent derivation of this formula with an eye on further applications. [381]

Example. Let

$$X = B^n \subset \mathbb{R}^n = \mathbb{R}^{n-1} \times \mathbb{R}, \ S = \partial B^n = S^{n-1},$$

be the unit ball, with the boundary sphere $S = S^{n-1}$ and let $Y_t = Y_t^{n-1} \subset B^n$, be the horizontal discs, that are the intersections

$$Y_t = B^n \cap (\mathbb{R}^{n-1} \times \{t\}), \ -1 < t < 1.$$

Let

$$\angle_t = \angle_{Z_t}(Y_t, S^{n-1})$$

be the dihedral angles between the hypersurfaces Y_t and S^{n-1} along their intersection

$$Z_t = \partial Y_t = Y_t \cap S^{n-1}$$

where, we agree that this angle is measured "below" Y_t; thus $\angle_{-1} = 0$ and $\angle_1 = \pi$, i.e. it is related to the height $t = t(s)$ of $Z_t \subset S^{n-1}$ by

$$t = \cos(\angle_t - \pi/2).$$

Next, let $\mu_\bullet = 0$ and let

$$\underline{\mu}_\partial = \underline{\mu}_\partial(s) = cos \angle_{t(s)},$$

where t is regarded here as the height function $t : S^{n-1} \to [-1, 1]$ for $S^{n-1} \subset \mathbb{R}^{n-1} \times [-1, 1] \subset \mathbb{R}^{n-1} \times \mathbb{R}$.

Then the *normal derivative* $\partial_\nu = \frac{d}{dt}$ of the volume of the discs $Y_t \subset B^n$ is expressed in terms of

$$|Z_t| = vol_{n-2}(Z_t), \text{ and the angle } \angle_t \in (0, \pi)$$

as follows

$$\partial_\nu vol_{n-1}(Y_t) = |Z_t| \cot \angle_t,$$

while the derivative of the $\underline{\mu}_\partial$-measure of the region $S_{\leq t} \subset S^{n-1}$ below $Z_t = \partial Y_t \subset S^{n-1}$ for $\mu = \underline{\mu}_\partial(s) = cos \angle_{t(s)}$ is

$$\partial_\nu \underline{\mu}_\partial(S_{\leq t})) = \frac{|Z_t| \underline{\mu}_\partial(s)}{\sin \angle_t} = |Z_t| \cot \angle_t.$$

Thus,

[380] Necessary existence and regularity of capillary hypersurfaces follow from [Simon-Spruck(capillary) 1976], [Gerhard(capillarity) 1976], [Liang(capillarity) 2005], [Philippis-Maggi(capillary) 2015] as it is indicated in [Li(comparison) 2017] and [Li(rigidity) 2019].

[381] The first version of this manuscript contained a computational error that lead to a most disappointing conclusion. I am thankful to Mike Anderson who encouraged me to double check my computation.

the derivatives of $vol_{n-1}(Y_t)$ and of $\underline{\mu}_\partial(S_{\leq t})$ by the field $\psi\nu$ for all C^1-smooth functions $\psi = \psi(y)$, $y \in Y_t$ satisfy

$$\partial_{\psi\nu} vol_{n-1}(Y_t) = \partial_{\psi\nu}\underline{\mu}_\partial(S_{\leq t})$$

and

$$\partial_{\psi\nu} vol_{n-1}^{[-\mu]}(Y) = 0,$$

which says that

Y_t are μ-bubbles for this $\mu = \underline{\mu}_\partial(s)$,

since they *are* stationary for the the functional $Y \mapsto vol_{n-1}^{[-\mu]}(Y)$.

Exercise. Let

$$Y_\rho \subset B^n(1) = \{x \in \mathbb{R}^n\}_{||x\leq 1||},\ 0 < \rho < 2,$$

be the intersections of the concentric ρ-spheres $S^{n-1}_{x_0}(\rho) \subset \mathbb{R}^n$ around $x_0 = (0,0,..,-1) \in \mathbb{R}^n$ with the unit ball $B^n \subset \mathbb{R}^n$.

Determine the measure $\mu = \mu_\bullet(x)dx + \mu_\partial(s)dy$, for which these Y_ρ serve as μ-bubbles.

Let us return to the general Riemannian manifold X with boundary $S = \partial X$, a hypersurface $Y \subset X$, where $Z = \partial Y \subset S = \partial X$ and let μ be a measure of the form $\mu = \mu_\bullet(x)dx + \mu_\partial(s)ds$ for continuous functions $\mu_\bullet(x)$ on X and $\mu_\partial(s)$ on S.

First Variation Formula for $vol_{n-1}^{[-\mu]}(Y)$. In order to define ∂_ν one needs to choose *a vector field* extending the "upward" normal field to Y, denoted, as earlier, ν, *from Y to a neighbourhood of Y* in X, that we do as follows.

Smoothly extend X beyond its boundary by a slightly greater Riemannian manifold X_+ of the same dimension and extend Y by a hypersurface $Y_+ \subset X_+$.

Move Y_+ in both normal directions by distance $|t|$ to $Y_{+,t} \subset X_+$, $-\varepsilon \leq t \leq \varepsilon$ for a small $\varepsilon > 0$ and let $Y_t \subset X$ be the intersection of the so moved Y_+ with $X \subset X_+$

$$Y_t = Y_{+,t} \cap X \subset X,$$

where, observe, $Y_0 = Y$, and where $Y_{\pm t}$ are t-equidistant hypersurfaces to Y_0 in X, except, maybe, for the $|t|$-neighbourhood of $S = \partial X$, where we agree that Y_t with $t < 0$ lies below Y i.e. in the domain $X_< \subset X$ and $Y_{t>0}$ are positioned over Y in X.

Now ∂_ν is understood as $\frac{d}{dt}|_{t=0}$ and $\partial_{\psi\nu}$ and $\partial^2_{\psi\nu}$ are understood accordingly.

Let $\angle_z \in (0,\pi)$ denote the angle between (the tangent spaces of) Y and S at $z \in Z = \partial Y = Y \cap S$ measured below Y, i.e. in $X_<$.

Then, clearly, for all smooth functions $\psi = \psi(y)$,

$$\partial_{\psi\nu} vol_{n-1}(Y) = \int_Y \psi(y)\cdot mean.curv(Y,y)dy + \int_Z \psi(z)\frac{\cos\angle_z}{\sin\angle_z}dz,$$

$$\partial_{\psi\nu}\mu_\bullet(Y) = \int_Y \psi(y)\cdot\mu_\bullet(y)dy$$

and

$$\partial_{\psi\nu}\mu_\partial(S_-) = \int_Z \frac{\psi(z)}{\sin\angle_z}\mu_\partial(z)dz,$$

where $\mu(S_<)$ stands for $\int_{S_<}\mu_\partial(s)ds$, where the "$Y$"-integrals are the ones we met earlier in section 5 for X without boundary and where the shape of the Z-integrals can be seen by looking at the above example.

Thus, the first variation $\partial_{\psi\nu} vol_{n-1}^{[-\mu]}(Y)$ equals the sum of two integrals, one over Y and the other one over $Z = \partial Y$,

$$\int_Y \psi(y)(mean.curv(Y,y) - \mu_\bullet(y))dy + \int_Z \psi(z)\left(\frac{\cos\angle_z}{\sin\angle_z} - \frac{1}{\sin\angle_z}\mu_\partial(z)\right)dz,$$

Therefore, $\partial_{\psi\nu} vol_{n-1}^{[-\mu]}(Y)$ vanishes for all smooth functions $\psi(y)$ and Y *is a (stationary) μ-bubble, if and only if*

$$mean.curv(Y) = \mu_\bullet \text{ and } \cos\angle_z = \mu_\partial(z),\ z \in Z = \partial Y.$$

∂Y-Contribution to the Second Variation Formula for $vol_{n-1}^{[-\mu]}(Y)$. Let us compute the second derivative (variation) $\partial_{\psi\nu}\partial_{\psi\nu} vol_{n-1}^{[-\mu]}(Y)$ on a *stationary* $Y = Y_0$, where the first variation vanishes and, thus,

$$\cos\angle_z = \mu_\partial(z).$$

To make it clear, we do it for the normal deformation Y_t of $Y = Y_0$ with $\psi = 1$, we ignore the contribution from the Y-integral and observe, that because of the identity $\cos\angle_z = \mu_\partial(z)$ on Y_0, the only non-zero term in the (Leibniz formula for the) derivative $\partial_\nu - \frac{d}{dt}$ of the above Z-integral is

$$\int_Z \frac{1}{\sin\angle_z}(\partial_\nu \cos\angle_z - \partial_\nu\mu_\partial(z))dz = -\int_Z \partial_\nu\angle_z dz + \partial_\nu\mu_\partial(z)dz,$$

where the derivative of the angle $\angle_z = \angle_z(t)$ is determined as follows.

Intersect $S = \partial X$ and $Y = Y_0$ with (a germ at z of) a surface $E_z \subset X_+ \supset X$, which is normal to $Z = S \cap Y = \partial Y$ at $z \in Z$ and is geodesic at z, e.g. being the image of the local exponential map from the normal plane $T_z^\perp(Z) \subset T_z(X) = T_z(X_+)$ to X_+, where X_+ is the above extension of X.

Let

$$\underline{Y} = \underline{Y}(z) = Y \cap E_z \text{ and } \underline{S} = \underline{S}(z) = S \cap E_z$$

be the intersection curves in this surface E_z, where we identify E_z with a small ball in the Euclidean plane $\mathbb{R}^2 = T_z^\perp(Z)$ and let $\underline{Y}_t \subset \mathbb{R}^2$ be t-equidistant curves to $\underline{Y} = \underline{Y}_0$.

Let $z_t = \underline{Y}_t \cap \underline{S}$, (thus $z_0 = z_{t=0} = z$) let $y_t \in \underline{Y}_0$ be the *normal projection of* $z_t \in \underline{Y}_t$ to $\underline{Y}_0$, which means that the straight segment $[y_t, z_t] \subset \mathbb{R}^2$ is normal to the curve $\underline{Y}_0$, (also normal to $\underline{Y}_t$ and having length $|t|$, since Y_t is equidistant to Y_0).

There are two summands that contribute to the difference $\angle_{z_t} - \angle_{z_0}$ between the angles between our curves at their intersection points.

(1) The first summand is due to the turn of the tangent lines to $\underline{S}$ along the segment $\underline{S}_{z_0,z_t} \subset \underline{S}$ between the points z_0 and z_t, which is equal to the integral of the curvature κ_S of S over this (curved) segment, where

$$\int_{\underline{S}_{z_0,z_t}} \kappa_{\underline{S}}(\underline{s})d\underline{s} = \kappa_{\underline{S}}(\underline{z}_0)|z_0 - z_t| + o(t) = \kappa_{\underline{S}}\cdot(\underline{z}_0)\frac{1}{\sin\angle_{z_0}} + o(t),\ t \to 0.$$

(2) The second contribution to $\angle_{z_t} - \angle_{z_0}$ comes from the curvature of the curve $\underline{Y} = \underline{Y}_0$ integrated over the segment $\underline{Y}_{y_t,z_0} \subset \underline{Y}$,

$$\int_{\underline{Y}_{y_t,z_0}} \kappa_{\underline{Y}}(\underline{y})d\underline{y} = \kappa_{\underline{Y}}(z_0)\cot\angle_{z_0} + o(t).$$

Summing up, the normal derivative of the the Z-integral in the first variation formula is expressed in terms of the curvatures of Y and S and the angle between them as follows.

$$\partial_\nu \int_Z \left(\frac{\cos\angle_z}{\sin\angle_z} - \frac{1}{\sin\angle_z}\mu_\partial(z)\right)dz = -\int_Z \frac{\kappa_{\underline{S}(z)}(z)}{\sin\angle_z}dz + \kappa_{\underline{Y}(z)}(z)\cdot\cot\angle_z dz + \partial_\nu\mu_\partial(z)dz,$$

where, recall,

(i) S is the boundary the ambient n-manifold X,

(ii) $Y \subset X$ is a hypersurface with boundary $Z = \partial Y = Y \cap S$,

(iii) $\underline{S}(z) \subset S$ and $\underline{Y}(z) \subset Y$ are intersections of S and Y with a germ of a surface $E_z \subset X$ normal to Z at z and geodesic at z[382], where $\kappa_{\underline{S}}$ and $\kappa_{\underline{Y}}$ denote the curvatures of these curves in E_z with the following sign convention:

($\pm$) *if the boundary $S = \partial X$ is convex, then $\kappa_{\underline{S}} \geq 0$; if the the "lower region" $X_< \subset X$ bounded by Y is convex, then $\kappa_{\underline{Y}} \geq 0$.*

(iv) $\angle_z$ is the angle between the tangent spaces $T_z(S)$ and $T_z(Y)$ in $T_z(X)$, which is measured in $X_<$ *under* Y,[383]

(v) the above formula is supposed to hold if Y is *stationary* for the functional

$$Y \mapsto vol_{n-1}^{[-\mu]}(Y) = vol_{n-1}(Y) - \int_{X_<} \mu_\bullet(x)dx - \int_{S_<} \mu_\partial(s)ds$$

where $\mu_\bullet(x)$ and $\mu_\partial(s)$ are continuous functions on X and on S and where $X_< \subset X$ is the just mentioned "lower region" in X bounded by Y and $S_< = S \cap X_<$.

The above formula for $\partial_\nu \int_Z \ldots$ can be neatly rewritten in terms of the mean curvatures M_S of S, M_Y of Y and M_Z of Z in Y by invoking the following.

Algebraic Identity.[384]

$$M_Z(z) = \frac{M_S(z)}{\sin\angle_z} + (\cot\angle_z)M_Y(z) - \frac{\kappa_{\underline{S}(z)}(z)}{\sin\angle_z} - (\cdot\cot\angle_z)\cdot\kappa_{\underline{Y}(z)}(z).$$

(What is significant is that the coefficients on the right hand side here are the same as in the above expression for the Z-term in the second variation formula.)

To prove this identity, let us express everything in terms of the traces second fundamental forms II_S, II_Y and II_Z, where

II_S is taken with outward normal field denoted $\nu_S = \overleftarrow{\nu}_S^\perp$

II_Y is taken with the upward field $\nu = \nu_Y = \nu_{Y\uparrow}^\perp$

and where

II_Z will be evaluated with two unit normal fields to Z, one of them ν_Y restricted to Z and the other one is tangent to Y and facing outward, call it $\nu_Z = \overleftarrow{\nu}_Z^\perp$. (If Y is *normal* to S at z, i.e. $\angle_z = \pi/2$, then $\nu_Z(z) = \nu_S(z)$.)

Observe that

ν_Y is *normal* to ν_Z and that

the angle between ν_S and ν_Y is *complementary* to the angle $\angle_z$ between S and Y at all $z \in Z = S \cap Y$,

$$\angle_z(\nu_S, \nu_Y) = \pi - \angle_z.$$

[382] It is better, as earlier, to think of E_z in $X_+ \supset X$ and take the image of the exponential map from a small disc in the normal plane $T_z^\perp(Z) \subset T_z(X_+)$ to X_+ for this E_z.

[383] This "under", together with ($\pm$), determines the signs of the integrants in the above formula that is crucial for our (potential) applications.

[384] Compare with 3.8 in [Li(comparison) 2017].

Therefore,

$$\nu_S(z) = (\sin \angle_z) \cdot \nu_Z(z) - (\cos \angle_z) \cdot \nu_Y(z)$$

and, by the linearity of the form II_Z and its trace in the normal vectors,

$$trace_Z(\mathrm{II}_S) = (\sin \angle_z) \cdot trace_{\nu_Z}(\mathrm{II}_Z) - (\cos \angle_z) \cdot trace_Z(\mathrm{II}_Y),$$

or

$$[1/\sin] \qquad M_Z = trace_{\nu_Z}(\mathrm{II}_Z) = \frac{1}{\sin \angle_z} trace_Z(\mathrm{II}_S) + (\cot \angle_z) \cdot trace_Z(\mathrm{II}_Y),$$

where $trace_Z(\mathrm{II}_S)$ denotes the trace of the restriction of the form II_S to (the tangent bundle of) $Z \subset S$, that is the same as the trace of the form II_Z with respect to the vector field ν_S restricted to Z, where $trace_Z(\mathrm{II}_Y)$ is understood similarly and where $trace_{\nu_Z}(\mathrm{II}_Z)$ denotes the trace with respect to the field ν_Z, where, indeed, it is equal to the mean curvature M_Z of Z in Y,

$$trace_{\nu_Z}(\mathrm{II}_Z) = M_Z.$$

Finally, we observe that the curvatures of the curves $\underline{S}(z)$ and $\underline{Y}(z)$ in E_z are equal to the traces of the form II_S and II_Y restricted to $\underline{S}(z) \subset S$ and $\underline{Y}(z) \subset Y$,

$$\kappa_{\underline{S}(z)}(z) = trace_{\underline{S}(z)}\mathrm{II}_S(z) \text{ and } \kappa_{\underline{Y}(z)}(z) = trace_{\underline{Y}(z)}\mathrm{II}_Y(z),$$

while

$$trace_{\underline{S}(z)}\mathrm{II}_S(z) + trace_Z\mathrm{II}_S(z) = trace_S\mathrm{II}_S(z) = M_S(z)$$

and

$$trace_{\underline{Y}(z)}\mathrm{II}_Y(z) + trace_Z\mathrm{II}_Y(z) = trace_S\mathrm{II}_Y(z) = M_Y(z)$$

These, combined with [1/ sin] yield the required algebraic identity which we write now as

$$-\frac{\kappa_{\underline{S}(z)}(z)}{\sin \angle_z} - (\cdot \cot \angle_z) \cdot \kappa_{\underline{Y}(z)}(z) = M_Z(z) - \frac{M_S(z)}{\sin \angle_z} + (\cot \angle_z) M_Y(z)$$

Substitute this into the above formula for the ∂_ν-derivative of the integral $\int_Z ...dz$ in the first variation formula for $vol^{[-\mu]}_{n-1}(Y)$ and express this derivative by the following

Mean Curvature Stability Relation.

$$\partial_\nu \int_Z ...dz = \int_Z \left(M_Z(z) - \frac{M_S(z)}{\sin \angle_z} + (\cot \angle_z) n \cdot M_Y \right)(z) - \partial_\nu \mu_\partial(z) dz.$$

Thus, for instance, if $\mu_\partial(z)$ is *constant* and Y is a *local minimizer* for $vol^{[-\mu]}_{n-1}(Y)$, then, this formula, which necessarily holds for the integrals over all subdomains $U \subset Y$, shows that

$$[\geq] \qquad M_Z(z) - \frac{M_S(z)}{\sin \angle_z} + (\cot \angle_z) n \cdot M_Y \geq 0,$$

which us most informative (and quite useful) if $\mu_\bullet = M_Y = 0$.

About the Signs. Consistency in the choices of signs in the definitions of the curvatures, and/or of normal vectors for the second fundamental forms is crucial for applications.

There is hardly a problem here with the sign of the $M_S/\sin$-term, since it is clearly visible by looking at the case where Y is *normal* to S, i.e. $\angle_z = \pi/2$; it is also instructive to go through the full calculation in the following.

Example/Exercise 1. Let $f(t) > 0$, $0 < t < \infty$ be a smooth function and $X \subset \mathbb{R}^{n-1} \times \mathbb{R}_+$ be the rotation body of the subgraph of the function f (i.e. the region below the graph) around the t-axes.

Let $\mu_\bullet = 0$ and μ_∂ is a constant, say $\mu_\partial(t) = c$.

Let $Y_t =\subset X$ be the $(n-1)$-balls of radii $f(t)$ normal to the t-axes and let us compute the the second variation, of $vol_{n-1}^{[-\mu]}(Y_t)$, at t where this ball is stationary, that is the second derivative

$$(bR^{n-1} - c \cdot vol_{n-1}(S_{<t})'',$$

where $b = b_{n-1}$ denotes the volume of the unit ball $B^{n-1} \subset \mathbb{R}^{n-1}$ and where $S_{<t} \subset S = \partial X$ is the part of this boundary below (or to the left from) t, where the stationary Y_t is where the first derivative vanishes, i.e.

$$(bR^{n-1} - c \cdot vol_{n-1}(S_{<t}))' = (n-1)bR'R^{n-2} - bc(n-1)R^{n-2}\sqrt{1+R'^2} = 0,$$

that is

$$\sqrt{1+R'^2} = R'/c.$$

Then an elementary calculation show that

$$(bR^{n-1} - c \cdot vol_{n-1}(S_{<t})'' = -vol(S^{n-2}(R))\frac{k_f(t)}{\sin \angle_t} = -vol_{n-2}(\partial Y_t)\frac{k_f(t)}{\sin \angle_t},$$

for $k_f(t)$ being the curvature of the graph of $f(t)$ which, according to our convention, is *negative* for $f(x) = x^2$, since the subgraph of x^2 is concave.

The computation becomes messier if Y is non-flat but it still durable in simple cases.

Example/Exercise 2. Let f and X be as above and let $Y_t \subset X$ be the intersections of X with the spheres of radii t centered at $\mathbf{0}$ that is the zero on the z-axes, where we assume that the intersections Z_t of these t-spheres with $S = S_f = \partial X$ are *non-empty connected transversal* for $t \geq 1$.

Let $\mu_\bullet(t)$ be equal to the mean curvature of Y_t, i.e. $\mu_\bullet(t) = \frac{n-2}{t}$ and let μ_∂ be constant, denoted $\mu_\partial = c$.

We invite the reader to evaluate the second variation of $vol_{n-1}^{[-\mu]}(Y_t)$ at stationary Y_t.

5.8.1 Capillary Warped Products Inequalities

Most (all?) extremality/rigidity properties of warped products proved in the earlier sections, as well as Gauss-Bonnet kind inequalities from the following section, have their counterparts for manifolds with boundaries, which are proven with "capillary" μ-bubbles with a use of the above inequality [≥].

We formulate below a a few examples and postpone a more thorough analysis and applications, e.g. to manifolds with corners [385] until another occasion.

[385] See [Li(comparison) 2019] in this regard.

Spherical Suspension Inequality. Let $\underline{X}_0 \subset S^{n-1} \subset S^n$ be a smooth convex domain in the equatorial sphere $S^{n-1} \subset S^n$ and let $\underline{X}_{\pm 1} = \underline{X}_{\pm 1}(r) \subset S^n$ be the the union of the geodesic segments between the north and the south poles of S^n through this domain. Remove the poles $\pm 1 \in \underline{X}_{\pm 1} \subset S^n$ from $\underline{X}_{\pm 1}$ and denote

$$\underline{X} = \underline{X}_{\pm 1} \setminus \{-1, +1\} \subset S^n \setminus \{-1, +1\}$$

Let X be a (non-compact) Riemannian manifold with a boundary and let $f : X \to \underline{X}$ be a smooth proper (boundary-to-boundary, infinity-to-infinity) 1-Lipschitz map $f : X \to \underline{X}$ of non-zero degree.

Let the scalar curvature of X is bounded from below by

$$Sc(X) \geq n(n-1) = Sc(S^n).$$

If X is spin, if $n \leq 7$, and if either n is odd or $\underline{X}_0$ is a ball,[386] *then there exists a point $x \in \partial X$, where the mean curvature of X at x is bounded by that of $\underline{X}$ at $f(x)$,*

$$mean.curv(X, x) \leq mean.curv(\underline{X}, f(x)).$$

Moreover,

if $mean.curv(X, x) \geq mean.curv(\underline{X}, f(x))$ at all $x \in X$, then $mean.curv(X, x) = mean.curv(\underline{X}, f(x))$ and the map f is an isometry.

About the Proof. The "right sign" [≥] at the boundary, allows carrying through the μ-bubble argument from 5.5 in the proof of the extremality/rigidity of double punctured spheres. This reduces the problem to the ⬬-comparison theorem for $Sc(V) \geq 0$ of log-concave warped products from section 3.1 and the proof follows.

About $n \geq 8$. Probably, the case $n = 8$ follows by a Natan Smale's kind of perturbation argument and if $n \geq 9$ generalizations of Lohkamp's and/or of Schoen-Yau's arguments would work, but singularities at capillary boundaries need additional care.

About Corners. The above theorem remains valid for non-smooth domains $\underline{X}_0 \subset S^{n-1}$ with properly understood generalized mean curvature, e.g. for convex k-gons in S^2, where the proof can be obtain by properly smoothing the corners.(See the next section.)

In general, the density function $\mu\partial(s)$ on the boundary $S = \partial X$ from the previous section may be (and typically is) discontinuous along the corners. In this case the smoothing argument introduces an unpleasant error and the behaviour of μ_∂-bubbles at the corners needs an additional care.[387]

Spin-Extremality of Doubly Punctured Balls. Let X be a compact manifold with *non-negative scalar curvature* and a *mean convex boundary* $S = \partial X$, let $P_-, P_+ \subset \partial X$ be two closed subsets and let $f : S \to \underline{S} = S^{n-1}$ be a 1-*Lipschitz* map of *non-zero degree* , such that the subsets $P_\pm$ *go to the North and the South poles* of the unit sphere $\underline{S} = S^{n-1} = \partial B^n \subset \mathbb{R}^n$.

If X is spin and if $n = dim(X) \leq 8$,[388] *then the mean curvature of $S = \partial X$*

[386] Probably these "if" are unnecessary.

[387] In the case of $\mu_\partial(s)$ constant on the faces of 3-dimensional domains, the proof of $C^{1,\alpha}$-regularity of capillary surfaces at the corners is indicated in [Li(rigidity) 2019], see next section.

[388] Both conditions are, probbaly, redundant, where dropping the the latter could be possible with the recent Lohkamp's techniques, while removing the former remains beyond the range of the present day knowledge.

outside P_- and P_+ can't be greater than that of S^{n-1},

$$\inf_{s\in S\smallsetminus(P_-\cup P_+)} mean.curv(S,s)\leq n-1.$$

Sketch of the Proof. Let $\underline{\mu}_\partial(\underline{s}) = \cos\angle_{t(\underline{s})}$, $\underline{s}\in\underline{S} = S^{n-1}$, be the function on S^{n-1} as in example from the previous section (where $\angle_{t(\underline{s})}$ are the angles between S^{n-1} and the (parallel) hyperplanes $\mathbb{R}^{n-1}\times\{t\}$) and let $\mu_\partial = \mu_\partial(s)$ be the composed function

$$S\overset{f}{\to}\underline{S}\overset{\underline{\mu}_\partial}{\to}\mathbb{R} \text{ on } S=\partial X, \text{ that is } \mu_\partial(s) = \underline{\mu}_\partial(f(s),\ s\in S.$$

.

Then, arguing as in the proof of the double puncture theorem for spheres S^n in sections 3.9 5.5, we conclude to the existence of a stable μ-bubble $Y\subset X$ for $\mu=\mu_\partial ds$, which is smooth up to the boundary for $n\leq 8$. (if $n=8$ one needs a version of Nathan Smale's generic regularity theorem.)

Next, the mean curvature stability relation from the previous section show that the mean curvature $M=M_Z$ of the boundary $Z=\partial Y\subset Y$ in Y and the norm of the differential of the natural map $\phi: Z\to S^{n-2}$ satisfy the inequality

$$\frac{M(Z,z)}{||d\phi(z)||}\geq n-2.$$

(Hopefully, there is no silly error in the computation)

Finally, the mean curvature spin-extremality theorem [389] from section 3.5 applies to $\mathbb{T}^\rtimes$-stabilized Y, that is to $Y\rtimes\mathbb{T}_1$, and the proof follows.

Remarks. (a) It is not hard to prove, as in the cases we encountered earlier, that the balls are rigid in this regard:

if

$$\inf_{s\in S\smallsetminus(P_-\cup P_+)} mean.curv(S,s) = n-1,$$

then X is isometric to B^n.

(b) The above argument generalizes to *complete* non-compact manifolds X, but, probbaly, completeness can be replaced by a weaker condition.

Corollary to the Proof:] Multi-Width Mean Curvature Inequality for non-Spin Manifolds.

Let X be a compact Riemannin n-manifold with a boundary, and let f be a continuous map from ∂X to the boundary of the n-cube with *non-zero degree,*

$$f:\partial X\to\partial[-1,1]^n,$$

such that the *distances between the pullbacks of the opposite faces of the cube are all $\geq\pi$,*

$$dist(\partial_{-,i},\partial_{+,i})\geq\pi,\ \ i=1,...,n.$$

[389]One needs here this theorem for maps to the convex hypersurfaces not only in Euclidean spaces but also in other Riemannian flat manifolds, specifically in $\mathbb{R}^m\times\mathbb{T}^N$ in the present case.

If X has non-negative scalar curvature, $Sc(X) \geq 0$, and if $n = dim(X) \leq 8$[390] *then*

$$\inf_{x \in \partial X} mean.curv(\partial X, x) \leq n - 1.$$

Proof. Apply the above argument to the f-pullbacks of a pair of opposite faces of the cube, say to

$$P_{\mp} = f^{-1}(\partial_{\mp,1}) \subset \partial X$$

let $Y \subset X$ be the corresponding stable μ-bubble which separates P_- from P_+. Then apply the same argument to $Y \rtimes \mathbb{T}^1$ and continue inductively as in the proof of the multi-width $\square^n$-Inequality $\sum_{i=1}^n \frac{1}{d_i^2} \geq \frac{n^2}{4\pi^2}$ in section3.8 thus reducing the problem to the case of $X^2 \rtimes \mathbb{T}^{n-2}$, where X^2 is a surface with $curv(\partial X^2) \geq 1$, where $Sc(X^2 \rtimes \mathbb{T}^{n-2}) \geq 0$, and where the proof follows by the proof of the mean curvature spin-extremality theorem.

Remarks (a) The above inequality improve non fill-is results 4(A) and (5) in section **??**

(b) If X is spin this inequality follows from the mean curvature spin-extremality theorem.

(c) One can improve this inequality with a better (iterated) warped product model manifold $\underline{S}$, and, probbaly, with the best such $\underline{S}$ the improved inequality, will not follow from the mean curvature extremality of spheres, even for spin manifolds S. (It is not impossible, that the the sphere S^{n-1} radially mapped to $\partial[-1,1]^n$ is extremal this inequality.)

Capillary Mean Curvature Separation Theorem. Let X be a compact manifold with boundary $S\partial X$ and let , let $P_-, P_+ \subset \partial X$ be two closed subsets sich that,

$\bullet_\sigma$ the scalar curvature of X is bounded from below by a given non-positive number,

$$Sc(X) \geq \sigma, \ \sigma \leq 0;$$

$\bullet_M$ the values of mean curvature of S in the subset $P_- \subset S$ and in its complement are bounded from below as follows

$$mean.curv(S,s) \geq M_-, \ s \in P_- \text{ and } mean.curv(S,s) \geq M_+, \ s \in S \setminus P_-,$$

for some M_-, M_+, where M_+ is *positive* while M_- can be *negative*.

Let M_+ be bounded from below in terms of $-\sigma$ and $-M_-$ according to the following inequality

$$M_+^2 \geq \max\left(\frac{n-1}{-n\sigma}, -M_-\right),$$

and let the distance D between P_- and P_+ measured in S, with respect to the induced Riemannian metric in $S \subset X$ be bounded in terms of M_+ as follows,

$$D \geq const_n \frac{1}{M_+} \text{ for } const_n \geq 100\pi.$$

Let $M' > 0$ be a given number and let the numbers = M_+ and D be *sufficiently large* depending on n, σ, M_- and M' – specific inequalities are indicated below.

[390] One can drop this if one extend Schoen-Yau's "desingularization" theorem for capillary hypersurfaces

Then, assuming $n = dimX \leq 8$, there exists a smooth compact hypersurface $Y \subset X$ with boundary $\partial Y \subset S = \partial X$, such that the mean curvature of the boundary of Y is bounded from below by $\frac{1}{2}M_+$,

$$mean.curv(\partial Y) \geq \frac{1}{2}M_+,$$

and the scalar curvature of some warped $\mathbb{T}^{\rtimes}$-extension of Y is non-negative,

$$Sc(Y \rtimes \mathbb{T}^1) \geq 0.$$

Sketch of the Proof. Let

$$\mu = \mu_\bullet(x)dx + \mu_\partial(s)dx$$

where $\mu_\bullet(x) = M_+$ and where $\mu_\partial(s)$ is "induced" as earlier from the function $\underline{\mu}_\partial(\underline{s}) = \cos_{t(\underline{s})}$ on S^n by a $\frac{\pi}{D}$-Lipschitz map $S \to S^{n-1}$, which sends P_+ to the North pole and P_+ to the South pole of S^{n-1}. (The existence of such a map is obvious.)

Then our conditions on the mean curvatures guarantee the existence of a stable μ-bubble $Y \subset X$, which separates P_- and P_+ and has (free) boundary in S and where the second variation formula along with the mean curvature stability relation from the previous section imply the desired properties of this Y.

Remarks. (i) Our bound on D is very rough. We suggests the reader would find a better estimate.

(ii) If one takes into account, besides $D = dist_S((P_-, P_+)$, the distance $d = dist_X(P_-, P_+)$ then, one can prove, with some $\mu_\partial ds + \mu_\bullet(x)dx$ for a suitable function $\mu_\bullet(x)$, a comprehensive separation theorem incorporating the above with ||| from section 3.7 for closed manifolds.

Problem. Find adequate version of the "log-convexity" condition on $\mu_\partial ds + \mu_\bullet(x)dx$ and find all "interesting" sharp capillary extremality/rigidity inequalities including all such inequalities presented in the previous sections.

5.9 3D Gauss Bonnet Inequalities

The simplest inequality of this kind, which applies to closed connected cooriented stable minimal surfaces Y in orientable Riemannian 3-manifolds $X = (X, g)$, is a bound on the integral of the scalar curvature of X over Y, that reads:

$$(A) \qquad \int_y Sc(X, y)dy \leq 8\pi,$$

where the equality holds for Riemannian products $Y_0 \times S^1$, for sufaces $Y_0 = (Y_0, h_0)$ homeomorphic to S^2. (Compare with *area exercises* in section 2.7.) *Proof.* Combine the inequality $[\star\star]$ involved in the second variation formula (section 2.5) with the Gauss-Bonnet theorem.

Corollary. If X is compact with $Sc(X) > 0$, then the 2-systole of X with the metric $g^*(x) = Sc(g, x)g(x)$ satisfy is bounded by 8π. Moreover

The 2-dimensional homology of X admits a basis represented by closed surfaces $Y \subset X$ with $area_{g^*}(Y) \leq 8\pi$.

Question. Can one directly bound the areas of g^*-minimal surfaces in X?

Using the Dirac Operator. Let us give a Dirac theoretic proof of this corollary, where, observe, this is the only known case where the Dirac operator goes in parallel with minimal surfaces.

To simplify, let X be homeomorphic to $S^2 \times S^1$ and to keep track of constants let us compare the metric g_* on this X with the Riemannian product $\underline{X} = (\underline{X}, \underline{g})$ of the circle S^1 the unit sphere S^2 with it's usual metric with $Sc = 2$.

Let $X^4 = X \times S^1$ and $\underline{X}^4 = \underline{X} \times S^1$ be the corresponding 4-manifolds, where the Dirac operator will be employed, let $\underline{L}^* \to \underline{X}^4$ be the line bundle induced from the Hopf bundle by the natural map $\underline{X}^4 \to S^2$ and let $\underline{L}^\circ_\varepsilon \to X^4$ be an ε-flat bundle induced by the natural map $X^4 \to S^1 \times S^1$ from an ε-flat bundle $L_\varepsilon \to S^1 \times S^1$, such that the first Chern class of L_ε doesn't vanish

Then the twisted Dirac $\mathcal{D}_{\otimes L^* \otimes L^\circ}$ on X^4 has *non-zero index*, and this nonvanishing of $ind(\mathcal{D}_{\otimes L^* \otimes L^\circ})$ persists for all Riemannian metrics c' on X^4.

On the other hand, if a line bundle $L \to X^4 = (X^4, g'_* = g' Sc(g'))$ has the the norms of its curvature ω bounded by curvature $\underline{\omega}$ of $\underline{L}^*$ according to the inequality

$$||\omega||_{g'_*} = ||\omega||_g{}'(x)/Sc(g,x) < ||\underline{\omega}||_{\underline{g}}(x)/Sc(\underline{g},x) = \frac{1}{8\pi},$$

then $ind(\mathcal{D}_{\otimes L^* \otimes L^\circ}) = 0$ as it follows from the twisted Lichnerowicz-Weitzenboeck-formula (and a little computation).

Now let us assume that the g_*-areas of all non-homologous to zero 2-cycles in X are bounded from below by $8\pi + \epsilon$.

Then, by the *Morse lemma for mass in codimension 1*, the mass of the generator of $H_2(X, \mathbb{R}$ is also bounded from below by $8\pi + \epsilon$, which, by duality, bounds the comass of the corresponding generator of $H^2(X; \mathbb{R}$ by $(8\pi + \epsilon)^{-1}$. Hence, there exists a 2-form ω_0 on X with g_*-norm $\leq (8\pi + \epsilon)_{-1}$ in the cohomology class of the curvature form of the line bundle induced from the Hopf bundle.

ω_0 by the curvature of a line bundle over X, lift this bundle $X^4 = X \times S^1$ and, confront its properties with the above discussion.

Then, by contradiction, we conclude that the g_*-areas of *certain non-homologous to zero* 2-cycles in X must be arbitrarily close to 8π. (One could go to the limit and get such cycles with areas $\leq 8\pi$, but doing this, which needs an additional, let it be a well known, argument, is unnecessary for our purpose.)

Let us return to minimal surfaces and formulate a version of the above (A) for (compact orientable Riemannian) 3-manifolds X *with boundaries*, denoted $S = \partial X$, which involves the integral of the mean curvature $M(S)$ over boundary curves of surfaces $Y \subset X$ with $Z = \partial Y = Y \cap S$. Namely,

connected cooriented cooriented surfaces $Y \subset X$ with non-empty boundaries $Z = \partial Y = Y \cap Z$ which are stable minimal for the free boundary condition, satisfy:

$$\frac{1}{2}\int_Y (Sc(X,y)dy + \int_Z M(S,z)dz \leq 2\pi.$$

About the Proof. This can be obtained by applying the above (A) to the double of X, or, alternatively, with a use of the second variation formula for manifolds with boundaries from section**??**, (where only the simplest case of $\mu = 0$ is needed here).

Corollary. Let $S \subset \mathbb{R}^3$ be a smooth embedded *non-simply connected* closed surface. Then there exists a closed *non-contractible* curve $Z \subset S$. such that $\int_Z M(S,z)dz \leq 2\pi$.

Question. Can one find such a curve in S without using minimal surfaces in the domain bounded by S?

Gauss-Bonnet Extremality of Truncated Cones. Let $\underline{X} \subset \mathbb{R}^3$ be a round truncated cone, the essential invariant of which is the angle β between the side surface $\underline{S}$ of this cone and the bottom, where as in section 5.8 we prefer do deal with the complementary angle $\alpha = \pi - \beta$ and where the inequality [≥] from section 5.8

$$[\geq] \qquad M_Z(z) - \frac{M_S(z)}{\sin \angle_z} + (\cot \angle_z) n \cdot M_Y \geq 0,$$

become an equality, where the curvature $M(\underline{Z})$ of the horizontal circles $\underline{Z} \subset \underline{S}$ is related to the mean curvature of of S along these circles by

$$M(\underline{Z}) = \frac{M(\underline{S})}{\sin \alpha}.$$

Now let X be a compact Riemannian 3-manifold with boundary which is divided in 3 parts bottom B, top T and the side surface S, which separates B from T and such that

the angle between B and S is everywhere $\leq \beta$ and the angle between S and T is everywhere $\leq \alpha$.

Let, moreover, B and S be *mean convex*, i.e. their mean curvatures with respect to the outward normals are *positive*.

Under this condition the functional $Y \mapsto area(Y) - \cos(\alpha) area(S_<)$ defined on surfaces $Y \subset X$ with boundaries $Z = \partial Y \subset S$ and separating B from T necessarily assumes minimum at a surface $Y \subset X$ with $\partial Y \subset S$, which satisfies according to [≥]:

$$\int_Y Sc(X,y)dy + \frac{1}{\sin \alpha}\int_Z M(S,z)dz \leq \pi.$$

As earlier, the most interesting case is for $Sc(X) \geq 0$ and $M(S) \geq 0$, already for domains $X \subset \mathbb{R}^3$, where the existence of a curve $Z \subset S$ separating the top from the bottom and having $\frac{1}{\sin \alpha}\int_Z M(S,z)dz \leq \pi$ seems non-obvious. (Am I missing a direct obvious proof?)

Also note that similar inequalities hold for manifolds X with more complicated corners (see section 5.4 in [G(billiards) 2014] and [Li(comparison) 2017]) but many such inequalities still reman conjectural.

Besides manifolds with $Sc \geq 0$, the above type Gauss-Bonnet inequalities yield geometric information for manifolds with scalar curvatures bounded from below by *negative constants σ*, where this information is somewhat opposite to that for manifolds X with $Sc(X) \geq \sigma > 0$.

Namely, in the later case one conclude that X must have representatives of non-zero homology classes by surfaces of area bounded by $const \cdot \sigma$. On the contrary, the bound $Sc(X) \geq \sigma$ for $\sigma < 0$, implies, under additional *topological conditions*, that X can't have such surfaces with small area.

Example. Let X be homeomorphic to $S_\chi \times S^1$, where S_χ is a closed connected orientable surface with the Euler characteristic $\chi < 0$.

If $Sc(X) \geq -2$, then all surfaces $Y \subset X$ in the homology class of $S_\chi \times \{s_0\} \subset X$ have

$$area(Y) \geq 2\pi|\chi(Y)|.$$

This is, of course, obvious. What is slightly more interesting is a similar inequality for "area minimizing" families of 2d-foliations in X, but these inequalities are inherently non-sharp in the key example of hyperbolic manifolds X (see [G(foliated) 1991]) for more about it).

What looks more promising are foliations by μ-bubbles using horospherical foliations for models, but the corresponding inequalities here is yet to be properly formulated and proved.

5.10 Topological Obstructions to $Sc > 0$ Issued from Minimal Hypersurfaces and μ-Bubbles

Start with recalling the proof of Schoen-Yau's Non-Existence&Rigidity Theorem by $\mathbb{T}^\rtimes$-stabilization argument (see section 1.6.4) applied to complete non-compact manifolds as follows.

0. Let a smooth open orientable manifold X contain a decreasing chain (flag) of oriented *properly embedded* (infinity-to-infinity) submanifolds

$$X \supset X_{-1} \supset \dots \supset X_{-i} \supset \dots \supset X_{-(n-2)},\ dim(X_{-i}) = n - i,$$

such that the homology classes $[X_{-i} \in H_{i,inf}(X)$ of X_{-i} with infinite supports are *non-zero* for all i and the class $[X_{-(n-2)}] \in H_{2,inf}(X)$ is not representable by a simply connected surface (i.e. by S^2 or $\mathbb{R}^2$).

If X supports a complete metric with $Sc \geq 0$, then X is isometric to the product $X = X_0 \times \mathbb{R}^1$, where X_0 is a flat manifold.[391]

Proof. By Jerry Kazdan's perturbation theorem and Cheeger-Gromoll splitting theorem, the case $Sc \geq 0$ reduces to that of $Sc > 0$, where the $\mathbb{T}^\rtimes$-symmetrization shows that X contains a properly embedded surface $Y \subset X$ in the(infinite) homology class of $X_{-(n-2)}$, such that some warped product $Y \rtimes \mathbb{T}^{n-2} = (Y \times \mathbb{T}^{n-2}, g^\rtimes = dy^2 + \phi^2(y)dt^2)$ has positive scalar curvature,

$$Sc(Y \rtimes \mathbb{T}^{n-2}) = Sc(g^\rtimes) > 0.$$

Hence, Y must be simply connected. Otherwise a covering $\tilde{Y}$ of Y with infinite cyclic fundamental group $\pi_1(\tilde{Y}) = \mathbb{Z}$ would allow an extra $\mathbb{T}^\rtimes$-symmetrization, and turn into a complete manifold $\mathbb{R} \times \mathbb{T}^{n-1}$ with a (warped product) metric $\tilde{g}^\rtimes = dx^2 + \varphi(x)^2 d\tilde{t}^2$, for $x\mathbb{R}^1$ and $t \in \mathbb{T}^{n-1}$, on $\mathbb{R} \times \mathbb{T}^{n-1}$ *invariant under the action of the torus* and such that $Sc(\tilde{g}^\rtimes) = 0$.

Thus, impossibility of this follows by the formula

$$Sc(g)(x,\tilde{t}) = -\frac{(n-1)(n-2)}{\varphi^2(y)}\left\|\frac{d\varphi(x)}{dx}\cdot\frac{1}{\varphi}\right\|^2 - \frac{2(n-1)}{\varphi(y)}\frac{d^2\varphi(x)}{dx^2},$$

[391] As usual, if $n \geq 8$, one has to appeal to "desingularization" results from [Lohkamp(smoothing) 2018] or from [SY(singularities) 2017]. (If X is spin and $H_1(X)$ has no torsion, then the results from section 6 in [GL(complete) 1983] apply.)

since no function $\varphi > 0$ can have negative second derivative.

Manifolds with Spines. Let us now turn to more general open manifolds X, including infinite coverings of *compact enlargeable* (e.g. admitting metrics with non-positive sectional curvatures) manifolds *with punctures* and on products of SYS-manifolds by enlargeable ones, where geometry depends on the distance to a distinguished closed subset $S \subset X$ called the *spine of* X.

Example. If X comes with a covering map to a compact manifold minus a point, $X \to X_0 \setminus x_0$, then relevant spines $S \subset X$ are the pullbacks of compact subsets $S_0 \subset X_0 \setminus x_0$.

S-Quasiproper Maps. Given a spine S in X, a continous map from X to a metric space, say $f : X \to \underline{X}$, is called *uniformly* S*-quasi-proper* if it is *constant on the connected components* of the complement $X \setminus S$ and if the restriction of f to S,

$$f_{|S} : S \to \underline{X}$$

is *uniformly proper*, i.e. the diameters of the $f_{|S}$-pullbacks of subsets from $\underline{X}$ are bounded in terms of the diameters of these subsets,

$$diam(f^{-1}(\underline{U}) \cap S) \leq \xi(diam(\underline{U})),$$

for some continuous function $\xi(d)$, $d \geq 0$ and all $\underline{U} \subset \underline{X}$.

Bounded Geometry along Spine. A Riemannian manifold X with a spine S is said to have *bounded* C^∞*-geometry along* S if there are continuous functions $\xi_i(d)$ and ξ_o such the i-th covariant derivatives of the curvature tensor of X satisfy

$$\square_{bnd} \qquad ||\partial_i curv(X,x)|| \leq \xi_i(dist(x,S)) \text{ and } \frac{1}{inj.rad(X_i,x)} \leq \xi_o(dist(x,S)).$$

Lemma: $\mathbb{R}^\times$-Symmetrization of Manifolds with Spines. Let X be a complete connected orientable Riemannian n-manifold with a spine $S \subset X$ and let $f : X \to \underline{X} = \underline{Y} \times \mathbb{R}^1$ be a uniformly S-quasi-proper 1-Lipschitz map.

Let the scalar curvature of X be bounded from below in terms of the distance function $d(x) = dist(x,S)$,

$$Sc(X,x) \geq \sigma(d(x))$$

for some continuous monotone decreasing function $\sigma(d)$ $d > 0$.

If $n = dim(X) \leq 7$ *and if* X *has bounded* C^∞*-geometry along* S,[392] *then there exists a smooth connected complete Riemannian warped product* n*-manifold* $X_1 = (Y_1 \times \mathbb{R}^1, dy^2 + \phi(y)^2 dt^2)$ *with a* $\mathbb{R}^1$*-invariant spine* $S_1 \subset X_1$ *and with a uniformly* S_1*-quasi-proper* $\mathbb{R}^1$*-equivariant 1-Lipschitz map*

$$f_1 : X_1 = Y_1 \times \mathbb{R} \to \underline{X} = \underline{Y} \times \mathbb{R}^1$$

for the obvious action of the group $\mathbb{R}^1$ *on both spaces, such that*

$\bullet_{bnd}$ X_1 has bounded C^∞ geometry along S_1;

[392] This C^∞ is a minor technicality: the geometry which is actually used in the proof below is that of the curvature itself and of the injectivity radius, where even these maybe redundant.

$\bullet_{Sc}$ the scalar curvature of X_1 is bounded from below by the same function $\sigma(d)$ as the the scalar curvature of X,

$$Sc(X_1, x_1) \geq \sigma(dist(x_1, S_1)).$$

$\bullet_{f_1}$ the topology of the map f_1 is "essentially the same" as that of f, where, in our case, we shall need two specific instances of this:

$\bullet_{deg}$ if $dim(X) = dim(\underline{X})$, then the map f_1 has the *same degree* as f;

$\bullet_{SYS}$ if $dim(X) = dim(\underline{X}) + 2$ and if the homology class of the f-pullbacks of generic points, $f^{-1}(\underline{x}) \subset X$, $\underline{x} \in \underline{X}$, is *spine detectably non-spherical*, i.e. all surfaces $\Sigma \subset X$ in this class contain closed curves in the intersection $\Sigma \cap S$, which are non-contractible in X then the the homology class of f-pullbacks of generic points, $f_1^{-1}(\underline{x}) \subset X_1$, is also *spine detectably non-spherical.*

Proof. Apply μ-bubble separation theorem from section 3.7 to the bands $X_{[-d,d]} \in X$ that are the pullbacks of the bands $\underline{Y} \times [d, d] \subset \underline{Y} \times \mathbb{R}^1$,

$$X_{[-d,d]} = f^{-1}(\underline{Y} \times [d, d]$$

for the segments $[d, d] \subset \mathbb{R}^1$, $d > 0$, and thus obtain hypersurfaces $Y = Y(d) \subset X_{[-d,d]} \subset X$ and warping functions $\phi_d(y)$, such that the manifolds $X^{\rtimes}(d) = (Y(d) \times \mathbb{R}^1, dy^2 + \phi_d(y)^2 dt^2)$ (obviously) satisfy all requirements of the lemma, except for $\bullet_{Sc}$ which is replaced by an ε_d-weaker inequality,

$$Sc(X_1, x_1) \geq \sigma(dist(x_1, S_1)) - \varepsilon_d,$$

where $\varepsilon_d \to 0$ for $d \to \infty$.

Now, the C^∞ -geometry of X is bounded along the spine $S \subset X$, the standard elliptic estimate implies that the C^∞-geometries of all $X^{\rtimes}(d)$ are uniformly, (i.e. independently of d) bounded along the spines of these manifolds; hence, some sequence $X^{\rtimes}(d_i)$ Hausdorff converges to the required X^1. QED.

Then we recall the "symmetry appendix" to the separation theorem and conclude that the $\mathbb{R}^{\rtimes}$-symmetrization is also compatible with extra symmetries and with the warper product structures as follows.

$\mathbb{R}^{\rtimes}$-Symmetrization in a Presence of a Group Action. *If the manifolds X and $\underline{Y}$ are isometrically acted upon by a group G, and if the map $f : X \to \underline{X} = \underline{Y} \times \mathbb{R}^1$ is G-equivariant, then X_1 comes with an isometric action of $G \times \mathbb{R}^1$ and the map $f_1 : X_1 \to X = \underline{X} = \underline{Y} \times \mathbb{R}^1$ is $G \times \mathbb{R}^1$-equivariant.*

Furthermore, if

- $G = \mathbb{R}^m$;
- $\underline{Y} = \underline{Z} \times \mathbb{R}^m$;
- $X = (Z \times \mathbb{R}^m, dz^2 + \psi(z)^2 dt^2$,

then $X_1 = Z \times \mathbb{R}^{m+1} dz^2 + \varphi(z)^2 dt^2$.

(dt^2 stands for the Riemannian metric in both Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^{m+1}$.)

Corollary A. Let X be a complete Riemannian n-manifold with a spine $S \subset X$ and $f : X \to \mathbb{R}^n$ be a uniformly S-quasi-proper 1-Lipschitz map. Then the scalar curvature of X *can't be uniformly positive along* S, i.e.

there is no positive function $\sigma(d) > 0$, such that $Sc(X, x) \geq \sigma(dist(x, S))$, $x \in X$.

Non-Existence/Rigidity Sub-Corollary A′. *If a complete orientable -n-manifold $\hat{X}$, $n \leq 7$, dominates with non-zero degree a compact orientable enlargeable manifold X_0,sf e.g. $\hat{X}$ is homeomorphic to X_0 minus a point, then $\hat{X}$ is a compact flat manifold.*

Corollary B. Let X be a complete Riemannian n-manifold with a spine $S \subset X$ and $f : X \to \mathbb{R}^{n-2}$ be a smooth uniformly S-quasi-proper 1-Lipschitz map, such that the homology class of the f-pullbacks of generic points, $f^{-1}(\underline{x}) \subset X$, $\underline{x} \in \underline{X}$, is *spine detectably non-spherical.*

Then the scalar curvature of X can't be uniformly positive along S.

Non-Existence/Rigidity Sub-Corollary B′. *If a complete orientable n-manifold $\hat{X}$, $n \leq 7$, dominates with degree one the product of an orientable enlargeable manifold by a SYS-manifold, then $\hat{X}$ is a compact flat manifold.*

Remarks on Rigidity, $n > 7$, and on SYS-Enlargeable manifolds.
(a) Corollaries **A** and **B** also extend to the case of $Sc \geq \sigma(dist(x, S)$, where the function σ is not strictly positive, $\sigma(d) \geq 0$, where the conclusion is that X is Riemannian flat: it is isometric to $\mathbb{R}^n$, in the case **A**, and to $\mathbb{R}^{m-2} \times T^2$, for a flat (possibly non-split) torus T^2 for **B**.

This can be proven either by adapting Jerry Kazdan's perturbation argument or arguing as in the proof of the rigidity of warped products in section 5.7

(b) If $n = 8$, then the conclusion of **A** and **B** remain intact, since the perturbations from Nathan Smale's argument are controlled by the bound on the C^∞ geometry.

Also, the rigidity sharpening of **A** and **B** remans valid, since the warped product rigidity proof compensates for Smale's perturbations.

(c) Probably – IF I understand the logic of Schoen-Yau's "desingularization" proof correctly – it, similarly to Smale's proof, extends to the present case and implies as much of the lemma as is needed for A and B, but proving rigidity for $n \geq 9$ seems technically more involved.

(d) It is unclear if the non-domination corollary **B′** for *SYS-enlargeable* manifolds (defined below) that are significantly more general then those in **B**' follows from the $\mathbb{R}^{\times}$-symmetrization lemma, because of to the "spine detectability" condition in this lemma that can (can it?) fail to be satisfied in the general case.

Definition. A Riemannian manifold X is *SYS-enlargeable*, if, for all $d > 0$, there exists a proper compact n-dimensional Riemannian band X_d with width $width(X_d) > d$, which admits a locally isometric immersion $X_d \to X$ and such that all compact hypersurfaces $Y \subset X$, which separate $\partial_-(X_d) \subset \partial X_d$ from $\partial_+(X_d) \subset \partial X_d$, are SYS, i.e. Schoen-Yau-Schick manifolds.

(A more general class of such manifolds is defined in [g(inequalities] 2018], but I admit, finding the "true definition" remains problematic.) definition.)

6 Generalisations, Speculations

The most tantalizing aspect of scalar curvature is that it serves as a meeting point between two different branches of analysis: the index theory and the geometric measure theory,

Each of the this theories, has its own domain of applicability to the scalar curvature problems (summarized below) with a significant overlaps and distinctions between the two domains.

This suggests, on the one hand,

a possible unification of these two theories

and, on the other hand,

a radical generalization, or several such generalizations,

of the concept of a space with the scalar curvature bounded from below.

This is a dream. In what follows, we indicate what seems realistic, something lying within the reach of the currently used techniques and ideas.

6.1 Dirac Operators versus Minimal Hypersurfaces

Let us briefly outline the relative borders of the domains of applicability of the two methods.

1. Spin/non-Spin. There is no single instance of *topological obstruction* for a metric with $Sc > 0$ on a closed manifold X, the *universal coverings* $\tilde{X}$ of which is *non-spin*[393] that is obtainable by the (known) Dirac operator methods.[394]

But the minimal hypersurface method delivers such obstructions for a class manifolds X, which admits continuous maps f to *aspherical spaces* $\underline{X}$, such that such an f doesn't annihilate the fundamental class $[X] \in H_n(\underline{X})$, $n = dim(X)$, i.e. where the image $f_*[X] \in H_n(\underline{X})$ doesn't vanish.

Example. The connected sum $X = \mathbb{T}^n \# \Sigma$, where Σ is a simply connected non-spin manifold are instance of such X with the universal coverings $\tilde{X}$ being non-spin.)

2. Homotopy/Smooth Invariants. The minimal hypersurface method alone can only deliver *homotopy theoretic* obstructions for the existence of metrics with $Sc > 0$ on X.

But $\hat{\alpha}(X)$, non-vanishing of which obstructs $Sc > 0$ according to the results by Lichnerowicz and Hitchin proven with *untwisted* Dirac operators is not homotopy invariant. (Non-vanishing of $\hat{\alpha}$ is *the only* obstruction for $Sc > 0$ for simply connected manifolds of dimension ≥ 5, see section 3.2.)

Here, observe, the spin condition is essential, but when it comes to twisted Dirac operators, those obstructions for the existence of metrics with $Sc > 0$, which are *essentially due to twisting* are also *homotopy invariant*, and, for all we know, the spin condition is redundant there.

Furthermore, minimal hypersurfaces can be applied together with that Dirac operators.

For example the product manifold $X = X_1 \times X_2$, where $\hat{\alpha}(X_1) \neq 0$ and $X_2 = \mathbb{T}^n \# \Sigma$, doesn't carry metrics with $Sc > -0$, which for $dim(X) \leq 8$ follows from Schoen-Yau's [SY(structure) 1979] (with a use Nathan Smale's generic non-singularity theorem for $n = 8$), while the general case needs Lohkamp's [Lohkamp(smoothing) 2018].

Notice that the twisted Dirac operator method also applies to these, $X = X_1 \times X_2$, provided that Σ is spin, or at least, the universal covering $\tilde{\Sigma}$ is spin.

[393]Relaxing the condition "X is spin" to "$\tilde{X}$ is spin" is achieved with (a version of) the Atiyah L_2-index theorem from [Atiyah(L_2) 1976], as it is explained in §§$9\frac{1}{9}, 9\frac{1}{8}$ of [G(positive) 1996].

[394]Never mind Seiberg-Witten equation for $n = 4$

3. SYS-Manifolds. The most challenging for the Dirac operator methods is Schoen-Yau's proof of non-existence of metrics with $Sc > 0$ on Schoen-Yau-Schick manifolds (see section 2.7), where the known Dirac operator methods, even in the spin case, don't apply.

And as far as the topological non-existence theorems go, the minimal hypersurface method remains silent on the issue of metrics with $Sc > 0$ on quasisymplectic manifolds X as in section 2.7, (e.g. closed aspherical 4-manifolds X with $H^2(X;\mathbb{Q}) \neq 0$.) And we can't rule out metrics with $Sc > 0$ on the connected sums $X\#\Sigma$ with any one of the present day methods, if the universal coverings $\tilde{\Sigma}$ are non-spin.

4. Area Inequalities. The main advantage of the twisted Dirac operator over minimal hypersurfaces is that geometric application of the latter to $Sc > 0$ depend on lower bounds on the sizes of Riemannian manifolds X, where these sizes are expressed in terms of the *distance functions* on X, while the twisted Dirac relies on the *area-wise lower bounds* on X.

The simplest (very rough) result in this regard says that every (possibly non-spin) smooth manifold X admits a Riemannian metric g_0, such that every *complete*[395] metric g on X, for which

$$area_g(S) \geq area_{g_0}(S)$$

for all smooth surfaces $S \subset X$, satisfies:

$$\inf_{x\in X} Sc(g,x) \leq 0$$

(see section 11 in [G(101) 2017]). More interestingly, there are better, some of them sharp, bounds on the area-wise size of manifolds with $Sc \geq \sigma > 0$, such as sharp area inequalities in section 3.4 and Cecchini's long neck theorem for maps of manifolds with boundaries to spheres 3.14.3.

These can't be obtained, in general, with the (present day) techniques of minimal hypersurfaces and stable μ-bubbles, but the following area bounds do follow by these techniques, yet they are unapproachable with Dirac operators.

(a) *Marcus-Neves' S^3 by S^2-Sweeping Theorem* [Marques-Neves(min-max spheres in 3d) 2011]] (section 3.10).

(b) *Zhu's $S^2 \times T^n$-Systole Theorem* [Zhu(rigidity) 2019], (see footnote in section 4.1)

(c) Richard's $S^2{\times}S^2$-Systole Theorem [Richard(2-systoles) 2020], (same footnote in section 5.5).

5. Inequalities for Metrics Normalized by Sc. Dirac operator arguments that yield geometric bounds on Riemannian manifolds $X = (X, g)$ with $Sc(X) \geq \sigma > 0$, e.g. on their spherical radii, in terms of σ, automatically deliver in most (all?) cases similar bounds on $Sc(X)\cdot X = (X, Sc(X,x)\cdot g(x))$.

For instance, Llarull's algebraic inequality (see section 4.2) not just implies that

$$Rad_{S^n}(X/\sigma) = Rad_{S^n}(X)/\sqrt{\sigma} \leq 1/\sqrt{n(n-1)}$$

[395] "Complete" is essential as it is seen already for $dim(X) = 2$. But if $area_g(S) \geq area_{g_0}(S)$ is strengthened to $g \geq g_0$ one can drop "complete", where the available proof goes via minimal hypersurfaces and where there is a realistic possibility of a Dirac operator proof as well.

for $\sigma = \inf_{x\in X} Sc(X,x)$, but in fact, that

$$Rad_{S^n}(Sc(X)\cdot X) \leq \sqrt{n(n-1)} = Rad_{S^n}(Sc(S^n)\cdot S^n))$$

for *all* compact spin manifolds X with positive scalar curvatures.

But it is unclear if such inequalities, let them be non-sharp ones, can be obtained with techniques of minimal hypersurfaces and stable bubbles and, the bound $Rad_{S^n}(Sc(X)\cdot X) \leq const_n$ remains *problematic* for *non-spin* manifolds X, while the inequality $Rad_{S^n}(X/\sigma) \leq const_n$ follows with minimal hypersurfaces (see section 12 in [GL(complete) 1983] and section 5.5, augmented by the regularity results from [Lohkamp(smoothing) 2018] and/or [SY(singularities) 2017] for $n \geq 9$.

6. **Families of Manifolds, Foliations and Homotopies of Metrics with $Sc > 0$.** Individual index formulas typically (always?) extends to families of operators and deliver harmonic spinors on members of appropriate families. But there is no (apparent?) counterpart of this for minimal hypersurfaces and/or for stable μ-bubbles that is partly due to discontinuity of minimal subvarieties under deformation of metrics in the ambient manifolds.

Consequently, non-triviality of homotopy groups (except for π_0) of spaces of metrics with $Sc > 0$ is undetectable by minimal hypersurfaces. Also the Sc-normalized (in the sense of 2.8) distance inequalities, as well as topological and geometric obstruction for $Sc > \sigma$ on foliations, escape the embrace of minimal hypersurfaces.[396]

7. **Non-Completeness and Boundaries.** Until recently, the major drawback of the Dirac operator methods was reliance on completeness of manifolds X it applied to,[397] but recent results by Zeidler, Cecchini, Lott and Guo-Xie-Yu on index theorems for manifolds with boundaries[398] have effectively extended the Dirac operator index theory to such manifolds.

Also minimal hypersurfaces and especially stable μ-bubbles in conjunction with twisted Dirac operators, fare better in non-complete manifolds, especially in manifolds with controlled mean curvature of their boundaries, as it is demonstrated in section **??** of this paper, but the recent articles by John Lott [Lott(boundary 2020] and Christian Bär with Bernhard Hanke [Bär]-Hanke(boundary) 2021] open here new possibilities for Dirac operators.

8. **$Sc \geq \sigma$ for $\sigma < 0$.** Both methods have more limited applications here than for $\sigma \geq 0$, where the most impressive performance of the Dirac operator is in the proof of the Ono-Davaux spectral inequality (stated in section 3.13), which also may be seen from a more geometric perspective of stable μ-bubbles, as it is suggested by the *Maz'ya-Cheeger inequality.*

9. **Singular Spaces.** Unlike Dirac operators, minimal varieties and μ-bubbles cn be defined for many relevant singular spaces, such as

(i) *pseudomanifolds* with piecewise linear or piecewise smooth metrics,

(ii) *Alexandrov spaces* with sectional curvatures bounded from below,

(iii) *singular minimal hypersurfaces* and related spaces, e.g. doubles of smooth manifolds over such hypersurfaces.

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[396] Possibly, this can be remedied by an extension of the Schoen-Yau inductive descent method to a class of discontinuous families.

[397] Our attempts to alleviate this limitation in section 4.6, remains unsatisfactory.

[398] See [Cecchini-Zeidler(Scalar&mean) 2021], [Guo-Xie-Yu(quantitative K-theory) 2020.

However, despite the recent progress in the papers [SY(singularities) 2017] and [Lohkamp(smoothing) 2018], there is neither a concept of $Sc \geq \sigma$ for such spaces X nor comprehensive theory of minimal hypersurfaces in X.

And it is not clear at all if there is room for Dirac operators on this kind of singular spaces X.

6.1.1 13 Proofs of non-Existence of Metrics with $Sc > 0$ on Tori

The present-day proofs can be divided according the techniques they are achieved with; these are

A. Dirac operators.

B. Minimal hypersurface and stable μ-bubbles.

C. Combination of A and B.

D. Harmonic maps in dimension 3.

E. Ricci flow in dimension 3.

(I am not certain if one can do something with the Seiberg-Witten equations.)

In what follows, X is a Riemannin manifold diffeomorphic to $\mathbb{T}^n$. We agree that two A-proofs of $Sc \ngtr 0$ on X are *different* if they rely on different variants of the index theorems and which deliver different harmonic spinors for generic metrics in X. Similarly, B-proofs are regarded different if the relevant minimal surfaces or μ-bubles are, generically, different. [399]

Here one notice that all proofs based on index theorems on compact manifolds X and relative index theorems on complete manifolds have their L_2-*counterparts on Galois coverings* $X_* \to X$ that result in *different* harmonic spinors[400] if the fundamental groups $\pi_1(X_1)$ and $\pi_1(X_2)$ are *non-commensurable.*

Shall we regard such proofs different?

SIX A-PROOFS WITH VARIATIONS

1. Lusztig's Kind of Proof. This, for n even, goes with the family of Dirac operators $\mathcal{D}$ on X twisted with unitary line bundles $l_\tau\mathbb{T}^n$ parametrized by the dual torus $hom(H_1(X) \to \mathbb{T}^1) \ni \tau$.

This proof can be rendered in the language of C^*-algebras (here this is the algebra of continous function on the dual torus) but, probbaly, the harmonic spinors will be the same.

If n is odd, besides the reduction to the even case, either for $X \times \mathbb{T}^1$ or or $X \times X$ (are these two proof different?) one, probably can proceed with the odd dimensional spectral flow argument. (I am not certain if, in a general C^*-algebraic K-theoretic setting, there is a distinction between what happens to even and to odd n.)

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[399]Difference between spinors and minimal hypersurfaces often disappears for flat metrics on tori and also two seemingly different spaces of spinors may, in fact, be canonically isomorphic, such as the space of spinors on the universal covering $\tilde{X}$ of an X and the space of spinors on X twisted with the flat bundle over X with the fiber $L_2(\pi_1(X))$ associated with the covering $\tilde{X} \to X$ via the regular representation of $\pi_1(X)$.

I must admit I haven't systematically traced such isomorphisms in all cases and some proofs in our list can be not different after all.

[400]To compare spinors on different coverings of X we lift them all to the universal covering $\tilde{X}$ of X. (For general X, this L_2 has an advantage of allowing one to relax the spin condition on X to that on $\tilde{X}$.)

2. $\wedge^2$-Hypersphericity of $\tilde{X}$. Here, never mind odd n, one uses the relative index theorem for the Dirac operator on the universal covering $\tilde{X}$ twisted with almost flat bundles $L_\varepsilon \to \tilde{X}$ on $\tilde{X}$ with compact supports .

3. Infinite K-Area/Cowaist$_2$. Since K-$cowaist_2(X) = \infty$, can use the ordinary index theorem on X for $\mathcal{D}$ on $\tilde{X}$ itself twisted with almost flat bundles over X.

This is close to but different from $\mathcal{D}$ twisted with Mishchenko's infinite dimensinal Fredholm bundles which also yields $Sc \not> 0$ on tori.

4 Quasi-symplectic Proof. This depends (n is even) on the L_2-index theorem applied $\mathcal{D}$ on $\tilde{X}$ twisted with fractional powers of a lift of a line bundle from X to $\tilde{X}/$ (I am not certain how to arrange a spectral flow argument for odd n in this case.)

5. Roe's Index Theorems. Since $\tilde{X}$ is hypereuclidean, the Roe's algebra index theorem applies to $\tilde{X}$. Also one may use Roe's partitioned index theorem applied to the half-cyclic cover of X (homeomorphic to $\mathbb{T}^{n-1} \times \mathbb{R}^1_+$ in our case)

6. Bounds on Widths of Bands. All infinite coverings X_* of X contains arbitrarily wide torical bands to which the index theorem by Zeidler-Cecchini and by Guo-Xie-Yu apply and yield $Sc \not> 0$

Fife B-Proofs

All these proofs rely on Schoen-Yau's Inductive Descent with Minimal Hypersurfaces or with μ-Bubbles. This is a list of these.

1. S-Y ID with Minimal Hypersurfaces in X and with Conformal Modification of Metrics via Kazdan-Warner's Theorem.

2. MH Inductive Descent in X with $\mathbb{T}^\rtimes$-Symmetrization.

(Both 1 and 2 apply to all SYS manifolds.)

3. ID in $\tilde{X}$ with $\mathbb{T}^\rtimes$-Symmetrization of Minimal Hypersurfaces in large balls in $\tilde{X}$ with prescribed boundaries.

(This proof applies to all, possibly non-complete) manifolds with large hyperspherical radii.)

4. Proofs via Bounds on Width of Bands in infinite Coverings of X either with MH or with μ-B. (Compare with A8.)

5. Exhaust $\tilde{X}$ by domains $U_i \subset \tilde{X}$ bounded by μ-Bubbles $Y_i = \partial U_i$ and Apply either 3 or 4 to $Y_i \rtimes \mathbb{T}^1$. (One can also use here ID with 5 itself applied in all dimension.)

Two C-Proofs

1. The above Y_i have their hyper-spherical radii $Rad_{S^{n-1}}(Y_i) \to \infty$, that is incompatible with $Sc(Y_i \rtimes \mathbb{T}^1) \geq \sigma > 0$ by the index theorem for the Dirac operators on $Y_i \rtimes \mathbb{T}^1)$ twisted with bundles induced from a complex vector bundle $\underline{L} \to \mathbb{S}^{n-1}$ with a non-zero top Chern class. (You know what to do if n is odd.)

2 Exhaust $\tilde{V}$ by domains Y'_i with $mean.curv(\partial U'_i) > 0$ and apply Lott's index theorem for maps from these U'_i to the hemisphere S^n_+, or use the Goette-Semmelmann's theorem for smoothed doubles of U'_i mapped to S^n.

(There are also variations of these proofs with exhaustions of $\tilde{V}$ by cubical domains but these, albeit especially useful in dimension 9, where 1 and 2 don't apply, are unbearably artificial.)

All these proofs, have different possibilities for generalizations to non-torical X and different ranges of applications. It would be pleasant to find a unifying framework for them.

6.1.2 On Positivity of $-\Delta+const\cdot Sc$, Kato's Inequality and Feynman-Kac Formula

1. *Question.* What are effects on the topology and/or metric geometry of a Riemannian manifold X played by *positivity* of the

$$L_\gamma : f(x) \mapsto -\Delta f(x) + \gamma \cdot Sc(X,x) f(x)$$

for a given constant $\gamma > 0$?

Observe that the greater the constant γ is, the stronger this effect should be. Indeed, since $-\Delta$ is a positive ,

$$-\Delta + \gamma_1 \cdot Sc(X) \geq 0 \Rightarrow -\Delta + \gamma_2 \cdot Sc(X), \text{ for } \gamma_1 \geq \gamma_2.$$

If $\gamma = \frac{1}{2}$, then the product $X \times \mathbb{T}^1$ admits a $\mathbb{T}^1$-invariant metric with $Sc \geq 0$, namely the warped product metric $g_\rtimes(x,t) = \phi^2 dx^2 + dt^2$, where ϕ is the lowest eigenfunction of the $-\Delta f(x) + \frac{1}{2} Sc(X)$ (see section 1.6.5).

Thus, all we know about geometry and topology of $\mathbb{T}^\rtimes$-stabilized manifolds with $Sc \geq 0$ applies to manifolds with positive $-\Delta f(x) + \frac{1}{2} Sc(X)$.

Yet, there can be (maybe not?) a difference between metric geometries of manifolds X with positive $-\Delta f(x) + \gamma \cdot Sc(X)$ for different $\gamma \geq \frac{1}{2}$.

Now, turning to small γ, observe the following.

2. *All compact smooth manifolds X of dimension $n \geq 3$ admit Riemannian metrics g for which the $-\Delta_g + \varepsilon Sc(g)$ is positive for some $\varepsilon = \varepsilon(X) > 0$.*

Idea of the Proof. Make a "thin connected sum" of (X, g_0) with a huge (volume-wise huge) topologically spherical manifold $X_\circ$, where $Sc(X_\circ) \geq 1$ and apply the following.

Lemma/Exercise. Let $s(x)$ be a continuous function on a compact connected manifold X, such that $\int_X s(x)dx > 0$, then the $-\Delta + \varepsilon s$ is positive for all sufficiently small $\varepsilon > 0$.

3. *Conjecture. There is a universal $\bar{\varepsilon} = \bar{\varepsilon}_n > 0$, such that all compact n-manifolds admit Riemannian metrics g_ε, for all $0 \leq \varepsilon < \bar{\varepsilon}$, such that*

$$-\Delta_{g_\varepsilon} + \varepsilon Sc(g_\varepsilon) \geq 0.$$

One knows in this respect that if such $\bar{\varepsilon}_n$ does exist, then it can't be greater than the conformal Kazdan-Warner constant,

$$\bar{\varepsilon}_n \leq \gamma_n = \frac{n-1}{4(n-2)}$$

and, for all we know, ε_n, may be equal to this γ_n.

But it would be more interesting to have $\bar{\varepsilon} = \bar{\varepsilon}(X)$ as a topological invariant which takes infinitely many different values on n-dimensional manifolds X.

If the $-\Delta_g + \gamma_n Sc(X)$, where $\gamma_n = \frac{n-1}{4(n-2)}$ then, by Kazdan-Warner theorem X admits a (conformal) metric with $Sc \geq 0$. hen

Moreover, there may exist a universal $\varepsilon > 0$ that serves all manifolds X or at least all X of a given dimension n, but all one can say at this point is that thisε must be $< \frac{n-2}{4(n-1)}$.

Remark. If $-\Delta+\gamma_n Sc(X)>0$, then, by Kazdan-Warner theorem, $X=X,(g_0)$ admits a metric g (conformal to g_0 with $Sc(g)>0$. In particular, if X is spin, it admits no g-harmonic spinors by Lichnerowicz-Hitchin vanishing theorem; thus, $\hat{\alpha}(X)=0$ by the Atiyah-Singer index theorem.

In fact, regardless of the sign of the scalar curvature, the existence of harmonic spinors is a conformal invariant by Hitchin's theorem, and this, applied to Dirac operators twisted with infinite dimensional unitary (almost) flat bundles, allows an extension of most (all) Dirac operator topological obstructions to $Sc>0$ to manifolds with positive operators $-\Delta+\gamma_n Sc(X)$.

But it feels a bit strange (have I confused the values of the constants?) that a natural alternative argument with the *refined Kato's inequality* (see below) and the Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) formula delivers such a conclusion *only* with $\gamma'_n=\frac{n-1}{4n}>\gamma_n=\frac{n-2}{4(n-1)}$.

4. *Kato's Inequality* [Hess-Schrader-Uhlenbrock(Kato) 1980]. Let $V\to X$ be a vector bundle with a unitary connection ∇ over a Riemannian manifold and let $f:X\to V$ be a smooth section.

Then, an elementary calculation shows that the *gradient of the norm of f is bounded by the norm of the covariant derivative of f,*

$$|d|f||\leq|\nabla f|,$$

where this inequality at the zero points of f is understood in the distribution sense.

5. *Corollary.* Let $S:V\to V$ be a selfadjoint endomorphism, i.e. a family of selfadjoint operators in the fibers, $S(x):V_x\to V_x$, and let $s(x)$ be the lowest eigenvalue of $S(x)$.

Then the lowest eigenvalue of the $f\mapsto\nabla^2 f(x)+S(x)f(x)$ is bounded from below by the lowest eigenvalue of the scalar $-\Delta+s(x)$ on X.

Equivalently,

if the $-\Delta+s_\lambda(x)$ on X for $s_\lambda=s-\lambda$ is positive for some real number λ, then the $\nabla^2+S_\lambda(x)$ for $S_\lambda=S-\lambda$ is also positive.

In fact, *non-positivity* of a selfadjoint means that there exists a *test vector* ϕ, such that $\langle A\phi,\phi\rangle<0$.

Thus , if ∇^2+S_λ is non-positive, then there exists a section $f:X\to V$, such that

$$0>\int_X(\langle\nabla^2 f(x),f(x)\rangle+\langle S_\lambda f(x),f(x)\rangle)dx=\int_X(|\nabla f(x)|^2+\langle S_\lambda f(x),f(x)\rangle)dx\geq$$

$$\geq\int_X(|\nabla f(x)|^2+s_\lambda|f(x)|^2)dx,$$

and, by Kato's inequality, the norm $|f(x)|$ serves as the test function for non-positivity of $-\Delta+s_\lambda$, for

$$\int_X(-\Delta|f(x)|+s_\lambda(x)|f(x)|)dx=\int_X|d|f(x)||^2+s_\lambda(x)|f(x)|^2)dx\leq$$

$$\leq\int_X|\nabla f(x)|^2+s_\lambda(x)|f(x)|^2)dx<0.$$

Bundles V relevant for applications to scalar curvature are spinor bundles $\mathbb{S}(X)\to X$ twisted with unitary bundles L with "small" curvatures, such as the following.

6. *Example*. Let X be a compact orientable Riemannian n-manifold, which admits a distance (or area) decreasing map to the unit sphere S^n with *non-zero degree.*

If X is spin, then the lowest eigenvalue $\lambda_1(X, \frac{Sc}{4})$ of the $f(x) \mapsto -\Delta f(x) + \frac{1}{4}Sc(X,x)f(x)$ is bounded by that for S^n,

$$\lambda_1\left(X, \frac{Sc}{4}\right) \le \frac{1}{4}n(n-1).$$

Exercise. Formulate and prove similar generalizations of other bounds on the size of Riemannian manifolds X by $\inf Sc(X)$, such as area (non)-contraction inequalities from sections 3. 3 and 3.4.

7. *Conjecture.* Probably, the Dirac operator proofs of geometric inequalities for non-complete manifolds with $Sc \ge \sigma > 0$ by Cecchini, Zeidler and Guo-Xie-Yu also extend to manifolds X with lower bounds on (properly defined) eigenvalues $\lambda_1\left(X, \frac{Sc}{4}\right)$.

Remark. No single results of this kind is available by the methods of the geometric measure theory, where one faces the following

8. *Open Problem.* Find counterexamples to the following claim.

Let $X = (X, g_0)$ be a compact Riemannian n-manifolds $X = (X, g_0)$, $n \ne 4$.

If *the universal covering of X is non-spin*, then for all $\gamma < \frac{1}{2}$ and $\lambda > 0$, there exists a Riemannian metrics $g = g_{\gamma,\lambda}$ on X, such that $g \ge g_0$, and such that the $-\Delta_g + \gamma \cdot Sc(g) - \lambda$ is *positive*.

9.*Exercises.* Denote by $\lambda_1(X, \gamma Sc) = \lambda_1(X, \gamma g)$ the bottom of the spectrum of the $-\Delta_g + \frac{1}{2}Sc(g)$ on a Riemannian manifold $X = (X, g)$.

(a) Show that $\lambda_1(X, \gamma Sc)$ is invariant under finite covering $\tilde{X} \to X$ of compact Riemannian manifolds,

$$\lambda_1(\tilde{X}, \gamma Sc) = \lambda_1(X, \gamma Sc).$$

(b) Show that $\lambda_1(X, \gamma Sc)$ is additive under Riemannian products of manifolds,

$$\lambda_1(X_1 \times X_2, \gamma Sc) = \lambda_1(X_1, \gamma Sc) + \lambda_1(X_2, \gamma Sc).$$

(c) Let $X = (X, g_0)$ be a compact (possibly non-spin) Riemannian manifold. Show that there is a constant $\lambda = \lambda(X)$, such that all Riemannian metrics g on X, which are *area wise greater than g*, i.e. such that $area_g(S) \ge area_{g_0}(S)$ for all smooth surfaces $S \subset X$ satisfy $\lambda_1(g, \frac{1}{2}Sc) \le \lambda$.

(d) Show that $\lambda_1(g, \gamma Sc)$ is semicontinuous under C^0-limits of Riemannian metrics:

if Riemannian metrics g_i uniformly converge to g then

$$\lambda_1(g, \gamma Sc) \ge \limsup \lambda_1(g_i, \gamma Sc) \textit{ for all } \gamma.$$

Also prove this for other eigenvalues of the operators $-\Delta_g + \gamma Sc$.

10. *Questions.* (a) Is there a geometric definition of $\lambda_1(g, \gamma Sc)$ (and of higher eigenvalues of $-\Delta_g + \gamma Sc$) applicable to *continuous* Riemannian metrics similarly to ■ and ⬬ from section 3.1.

(b) Is there any kind of semicontinuity of the spectra of Dirac operators $\mathcal{D} : \mathbb{S}(\mathcal{X}) \to \mathbb{S}(\mathcal{X})$ under "weak" limits of Riemannian metrics g on $X = (X, g)$ and/or under "weak" limits of connections of "twisting" vector bundles L in $\mathcal{D}_{\otimes L} : \mathbb{S}(X) \otimes L \to \mathbb{S}(X) \otimes L$?

11. *Refined Kato's Inequality* [Herzlich(Kato) 2000)], [Davaux(spectrum) 2003]. This improves **4** in the case where V is a twisted spin bundle and f is in the kernel of the twisted Dirac operator as follows.

$$|d|f|| \le \sqrt{\frac{n-1}{n}}|\nabla f|.$$

Accordingly, the above **6** and **7** *should hold* for $\lambda_1(X, \frac{(n-1)Sc}{4n}$, but I didn't check it carefully.

12. *Feynman- Kac Formula.* The Kato inequality, implies further bounds on the spectrum of the ∇^2 that acts on sections of the bundle $V \to X$, by the spectrum of $-\Delta$, namely the inequality

$$trace(\exp -t\nabla^2) \le rank(V) \cdot trace(\exp t\nabla^2),\ t > 0,$$

which follows from the point-wise inequality between the corresponding heat kernels.

Remarkably, the latter (trivially!) follows from an identity – *Feynman-Kac formula*, that says that

the heat value $H_{\nabla^2}(x_0, x_1) : V_{x_0} \to V_{x_1}$ is equal to the *average of the parallel transport s* from the fiber V_{x_1} to V_{x_1} along "all" paths between these points, where this average is taken with respect to the *Wiener measure* on the space of paths between x_0 and x_1 in X.

13. *Question.* Can geometric inequalities on scalar curvatures of Riemannian manifolds X, at least those proven with Dirac operators, be derived from *integral identities* for natural measures in spaces of maps from graphs to X?

6.2 Logic of Propositions about the Scalar Curvature

Propositions/properties $\mathcal{P}|Sc$ concerning the scalar curvatures of Riemannian manifolds or related invariants, makes a kind of an "algebra", vaguely similar to how it is in algebraic topology, where properties of invariants $\mathcal{P}|Sc$ can be modified, generalized, stabilized in a systematic manner, e.g. those concerning X and Y, can be coupled to corresponding propositions, let them be only conjectural, concerning the *Riemannian products* $X \times Y$.

Then these hybridised propositions can be developed/generalized to statements on

fibrations over Y *with* X*-like fibers*

and then further to

foliations with X*-leaves, where a properly understood (non-commutative?) space of leaves is taken for* Y*.*

Conjectural Example: Lichnerowicz × Llarull × Min-Oo. Let $\underline{X}$ be the product of the the hyperbolic space by the unit sphere,

$$\underline{X} = \mathbf{H}^n \times S^n.$$

Let X be a complete orientable spin Riemannian manifold, such that $Sc(X) \geq 0$. Let $f : X \to \underline{X}$ be a smooth proper map with the following two properties.

$\bullet_{S^n}$ The S^n-component $f_{S^n} : X \to S^n$ of f, that is the composition of f with the projection $\underline{X} = \mathbf{H}^n \times S^n \to S^n$, is an *area contracting*, e.g. 1-*Lipschitz* map.

$\bullet_{\mathbf{H}^n}$ The $\mathbf{H}^n$-component of f is a *Riemannian submersion at infinity*:

the map $f_{\mathbf{H}^n} : X \to \mathbf{H}^n$ is a *submersion* outside a compact subset in X, where the differential $df_{\mathbf{H}^n} : T(X) \to T(\mathbf{H}^n)$ is *isometric* on the orthogonal complement to the kernel of $df_{\mathbf{H}^n}$.

Then either $Sc(X) = 0$, or the $\hat{A}$-genera of the pullbacks $f^{-1}(\underline{x}) \subset X$ of generic points $\underline{x} \in \underline{X}$ vanish.

In particular, if $dim(X) = 2n$ and $Sc(X, x_0) > 0$ at some $x_0 \in X$, then $deg(f) = 0$.

Theorem Generalisation. There other avenues for generalizations of results on scalar curvature. Below we indicate directions of some of these "avenues" mentioned in the previous sections.

• from manifolds to distance and area controlled maps between manifolds • from closed manifolds to manifolds with boundaries, where the mean curvature is bounded from below;

• from manifolds to manifolds with boundaries to manifolds with corners;

• from (X, g), where $Sc(g) > 0$ to to $(X, Sc(g) \cdot g$;

• from X to $X \times \mathbb{R}^N$;

• from complete to non-complete manifolds with long necks;

• from properties of compact manifolds Y with $Sc(X) \geq \sigma$ to similar properties of generic point-pullbacks $Y = f^{-1}(\underline{x})$ of smooth proper distance decreasing maps $f : X \to \underline{X}$, $Sc(X) \geq \sigma$ and $\underline{X}$ is a "large" manifold, e.g. $\underline{X} = \mathbb{R}^m$.

Suggestions to the Reader. Hybridize/generalize various theorems/inequalities from the previous as well as of the following sections. More specifically, formulate and prove whenever possible counterparts of results for n-dimensional manifolds with $Sc \geq \sigma$ to $n - N$ dimensional ones with $Sc \geq \sigma$ and which admit an isometric (possibly non-free) action of the torus $\mathbb{T}^N$.

6.3 Almost flat Fibrations, K-Cowaist and *max*-Scalar Curvature

Much of what follows in this section and in 6.4 and 6.5 represents an attempt to find geometric counterparts to the foliated $Sc \geq 0$ non-existence theorems based on the *Connes' fibration* idea.[401]

Let let P and Q be Riemannian manifolds, let $F : P \to Q$ be a smooth fibration. and let $\underline{\nabla}$ be the connection defined by the *horizontal tangent (sub) bundle* on P that is the orthogonal complement to the *vertical* subbundle of $T(P)$, where "vertical" means "tangent to the fibers" called $S_q = F^{-1}(q) \subset P$, $q \in Q$.

[401] See [Connes(cyclic cohomology-foliation) 1986], [Bern-Heit(enlargeability-foliations) 2018], [Zhang(foliations) 2016] and also [Zhang(foliations:enlargeability) 2018], [Su(foliations) 2018] and [Su-Wang-Zhang(area decreasing foliations) 2021] for a definite results in his direction.

Problem. Find relations between the K-cowaists$_2$ and between max-scalar curvatures of P, Q and the fibers $F^{-1}(q)$ for fibrations with "small" curvatures $|curv|(\underline{\nabla})$.[402]

We already know in this regard the following

(A) If $P \to Q$ is a *unitary vector bundle with a non-trivial Chern number*, then, by its very definition, *K-cowaist$_2(Q)$ is bounded from below* by $\frac{const_n}{|curv|(\underline{\nabla})}$.

(B) There is a fair bound on $Sc^{\max}$ of product spaces $P = Q \times S$, such as the rectangular solids, for instance, as is shown by methods of minimal hypersurfaces and of stable μ-bubbles in section 5.4.

In what follows, we say a few words about (A) for non-unitary bundles in the next section and then turn to several extensions of (B) to non-trivial fibrations.

6.3.1 Unitarization of Flat and Almost Flat Bundles.

Let Q be a closed oriented manifold and start with the case where $L \to Q$ is a *flat* vector bundle with a structure group G, e.g. the orthogonal group $O(N_1, N_2)$.

Let some characteristic number of L be non-zero, which means that the classifying map $f : Q \to \mathsf{B}(G)$ sends the fundamental class $[P]_{\mathbb{Q}}$ to a non-zero element in $H_n(\mathsf{B}(G); \mathbb{Q})$.[403]

Then X admits no metric with $Sc > 0$.

First Proof. Let $\Gamma \subset G$ be the monodromy group of L and recall (see section 4.1.2) that Γ properly and discretely acts on a product $\underline{X}$ of Bruhat-Tits building. Since this $\underline{X}$ is $CAT(0)$ and $Sc(P) > 0$, the homology homomorphism $H_n(P; \mathbb{Q}) \to H_n(\mathsf{B}(\Gamma); \mathbb{Q})$ induced by the classifying map $f_\Gamma : P \to \mathsf{B}\Gamma$ is zero.

Since the classifying map $f : Q \to \mathsf{B}(G)$ factors through $f_\Gamma : P \to \mathsf{B}\Gamma$ via the embedding $\Gamma \hookrightarrow G$, the homomorphism $H_n(P; \mathbb{Q}) \to H_n(\mathsf{B}(G); \mathbb{Q})$ is zero as well and the proof follows.

Second Proof? Let $K \subset G$ be the maximal compact subgroup and let S be the quotient space, $S = G/K$ endowed with a G-invariant Riemannian metric.

Let $\mathcal{S}_*$ be the space of L_2-spinors on S twisted with some bundle $L_* \to S$ associated with the tangent bundle of S and let $\mathscr{S}_* \to Q$ be the corresponding Hilbert bundle over Q with the fiber $\mathcal{S}_*$.

Apparently, an argument by Kasparov (see below) implies that, at least under favorable conditions on G, a certain generalized *index of the Dirac operator* on Q twisted with $\mathscr{S}_* \to Q$ is *non-zero*; hence, Q carries a *non-zero harmonic* (possibly almost harmonic) *spinor* and the proof follows by revoking the Schroedinger-Lichnerowicz-Weitzenboeck formula.

Kasparov KK-Construction. Let G be semisimple, and observe that the quotient space $S = G/K$ carries a G-invariant metric with non-positive sectional curvature.

[402] Recall that the K-cowaists$_2$ defined in section 4.1.4 measure area-wise sizes of spaces, e.g. K-$cowaist_2(S) = area(S)$ for simply connected surfaces and K-$cowaist_2(S^n) = 4\pi$, while max-scalar curvature of a metric space P defined in section 5.4.1 is the supremum of scalar curvatures of Riemannian manifolds X that are in a certain sense are greater than P.

[403] If G is compact, or if $G = GL_N(C)$, then $H_n(\mathsf{B}(G); \mathbb{Q})$, then the homology homomorphism $f_* : H_i(Q, \mathbb{Q}) \to H_i(\mathsf{B}(G); \mathbb{Q})$, $i > 0$, for *flat* bundles L, but it is not so, for instance, if $G = O(N_1, N_2)$ with $N_1, N_2 > 0$.

Take a point $s_0 \in S$ and let $\tau_0(s) = \tau_{s_0}(s)$ be the gradient of the distance function $s \mapsto dist(s, s_0)$ on S regularized at r_0 by smoothly interpolating between $r \mapsto dist(s, s_0)^2$ in a small ball around s_0 with $dist(s, s_0)$ outside such a ball.

Let $\tau_0^\bullet : \mathcal{S}_* \to \mathcal{S}_*$ be the Clifford multiplication by $\tau_0(r)$, that is $\tau_0^\bullet : s \mapsto \tau_0(r) \bullet s$, $s \in \mathcal{S}_*$.

Discreetness Assumption. Let the monodromy subgroup $\Gamma \subset G$ be discrete and let us restrict the space $\mathcal{S}_*$ and the $\tau_0^\bullet$ to a Γ orbit $\Gamma(s) \subset S$ for a point $r \in R$ different from r_0

Then, according to an *observation by Mishchenko* [Mishchenko(infinite-dimensional) 1974] the resulting on the space of spinors restricted to $\Gamma(s)$,

$$\tau^\bullet_{s_0,\Gamma} = \tau^\bullet_{s_0|\Gamma(s)} : \mathcal{S}_{*|\Gamma(s)} \to \mathcal{S}_{*|\Gamma(s)},$$

has the following properties:

$(\star)$ $\tau^\bullet_{s_0,\Gamma}$ *is Fredholm;*

$(\star\star)$ $\tau^\bullet_{s_0,\Gamma}$ *commutes with the action of* Γ *modulo compact operators in the following sense: the operators*

$$\tau^\bullet_{\gamma(s_0),\Gamma} - \tau^\bullet_{s_0,\Gamma} : \mathcal{S}_{*|\Gamma(s)} \to \mathcal{S}_{*|\Gamma(s)}$$

are compact for all $r \notin \Gamma(s_0)$ *and all* $\gamma \in \Gamma$.

These properties and the contractibility of S, show, by an elementary extension by skeleta argument [Mishchenko(infinite-dimensional) 1974], that

$(\star\star\star)$ *the (graded) Hilbert bundle* $\mathcal{S}_{*|\Gamma} \to Q$ *admits a Fredholm endomorphism homotopically compatible with* $\tau^\bullet_{s_0,\Gamma}$.

Finally, a *K-theoretic index computation* in [Kasparov(index) 1973], [Kasparov(elliptic) 1975] and/or in [Mishch 1974] yields

$(\star\star\star\star)$ *non-vanishing of the index of the Dirac operator on* Q *twisted with* $\mathscr{S}_{*|\Gamma}$ in relevant cases (which delivers non-zero harmonic spinors on Q and the issuing $Sc(Q) \not>)$ conclusion in our case). [404]

Now, let us *drop the discreetness assumption* and make the above (Γ-equivariant) construction(s) fully G-equivariant.

The (unrestricted to an orbit $\Gamma(s) \subset S$) $\tau_0^\bullet : \mathcal{S}_* \to \mathcal{S}_*$ seems at the first sight no good for tis purpose:

the properties $(\star)$ and $(\star\star)$ fails to be true for it, since the space $\mathcal{S}_*$ of L_2-spinors on S is too large and "flabby".

On the positive side, the space $\mathcal{S}_*$ may contain a G-invariant subspace, roughly as large as $\mathcal{S}_{*|\Gamma}$, namely the subspace of *harmonic* spinors in it. But the $\tau_0^\bullet$ doesn't, not even approximately, keeps this space invariant. However – this is an idea of Kasparov, I presume, – one can go around this problem by invoking the full Dirac $\mathcal{D} : \mathcal{S}_* \to \mathcal{S}_*$, rather than its kernel alone.

[404] The properties $(\star)$ and $(\star\star)$, however simple, establish the key link between geometry and the index theory. These were discovered and used by Mishchenko in the ambience of the Novikov higher signatures conjecture and the Hodge, rather than the Dirac, operator on manifolds with non-positive sectional curvatures.

It seems, no essentially new geometry-analysis connection has be been discovered since, while $(\star\star\star\star)$ grew into a fast field of the KK-theory of C^*-algebras in the realm of the non-commutative geometry.

Namely, we add the following extra structure to $\mathcal{S}_*$:

(A) the action of the Dirac operator $\mathcal{D}$ or rather of the technically more convenient first order operator

$$\mathcal{E} = \mathcal{D}(1 - \mathcal{D}^2)^{\frac{1}{2}} : \mathcal{S}_* \to \mathcal{S}_*$$

:

(B) the action of continuous functions ϕ with compact supports in S.

These functions $\phi(s)$ act on spinors by multiplication, where this action, besides *commuting with the action by* G,

commute with $\mathcal{E}$ *modulo compact s.*

Now, because of (A) and(B), a suitably generalized index theorem applies, I guess, and, under suitable topological conditions, yields non-zero (almost) harmonic spinors on Q.[405]

Problem. Does the above (assuming it is correct) generalises to non-flat bundles $L \to Q$?

Namely,

is there a natural Hilbert bundle $\mathscr{S} \to Q$ associated with L and having its curvature bounded in terms of that of L and such that $\mathscr{S}$ carries an additional structure, such as a (graded) Fredholm endomorphism, that would yield, under some topological conditions, *non-zero harmonic* (or almost harmonic) $\mathscr{S}$-twisted spinors on Q via a suitable index theorem?[406]

Generalized Problem. Does the above generalizes further to fibrations with variable fibers with nonpositive sectional curvatures?

Namely, let $F : P \to Q$ be a smooth fibrations between complete Riemannian manifolds, where the fibers $S_q = f^{-1}(q) \subset P$ are simply connected and the induced metrics in which have non-positive sectional curvatures.

Let a connection in this fibration be given by a horizontal subbundle $T^{hor} \subset T(P)$, that is the orthogonal complement to the vertical bundle – the kernel of the differential $dF : T(P) \to T(Q)$.

Let $[q, q'] \subset Q$ be a (short) geodesic segment between $q, q' \in Q$ and let $[p, p']^\sim \subset P$ be a horizontal lift of $[q, q']$.

We don't assume that the holonomy transformations $S_q \to R_{q'}$ are isometric and let

(1) $maxdil_p(\varepsilon)$ be the supremum of the norm of the differentials of the transformations $S_q \to S_{q'}$ at $p \in S_q$ for all horizontal path $[p, p']^\sim \subset P$ of length $\leq \varepsilon$ issuing from $p \in P$;

and

(2) $maxhol_p(\varepsilon, \delta)$ be the supremum of $dist(p, p')$ for all horizontal paths $[p, p']^\sim$ of length$\leq \varepsilon$, where p' lies in the fiber of p, i.e. $F(p') = F(p) = q$ and where there is a smooth surface $S \subset P$ the boundary of which is contained in the union of the path $[p, p']^\sim$ and the fiber F_q which contains p and p' and such that $area((S) \leq \delta^2$.

[405] I couldn't find any explicit statement of this kind in the literature, but it must be buried somewhere under several layers of KK-theoretic formalism, which fills pages of the books and articles I looked into.

(In my article [G(positive) 1996]), §$8\frac{1}{2}$, I mistakenly use a simplified argument of composing $\tau_0^\bullet$ with a projection on $ker(\mathcal{D})$)

[406] "Almost flat" generalizations of the "flat" Lusztig signature theorem are given in §§$8\frac{3}{4}, 8\frac{8}{9}$ of [G(positive) 1996].

Can one bound $\inf_q Sc(Q,q)$, or, more generally, $max\text{-}Sc(Q)$ in terms of bounds on the functions $\log maxdil_p(\varepsilon)$ and $maxhol_p(\varepsilon,\delta)$, for all (small) $\varepsilon,\delta>0$ and all $p\in P$?

6.3.2 Comparison between Hyperspherical Radii and K-cowaists of Fibered Spaces

A. The methods of minimal hypersurfaces and of stable μ-bubbles from section 5.4 that deliver fair bounds on $Sc^{\max}$ of product spaces P, such as the rectangular solids, for instance, dramatically fail (unless I miss something obvious) for fibrations with *non-flat connections* because of the following.

Distortion Phenomenon. What may happen, even for (the total spaces of) *unit m-sphere bundles* P with orthogonal connections $\underline{\nabla}$ over closed Riemannian manifolds Q, where *the hyperspherical radius is large, and the curvature is small,* say

$$Rad_{S^n}(Q)=1,\ n=dim(Q),\ \text{and}\ |curv|(\underline{\nabla})\leq\varepsilon,$$

is that, at the same time,

$$Rad_{S^{m+n}}(P)\leq\delta,\ \ m+n=dim(P),$$

where $\varepsilon>0$ and $\delta>0$ can be *arbitrarily small.*[407]

This possibility is due to the fact that, in general, P admits *no Lipschitz controlled retractions* to the spherical fibers of our fibration, even if the fibration is topologically trivial and continuous retractions (with uncontrollably large Lipschitz constants) do exits, where

non-triviality of monodromy, say at $q\in Q$ can make the distance function $dist_P$ on the fiber $S^m_q\subset P$ *significantly smaller* than the (intrinsic) spherical metric.

Example. Let Q be obtained from the unit sphere S^2 by adding ε-small handles at finitely many points which are together ε-dense in S^2 and such that Q goes to S^2 by a 1-Lipshitz map of degree one.[408]

Let $P\to Q$ be a topologically trivial flat unit circle bundle, such that the monodromy rotations $\alpha\in\mathbb{T}^1$ of he fiber $S_q=S^1$ around the loops at $q\in Q$ of length $\leq\delta$ are δ-dense in the group $\mathbb{T}^1$ for all $q\in Q$.

Then, clearly, $Rad_{S^3}(P)\leq 10\delta$, where δ can be made arbitrarily small for $\varepsilon\to 0$, whilst the trivial fibration has large hyperspherical radius, namely, $Rad_{S^3}(Q\times S^1)=1$.

B. Metric distortion of the fibers of the fibration $P\to Q$ has, however, little effect on the K-cowaist of P, that can be used, instead of the hyperspherical radius, as a measure of the size of P and that allows non-trivial bounds on $Sc^{\max}(P)$ for *spin* manifolds P with a use of twisted Dirac operators.

In practice, to make this work, one needs vector bundles with unitary connections over the base Q and over the manifold S isometric to the fibers $S_q\subset P$, call these bundles $L_Q\to Q$ and $L_S\to S=S_q$, where the following properties of these bundles are essential.

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[407] This doesn't happen if the action of the structure group on the fiber of our fibration has bounded displacement, see (2) in section 6.3.1.

[408] E.g. let the handles lie outside (the ball bounded by) the sphere $S^2\subset\mathbb{R}^3$ and let our map be the normal projection $Q\to S^2$.

$\bullet_{\rm I}$ *Monodromy Invariance of* L_S. The bundle $L_S \to S$, where S is isometric to the fibers S_q of the fibration $P \to Q$, must be *equivariant* under the action of the monodromy group G of the connection $\underline{\nabla}$ on the fibers S_q of the fibration $P \to Q$.

(Recall that an equivariance structure on a bundle L over a G space S is an *equivariant lift* of the action of G on S to an action of G on L.)

If a bundle $L_S \to L$ is G-equivariant, it extends fiberwise to a bundle over P, call it $L_\updownarrow \to P$.

(An archetypical example of this is the tangent bundle $T(S)$ which extends to what is called call the vertical tangent bundle for all fibration with S-fibers. But, in general, actions of groups G on S do not lift to vector bundles $L \to S$. However, such lifts may become possible for suitably modified spaces S and/or bundles over them.)

$\bullet_{\rm II}$ *Homologically Substantiality of the two Vector Bundles.* Some Chern numbers. of the bundles L_S and L_Q must be *non-zero.*

$\bullet_{\rm III}$ *Non-vanishing of* $F^*[Q]^\circ_{\mathbb{Q}} \in H^n(P;\mathbb{Q})$. The image of the fundamental cohomology class $[Q]^\circ \in H^n(Q)$, $n = dim(Q)$, under the rational cohomology homomorphism induced by $F : P \to Q$ doesn't vanish,

$$F^*[Q]^\circ \neq 0.$$

(This is satisfied, for instance, if the fibration $P \to Q$ admits a section $Q \to P$.)

Granted $\bullet_{\rm I}$-$\bullet_{\rm II}$-$\bullet_{\rm III}$, there exists a vector bundle $L^\times \to P$, which is equal to a tensor product of exterior powers of the "vertical bundle" $L_\updownarrow \to P$ and $F^*(L_Q) \to P$ (that is F-pull back of L_Q) and such that a *suitable* Chern number of $L^\times$ doesn't vanish.

Here "suitable" is what ensures *non-vanishing of the index* of the twisted Dirac operators $\mathcal{D}_{\otimes f^*(L^\times)}$ on manifolds X mapped to P by maps $f : X \to P$ with *non-zero degrees.* (Compare with $5\frac{1}{4}$ in [G(positive) 2016].)

Then bounds on curvatures of the bundles L_S and L_Q together with such a bound for $\underline{\nabla}$ and also a bound on *parallel displacement of the G action on S* (see below) yield a bound on $|curv|(L^\times)$, which implies a bound on $Sc^{\max}(P)$ according to the twisted Schroedinger-Lichnerowicz-Weitzenboeck formula applied to the operators $\mathcal{D}_{\otimes f^*(L^\times)}$ on manifolds X mapped to P by (smoothed) *1-Lipschitz maps* $f : X \to P$, used in the definition of $Sc^{\max}(P)$.

Parallel Displacement. The geometry of a G-equivariant unitary bundle $L = (L, \nabla)$ over a Riemannian G-space S is characterized, besides the (norm of the) curvature of ∇, by the difference between the parallel transform and transformations by small $g \in G$.

To define this, fix a norm in the Lie algebra of G and let $|g|$, $g \in G$ denote the distance from g to the identity in the corresponding "left" invariant Riemannian metric in G.

Then, given a transformation $g : S \to S$ and a lift $\hat{g} : L \to L$ of it to L, compose it with the parallel translate of it back to L along shortest curves (geodesics for complete S) between all pairs $s, g(s) \in S$. Denote by $\hat{g} \div \nabla : L \to L$ the resulting endomorphism and let

$$|\hat{G} \div \nabla| = \limsup_{|g| \to 0} \frac{||(\hat{g} \div \nabla) - \mathbf{1}||}{|g|},$$

where $\mathbf{1} : L \to L$ is the identity endomorphism () and $||...||$ denotes the norm.

Notice at this point that the curvature of the connection $\underline{\nabla}$ takes values in the Lie algebra of G and the norm $|curv|(\underline{\nabla})$, similarly to the above "parallel displacement", depends on a choice of the norm in this Lie algebra.

If S is compact, we agree to use the norm equal to the sup-norms of the corresponding vector fields on S, but one must be careful in the case of non-compact S. (Compare with (2) in section 6.3.1.)

6.3.3 $Sc^{\max}$ and $Sc^{\max}_{sp}$ for Fibrations with Flat Connections

Let P and Q be closed orientable Riemannian manifolds and let us observe that what happens to the non-spin and spin *max*-scalar curvatures and of the K-cowaists[409] of fibrations $P \to Q$ with flat connections, follows from what we know for trivial fibrations over covering spaces $\tilde{Q} \to Q$. [410]

(A) *If the monodromy group of a flat fibration of) $F : P \to Q$ is* finite *and the map F is* 1-Lipschitz, *then*

$$\star_{waist_2} \qquad Sc^{\max}_{sp}(P) \le const_{m+n} \cdot \max\left(\frac{1}{K\text{-}cowaist_2(Q)}, \frac{1}{K\text{-}wast_2(S)}\right),$$

$$\star_{Rad^2} \qquad Sc^{\max}(P) \le const'_{m+n} \cdot \max\left(\frac{1}{Rad^2_{S^n}(Q)}, \frac{1}{Rad^2_{S^m}(S)}\right)$$

and

$$\star_{sp,Rad^2} \qquad Sc^{\max}_{sp}(P) \le (m+n)(m+n-1) \cdot \max\left(\frac{1}{Rad^2_{S^n}(Q)}, \frac{1}{Rad^2_{S^m}(S)}\right)$$

for $n = dim(Q)$ and $m = dim(S)$, where S is the fiber of our fibration $P \to Q$.

In fact, these reduce to the corresponding inequalities for the product $\tilde{P} = \tilde{Q} \times S$ for the finite(!) covering $\tilde{P}$ of P, induced from the monodromy covering $\tilde{Q} \to Q$, where

• in the case $\star_{waist_2}$, one uses the tensor product of the relevant vector bundles over $\tilde{Q}$ and S and where the $\otimes$-product bundle can be pushed forward from $\tilde{P}$ back to P, if one wishes so;

[409] K-cowaist$_2(P)$ is the reciprocal of the *infimum of the norms of the curvatures* of unitary bundles over P with *non-zero* Chern numbers.

$Sc^{max}(P)$ is the *supremum* of σ, such that P admits an equidimensional *1-Lipschitz* map with *non-zero* degree from a closed Riemannian manifold X with $Sc \ge \sigma$, and where X in the definition of Sc^{max}_{sp} must be *spin*).

The hyperspherical radius $Rad_{S^N}(P)$, $N = dimP$, the *supremum* R_{max} of *radii of the spheres* $S^N(R)$, which receive *1-Lipshitz* maps from P of *non-zero* degree.

It is (almost) 100% obvious that $Rad_{S^N}(S^N) = 1$, it is not hard to show that K-cowaist$_2(P)$ is 4π, that the equality $Sc^{max}_{sp}(S^N) = Sc(S^N) = N(N-1)$ follows from Llarull's' inequality for twisted Dirac operators and it remains unknown if $Sc^{max}(S^N) = Sc^{max}_{sp}(S^N) = Sc(S^N) = N(N-1)$ for $N \ge 5$ (see section 5.5 for $N = 4$).

[410] *A flat structure (connection)* in a fibration $F : P \to Q$ with S-fibers is defined for arbitrary topological spaces Q, S and P, as a Γ-equivariant splitting $\tilde{F} : \tilde{P} = \tilde{Q} \times S \to \tilde{Q}$ for some Γ-covering $\tilde{Q} \to Q$ and the induced covering $\tilde{P} \to P$.

In the present case we assume that our Q and S, hence P, are compact *orientable pseudo-manifolds* with piecewise smooth Riemannian metrics, where $\tilde{P} = \tilde{Q} \times S$ carries the (piecewise) Riemannian product metric and the action of Γ on $\tilde{P}$ is isometric.

• in the case $\star_{Rad^2}$, the (obvious) inequalities

$$Rad_{S^{n+m}}(\tilde{P}) \geq Rad_{S^{n+m}}(P)$$

– the finiteness of monodromy is crucial in this one – and

$$Rad_{S^{n+m}}(\tilde{Q} \times S) \geq min(Rad_{S^n}(\tilde{Q}), Rad_{S^m}(S))$$

allows a use of the "cubical bounds" from the previous section, which need no spin condition, while the corresponding sharp inequality $\star_{sp,Rad^2}$ for spin manifolds P follows from Llarull's theorem.

(B) *If the monodromy group Γ of the fibration $P \to Q$ is* infinite*, then the above argument yields the following modifications of the inequalities $\star_{sp,Rad^2}$, $\star_{sp,Rad^2}$ and $\star_{waist_2}$.*

$\star^{\infty}_{Rad^2}$ The two Rad^2 inequalities $\star_{Rad^2}$ and $\star_{sp,Rad^2}$ for spin manifolds P remain valid for infinite monodromy, if $Rad_{S^n}(Q)$ *is replaced in these inequalities by* $Rad_{S^n}(\tilde{Q})$ *for a (now infinite)* Γ*-covering* $\tilde{Q}$ *of* Q.

(The universal covering of Q serves this purpose but the monodromy covering gives an a priori sharper result.)

$\star^{\infty}_{waist_2}$ *One keeps $\star_{waist_2}$ valid for infinite $\underline{\nabla}$-monodromy by replacing K-$cowaist_2(Q)$ by K-$waist_2(\tilde{Q})$.*[411]

Remarks. (a) *Sharpening the Constants.* Our argument allows improvements of the above inequalities as we shall see, at least for $\star_{sp,Rad^2}$, in the following sections.

(b) *On Displacement and Distortion.* None of the above inequalities contains corrections terms for parallel displacement defined earlier in section 6.3.2, albeit it may result in a decrease of the hyperspherical radii of P due to distortion of the fibers $S \subset P$ as the example in section 6.3.2 shows.

Notice at this point that the presence of large distortion is inevitable for fibrations with non-compact fibers, where the monodromy along short loops has unbounded displacement.

Example. Let Q be a surface and $P \to Q$ an $\mathbb{R}^2$-bundle with an orthogonal connection, the curvature form of which doesn't vanish, and let g be a Riemannian metric on P which agrees with the Euclidean metrics in the $\mathbb{R}^2$-fibers and such that the map $P \to Q$ is a Riemannian fibration, i.e. it is isometric on the horizontal subbundle in $T(P)$ corresponding to the connection.

The the Euclidean distance between points in the fibers,

$$p_1, p_2 \in \mathbb{R}^2_q \subset P, \; q \in Q$$

is related to the g-distance in P as follows

$$dist_{\mathbb{R}^2}(p_1, p_2) \sim (dist_P(p_1, p_2))^2 \text{ for } dist_{\mathbb{R}^2}(p_1, p_2) \to \infty.$$

(This is the same phenomenon as the distortion of central subgroups in two-step nilpotent groups.)

[411] It is known [Brun-Han(large and small) 2009] that the hyperspherical radius can drastically decrease under infinite coverings but the situation with K-$cowaist_2$ remains unclear.

6.3.4 Even and Odd Dimensional Sphere Bundles

$Sc^{\max}_{sp}$-Bound for Sphere Bundles. Let P and Q be closed orientable spin manifolds, where P serves as the total space of a unit m-sphere bundle $F : P \to Q$ with an orthogonal connection $\underline{\nabla}$.

If the map $F : P \to Q$ is 1-Lipschitz[412] *and if the cohomology class*

$$F^*[Q]^\circ_{\mathbb{Q}} \in H^n(P;\mathbb{Q}),\ n = dim(Q),$$

doesn't vanish (as in $\bullet_{\rm III}$ in section 6.3.2), then the spin max-scalar curvature of P (defines with spin manifolds X mapped to P) is bounded in terms of the hyperspherical radius $R = Rad_{S^n}(Q)$ and of the norm of the curvature of $\underline{\nabla}$ as follows:

$$[\rtimes S^m] \qquad Sc^{\max}_{sp}[P] \le const \cdot (1 + \underline{\epsilon}) \cdot (Sc(S^n(R)) + Sc(S^m)),$$

where, recall, $Sc(S^n(R)) = \frac{n(n-1)}{R^2}$, $Sc(S^m) = m(m-1)$, where $const = const_{m+n}$ is a universal constant (specified later) and where *$\underline{\epsilon}$ is a certain positive function $\underline{\epsilon} = \underline{\epsilon}_{m+n}(\underline{c})$, for $\underline{c} = |curv|(\underline{\nabla})$, such that*

$$\underline{\epsilon}_{m+n}(\underline{c}) \to 0 \text{ for } \underline{c} \to 0.$$

Proof. Start by observing that if either $m = 0$ or $n = 0$, then $[\rtimes S^m]$ with $const = 1$ reduces to Llarull's inequality, which says in these terms, e.g. for Q, that

$$Sc^{\max}(Q) \le \frac{n(n-1)}{Rad^2_n(Q)} = Sc(S^n(R)).$$

What we need in the general case if we want $const = 1$ is a complex vector bundle $L \to P$ with non-zero top Chern number and such that the normalised curvature (defined in section 2.8.) satisfies

$$|curv|_{\otimes\mathbb{S}}(L) \le \frac{Sc(S^n(R)) + Sc(S^m(1)}{4} + const' \cdot \underline{\epsilon}.$$

Now, let $m = dim(S = S^m)$ and $n = dim(Q)$ be even and observe that the non-vanishing condition $F^*[Q]^\circ_{\mathbb{Q}} \ne 0$ always holds for *even dimensional* sphere bundles.

Also observe that S^m and Q support bundles needed for our purpose, call them L_S and L_Q, where L_S is the positive spinor bundle $\mathbb{S}^+(S^m) \to S = S^m$ and $L_Q \to Q$ is induced from the spinor bundle $\mathbb{S}^+(S^n(R))$ by a 1-Lipschitz map $Q \to S^n(R)$ with non-zero degree.

One knows that the top Chern numbers of these bundle don't vanish and, according to Llarull's calculation,

$$|curv|_{\otimes\mathbb{S}}(L_S) = \frac{1}{4} Sc(S^m) = \frac{1}{4} m(m-1)$$

and

$$|curv|_{\otimes\mathbb{S}}(L_Q) \le \frac{1}{4}(Sc(S^n(R)) = \frac{n(n-1)}{4R^2}.$$

[412]The role of this "1-Lipschitz" is seen by looking at the trivial fibrations $P = Q \times S \to Q$ and also at *Riemannian* fibrations $F : P \to Q$ (the differentials of) which are *isometric* on the horizontal (sub)bundle. In general, when the metrics in the horizontal tangent spaces may vary, estimates on $Sc^{\max}(P)$ should incorporate along with , besides $curv(\underline{\nabla})$, (a certain function of) these metrics. (Observe, that the scalar curvature of P itself is influenced by the first and second "logarithmic derivatives" of these metrics.)

Since the (unitary) bundle $L_S \to S^m$ is *invariant* under the action of the spin group, that is the double covering of $SO(m)$, [413] it defines a bundle $L_\updownarrow \to P$, the curvature of which satisfies

$$|curv|(L_\updownarrow) = |curv|(L_S) + O(\underline{\epsilon}).$$

Then all one needs to show is that the tensor product of

$$L = L^{\rtimes} = L_\updownarrow \otimes F^*(L_Q),$$

satisfies

$$|curv|_{\otimes\mathbb{S}}(L) \le \frac{Sc(S^n(R)) + Sc(S^m(1))}{4} + const' \cdot \underline{\epsilon}.$$

This follows by a multilinear-algebraic computation similar to what goes on in the paper by Llarull, where, I admit, I didn't carefully check this computation.

But if one doesn't care for sharpness of *const*, then a direct appeal to the $\otimes_\varepsilon$-Twisting Principle formulated in section 3.3 correctedsuffices.

Remark. Even the non-sharp version of $[\rtimes S^m]$, unlike how it is with a non-sharp bound $Rad_{S^n}(X) \le const_n(\inf_x Sc(X,x))^{-\frac{1}{2}}$, $n = dim(X)$, *can't be proved* at the present moment *without Dirac operators*, which necessitate spin as well as compactness (sometimes completeness) of our manifolds.

Odd Dimensions. If $n = dim(Q)$ is odd, multiply P and Q by a long circle, and then either of the three arguments, used in the odd case of Llarull's theorem which are mentioned in section 3.4.1 and referred to [Llarull(sharp estimates) 1998], [Listing(symmetric spaces) 2010] and [G(inequalities) 2018], applies here.

Now let n be even and the dimension m of the fiber be odd. Here we multiply the fiber S, and thus P by $\mathbb{R}$, and endow the new fiber, call it $S' = S^m \times \mathbb{R}$ with the bundle $L_{S'}$ over it, which is induced by an $O(m+1)$-*equivariant* 1-Lipschitz map $S^m \times \mathbb{R} \to S^{m+1}$, which is *locally constant at infinity.* Since the curvature of the new fibration $P' = P \times \mathbb{R} \to Q$ is equal to that of the original one of $\underline{\nabla}$ in $P \to Q$, the proof follows via the relative index theorem.

Remarks/Questions. (a) Is there an alternative argument, where, instead of $\mathbb{R}$, one multiplies the fiber S with the circle $\mathbb{T}$, and uses, in the spirit of Lusztig's argument, the obvious $\mathbb{T}$-family of flat connection in it.

(b) Is there a version of the inequality $[\rtimes S^m]$, which is sharp for $|curv|(\underline{\nabla})$ far from zero?

(c) What are $Sc^{\max}_{sp}$ of the Stiefel manifolds of orthonormal 2-frames in the Euclidean $\mathbb{R}^n$, Hermitian $\mathbb{C}^n$ and quaternion $\mathbb{H}^n$?[414]

6.3.5 K-Cowaist and $Sc^{\max}$ of Iterated Sphere Bundles, of Compact Lie Groups and of Fibrations with Compact Fibers

Classical compact Lie groups are equivariantly homeomorphic to iterated sphere bundles.

[413] This bundle is *not* $SO(m)$ -invariant, but I am not certain if this is truly relevant.

[414] Notice that $St_2(\mathbb{C}^2) = S^3$ and $St_2(\mathbb{H}^2) = S^7$, but not all invariant metrics on Stiefel manifolds are symmetric.

Also notice that the corresponding (Hopf) fibrations $F : P = S^3 \to Q = S^2$ and $F : P = S^7 \to Q = S^4$ have $F^*[Q]^\circ = 0$ in disagreement with the above condition $\bullet_{\mathrm{III}}$; this makes one wonder whether this condition is essential.

For instance, $U(k)$ is equal to the complex Stiefel manifold of Hermitian orthonormal k-frames $St_k(\mathbb{C}^k)$, where $St_i(\mathbb{C}^n)$ fibers over $St_{i-1}(\mathbb{C}^n)$ with fibres $S^{2(k-i)-1}$ for all $i = 1, ..., k$.

Since the rational cohomology of $U(k)$ is the same as of the product $S^1 \times S^3 \times ... \times S^{2k-1}$, these fibrations satisfy the above non-vanishing condition $\bullet_{III}$, which implies by the above $[\rtimes S^m]$ that

the product $U(k) \times \mathbb{R}^k$ carries a $U(k)$-invariant bundle, which is trivialized at infinity, such that the top Chern number of it is non-zero.

This, by the argument from the previous section, delivers

complex vector bundles with *curvature controlled* unitary connections and *non-vanishing* top Chern classes over total spaces P of principal $U(k)$-fibrations $F : P \to Q$, *provided* $F^*[Q]^\circ_{\mathbb{Q}} \neq 0$ (that is the above $\bullet_{III}$).

This yields

a lower bound on the K-cowaist of $P \times \mathbb{T}^k$,

which, in turn, implies, the following.

Corollary 1. Let $F : P \to Q$ be a principal $U(k)$-fibration with a unitary connection $\underline{\nabla}$, where the map F is 1-Lipschitz and $F^*[Q]^\circ_{\mathbb{Q}} \neq 0$.[415]

Then

$$[\rtimes U(k)], \qquad Sc^{\max}_{sp}[P] \leq const_{m+k} \cdot (1+\underline{\epsilon}) \cdot \left(\frac{n(n-1)}{Rad_{S^n}(Q)^2} + const_k\right),$$

where $\underline{\epsilon}$ is a certain positive function $\underline{\epsilon} = \underline{\epsilon}_{k+n}(\underline{c})$, for $\underline{c} = |curv|(\underline{\nabla})$, such that

$$\underline{\epsilon}_{k+n}(\underline{c}) \to 0 \text{ for } \underline{c} \to 0.$$

Now let us state and prove a similar inequality for topologically trivial fibrations with *arbitrary compact holonomy groups G.*

Corollary 2. Let S and Q be compact connected orientable Riemannian manifolds of dimensions $m = dim(S)$ and $n = dim(Q)$ and let G be a compact isometry group of S endowed with a biinvariant Riemannian metric.[416]

Let $F_{pr} : P_{pr} \to Q$ be a principal G-fibration with a G-connection $\underline{\nabla}$ and with a Riemannian metric on P_{pr}, which agrees with our metric on the G-fibers, for which the action of G is isometric and for which the differential of the map F_{pr} is isometric on the $\underline{\nabla}$-horisontal tangent bundle $T_{hor}(P_{pr}) \subset T(P_{pr})$.

Let $F : P \to Q$ be an associated S-fibration that is

$$P = (P_{pr} \times S)/G$$

where the quotient is taken for the diagonal action of G.

Endow P with with the Riemannian quotient metric.

$[\rtimes S_G]$ *Let $F_{pr} : P_{pr} \to Q$ be a topologically* (but not, in general geometrically) *trivial fibration* (i.e. $P_{pr} = Q \times G$ with the obvious action by G).

There exists a positive constant $\underline{c}_0$ and a function $\underline{\varepsilon} = \underline{\varepsilon}_{m+n}(\underline{c})$, $0 \leq \underline{c} \leq \underline{c}_0$, where $\underline{\varepsilon} \to 0$ for $\underline{c} \to 0$, and such that if $|curv|(\underline{\nabla}) = \underline{c} \leq \underline{c}_0$, then the spin max-scalar curvature of P is bounded by

[415] For a principal fibration, this is a very strong condition, saying, in effect, that the fibration is "rationally trivial".

[416] If G is disconnected "Riemannian" refers to the connected components of G.

$$Sc^{\max}_{sp}[P] \leq const_* \cdot (1+\underline{\epsilon}) \cdot \left(\frac{n(n-1)}{Rad_{S^n}(Q)^2} + \frac{m(m-1)}{Rad_{S^m}(S)^2} + const_G \right).$$ [417]

Proof. Embed G to a unitary group $U(k)$ and let $F_U : P_U \to Q$ be the fibration with the fiber $U = U(k)$ associated to $F_{pr} : P_{pr} \to Q$.

Let $P^U \to Q$ be the fibration with the fibers $S_q \times U_q$, $q \in Q$ and observe that this P^U fibers over P with U-fibers and over P_U with S-fibers, where the latter is a *trivial fibration.*

To show this it is enough to consider the case, where P is the principal fibration P_{pr} for which $P^U = P_{pr} \times U$ and P_U is the quotient space, $P_U = (P_{pr} \times U)/G$ for the diagonal action of G.

Then the triviality of the principal G-fibration $P^U \to P_U$ is seen with the map $P^U \to U = U(k)$ for $\{G_q \times U_q\} \mapsto U_q = U$ which sends the diagonal G-orbits from all $G_q \times U_q$ to $G \subset U(k) = U$.

Thus, assuming $m = dim(S)$ is even (the odd case is handled by multiplying by the circle as earlier) we obtain an *upper bound* on spin max-scalar curvature of $P^U = P_U \times S$ in terms of the K-cowaist of P_U and $Rad_{S^m}(S)$.

On the other hand, if the fibration $P \to Q$ has curvature bounded by $\underline{c}$, the same applies to the induced fibration $P^U \to P$ with U-fibers, and since the (biinvariant metric in the) unitary group $U = U(k)$ has positive scalar curvature, the max-scalar curvature of P^U is *bounded from below by one half of that for* P for all sufficiently small $\underline{c}$ and when $\underline{c} \to 0$ these estimate converge to what happens to Riemannian product $P = Q \times S$.

Confronting these upper and lower bounds yields a qualitative version of [$\rtimes S_G$], while completing the proof of the full quantitative statement is left to the reader.

About the Constants. A Llarull's kind of computation seems to show that the above inequalities hold with $const_{m+n} = const_* = 1$.

6.4 K-Cowaist and Max-Scalar Curvature for Fibration with Non-compact Fibers

Let $P \to Q$ be a Riemannian fibration where the fiber S is a complete contractible manifold with non-positive sectional curvature and such that the monodromy of the natural connection $\underline{\nabla}$ in this fibration (defined by the horizontal tangent subbundle $T^{hor} \subset T(P)$) *isometrically* acts on S.

Problem. (Compare with "Generalized Problem" in section 6.3.1.) Is there a lower bound on the K-$cowaist_2(P)$ in terms of such a bound on K-$cowaist_2(Q)$ and on an upper bound on the norm of the curvature of $\underline{\nabla}$ that can be represented by the function $maxhol_p(\varepsilon, \delta)$ as in (2) of section 6.3.1?

6.4.1 Stable Harmonic Spinors and Index Theorems.

Our primarily interest in such a lower bound is that it would yield an *upper bound* on the *proper spin max-scalar* curvature of P.

[417] I apologise for the length of this statement that is due to so many, probably redundant, conditions needed for the proof.

This "proper spin max-scalar" is defined via proper 1-Lipschitz maps of open spin manifolds X to P, section 5.4.1 where following recipes $\bullet_{\rm I}$, $\bullet_{\rm II}$, $\bullet_{\rm III}$, from **B** in section 6.3.2 one has to construct a (finite or infinite dimensional graded) with a unitary connection vector bundle $\mathcal{L} \to S$, which is

$\star_{\rm I}$ *invariant (modulo compact operators?) under isometries of S* (compare with $\bullet_{\rm I}$ in section 6.3.2).

and

$\star_{\rm II}$ *homologically substantial*, where this substantiality must generalize that of $\bullet_{\rm II}$ by properly *incorporating the action of the isometry group G of S.* (An inviting possibility is the above $L^{\otimes_N}$.)

What one eventually needs is not such a bundle $\mathcal{L} \to S$ per se, but rather some Hilbert space of sections for a class of related bundles over P, where

(i) a suitable *index theorem,* e.g. in the spirit of our the second "proof" in section 6.3.1 (with a *Hilbert C^*-module $\mathscr{H}$* over the *reduced C^*-algebra* of the group G being utilized),

and where

(ii) *the Schroedinger-Lichnerowicz-Weitzenboeck formula* applies to twisted harmonic L_2-spinors delivered by such a theorem and provides a bound on the scalar curvature of P.

Who is Stable? Harmonic spinors delivered by index theorems (and also spinors with a given asymptotic behaviour as in Witten's and Min-Oo's arguments) are stable under certain deformations (and some discontinuous modifications, such as surgeries) of the metrics and bundles in questions, albeit the exact range of these perturbation on non-compact manifolds is not fully understood.

But the Schroedinger-Lichnerowicz-Weitzenboeck formula doesn't use, at least not in a visible way, this stability, which is unlike how it is with stable minimal hypersurfaces and stable μ-bubbles.

One wonders, however,

whether there is a common ground for these two stabilities in our context.

6.4.2 Euclidean Fibrations

Let us indicate an elementary approach to the above *problem* in the case where the fiberes S of the fibration $F : P \to Q$ are isometric to the *Euclidean space.*

(1) Start with the case where the (isometric!) action of the (structure) group G on the fiber S of the fibration $P \to Q$ has a fixed point, then assume $m = dim(S)$ is even and observe that radial maps $S \to S^m$, which are constant at infinity and have degrees one, induce homologically substantial G-invariant bundles $L = L_S$ bundles on S.

Since $S = \mathbb{R}^m$, such maps can be chosen with arbitrarily small Lipschitz constants, thus making the curvatures of these bundles arbitrarily small, namely, (this is obvious) with the supports in the R-balls $B_{s_0}(R) \subset S$, around the fixed point $s_0 \in S$ for the G-action and with curvatures of our (induced from $\mathbb{S}(S^m)$) bundles $L_S = L_{S,s_0,R} \to S$ bounded by $\frac{1}{R^2}$.[418]

Then we see as earlier that in the limit for $R \to \infty$, the curvature of the bundle $L_\updownarrow \to P$, which is on the fibers $S = S_q \subset P$ is equal to $L_S \to S$, (see $\bullet_{\rm I}$

[418]It suffices to have the universal covering $\tilde{S}$ of S isometric to $\mathbb{R}^m$, where radial bundles on $\tilde{S}$ can be pushed forward to Fredholm bundles on S.

in **B** of section 6.3.2) will be bounded by the curvature of the connection $\underline{\nabla}$ on $P \to Q$, provided the map $P \to Q$ is 1-Lipschitz.[419]

Consequently,

the K-cowaist$_2$ *of* P *is bounded from below by the minimum of the K-cowaist*$_2$ *of* Q *and the reciprocal of the curvature* $|curv|(\underline{\nabla})$

(2) Next, let us deal with the opposite case, where the structure group $G = \mathbb{R}^m$, i.e. the Euclidean space $\mathbb{R}^m$ acts on itself by parallel translations.

Then, topologically speaking, the fibration $F : P \to Q$ is trivial, but the above doesn't, apply since this $P \to Q$ typically admits *no parallel section.*

But since the $\underline{\nabla}$-monodromy transformations, that are parallel translations on the fiber $S = \mathbb{R}^m$, have bounded displacements, there exists a continuous *trivialization map*

$$G : P \to Q \times \mathbb{R}^n,$$

which, assuming Q is compact, (obviously) has the following properties.

(i) The fibers $\mathbb{R}^m_q \subset P$ are *isometrically* sent by G to $\mathbb{R}^m = \{q\} \times \mathbb{R}^m \subset Q \times \mathbb{R}^m$ for all $q \in Q$.

(ii) The composition of G with the projection $Q \times \mathbb{R}^m \to \mathbb{R}^m$, call it

$$G_{\mathbb{R}^m} : P \to \mathbb{R}^M$$

is *1-Lipshitz on the large scale,*

$$dist(G^m_{\mathbb{R}}(q_1, q_2)) \leq dist(q_1, q_2) = cost.$$

It follows by a standard *Lipschitz extension* argument, that, for an arbitrary $\varepsilon > 0$, there exists a smooth map

$$G'_\varepsilon : P \to Q \times \mathbb{R}^m,\ \varepsilon > 0,$$

which is properly homotopic to G and such that the corresponding map

$$G'_{\varepsilon, \mathbb{R}^m} : P \to \mathbb{R}^m$$

is λ-Lipschitz for $\lambda \leq m + n + \varepsilon$, where, recall, $m + n = dim(P)$

Now, the concern expressed in **A** of section 6.3.2 notwithstanding, the μ-bubble splitting argument from section 5.3 applies and shows that

(a) *the stabilized max-scalar curvature of* P defined via products of P with flat tori *is bounded, up to a multiplicative constant, by that of* Q.

Besides, the existence of fiberwise contracting scalings of P, which fix a given section $Q \to P$, show that

(b) *if* Q *is compact and if* m *is even, then the K-cowaist*$_2$ *of* P *is bounded from below, by that of* Q.

Notice here, that

unlike most previous occasions, neither a bound on the curvature of the fibration $P \to Q$ is required, nor the manifold X in the definition of the max-scalar curvature mapped to P need to be spin.

And besides dispensing of the spin condition, one may allow here

[419]The parallel displacement contribution to the curvature of $L_\updownarrow$ (see **B** of section 6.3.2)) cancels away by an easy argument.

non-complete manifolds Q and X and/or manifolds in (a) and compact manifolds Q with boundaries in (b).

(3) Finally, let us turn to the general case where the structure group of a fibration $P \to Q$ with the fiber $S = \mathbb{R}^m$ is the full isometry group G of the Euclidean space $\mathbb{R}^m$.

Recall that G is a the semidirect product, $G = O(m) \rtimes \mathbb{R}^m$, let $P_G \to Q$ be the principal bundle with fiber G associated with $P \to Q$ and let $P_O \to P$ be the associated $O(m)$ bundle. Let

$$P_O \leftarrow P_G \to P$$

be the obvious fibrations.

Now, granted a bound on the Lipschitz constant of $F : P \to Q$ and the curvature of this fibration, we obtain

(i) *a bound on the max-scalar curvature of the space P_G in terms of such a bound on P*

In fact, the curvature of the fibration $P_G \to P$ as well as its Lipschitz constant are bounded by those of $F : P \to Q$ and our bound (i) follows from non-negativity of the scalar curvature of the fiber $O(m)$ of this fibration by the (obvious) argument used in section 6.3.5.

Then we look at the fibrations $P_G \to P_O \to Q$ and observe that

(ii) the fibration $P_O \to Q$ has $O(m)$-fibers and, thus the K-$cowaist_2(P_O)$ is bounded from below by that of Q as it was shown in section 6.3.5;

(iii) the fibration $P_G \to P_O$ has $\mathbb{R}^m$-fibers and the structure group $\mathbb{R}^m$ and, by the above (2), the K-$cowaist_2$ of P_G is bounded from below by that of Q; hence

K-$cowaist_2(P_G)$ of P_G is bounded by K-$cowaist_2(Q)$.

We recall at this point the basic bound on $Sc^m_{sp}ax(P_G)$ by the reciprocal of the K-$cowaist_2(P_G)$, confront (i) with (iii) and conclude (similarly to how it was done in section 6.3.5) to the final result of this section.

✎✐ Let $F : P \to Q$ be a smooth fibration between Riemannian manifolds with fibers $S_q = \mathbb{R}^m$ and a connection $\underline{\nabla}$, the monodromy of which isometrically acts on the fibers. If the map F is 1-Lipschitz, then

the proper spin max-scalar curvature of P is bounded in terms of the curvature $|curv|(\underline{\nabla})$ and the reciprocal to K-$cowaist_2(Q)$.

Corollary. Let Q admit a constant at infinity area decreasing map to S^n, $n = dim(Q)$, of non-zero degree.

Let the norm of the curvature of (the connection $\underline{\nabla}$ on) a bundle $P \to Q$ with $\mathbb{R}^m$-fibers is bounded by $\underline{c}$.

Let a complete orientable Riemannian spin manifold X of dimension $m+n$ admit a proper area decreasing map to P.

Then

$$\inf_{x \in X} Sc(X, x) \le \Psi(\underline{c}),$$

where, $\Psi = \Psi_{m+n}$ is an effectively describable positive function; in fact, the above proof of ✎✐ shows that one may take

$$\Psi(\underline{c}) = (m+n)(m+n-1) + const_m \underline{c}$$

and where, probably, $(m+n)(m+n-1)$ can be replaced by $n(n-1)$.

6.4.3 Spin Harmonic Area of Fibrations With Riemannian Symmetric Fibers

Let S be a complete Riemannian manifold with a transitive isometric action of a group G which equivariantly lifts to a vector bundle $\mathbf{L}_S \to S$ with a unitary connection, such that the integrant in the local formula for the index of the twisted Dirac $\mathcal{D}_{\otimes L}$ doesn't vanish. Then a certain generalized analytic L_2-index of $\mathcal{D}_{\otimes L}$ doesn't vanish as well,[420] which implies the existence of non-zero harmonic L-twisted square summable spinors on S.

Example: Hyperbolic and Hermitian Symmetric spaces.

(a) The hyperbolic space $S = \mathbf{H}^{2m}_{-1}$ admits a non-zero harmonic L_2-spinors twisted with the spin bundle $L_S = \mathbb{S}^+(\mathbf{H}^{2m}_{-1})$ (compare with section 4.6.4).

(b) Hermitian symmetric spaces S, e.g. products of hyperbolic planes or the quotient space of the symplectic group $Sp(2k,\mathbb{R})$ by $U(k) \subset Sp(2k,\mathbb{R})$, admit non-zero harmonic L_2-spinors twisted with tensorial powers of the canonical line bundles.

Questions. (a) What are (most) general *local* conditions on pairs (X, L), where X[421] is complete Riemannian manifold and $L \to X$ is a vector bundle with a unitary connection, such that X would support non-zero L-twisted harmonic L_2-spinors, or, at least, the $\mathcal{D}^2_{\otimes L}$ would contain zero in its spectrum?[422]

(b) What happens, for example, to non-vanishing (twisted) harmonic L_2-spinors on homogenous spaces X under (small and/or big) non-homogeneous deformations of the metrics on X?

(c) Do non-vanishing harmonic L_2-spinors twisted with the spinor bundle $\mathbb{S}(X)$ exist on Riemannian manifolds X which are bi-Lipschitz homeomorphic to even dimensional hyperbolic spaces with constant sectional curvatures?

(d) Do complete simply connected Riemannian manifolds X of even dimension n with their sectional curvatures pinched between -1 and $-1-\varepsilon_n$ for a small $\varepsilon_n > 0$ carry such spinors?

Example of an Application. Let a complete oriented Riemannian n-dimensional spin manifold X admit a smooth area decreasing map to the unit sphere, $f : X \to S^n$, such that the pullback of the oriented volume form ω_{S^n} is *non-negative* on X,

$$\frac{f^*(\omega_{S^n})}{\omega_X} \geq 0,$$

and the pullback of the Riemannian metric from S^n to X is *complete*, that is the f-images of *unbounded connected* curves from X have *infinite lengths* in S^n.

Conjecture. The scalar curvature of X is bounded by that of S^n,

$$\inf_{x\in X} Sc(X, x) \leq n(n-1).$$

[420] As we have already mentioned in section 4.6.4, if S admits a free discrete cocompact isometric action of a group Γ, this is equivalent to the non-vanishing of the index of the corresponding on S/Γ [Atiyah (L2) 1976]; in general, this index is defined by Connes and Moscovici in[Connes-Moscovici($L_2 - index$ for homogeneous) 1982].

[421] We return to the notation X instead of S, since, in general, this X doesn't have to be anybody's fiber.

[422] Possibly, the answer is in [NaSchSt(localization) 2001], but I haven't read this paper and the book with the same title.

Remark. If *area-decreasing* is strengthened to ε*-Lipschitz* for a small $\varepsilon = \varepsilon_n > 0$, then this conjecture (without the *spin* assumption) might follow in many (all?) cases by the geometric techniques of section 5.

Back to Fibrations. Let $F : P \to Q$ be a fibration with the fiber S and the structure group G, let P be endowed with a complete Riemannian metric and let $L_{\updownarrow} \to P$ be the natural extension of the (G-equivariant!) bundle L_S to P (compare with section 6.3.2).

Let $L_Q \to Q$ be a vector bundle with a unitary connection. and let

$$L^{\rtimes} = F^*(L_Q) \otimes L_{\updownarrow} \to P$$

.

Conceivably there must exist (already exists) an index theorem for the Dirac operator on P twisted with the bundle $L^{\rtimes}$ that would ensure the existence of non-zero twisted harmonic L_2-spinors on P under favorable topological and geometric conditions.

For instance, if Q is a complete Riemannian of *even* dimension n, if the bundle L_Q is induced from the spin bundle $\mathbb{S}^+(S^n)$ by a smooth constant at infinity map $Q \to S^n$ of positive degree, if P is spin and if the map $F : P \to Q$ is isometric on the horizontal subbundle in $T(P)$, then, *conjecturally*,

the manifold P supports a non-zero $L^{\rtimes}$-twisted harmonic L_2-spinor.

In fact this easy if the fibration is flat, e.g. if the fibration $P = Q \times S$ and, if the curvature of this fibration is (very) small, then a trivial perturbation argument as in section 4.6.4 yields almost harmonic spinors on large domains $P_R \subset P$.

But what we truly wish is the solutions of the following counterparts to (A) and (B) from section 4.6.4.

Let $F : P_R \to Q_R$ be a submersion between compact Riemannian manifolds with boundaries, where

$$R = \sup_{p \in P} dist(p, \partial P)$$

and where the local geometries of the fibers are δ-close (in a reasonable sense) to the geometry of an above homogeneous S and let $L_{\rtimes,R} \to P_R$ be a vector bundle, also δ-close (in a reasonable sense) to an above $L_{\rtimes}$.

(A_F) When does P_R support a ε-harmonic $L_{\rtimes,R}$-twisted spinor which vanishes on the boundary of P?

(B_F) When does a similar spinor exist on a manifold $\overline{P}_R$, which admits a map to P_R with non-zero degree and with a controlled metric distorsion? (See section 4.6.4 for a specific conjecture in this direction.)

6.5 Scalar Curvatures of Foliations

Let X be a smooth n-dimensional manifold and $\mathscr{L}$ a smooth foliation of X that is a smooth partition of X into $(n-k)$-dimensional leaves, denoted $\mathcal{L}$.

Let $T(\mathscr{L}) \subset T(X)$ denote the tangent bundle of $\mathscr{L}$ and Recall that the *transversal (quotient) bundle* $T(X)/T(\mathscr{L})$ carries a natural *leaf-wise flat affine connection* denoted $\nabla^{\perp}_{\mathcal{L}}$, where the parallel transport is called *monodromy.*

This $\nabla^{\perp}_{\mathcal{L}}$ can be (obviously but non-uniquely) extended to an actual (non-flat) connection on the bundle $T(X)/T(\mathscr{L}) \to X$, which is called *Bott connection.*

Two Examples (1) Let $\mathscr{L}$ admit a transversal k dimensional foliation, say $\mathscr{K}$ and observe that the bundle $T(X)/T(\mathscr{L}) \to X$ is canonically (and obviously) isomorphic to the tangent bundle $T(\mathscr{K})$.

Thus, every $\mathscr{K}$-leaf-wise connection in the tangent bundle $T(\mathscr{K})$, e.g. the Levi-Civita connection for a leaf-wise Riemannian metric in $\mathscr{K}$, defines a $\mathscr{K}$-leaf-wise connection, say $\nabla_{\mathscr{K}}$ of $T(X)/T(\mathscr{L})$

Then there is a unique connection on the bundle $T(X)/T(\mathscr{L}) \to X$, which agrees with $\nabla^{\perp}_{\mathcal{L}}$ on the $\mathscr{L}$-leaves and with $\nabla_{\mathscr{K}}$ on the $\mathscr{K}$-leaves, that is the Bott connection.

(2) Let the bundle $T(X)/T(\mathscr{L}) \to X$ be topologically trivial and let $\partial_i : X \to T(X)$, $i = 1, ..., k$, be linearly independent vector fields transversal to $\mathscr{L}$. Then there exists a unique Bott connection, for which the projection of ∂_i to $T(X)/T(\mathscr{L})$ is parallel for the translations along the orbits of the field ∂_i for all $i = 1, ..., k$.

In what follows, we choose a Bott connection on the bundle $T(X)/T(\mathscr{L}) \to X$ and denote it $\nabla^{\perp}_{X}$.

Also we choose a subbundle $T^{\perp} \subset T(X)$ complementary to $T(\mathscr{L})$, which, observe, is canonically isomorphic to $T(X)/T(\mathscr{L})$, where this isomorphism is implemented by the quotient homomorphism $T^{\perp} \subset T(X) \to T(X)/T(\mathscr{L})$.

With this isomorphism, we transport the connections $\nabla^{\perp}_{\mathcal{L}}$ and $\nabla^{\perp}_{X}$ from $T(X)/T(\mathscr{L}$ to $\nabla^{\perp}_{X}$ keeping the notations unchanged. (Hopefully, this will bring no confusion.)

6.5.1 Blow-up of Transversal Metrics on Foliations

Let $g = g_{\mathscr{L}}$ be a leaf-wise Riemannian metric on the foliation $\mathscr{L}$, that is a positive quadratic form on the bundle $T(\mathscr{L})$, let $g^{\perp}$ be such a form on $T^{\perp}$ and observe that the sum of the two $g^{\oplus} = g \oplus g^{\perp}$ makes a Riemannian metric on the manifold X.

This metric itself doesn't tell you much about our foliation $\mathscr{L}$, but the family

$$g^{\oplus}_{e} = g \oplus e^2 g^{\perp},\ \ e > 0,$$

is more informative in this respect, especially for $e \to \infty$. For instance,

[a] if the metric $g = g_{\mathscr{L}}$ has *strictly positive scalar curvature*, i.e. $Sc_g(\mathcal{L}) > 0$ for all leaves $\mathcal{L}$ of $\mathscr{L}$, and, this is essential, if the metric $g^{\perp}$ is *invariant under the monodromy along the leaves* $\mathcal{L}$ – foliations which comes with such a $g^{\perp}$ are called *transversally Riemannian*, – then, assuming X is *compact*,

$$Sc(g^{\oplus}_{e}) > 0$$

for all sufficiently large $e > 0$.

Proof of [a]. Let $x_0 \in X$, let $\mathcal{L}_0 = \mathcal{L}_{x_0} \subset X$ be the leaf which contains x_0 and observe that the pairs pointed Riemannian manifolds $(X_e, \mathcal{L}_0 \ni x_0)$ for $X_e = (X, g^{\oplus}_{e})$ converge to the (total space of the) Euclidean vector bundle $T^{\perp}$ restricted to $\mathcal{L}_0$ with the metric

[⊕] $$g_{\lim} = g_{\mathcal{L}_0} \oplus g^{\perp}_{Eu},$$

where $g_{\mathcal{L}} = g_{\mathscr{L}}|\mathcal{L}_0$, where $g^{\perp}_{Eu} = g^{\perp}_{Eu}(l)$, $l \in \mathcal{L}_0$, is the a family of the Euclidean metrics in the fibers of the bundle $T^{\perp}|\mathcal{L}_0$ corresponding to $g^{\perp}$ on $\mathcal{L}_0$, and where "$\oplus$" refers to the local splitting of this bundle via the (flat!) connection $\nabla^{\perp}_{\mathscr{L}}|\mathcal{L}_0$.[423] The scalar curvature of the metric $g_{\mathcal{L}_0} \oplus g^{\perp}_{Eu}$ is determined by

the scalar curvature of the leaf $\mathcal{L}_0$ and the first and second (covariant) logarithmic derivatives of $g^{\perp}_{Eu}(l)$,

where $g^{\perp}_{Eu}(l)$ is regarded as a function on $\mathcal{L}_0$ with values in the space of (positive) quadratic forms on $\mathbb{R}^k$, which in the case $g^{\perp}_{Eu}(l) = \varphi(l)^2 g_0$ reduces to the "higher warped product formula" from section 2.4.1:

$$(\star\star_{\mathcal{L}}) \qquad Sc(\varphi(l)^2 g_0)(l,r) = Sc(\mathcal{L}_0)(l) - \frac{k(k-1)}{\varphi^2(l)}\|\nabla\varphi(l)\|^2 - \frac{2k}{\varphi(l)}\Delta\varphi(l),$$

where $(l,r) \in \mathcal{L}_0 \times \mathbb{R}^k$ and $\Delta = \sum \nabla_{i,i}$ is the Laplace on $\mathcal{L}_0$.

Since, in general, these "logarithmic derivatives" denoted $g^{\perp}_{Eu}(l)'/g^{\perp}_{Eu}(l)$ and $g^{\perp}_{Eu}(l)''/g^{\perp}_{Eu}(l)$ are the same as of the original (prelimit) metric $g^{\perp}(l)$, it follows, that

$(\star\star_{Sc})$
$$Sc(g_{\mathcal{L}_0} \oplus g^{\perp}_{Eu}) \geq Sc(g_{\mathcal{L}_0}) - const_n\left(\|(g^{\perp}(l)'/g^{\perp}(l))^2\| + \|g^{\perp}(l)''/g^{\perp}(l)\|\right).$$

In particular, if $g^{\perp}$ is constant with respect to $\nabla^{\perp}_{\mathscr{L}}|\mathcal{L}_0$, then the limit metric $g_{\lim}$ locally is the Riemannian product $(\mathscr{L}, g_{\mathscr{L}}) \times \mathbb{R}^k$ with the scalar curvature equal to that of $\mathscr{L}$. QED.

However obvious, this immediately implies

[a1] *vanishing of the $\hat{A}$-genus as well as of its products with the Pontryagin classes of $T^{\perp}$* for *transversally Riemannian* foliations on closed spin manifolds X, where the "product part" of this claim follows from the *twisted* Schroedinger-Lichnerowicz-Weitzenboeck formula for the Dirac operator $\mathcal{D}_{\otimes T^{\perp}}$, since the *curvature of the (Bott connection in the) bundle $T^{\perp} \to X$ converges to zero for $e \to \infty$*.

(This is not *formally* covered by Connes' theorem stated in section 3.15, where the spin condition must be satisfied by $\mathscr{L}$ rather than X itself as it is required here; but it can be easily derived from Connes' theorem.)

Another equally obvious corollary of [$\oplus$] is as follows.

[a2] *If $Sc(\mathscr{L}) > n(n-1)$ and if X is closed orientable spin, then X admits no map $f : X \to S^n$, such that $deg(f) \neq 0$ and such that the restrictions of f to the leaves of $\mathscr{L}$ are 1-Lipschitz.*

But this is not fully satisfactory, since it it *remains unclear*

if one truly needs the inequality $Sc(\mathscr{L}) > n(n-1)$ or $Sc(\mathscr{L}) > (n-k)(n-k-1)$ for $n-k = dim(\mathscr{L})$ will suffice?

Exercise. Show that $Sc(\mathscr{L}) > 2$ does suffice for 2-dimensional foliations.

Flags of Foliations. Let

$$\mathscr{L} = \mathscr{L}_0 \prec \mathscr{L}_1 \prec \dots \prec \mathscr{L}_j,$$

[423] The limit space $(T^{\perp}, g_{lim})$ can be regarded as the *tangent cone of X at $\mathcal{L}_0 \subset X$*, where the characteristic feature of this cone is its scale invariance under multiplication of the metric g_{lim} normally to $\mathcal{L}_0$ by constants.

where the relation $\mathscr{L}_{i-1} \prec \mathscr{L}_i$ signifies that $\mathscr{L}_i$ *refines* $\mathscr{L}_{i-1}$, which means the inclusions between their leaves,

$$\mathcal{L}_i \subset \mathcal{L}_{i-1},$$

and where $\mathscr{L}_0$ is the bottom foliation with a single leaf equal X.:

Let $T_i^\perp = T_i^\perp \subset T(\mathscr{L}_{i-1})$, $i = 1, 2, ..., j$ be transversal subbundles isomorphic to $T(\mathscr{L}_{i-1})/T(\mathscr{L}_i)$, let $g_j = g_{\mathscr{L}_j}$ be a $\mathscr{L}_j$-leaf-wise Riemannian metric, let $g_i^\perp$, $i = 1...j$,, be Riemannian metrics on $T_i^\perp$ and let

$$g^\oplus_{e_1,...,e_j} = g_0 \oplus e_1 g_1^\perp \oplus ... \oplus e_j g_j^\perp, \ e_i > 0, .$$

[b] *If the metrics in the quotient bundles $T(\mathscr{L}_{i-1})/T(\mathscr{L}_i)$, $i = 1, ..., j$, which corresponds to $g_i^\perp$, are invariant under holonomies along the leaves of $\mathscr{L}_j$, if $e_i \to \infty$, then*

$$Sc(g^\oplus_{e_1,...,e_j}) \to Sc(g_j),$$

where this convergence is uniform on compact subsets in X.

Proof. Since

the logarithmic derivatives of maps from Riemannian manifolds to the Euclidean spaces tend to zero as the metrics in these manifolds are scaled by constants $\to \infty$,

the above $(\star\star_{Sc})$ implies the following.

[$\mathbf{b}_{lim}$] The pair of pointed Riemannian manifolds $(X_{e_1,...,e_j}, \mathcal{L}_j \ni x_j)$, for all leaves $\mathcal{L}_j$ of $\mathscr{L}_j$ and all $x_j \in L_j$, converges to the (total space of the) flat Euclidean vector bundle $T_1^\perp \oplus ... \oplus T_j^\perp \to \mathcal{L}_j$, where

the limit metric on (the total space of) $T_1^\perp \oplus ... \oplus T_j^\perp$ locally splits as

[$\oplus_i\perp$], $$g_{\lim} = g_{\mathcal{L}_j} \oplus g_{Eu} \otimes g_{Eu,1(l)},$$

where g_{Eu} is the Euclidean metric on $\mathbb{R}^{k_2+...k_i+...+k_j}$ for $k_i = rank(T_i^\perp)$ and $g_{Eu,1(l)}$, $l \in \mathcal{L}_j$ is a family of Euclidean metrics in the fibers of the bundle $T_1^\perp \to X$ restricted to $\mathcal{L}_j$, where the logarithmic derivatives of these metrics are equal these for the original (prelimit) metrics in the bundle $T_1^\perp$ over $\mathcal{L}_j$.

Now, we see, as earlier, that [$\mathbf{b}_{lim}$] $\Rightarrow$ [b] and the proof follows.

Thus, the above [a1] and [a2] generalize to transversally Riemannian flags of foliations

6.5.2 Connes' Fibration

Let the "normal" bundle $T^\perp \to X$ to a foliation $\mathscr{L}$ on X admits a smooth G-structure for a subgroup G of the linear group $GL(k)$, $k = codim(\mathscr{L})$, which (essentially) means that the monodromy transformation for the above canonical flat leaf-wise connection $\nabla^\perp_{\mathscr{L}}$ are contained in G.

For instance, being Riemannian for a foliation is the same as to admit $G = O(k)$ and $G = GL(k)$ serves all foliation.

Let G *isometrically* act on a Riemannian manifold S and let $P \to X$ be a fibration associated to $T^\perp \to X$.

Then the monodromy of $\nabla^\perp_{\mathscr{L}}$ is isometric on the fibers $S_x \subset P$.

Principal Example.[Con(cyclic cohomology) 1986] Let

$$G = GL(k) \text{ and } S = GL(k)/O(k)$$

and let us identify the fiber S_x, for all $x \in X$, with the space of Euclidean structures, i.e. of positive definite quadratic forms, in the linear space $T_x^\perp$.

Clearly, this S canonically splits as

$$S = R \times \mathbb{R} \text{ for } R = SL(k)/SO(k),$$

where, observe, R carries a unique up to scaling $SO(k)$-invariant Riemannian (symmetric) metrics with non-positive sectional curvature and where the $\mathbb{R}$-factor is the logarithm of the central multiplicative subgroup $\mathbb{R}_+^\times \subset GL(k)$.

Thus, $S = R \times \mathbb{R}$ carries an invariant Riemannian product metric, call it g_S, which is unique up-to scaling of the factors.

Next, observe that the tangent bundle $T(P)$ splits as usual

$$T(P) = T^{vert} \oplus T^{hor}$$

where T^{vert} consists of the vectors tangent to the fibers $S_x \subset P$, $x \in X$, and where T^{hor}

is the horizontal subbundle corresponding to the Bott connection, and where the splitting $T(X) = T(\mathscr{L}) \oplus T^\perp$ lifts to a splitting of T^{hor}, denoted

$$T^{hor} = \tilde{T}(\mathscr{L}) \oplus \tilde{T}^\perp.$$

Thus, the tangent bundle $T(P)$ splits into sum of three bundles,

$$T(P) = T^{vert} \oplus \tilde{T}(\mathscr{L}) \oplus \tilde{T}^\perp,$$

where, to keep track of things, recall that

$$rank(\tilde{T}(\mathscr{L})) = dim(\mathscr{L}) = n - k,\ \ rank(\tilde{T}^\perp) = codim(\mathscr{L}) = k$$

and

$$rank(T^{vert}) = dim(GL(k)/O(k)) = \frac{k(k+1)}{2}.$$

Let us record the essential features of these three bundles and their roles in the geometry of the space P (see [Connes(cyclic cohomology-foliation) 1986] and compare with §$1\frac{7}{8}$ in [G(positive) 1996]).

(1) *Metric $\tilde{g}^\perp$ in $\tilde{T}^\perp$.* The (sub)bundle $\tilde{T}^\perp \subset T(P)$ carries a *tautological metric* call it $\tilde{g}^\perp$, which, in the fiber $\tilde{T}_p^\perp \subset \tilde{T}^\perp$ for $p \in P$ over $x \in X$, is equal to this very $p \in P_x$ regarded as a metric in $T_x^\perp \subset T^\perp \to X$.

(2) *Foliation $\mathscr{L}^+$ of P.* The leaves $\mathcal{L}^+ \subset P$ of this foliations are the pullbacks of the leaves $\mathcal{L}$ of $\mathscr{L}$ under the map $P \to X$. These $\mathcal{L}^+$ have dimensions $n - k + \frac{k(k+1)}{2}$ and the tangent bundle $T(\mathscr{L}^+)$ is canonically isomorphic to $\tilde{T}(\mathscr{L}) \oplus T^{vert}$.

(3) Foliation $\tilde{\mathscr{L}}$ of P. This is the natural lift of the original foliation $\mathscr{L}$ of X:

the leaf $\tilde{\mathcal{L}}_p$ of $\tilde{\mathscr{L}}$ through a given point $p \in P$ over an $x \in X$ is equal to the set of the Euclidean metrics in the fibers $T_l^\perp \subset T^\perp \to X$ for all $l \in \mathcal{L}_x \subset X$, which are obtained from p, regarded as such a metric in $T_x^\perp \subset T^\perp \to X$, by the monodromy along the leaf $\mathcal{L}_x$ of the foliation $\mathscr{L}$ of X.

This foliation can be equivalently defined via its tangent (sub)bundle, that is

$$T(\tilde{\mathscr{L}}) = \tilde{T}(\mathscr{L}) \subset T(P).$$

Also observe that this $\tilde{\mathscr{L}}$ refines $\mathscr{L}$, written as $\tilde{\mathscr{L}} > \mathscr{L}^+$, where, in fact, the leaves of $\mathscr{L}^+$ are products of the monodromy covers of the leaves of $\mathscr{L}$ by S.

(4) $\tilde{\mathscr{L}}$-Monodromy Invariance of the Metric $\tilde{g}^\perp$. The bundle $\tilde{T}^\perp \subset T(P)$, where the metric $\tilde{g}^\perp$ resides, is naturally isomorphic to the "normal" bundle $T(P)/T(\mathscr{L}^+)$, but this metric is *not invariant* under the monodromy of the foliation $\mathscr{L}^+$.

However, $\tilde{g}^\perp$ is *invariant under the monodromy of the sub-foliation* $\tilde{\mathscr{L}} > \mathscr{L}^+$ with the leaves $\tilde{\mathcal{L}} \subset \mathcal{L}^+$ as it follows from the above description of the leaves $\tilde{\mathcal{L}}_p$ of $\tilde{\mathscr{L}}$.

(5) $\tilde{\mathscr{L}}$-Monodromy Invariance of $\tilde{g}_S$ in the Bundle T^{vert}. Since the fibration $P \to X$ with the fiber $S = GL(k)/O(k)$ is associated with $T^\perp \to X$, every $GL(k)$ metric g_S on S gives rise to a *monodromy invariant metric in the fiberes of this fibration*, which is denoted $\tilde{g}_S$ and regarded as the metric in the subbundle $T^{vert} \subset T(P)$, which made of the vectors tangent to the S-fibers and which is canonically isomorphic to $T(\mathscr{L}^+)/T(\tilde{\mathcal{L}})$.

Clearly,

this metric $\tilde{g}_S$ is invariant under the monodromy along the leaves of the foliations $\tilde{\mathscr{L}}$ on P.

(6) *Scalar Curvature under Blow-up of Metrics in $T(P)$*. Let $g = g_{\mathscr{L}}$ be a Riemannian metric in the tangent bundle $T(\mathscr{L}) \subset T(X)$ of a foliation $\mathscr{L}$ of X as earlier and let $\tilde{g}$ be its lift to the bundle $\tilde{T}(\mathscr{L}) = T(\tilde{\mathscr{L}}) \subset T(P)$.

Let $\tilde{g}_{e_S,e_\perp}$, $e_S, e_\perp > 0$, be the Riemannian metric on the manifold P that is the metric in the bundle

$$T(P) = \tilde{T}(\mathscr{L}) \oplus T^{vert} \oplus \tilde{T}^\perp.$$

where this $\tilde{g}_{e_S,e^\perp}$ is split into the sum of the metrics from th above (5) and (4). which are taken here with (large) positive e-weights as follows.

$$\tilde{g}_{e_S,e^\perp} = \tilde{g} + e_S\tilde{g}_S + e_\perp\tilde{g}^\perp.$$

Then it follows from the above **[b]**, that if

$$e_S, e_\perp \to \infty,$$

then

[$\Uparrow_{Sc}$] *the scalar curvature of the metric $\tilde{g}_{e_S,e_\perp}$ at $p \in P$ over $x \in X$ converges to that of g on the leaf $\mathcal{L}_x \ni x$ at x, where*

this convergence is uniform on the compact subsets in P.[424]

[424]This convergence property, which is implicit in [Connes(cyclic cohomology-foliation) 1986], is used in §$1\frac{7}{8}$ of [G(positive) 1996] and in "adiabatic" terms in Proposition 1.4 of [Zhang(foliations) 2016], where it is required that $e_\perp/e_S$ is large, since the shape of the compact domain in P where the scalar curvature of the metric $\tilde{g}_{e_S,e_1}$ becomes ε-close to that of g, depends on the ratio $e_\perp/e_S$, (see section 6.5.4.)

Generalizations. Much of the above (1) - (6) applies to foliations with monodromy groups G not necessarily equal to $GL(k)$ and with fibrations with the fibers that may be different from G/K, which we will approach in the following sections on the case-by-case basis.

6.5.3 Foliations with Abelian Monodromies

Let a foliation $\mathscr{L}$ of an orientable n-dimensional Riemannian manifold X admit a smooth G-structure invariant under the monodromy, where the group G is Abelian and let the scalar curvatures of the leaves with the indices Riemannian metrics are bounded from below by $\sigma > n(n-1)$.

$\square\bigcirc$. *The hyperspherical radius of X is bounded by one,*

$$Rad_{S^n}(X) \leq 1.$$

That is, if $R > 1$, then

X admits no 1-Lipschitz map to the sphere $S^n(R)$, which is constant at infinity and which has non-zero degree.

Prior to turning to the proof, that is an easy corollary of what we discussed about $\mathbb{R}^k$-fibration in section 6.4.2, we'll clarify a couple of points.

1. We don't assume here that the manifold X is compact or complete, nor do we require it is being spin.

2. We don't know if our Abelian assumption on G is essential. It is conceivable that

$\square\bigcirc$ holds for all foliations, i.e. for $G = GL(k)$, $k = codim(\mathscr{L})$, and, moreover, with the bound $Sc(\mathscr{L}) \geq (n-k)(n-k-1)$.

2. Examples of foliations with Abelian G, include:

foliations with transversal conformal structure, e.g. (orientable) foliations of codimension one, where G is the multiplicative group $\mathbb{R}^\times$;

flags of codimension one foliations (where $G = (\mathbb{R}^\times)^k$) and/or of foliations with transversal conformal structures.

Proof of $\square\bigcirc$. Let $P \to X$ be the principal fibration associated with the bundle $T(X)/T(\mathscr{L})$ and by blowing up the metric of P transversally to the lift $\tilde{\mathscr{L}}$ to P as in the previous section, make the scalar curvature of P on a given compact domain $P_\varepsilon \subset P$ greater than $n(n-1) - \varepsilon$ for a given $\varepsilon > 0$.

Also with this blow-up, make the Lipschitz constant of the map $P \to X$ as small as you want.

(A possibility of this formally follows from the above (1) - (6) for foliations of codimension one, while the proof in general case amounts to replaying (1) - (6) word-for-word in the present case.)

Next, let $G = \mathbb{R}^m$, observe as in (2) in section 6.4.2 that P_ε admits a $(1+\varepsilon)$-Lipschitz map of degree one from P_0 to $X \times [0, L]^k$ for an arbitrary large L and apply the maximality/extremality theorem for punctured spheres from sections 3.9 and 5.5.

This concludes the proof for $G = \mathbb{R}^m$ and the case of the general Abelian G follows by passing to the quotient of G by the maximal compact subgroup.

To get an idea why one can control the geometry of the blow-up only on compact subsets in P, look at the following.

Geometric Example. Let (Y, g) be a Riemannian manifold and let $P_Y \to Y$ be the fibration, with the fibers S_y, $y \in Y$, equal to the spaces of quadratic forms in the tangent spaces $T_y(Y)$ of the form $c \cdot g_y$, $c > 0$. Thus, $P_Y = Y \times \mathbb{R}$, for $\mathbb{R} = \log \mathbb{R}_+^\times$ with the metric $e^{2r}dy^2 + dr^2$.

When $r \to +\infty$ and the curvature of $e^{2r}g$ tends to zero, then the metric $e^{2r}dy^2 + dr^2$ converges to the hyperbolic one with constant curvature -1, but when $r \to -\infty$, then the curvatures of $e^{2r}g$ and of $e^{2r}dy^2 + dr^2$ blow up at all points $y \in Y$, where the curvature of g doesn't vanish.

And if apply this to the fibration $P = P_Y \times \mathcal{L} \to X = Y \times \mathcal{L}$ with the same $\mathbb{R}$-fibers, then we see that the convergence of the scalar curvatures of the blown-up P to those of $\mathcal{L}$ is by no means uniform.

6.5.4 Hermitian Connes' Fibration

Let $\mathscr{L}$ be a foliation on X of codimension k as earlier with a transversal (sub)bundle $T^\perp \subset T(X)$ and a Bott connection in it. Let $T^{\bowtie}$ be the sum of $T^\perp$ with its dual bundle and endow $T^{\bowtie}$ with the natural, hence monodromy invariant, symplectic structure.

Let S_x denote the space of Hermitian structures in the space $T_x^{\bowtie}$, for all $x \in X$, and let $P \to X$ be the corresponding fibration, that is the fibration associated with $T_x^{\bowtie}$ with the fiber $S = Sp(2k, \mathbb{R})/U(k)$.

Equivalently, this fibration $P \to X$ is associated to $T^\perp \to X$ via the action of $GL(k)$ on $S = Sp(2k, \mathbb{R})/U(k)$ for the natural embedding of the linear group $GL(k)$ to the symplectic $Sp(2k, \mathbb{R})$

Besides sharing the properties (1)-(6) of the original Connes' bundle formulated in section **??**, this new $P \to X$ has, a lovely additional feature: >??

S is a *Hermitian* (irreducible) symmetric space, which implies (see section 6.4.3) non-vanishing of the index of some twisted Dirac on S that is invariant under the isometry group (that is $Sp(2k, \mathbb{R})$ of S.

This, as it was stated in section 6.4.3 must imply the existence of twisted harmonic L_2-spinors on fibrations with S-fibers, which we formulate below in the form relevant to foliations positive scalar curvatures and which, besides being interesting in its own right, would simplify the proof by Connes in [Connes(cyclic cohomology-foliation) 1986] as well as the arguments from [Zhang(foliations) 2016].

L et $Y = (Y, \omega)$ be a closed symplectic manifold of dimension $2k$ and let $F : P_Y \to Y$ be the fibration associated with the tangent bundle $T(Y)$ with the fiber $S = Sp(2k, \mathbb{R})/U(k)$.

Observe that the quotient bundle $T(P)/T^{vert}$ carries a tautological Hermitian metric $g_{\bowtie}$, and a granted $Sp(2k, \mathbb{R})$-connection in the tangent bundle $T(Y)$, that is a horizontal subbundle $T^{hor} \subset T(P)$, one obtains a Riemannian metric g_P in the tangent bundle $T(P) = T^{vert} \oplus T^{hor}$ that is

$$g_P = g_S + g_{\bowtie}$$

where g_S is a $Sp(2k, \mathbb{R})$-invariant Hermitian metric in S, which is unique up to scaling.

Let the symplectic form ω be integer and thus serves as the curvature of a unitary line bundle $L \to Y$.

Conjecture 1 *The bundle of spinors on P twisted with some tensorial power of the bundle $F^*(L) \to P$ admits a non-zero harmonic L_2-section on P.*

Remarks and Examples. (a) The geometry of this P, unlike of what we met in section 6.4.3, is as far from being a product as in P_Y from the geometric example in section 6.5.3.

(b) The simplest instance of Y is that of an even dimensional torus $\mathbb{T}^{2k}$ with an invariant symplectic form ω and trivial flat symplectic connection.

In this case, the universal covering $\tilde{P}_Y$ of the manifold P_Y is Riemannian homogeneous; moreover, the (local) index integrant is homogeneous as well. It is probable, that a version of the Connes-Moscovici theorem applies in this case and yields twisted harmonic L_2-spinors on $\tilde{P}_Y$ and, eventually, on P_Y.

(c) It would be most amusing to find a link between the symplectic geometry of (Y,ω). and twisted Dirac operators on P_Y or their non-linear modifications.

Let us modify the above conjecture to make it applicable to foliations.

Let $S = G/K$ be a symmetric space, where the index of the Dirac twisted with some bundle $L_S \to S$ associated with the K-bundle $G \to S$ doesn't vanish, e.g. $S = Sp(2k)/U(k)$.

Let $F : P \to X$ be a smooth S-fibration with a smooth G-connection $\underline{\nabla}$ and let $T^{hor} \subset T(P)$ be the corresponding horizontal subbundle.

Let $L_\updownarrow \to P$ be the bundle the restriction of which to the fibers $S = S_x \subset P$, $x \in X$ are equal to L_S.

Let g^{hor} be a smooth Riemannian metrics (positive quadratic forms) in T_{hor} and $g_P = s^{hor} + g_S$ be the sum of this metric with a G-invariant metric in the fiber.

Let $L_X \to X$ be a vector bundle with a unitary connection ∇_X trivialized at infinity (which is relevant for non-compact manifolds X) and let $L^* = (L^*, \nabla^*) \to P$ be the bundle pulled back by F from L_X along with the connection ∇_X, that is $L^* = F^*(L_X, \nabla_X)$.

Conjecture 2. If X is complete and if some Chern number of L_X doesn't vanish, then P supports a non-zero harmonic L_2-spinor $s = s(p)$ twisted with $L_\updownarrow$ and with (i.e. (tensored with) some bundle associated with L^*.

Moreover, there exists such a non-zero spinor $s(p)$, the rate of the decay of which at infinity is independent of the metric g^{hor}:

given an exhaustion of P by compact domains, $P_1 \subset ... \subset P_i \subset ...P$, then

$$(L_2\searrow) \qquad \frac{\int_P \|s(p)\|^2 dp}{\int_{P_i} \|s(p)\|^2 dp} \geq 1 - \varepsilon(i) \underset{i\to\infty}{\to} 1,$$

where the function $\varepsilon(i) > 0$ may depends on P_i, X, F, $\underline{\nabla}$, ∇_X and g_S, but which is independent of the metric g^{hor}.

6.5.5 Hermitian Connes' Fibrations over Foliations with Positive Scalar Curvature

Let $F : P \to X$ be the Hermitian Connes' fibration with the S-fibers, $S = Sp(2k)/U(k)$, over a Riemannian manifold $X = (X, g)$ with a foliation $\mathscr{L}$ of codimension k on it , as in the previous section, let the subbundle $T^{hor} \subset T(P)$ corresponds to a Bott connection on the bundle $T^\perp \to X$ normal to the tangent subbundle $T(\mathscr{L}) \subset T(X)$ and let us lift the splitting $T(X) = T(\mathscr{L}) \oplus T^{\perp}$ lifts to the corresponding splitting $T^{hor} = \tilde{T}(\mathscr{L}) \oplus \tilde{T}^{\perp}$.

Recall that the points $p \in P$ correspond to Hermitian structures in the symplectic spaces $T^{\bowtie}_{F(p)} \supset T^{\perp}_{F(p)}$, the real parts of which give Riemannian/Euclidean structures to $T^{\perp}_{F(p)}$ and which then pass to the spaces T^{hor}_p via the the differentials $dF_p : T_p(P) \to T_{F(p)}(X)$ which are isomorphic on the fibers of $T^{hor}_p \subset T_p(P)$.

Thus, the bundle $\tilde{T}^{\perp} \to P$ carries a canonical Riemannian metric, which we call $\tilde{g}^{\perp}$.

Next, let $\tilde{g}_{\mathscr{L}}$ be the metric on the bundle $\tilde{T}(\mathscr{L})$ that is induced from the Riemannian metric $g_{\mathscr{L}}$ that is the metric g on X restricted to the bundle $T(\mathscr{L})$ and let us endow the bundle $T(P) = \tilde{T}(\mathscr{L}) \oplus \tilde{T}^{\perp} \oplus T^{vert}$, where $T^{vert} \subset T(P)$ is the bundle tangent to the S-fibers of the fibration $F : P \to Q$, with the metrics

$$\tilde{g}_{e_S,e^{\perp}} = \tilde{g}_{\mathscr{L}} + e_{\perp} \cdot \tilde{g}^{\perp} + e_S \cdot g_S, \ \ e_S, e_{\perp} > 0,$$

as this was done in (6) of section 6.5.2, except that now the fibers S are isometric to $Sp(2k)/U(k)$ with a (unique up to scaling) $Sp(2k)$-invariant metric g_S, rather than to $GL(k)/O(k)$ as in section 6.5.2.

Now as in $[\Uparrow_{Sc}]$ of section 6.5.2, we observe the following.

Effect of g_S-Blow-up . If the constant e_S is much greater than $\frac{1}{\sigma}$, then the scalar curvatures of the leaves $\mathcal{L}^+ \subset P$, which are the pullbacks of the leaves $\mathcal{L} \subset X$ under the map $P \to X$ (see (2) in section 6.5.2), become close to those of the underlying leaves $\mathcal{L}$, hence $\geq \sigma - \varepsilon > 0$, while the norms of the logarithmic covariant derivatives $\nabla_{\mathcal{L}^+}(\log \tilde{g}^{\perp})$ of the transversal metric $\tilde{g}^{\perp}$ along the leaves $\mathcal{L}^+$ with respect to the metrics $\tilde{g}_{\mathscr{L}} + e_S \cdot g_S$, becomes $\leq \varepsilon$ on the leaves $\mathcal{L}^+ \subset P$, where observe, that, given an $\varepsilon > 0$,

if $e_S = e_S(\varepsilon)$ is sufficiently large, then these two ε-bounds hold on all of P.

$\tilde{g}^{\perp}_S$-Blow-up. This has two effects on the geometry of P.

$\mathbf{1}_{Sc}$ If the scalar curvatures of the leaves $\mathcal{L}^+$ are bounded from below by $\sigma - \varepsilon$ and if the norm of $\nabla_{\mathcal{L}^+}(\log \tilde{g}^{\perp})$ is bounded by ε, then

$$Sc(P) \underset{e_{\perp}\to\infty}{\to} \sigma - \epsilon$$

where

$$\epsilon \underset{\varepsilon\to 0}{\to} 0$$

and where the convergence $Sc(P) \to \sigma - \epsilon$ is uniform on compact subsets in P (but not on all of P).

$\mathbf{2}_{Lip}$ If $e_{\perp} \to \infty$, then (regardless of e_s) the Lipschitz constant of the map $F : P \to X$ tends to zero uniformly on compact subsets in P.

Conclusion. Granted Conjecture 2 from the previous section, we see that, as far as the the Dirac operators are concerned, the positivity of the scalar curvature of $g|\mathscr{L}$ has the same effect as of the metric g on X itself.

For example, Conjecture 2 implies the following.[425]

[425] Here we use the fact that if X is spin then P is also spin, where, if you are in doubt, this implication can be achieved by taking $S \times S$ instead of S.

★ *Let $X = (X, g)$ be a complete orientable spin Riemannian n-manifold and let $\mathscr{L}$, be a smooth foliation on $X$of codimension k, such that the scalar curvature of g restricted to the leaves of $\mathscr{L}$ satisfies*

$$Sc(g_{|\mathscr{L}}) > n(n-1).$$

Then every 1-Lipschitz map $X \to S^n$ locally constant at infinity has zero degree.

Let us spell it all out again.

Let $\mathscr{D}$ be a Dirac operator on P, which is

(a) twisted with an S-fiber bundle that makes the local index of the corresponding Dirac operator on S non-zero;

(b) on the top of that the $\mathscr{D}$ is twisted with a bundle induced from a bundle L_X on X, with a non-zero Chern number, where one may assume that the integrant in the local formula for the index of $\mathscr{D}$ doesn't vanish on P.

The above shows, that if $Sc(\mathscr{L}) \geq \sigma > 0$ and the curvature of L_X is small, then, given a compact domain $P_0 \subset P$, the spectrum of the $\mathscr{D}^2_{e_S, e_\perp}$ on P_0 (that is $\mathscr{D}^2$ for the metric $\tilde{g}_{e_S, e_\perp}$ on P_0) with the zero boundary condition can be made uniformly separated away from zero, that is $\lambda_1 \geq \delta = \delta(n, \sigma) > 0$ by taking sufficiently large e_S and $e_\perp$.

But this contradicts Conjecture 2, which implies that

given $e_S \geq 1$ and $\delta > 0$, there exists a compact domain $P_0 \subset P$, such that the first eigenvalue of $\mathscr{D}^2_{e_S, e_\perp}$ on P_0 satisfies

$$\lambda_1 = \lambda_1(P_0, \mathscr{D}^2_{e_S, e_\perp}) \leq \delta.$$

To prove such a bound on λ_1, one needs to construct an almost , $\mathscr{D}$-harmonic spinor with support in P_0, where a natural pathway to this end goes along the lines of the local proof of the index theorem, roughly, as follows.

Let $\mathcal{K}$ be an that is a function of the (unbounded self-adjoint) $\mathscr{D}$ (we suppress the subindices e_S and $e_\perp$), which can be represented by a smooth kernel $\mathcal{K}(p, q)$, $p, q \in P$ supported in a d-neighbourhood of the diagonal, where this the values of $K(p, q)$ at all pints $(p, q) \in P \times P$, depends on the metric in P only in the ball of $B_{p,q}(4d) \subset P \times P$ and such that the super-trace of $\mathcal{K}$ and of all its powers $\mathcal{K}^i$ is equal to the index of $\mathscr{D}$, whenever this construction is applied to compact manifolds P.

Then, as $i \to \infty$, this converges to the projection to the kernel of $\mathscr{D}$, that is the space of harmonic L_2-spinors, and the only issue to settle in the present case is certain uniformity of this convergence, for $e_\perp \to \infty$.

What may facilitate the estimates needed for the proof of this is the uniform bound on geometries of the metrics $\tilde{g}_{e_S, e_\perp}$ for $e_S, e_\perp \to infty$, probably, where, possibly, one can get a fair representation/approximation of $\mathcal{K}^i_{e_s, e_\perp}$ by a (singular) perturbation argument at $e_\perp = \infty$.

Remark. Even if the above argument is carried thought it, as we mentioned in section 3.15, it will be not deliver what, probably, follows by Connes' argument:

"*no* 1*-Lipschitz map $f : X \to S^n$ with $deg(f) \neq 0$*" for $Sc(g_{|\mathscr{L}}) > (n-k)(n-k-1)$. [426]

[426] According to what was explained to me by Jean-Michel Bismut, the same may apply to Zhang's argument.

But it seems beyond the present day methods to drop the spin assumption in ★ for $k \geq 2$.

On Non-Integrable Generalization. Let $X = (X, g)$ be a Riemannian manifold, let $\Theta \subset T(X)$ be a smooth subbundle of codimension k and let

$$\Lambda = \Lambda(\Theta) : \wedge^2\Theta \to \Theta^{\perp}$$

be its curvature, that is the 2-form on Θ with values in the normal subbundle (identified with the quotient bundle $T(X)/\Theta$), which is defined by the normal components of commutators of pairs of tangent fields $X \to \Theta$. x $Sc(g|\Theta, x)$, $x \in X$, be the sum of the sectional curvatures over the pairs of vectors in an orthonormal basis in Θ_x.

It seems probable, that most (all?) we know and/or conjecture about (tangent subbundles of) foliations with $Sc \geq \sigma > 0$

extends to Θ with $Sc(g|\Theta) \geq \sigma > 0$, if $||\Lambda(\Theta)||$ is much smaller than σ.

(Homogeneous Θ, e.g. on spheres of dimensions $2m+1$ and $4m+1$ may serve as extremal cases of the corresponding inequalities.)

6.5.6 Geometry and Dynamics of Foliations with Positive Scalar Curvatures

Let us formulate a few versions of the width/waist conjecture from section 3.10 for foliations with $Sc > 0$.

Let X be a complete Riemannian n-manifold with a foliation $\mathscr{L}$ of codimension k, where $n - k \geq 2$ and where the scalar curvatures of the induced Riemannian metrics in the leaves satisfy $Sc \geq (n-k)(n-k-1)$.

★★ (Unrealistically?) *Strong Foliated Width/Waist Conjecture.* There exists a continuous map from X to an $(n-2)$-dimensional polyhedral space, say $F : X \to P^{n-2}$, such that

the pullback $F^{-1}(p) \subset X$ is contained in a single leaf of $\mathscr{L}$ for all $p \in P$ and

$$diam(F^{-1}(p)) \leq const_n \text{ and } vol_{n-2}(F^{-1}(p)) \leq const'_n \text{ for all } p \in P^{n-2},$$

where, conceivably, these constants don't even depend on n, e.g. with a possibility $const' = 4\pi$.

An Impossible Proof of ★★. Ideally, one would like to have a continuous family of metrics in the leaves that would eventually simultaneously collapse all leaves this factorizing X to something $n-2$-dimensional.

But the obvious candidate for this – Hamilton's Ricci flow, even if it is defined for all time, doesn't collapse X fast enough to bound $width_{n-2}$ or $waist_{n-2}$.

In fact, what happens is better seen for the mean curvature flow, where the collapsing map for ellipsoids with the principal axes of the lengths $1, 1, d$, moves this ellipsoid by distance $\sim d$, which may be arbitrary large.

Here is another way to look at this problem.

(Provisional) *Bounded Distance Deformation Question.* Let (X, g_1) be a complete Riemannian manifold with $Sc \geq 1\sigma > 0$.

Does there exist a Riemannian metric g_2 on X, such that $Sc(g_2) \geq 2$ and

$$|dist_{g_1}(x,y) - dist_{g_2}(x,y)| \leq const_n$$

for all $x, y \in X$ and some $const_n < \infty$?

3D-Foliations. Let $X = (X, g)$ be a complete Riemannian manifold and $\mathscr{L}$ be a smooth 3-dimensional foliation such that the restrictions of g to all leaves $\mathcal{L}$ of *3D-Foliations.* Let $X = (X, g)$ be a complete Riemannian manifold and $\mathscr{L}$ have $Sc_g(\mathcal{L}) \geq \sigma > 0$.

Then there exist continuous maps from $\mathcal{L}$ to locally finite 1-dimensional polyhedra, say $F : \mathcal{L} \to P^1$, such that $diam(F^{-1}(p)) \leq c < \infty$ for all leaves $\mathcal{L}$ of $\mathscr{L}$.

(Such maps exist with $c = 2\pi\sqrt{\frac{36}{\sigma}}$, see section 3.10, but what is relevant at the moment is the universal bound $c = c(\sigma) < \infty$.)

Moreover, since such maps also exist for the *universal coverings of the leaves*, the maps F can be coherently chosen on all leaves simultaneously by adapting the geometric proof of *the Stallings Ends of the Groups Theorem* to foliations (compare with §$2\frac{2}{3}$ of [G(positive) 1996]).

Let us recall this proof in the simplest case where X is a smooth manifold with a discrete cocompact action of a group Γ [427] and show that

Proof. (Compare [G(infinite) 1984]) Let $Y \subset X$ be a connected *volume minimizing* compact hypersurface which *separates two ends* of X. Then, because of minimality, no γ-translate $\gamma(Y) \subset X$, intersects Y unless $\gamma(Y) = Y$.

Then the proof easily follows, since, due to the the end separation property, the complement to the set of the translates of Y, that is

$$X \setminus \bigcup_{\gamma \in \Gamma} \gamma(Y),$$

is the union of *mutually non-intersecting subsets of diameters* $\leq const$.

Similar argument applies to foliations where one similarly achieves invariance of the relevant maps under the monodromy groupoid instead of Γ (see §$2\frac{2}{3}$ of [G(positive) 1996]) [428]

which implies, for instance the following.

$(\star_3)$ *If a compact orientable Riemannian n-manifold $X = (X, g)$ carries a 3-dimensional foliation, where the leaves have positive scalar curvatures,*

$$Sc_g(\mathcal{L}) > 0,$$

then X admits no maps of non-zero degrees to aspherical n-manifolds.

Conclude with the following purely foliation theoretic question, the positive answer to which, that, I think, is unlikely, motivated the above conjectures.

Is it true that *no smooth foliation* on $\mathbb{R}^n$ of positive dimension invariant under the action of $\mathbb{Z}^n$ [429] can have the diameters of all leaves bounded by a common constant $C < \infty$?

[427] Contrary to the statements found sometimes in the literature, all versions of Stallings' theorem, as well as of its refinements and generalizations, effortlessly follow with a *proper* use of minimal hypersurfaces.

[428] The argument from [G(positive) 1996] becomes more transparent, if one makes the metric g on X *generic* by a small perturbation, for which all compact locally minimizing hypersurfaces Y in the leaves are isolated; hence stable under small transversal deformations of leaves.

This allows a sufficient quantity of compact volume minimizing hypersurfaces $\tilde{Y}$ with $diam(\tilde{Y}) \leq const$ in the leaves $\tilde{\mathcal{L}}$ of the lift $\tilde{\mathscr{L}}$ of $\mathscr{L}$ to the universal covering $\tilde{X}$, such that the intersections of the leaves $\tilde{\mathcal{L}}$ with the complement of the union of these $\tilde{Y}$ have the diameters of all their connected components also uniformly bounded, say by $\leq const'$.

[429] Ideally, one would like to drop this invariance condition.

(An approach to counterexamples may be found in [EM(wrinkling) 1998].)

6.6 Moduli Spaces Everywhere

All topological and geometric constraints on metrics with $Sc \geq \sigma$ are accompanied by non-trivial homotopy theoretic properties of spaces of such metrics.

A manifestation of this principle is seen in how topological obstructions for the existence of metrics with $Sc > 0$ on closed manifolds X of dimension $n \geq 5$ give rise to

pairs (h_0, h_1) of metrics with $Sc \geq \sigma > 0$ on closed hypersurfaces $Y \subset X$ which *can't be joined by homotopies h_t with $Sc(h_t) > 0$.*

The elementary argument used for the proof of this (see section 3.17) also shows that (known) constraints on *geometry*, not only on topology, of manifolds with $Sc \geq \sigma$ play a similar role.

For instance, assuming for notational simplicity, $\sigma = n(n-1)$, and recalling the $\frac{2\pi}{n}$-inequality from section 3.6, we see that

(a) *if $l \geq \frac{2\pi}{n}$, then the pairs of metrics $h_0 \oplus dt^2$ and $h_1 \oplus dt^2$ on the cylinder $Y \times [-l, l]$, for the above Y and $l \geq \frac{2\pi}{n}$, can't be joined by homotopies of metrics h_t with $Sc(h_t) \geq n(n-1)$ and with $dist_{h_t}(Y \times \{-l\}, Y \times \{l\}) \geq \frac{2\pi}{n}$.*

This phenomenon is also observed for manifolds with *controlled mean curvatures of their boundaries*, e.g. for Riemannian bands X with $mean.curv(\partial_\mp X) \geq \mu_\mp$ and with $Sc(X) \geq \sigma$, whenever these inequalities imply that $dist(\partial_- X, \partial_+ X) \leq d = d(n, \sigma, \mu_\mp)$. (One may have $\sigma < 0$ here in some cases.)

Namely,

(b) *certain sub-bands $Y \subset X$ of codimension one with $\partial_\mp(Y) \subset \partial_\mp(X)$ admit pairs of metrics (h_0, h_1), such that $mean.curv_{h_0,h_1}(\partial_\mp Y) \geq \mu_\mp$ and $Sc_{h_0,h_1}(Y) \geq \sigma$ while $dist_{h_0,h_1}(\partial_-, \partial_+) \geq D$ for a given $D \geq d$. But these metrics can't be joined by homotopies h_t , which would keep these inequalities on the scalar and on the mean curvatures and have $dist_{h_t}(\partial_-, \partial_+) \geq d$ for all $t \in [0,1]$.*

(c) This seems to persist (I haven't carefully checked it) for manifolds with corners, e.g. for cube-shaped manifolds X: these, apparently contain hypersurfaces $Y \subset X$, the boundaries of which $\partial Y \subset \partial X$ inherit the corner structure from that in X, and which admit pairs of "large" metrics h_0, h_1, which also have "large" scalar curvatures, "large" mean curvatures of the codimension one faces F_i in Y and "large" complementary $(\pi - \angle_{ij})$ dihedral angles along the codimension two faces F_{ij}, but where these h_0, h_1 can't be joint by homotopies of metrics h_t with comparable "largeness" properties.

It is unclear, in general, how to extend the π_0-non-triviality (disconnectedness) of our spaces of metrics to the higher homotopy groups, since the techniques currently used for this purpose rely entirely on the Dirac theoretic techniques (see [Ebert-Williams(infinite loop spaces) 2017] and references therein), which are poorly adapted to manifolds with boundaries. But some of this is possible for closed manifolds.

For instance, let Y be a smooth closed spin manifold, and h_p, $p \in P$, be a homotopically non-trivial family of metrics with $Sc(h_p) \geq \sigma > 0$, where, for instance, P can be a k-dimensional sphere and non-triviality means non-contractibility.

Let $\mathcal{S}^m_\sigma(S^m\times Y)$ denote the space of pairs (g,f), where g is a Riemannian metric on $S^m\times Y$ with $Sc(g)\geq\sigma$ and $f:(S^m\times Y,g)\to S^m$ is a distance decreasing map homotopic to the projection $f_o:S^m\times Y\to S^m$.

If non-contractibility of the family h_p follows from non-vanishing of the index of some Dirac operator, then (the proof of) Llarull's theorem suggests that the corresponding family $(h_p+ds^2,f_o)\in\mathcal{S}^m_{\sigma_+}(S^m\times Y)$ for $\sigma_+=\sigma+m(m-1)$ is non-contractible in the space

$$\mathcal{S}^m_{m(m-1)}(S^m\times Y)\supset\mathcal{S}^m_{\sigma_+}(S^m\times Y).$$

This is quite transparent in many cases, e.g. if $h_p=\{h_0,h_1\}$ is an above kind of pair of metrics with $Sc>0$, say an embedded codimension one sphere in a Hitchin's homotopy sphere.

Remarks. (i) If *"distance decreasing"* of f is strengthened to *"ε_n-Lipschitz"* for a sufficiently small $\varepsilon_n>0$, then the above disconnectedness of the space of pairs (g,f) follows for all X with a use of minimal hypersurfaces instead of Dirac operators.

(ii) The above definition of the space $\mathcal{S}^m_\sigma$ makes sense for all manifolds X instead of $S^m\times Y$, where one may allow $dim(X)<m$ as well as $>m$.

However, the following remains problematic in most cases.

For which closed manifolds X and numbers m, σ_1 and $\sigma_2>\sigma_1>0$ is the inclusion $\mathcal{S}^m_{\sigma_2}(X)\leq\mathcal{S}^m_{\sigma_1}(X)$ homotopy equivalence?

Suggestion to the Reader. Browse through all theorems/inequalities in the previous as well as in the following sections, formulate their possible homotopy parametric versions and try to prove some of them.

6.7 Corners, Categories and Classifying Spaces.

It seems (I may be mistaken) that all known results concerning the homotopies of spaces with metrics $Sc>0$ are about *iterated (co)bordisms* of manifolds with $Sc>0$ and/or about *cobordism categories with $Sc>0$* in the spirit of [EbR-W(cobordism categories) 2019], rather than about spaces of metrics per se.[430]

To explain this, start with thinking of morphisms $a\to b$ in a category as members of class of *labeled* (directed) *edges/arrows* [0,1] with the 0-ends labeled by a and the 1-end labeled by b.

Then define a *cubical category* $\mathcal{C}$ (I guess there is a standard term but I don't know it) as a class of *labeled combinatorial cubes* of all dimensions, $[0,1]^i$, $i=1,2,...$, where all faces are labeled by members of some class and which satisfied the obvious generalisations of the axioms of the ordinary categories: associativity and the presence of the identity morphisms.

Example. Let $\mathcal{C}=A^\square$ consist of continuous maps from cubes to a topological space A, e.g. to the space $A=G_+=G_+(X)$ of metrics with positive scalar curvature on a given manifold X, where these maps are regarded as labels on the cubes they apply to.

If we glue all such cubes along faces with equal labels, we obtain a cubical complex, call it $|\mathcal{C}|$, which is (weekly) homotopy equivalent to A, where possible

[430]See [Kaz(4-manifolds) 2019] for a computation of such cobordisms in dimension 3.

degeneration of cubes.e.g. gluing two faces of the same cube, is offset by possibility of unlimited subdivision of cubes by means of cubical identity morphisms.

Next, given a smooth closed manifold X, consider "all" Riemannian manifolds of the form $(X\times[0,1]^i,g)$, $i=0,1,2,...$, such that $Sc(g)>0$, and such that the metrics g in small neighbourhoods of all "X-faces" $X\times F_j$, where F_j is are ($(i-1)$-cubical) codimension one faces in the cube $[0,1]^i$), split as Riemannian products: $g=g_{X\times F_j}\otimes dt^2$. Denote the resulting cubical category by $XG_+^\square$ and observe that there is a natural cubical map

$$\Xi:|G_+(X)^\square|\hookrightarrow|XG_+^\square|.$$

Now we can express the above "iterated cobordism" statement by saying that the only part of the homotopy invariants of $G_+(X)$ (which is homotopy equivalent to $G_+(X)$), e.g of its homotopy groups, which is detectable by the present methods is what remains non-zero in $|XG_+^\square|$ under Ξ.

Similarly one can enlarge other spaces of Riemannian metrics on non-closed manifolds from the previous section with lower bounds on their curvatures and their sizes, where the latter can be expressed with maps $f:(X,g)\to\underline{X}$, with controlled Lipschitz constants with respect to g, or with respect to the Sc-*normalised metric* $Sc(X)\cdot g$.

There is yet another way of enlarging the cubical category $XG_+^\square$, namely by $B^*G^\square(\mathsf{D})$, where D is topological, e.g. metric space and where

$\bullet_0$ closed oriented Riemannian manifolds X of all dimensions n along with continuous maps $X\to\mathsf{D}$ stand for 0-cubes - "vertices",

$\bullet_1$ "edges" ; i.e 1-cubes are cobordisms W^{n+1} between X_0,X_1, with Riemannian metrics split near their boundaries $\partial W^{n+1}=X_0\sqcup -X_1$, and continuous maps to D extending those from $X_0 and X_1$,

$\bullet_2$ "squares", are (rectangularly cornered $(n+2)$-dimensional) cobordisms between W-cobordisms with maps to D, etc.

The actual cubical subcategory of $B^*G^\square(\mathsf{D})$, which is relevant for the study of the space $|XG_+^\square|$ (that is, essentially, the space of metrics with $Sc>0$ on X) is where all manifolds in the picture are spin, the scalar curvatures of their metrics are positive, D is the classifying space of a group Π and where one may assume the fundamental groups of all X to be coherently (with inclusion homomorphisms) to be isomorphic Π [431] (compare [Ebert-Williams(infinite loop spaces) 2017], [BoEW(infinite loop spaces) 2014], [HaSchSt(space of metrics)2014] ,

Question. What are possible generalizations of the above to manifolds with corners, which are far from being either cubical or rectangular?

For instance, prior to speaking of spaces of metrics and of categories of cobordisms, let X be an individual manifold with corners, say a (smoothly) topological n-simplex or a dodecahedron, let $(\infty<\sigma<\infty)$, let $(\infty<\mu_i<\infty)$ be numbers assigned to the codimension one faces F_i of X and $0<\beta_i j<\pi$ be assigned to the codimension two faces of the kind $F_i\cap F_j$.

When does X admit a Riemannian metric g such that

$$Sc_g\geq\sigma,\ mean.curv_g(F_i)\geq\mu_i \text{ and } \angle_g(F_1,F_j)\leq\pi-\beta_{ij}?$$

[431]This "assume" relies on the codimension two surgery of manifolds with $Sc>0$, which is possible for making the fundamental groups of n-manifolds isomorphic to Π if $n\geq4$ and where more serious topological conclusions need $n\geq5$.

Let moreover, $D \subset \mathbb{R}^N_+$, where the N Euclidean coordinates are associated with the faces F_i of X, be a closed convex subset, introduce the following additional condition on g:

the N- vector of distances $\{d_i(x) = dist_g(x, F_i)\}$ is in D for all $x \in X$.

We ask when does there exist a g with this additional condition and also

what is the homotopy type of the space of metrics g on X, such that

$$Sc_g \geq \sigma,\ mean.curv_g(F_i) \geq \mu_i,\ \angle_g(F_1, F_j) \leq \pi - \beta_{ij} \text{ and } \{d_i(x)\} \in D?$$

(For instance, if X is a topological n-simplex, then an "interesting" D is defined by $\sum_i d_i(x) \geq const.$)

One may also try to generalize the concept of cubical category by allowing all kinds of combinatorial types of manifolds X with corners and of attachments of X to X' along isometric codimension one faces $X \supset F \leftrightarrow F' \subset X'$, where the isometries $F \leftrightarrow F'$, must match the mean curvatures of the faces:

$mean, curv(F') = -mean.curv(F)$ which is equivalent to the natural metric on

$$X \bigcup_{F \leftrightarrow F'} X$$

being C^1-smooth.

Is there a coherent category-style theory along these lines of thought?

6.8 Scalar Curvature under Weak Limits of Manifolds

We show in this section by means of examples how the scalar curvature may behave under limit of sequences of Riemannian manifolds.[432]

We saw in section 3.19 how a Riemannian manifold X "emerges" as "bubble-limit" from a "foam" (sequence) X_i obtained by taking thin connected sums of X with compact Riemannin manifolds $X_{i,\circ}$, where this "emergence" becomes *Hausdorff* or *intrinsic flat Sormani-Wenger* convergence under suitable conditions imposed on $X_{i,\circ}$ and where the scalar curvatures of X_i subconverge to that of X in these cases.

Counter examples. The inequality $Sc(X_i) \geq \sigma$ is *not always preserved* by the Hausdorff and by the intrinsic flat limits.

In fact,

all Riemannian manifolds X of dimensions $n \geq 3$ can be approximated by n-dimensional X_i with $Sc(X_i) \geq 1$

(a) in the Hausdorff metric,

(b) in the intrinsic flat metric. (Here on speaks of closed oriented manifolds.)

Proof of (a). All Riemannian manifolds X can be Hausdorff approximated by graphs Γ and boundaries of suitable small neighbourhoods of these graphs embedded to $\mathbb{R}^{n+1}$ for $n \geq 3$, have arbitrarily large scalar curvatures (see section 1.3).

[432] Our examples a similar to these from [Sormani(scalar curvature-convergence) 2016] and [Lee-Naber-Neumayer](convergence) 2019].

Proof of (b) Assume X bounds an orientable $(n+1)$-manifold V (otherwise take the connected sum of X with a small copy of X with reverse orientation) and endow V with a Riemannian metric and let for which the embedding $X \to V$ is distance preserving.

Let $U \subset V$ be a union of small balls, or just of small "sufficiently convex" subsets U_j, the scalar curvatures of the boundaries of which satisfy $Sc(\partial U_j) \geq 2$.

If $vol(U) \geq vol(V) - \varepsilon$, that is easily achievable, then the *flat distance* between X and the boundary $\partial U = \cup_j \partial U_j$ is also $\leq \varepsilon$.

What remains in order to satisfy the definition of the intrinsic flat distance from [Sormani-Wenger(intrinsic flat) 2011] is to modify ∂U and the metric in V in order to have the embedding from ∂U to the complement of the interior of U, denoted $W = V \setminus int(U)$, *isometric.*

To do this, let $\delta = dist_V(X, U) > 0$ and this, take a finite δ'-dense subset $K \subset \partial U$ for δ' much smaller than δ and let $\{[k, k']\} \subset V$, be the set of those geodesic segments in V between the points $k \in K$ and$K \neq k' \in K$ which don't intersect the interior of U.

Assume that the segments $[k_i, k_j]$ are mutually disjoint and that their length are much smaller that δ, say of order δ'; other wise , add extra small ball to U.

Now, perform the (very) thin surgery along $[k_i, k_j]$, that is attach thin 1-handles to U, keeping the scalar curvature of the boundary of the resulting U' essentially as positive as that of ∂U, let $W' = V \setminus U'$ and observe that the oriented boundary of W' is

$$\partial W' = X - \partial U'$$

and that $vol(W') \leq \varepsilon$.

Since $\delta' << dist(U', X) \approx \delta$, and since the additive difference between the "intrinsic" metrics in $\partial U'$ and the "extrinsic" one, both defined by shortest paths, the former in in $\partial U'$ and in W respectively, is of order δ', one can enlarge the metric of W that would make it equal to the intrinsic metric in $\partial U'$ without changing the metric on $X \subset W$, and also only slightly changing the volume of W.

This makes the intrinsic flat distance between X and $\partial U'$ smaller than 2ε and the proof of (b) is concluded.

The examples (a) and (b) suggest the following.

Definitions. *A Riemannian* (α, β)*-cobordism* between closed oriented Riemannian n-manifolds X_1 and X_2 is an oriented Riemannian $(n+1)$-manifold

$$W = W_{\alpha,\beta} = \overleftrightarrow{W}_{\alpha,\beta}$$

with oriented boundary $\partial W = X_1 - X_2$,such that the Hausdorff distance between X_1 and X_2 in W satisfies

$$dist_{Hau}(X_1, X_2) \leq \alpha$$

and the volume of W is

$$vol(W) \leq \beta.$$

Such a cobordism can be regarded as a morphism $W : X_1 \to X_2$ with an obvious composition for $X_1 \overset{W_{\alpha_1,\beta_1}}{\to} X_2 \overset{W_{\alpha_2,\beta_2}}{\to} X_3$:

$$W_{\alpha_1,\beta_1} \circ W_{\alpha_2,\beta_2} = W_{\alpha_1+\alpha_2,\beta_1+\beta_2} : X_1 \to X_3.$$

A Riemannian (α, β, λ)*-cobordism* between X_1 and X_2, denoted

$$W = W_{\alpha_1,\beta_1,\lambda} = \overrightarrow{W}_{\alpha_1,\beta_1,\lambda},$$

is an (α, β)-cobordism with a λ-Lipshitz retraction $W \to X \subset W$.

Here, the arrows are not invertible and the composition for $X_1 \to X_2 \to X_3$ is multiplicative in λ,

$$W_{\alpha_1,\beta_1,\lambda_1} \circ W_{\alpha_2,\beta_2,\lambda_2} = W_{\alpha_1+\alpha_2,\beta_1+\beta_2,\lambda_1\cdot\lambda_2} : X_1 \to X_3.$$

Two Observations.

(i) Given a Riemannian manifold X (which corresponds to X_2 from our definition), there exists an $\varepsilon = \varepsilon(X) > 0$ such that the ε-neighbourhood $U_\varepsilon(X) \subset W$ of X in W admits a continuous retraction $F : U_\varepsilon \to X$, which is $(1 + 4\varepsilon)$-Lipschitz on the scale $>> \varepsilon$. Moreover,

$$dist(F(u_1), F(u_2)) \le dist(u_1, u_2) + 5\varepsilon \text{ for all } u_1, u_2 \in U_\varepsilon(X).$$

Indeed, there is such an F which sends each $u \in U_\varepsilon$ to an almost nearest point $x = x(u) \in X$, namely, such that $dist(u, x(u)) \le 2\varepsilon$.

(*Probably irrelevant*)*Remark.* It is not hard to show (an exercise to the reader) that there exist such retractions $F : U_\varepsilon(X) \to X$, that are $(\sqrt{N}+C_X\cdot\varepsilon)$-Lipschitz on all scales, where C_X is a constant which depends only on X. (If ε were allowed to depend on $W \supset X$,the map F could be made $1+C_W\varepsilon$-Lipschitz.)

(ii) *From* (α, β) *to* $(\alpha \le \varepsilon, \beta)$. A regularized ε-neighbourhood $W_\varepsilon \subset W$ of $X \subset W$ is not quite a (ε, β)-cobordism, since the embedding of the new boundary component to W_ε, say $X_\varepsilon \subset W_\varepsilon$ is not isometric.

But if β is much smaller than ε, this error can localized, by making ε smaller if necessary, on a small part $X'_\varepsilon \subset X_\varepsilon$, namely on the difference $X'_\varepsilon = X_\varepsilon \setminus \partial W = \partial W_\varepsilon \setminus \partial W$, since, by the coarea formula

$$\int_0^\varepsilon vol(X'_\varepsilon)d\varepsilon \le \frac{\beta}{\varepsilon}.$$

Moreover, if most of the volume of $X_1 = \partial W \setminus X(= X_2)$ is concentrated near X, namely,

$$vol(X_1 \setminus W_\delta) << \varepsilon \text{ for } \delta << \varepsilon,$$

e.g. if

$$vol(X_1) - vol(X_2) << \delta,$$

then, by the coarea inequality, the boundary of X'_ε can be also made small. Then, by filling in $\partial X'_\varepsilon$ by a X''_ε of small volume according to *the filling inequality* and then by applying the filling inequality to $X'_\varepsilon \cup X''_\varepsilon$, one modifies the metric in W_ε such that the embedding $X_\varepsilon \to W_\varepsilon$ becomes distance preserving.[433]

$(\alpha, \beta, \lambda, \sigma)$*-Problem.* Given a closed oriented Riemannian n-manifold X and numbers $(\alpha > 0, \beta > 0, \lambda \ge 1, \sigma > -\infty)$. Does there exist a cobordism $W_{\alpha,\beta} : X_1 \to X$ or $W_{\alpha,\beta,\lambda} : X_1 \to X$, where $Sc(X_1) \ge \sigma$?

Open Manifolds The definitions of $(\alpha...)$ cobordisms $W : X_1 \to X_2$ generalize to open manifolds and manifolds with boundaries, where in the latter case W

[433] I didn't check the details.

comes with a corner structure, organized as that of cylinders $X \times [1,2]$ regarded as cobordisms between $X \times 1$ and $X \times 2$, where the flat distance between X_1 and X_2 defined by such a W incorporates, besides $vol_{n+1}(W)$, the n-volume of the "side boundary" of W, that is $\partial_{side} W = \partial W \smallsetminus (X_1 \cup X_2)$.

Two Conjectures. (1) Let the sequence $W_{\alpha_i,\beta_i} : X_i \to X$ defines a α,β-convergence of X_i to X, for

$$\alpha_i, \beta_i \to_{i\to\infty} 0.$$

If the scalar curvatures of all X_i satisfy $Sc(X_i) \geq \sigma$, then also $Sc(X) \geq \sigma$.

(2) Let X_i converge to X via (α,β,λ)-cobordisms, that is a sequence $W_{\alpha_i,\beta_i,\lambda} : X_i \to X$,

$$\beta_i \to 0 \text{ and } \lambda_i \to 1.$$

(The Hausdorff distance $dist_{Hau}(X, X_i)$ and its bound α play no role here.)

If $Sc(X_i) \geq \sigma$ for all $i = 1, 2, ...$, then $Sc(X) \geq \sigma$ as well.

How to prove and how to improve, how to modify, and how to generalize. A natural approach to the proof of (1) and (2) could be as follows.

Let $\sigma \geq 0$, assume $Sc(X, x_0) < 0$, take a $\blacksquare$-neighbourhood $\blacksquare^n \subset X$ of x_0 that violates the $\blacksquare$ criterion for $Sc \geq 0$, and then approximate $\blacksquare^n$ by neighbourhoods $\blacksquare^n_i \subset X_i$, which violate the $\blacksquare$ criterion as well.

To appreciate the issue, let $Y \subset X$ be a closed volume minimizing hypersurface and try to find minimizing hypersurfaces in X_i that converge to Y for $i \to \infty$.

To do this, start with $Y_i \subset X_i$ that approximates X_i for $i \to \infty$ and which can't be *fully* moved away from their small neighbourhood in X_i, but, in the course of volume minimization, these Y_i may, a priori, develop"thin fingers" protruding far away from the original Y_i and carrying tiny, yet definite positive, amounts of volume.

The latter problem can be ruled out by imposing additional geometric condition(s) on X_i (which is automatic in the case of C^0-convergence, as in section 10 of [G(Hilbert) 2012] and section 4 of [G(billiards) 2014]), but in general, one has to accept these fingers that would allow only *weak approximation* of Y by $Y_{i,min}$. (This doesn't seem to create a serious problem for closed manifolds X, but may need a modification of the $\blacksquare$ criterion for open ones.)

Possibly, the validity of these conjectures needs additional conditions on X_i. e.g. the convergence of volumes $vol(X_i) \to vol(X)$ as in section 10 of [G(Hilbert) 2012].

On the other hand, the bubble example suggests that even a more general convergence may preserve positivity of the scalar curvature.

About Singular X and W. The above "convergence assisted by cobordisms" makes sense for *pseudomanifolds* X, X_i and W_i with piecewise smooth metrics on them.[434]

This suggests a provisional definition of $Sc^?(X)$, where $Sc^?(X,x) > \sigma$, $x \in X$, if and only if

there exists a closed neighbourhood $U \in X$ of x with piecewise smooth boundary, where U admits an (α,β)-approximation by Riemannian manifolds U_i, $i \to \infty$, with $Sc(U_i) \geq \sigma' > \sigma$.

[434]When it come to proofs, one needs to deal with *integral current spaces*, (see [Allen-Sormani(convergence) 2020], [Sormani(conjectures on convergence) 2021] and references therein) but as far as our geometric statements are concerned, pseudomanifolds will do.

Namely,

there exist cobordisms $W_i = W_{\alpha_i,\beta_i} = W_{U,\alpha_i,\beta_i}$, which are pseudomanifolds with "cornered" boundaries

$$\partial W_i = X \cup X_i \cup \partial_{side} W \text{ with } \partial\partial_{side} W = \partial X \cup \partial X_i,$$

where

- $Sc(X_i) \geq \sigma_i \to \sigma$,
- $dist_{Hau}(X_i, X) \leq \alpha_i \to 0$,
- $dist_{flat}(X_i, X) = vol_{n+1}(W) + vol_n(\partial_{side} W) \leq \beta_i \to 0$.

Observe that psedomanifold X, obtained, by ε-thin surgery with $\varepsilon \to 0$ (see [BaDoSo(sewing Riemannian manifolds) 2018] and [BaSo(sequences) 2019]) may have nasty singularities of codimensions $k \geq 3$, such e.g. as in joins $X_1 \vee X_2$, which are limits of thin connected sums of manifolds of dimensions $n \geq 3$.

Nevertheless, singular X with $Sc^? \geq \sigma$ in these example satisfy the ■-criterion and for all we know, enjoy all essential geometric properties known for smooth manifolds with $Sc \geq \sigma$.

But none of this is known at the present moment for general limit spaces X with the following questions remaining unresolved.

1. Does the inequality $Sc^?(X) \geq \sigma$, that is *local approximability* of X at all points $x \in X$ by (small open) manifolds $X_i = X_i(x)$ with $Sc(X_i) \geq \sigma$, imply the existence of *global approximation* of X by manifolds with $Sc \geq \sigma$?
2. Is $Sc^?$ satisfy the additivity relation $Sc^?(X \times Y) = Sc^?(X) + Sc^?(Y)$?
3. Do X which admit global approximations by manifolds with $Sc \geq \sigma$ satisfy the ■-criterion?

Notice that spaces X, which do admit global approximation by manifolds X_i with $Sc \geq \sigma > 0$, satisfy (essentially) the same geometric bounds as X_i, because the retractions $W_i \to X$ indicated in he above (i) defines maps $X_i \to X$ of degrees 1, which are λ_i-Lipschitz on the scale $\geq \varepsilon_i$, where $\lambda_i \to 1$ and $\varepsilon_i \to 0$ for $i \to \infty$. (If not for "scale>0", these X would have $Sc^{\max} \geq \sigma$, see section 5.4.1.)

It is unproven at the present moment that the limits g of *in measure convergence* sequences $g_i \to g$ inherit positivity of scalar curvature from g_i, but, probably, the ■-criterion can be used to do this. (This must be easy, if $|\log g/g_i| \leq const < \infty$, that is if the Lipschitz constants $Lip_g(g_i)$ and $Lip_{g_i}(g)$ of the identity maps $(X, g_i) \to X, g)$ and $(X, g) \to X, g_i)$ are uniformly bounded.)

6.9 Scalar Curvature beyond Manifolds Limits

There (at least) three different avenues of thought on generalization the concepts $Sc \geq \sigma$.

I. Finding workable classes of (singular) metric spaces that share their properties with smooth manifolds with $Sc \geq \sigma$, e.g. for $\sigma = 0$.

I.A. An attractive class of such spaces X, that have been already mentioned in section 3.19, is where the generalized sectional curvatures in the sense of Alexandrov satisfy $sect.curv(X) \geq \kappa > -\infty$, and where $Sc \geq \sigma$ at all C^2-smooth points of these X.[435]

[435] Alexandrov spaces with $sect.curv(X) \geq \kappa$ seem to provide a perfect playground for the geometric measure theory in all dimensions and codimensions as examples with conical sin-

Conjecturally, the basic properties of minimal subvarieties of all codimensions extend to these spaces, where such subvarieties of codimension one, as well as stationary μ-bubbles, serve for proving geometric inequalities similar to the ones we have for smooth manifolds.

In fact, this is not hard to prove for spaces X with *isolated conical singularities*, where, as far as minimal hypersurfaces are concerned, the positivity of the sectional curvatures of the the links (bases) of the singularities can be relaxed to positivity of the Ricci curvatures.

I.B. Another class, that immediately jumps to one's mimd is that of piecewise smooth, e.g. spaces with iterated conical singularities, such as piecewise flat spaces, where the key issue is working out a condition for $Sc \geq \sigma$ at conical singularities, where it may prudent to to require these spaces to be *rational homology manifolds.*

I.C. It seems plausible that (stationary?) minimal hypersurfaces in smooth manifolds have some generalized scalar curvatures $\geq -\infty$.

Also doubles $\mathbb{D}$of domains bounded by (possibly singular) minimal hypersurfaces in smooth manifolds X must have (generalized) scalar curvatures bounded from below by

$$\mathrm{Sc}(\mathbb{D}) \geq ScX.$$

I.D. Topologies in Spaces of Riemannian Manifolds Associated with Scalar Curvature It remains unclear what is the weakest topology in the space of isometry classes of Riemannian manifolds for which the condition $Sc \geq \sigma$ is closed under limits. [436]

Besides, properly defined weak limits X_∞ of spaces X_i with $Sc \geq \sigma$, even for singular X_∞ must have a suitably defined scalar curvature $\geq \sigma$ as well, where the following may be instructive.

Example 1. Infinite geometric connected sums

$$X_\infty = \lim_{i\to\infty} X_i \# Y_{i+1},$$

where Y_i are closed Riemannian n-manifolds with $Sc(Y_i) > \sigma$, such that

$$\sum_{i=1}^{\infty} diam(Y_i) < \infty,$$

must have (possibly under extra conditions on the geometries of Y_i)

$$Sc(X_\infty) \geq \sigma.$$

Recollection. A *thin* (geometric) connected sum (see section 1.3) $X_1 \# X_2$ is an abbreviation for a family of Riemannian manifolds $X_\varepsilon = X_1 \#_\varepsilon X_2$, for small positive $\varepsilon \to 0$, which Hausdorff converge to the join $X = (X, x) = (X_1, x_1) \vee (X_2, x_2)$, where the tube $T = T_\varepsilon \subset X$ (homeomorphic to $S^{n-1} \times [1,2]$) joining the two manifolds is

gularities show. Thus, for positive κ, they probably enjoy *Almgren's sharp isoperimetric inequality* in all codimensions and *Almgren's waist estimate.*

And as far as the scalar curvature and minimal hypersurfaces are concerned one may try more general singular spaces with the Ricci curvatures bounded from below.

[436]See [Sormani-Wenger(intrinsic flat) 2011], [Sormani(scalar curvature-convergence) 2016], [Allen-Sormani(convergence) 2020], [Sormani(conjectures on convergence) 2021] and section 10.1 in [G(Hilbert) 2012] for something about it.

based on small, say of radii $\frac{\varepsilon}{10}$, spheres in X_1 and X_2 around x_1 and x_2, and such that

• the complement to the $\frac{\varepsilon}{2}$-neighbourhood of T in X,

$$X \setminus U_{\frac{\varepsilon}{2}}(T) \subset X$$

is isometric to the disjoint union of the complements to the ε-balls $B_{x_1}(\varepsilon) \subset X_1$ and $B_{x_2}(\varepsilon) \subset X_2$,

and where – this can be arranged –

• the scalar curvature of $Sc(X_1 \#_\varepsilon X_2) \geq \sigma - \varepsilon$, in this neighbourhood is almost bounded from below by the scalar curvatures of X_1 and X_2 at the points x_1 and x_2,

$$Sc(U_{\frac{\varepsilon}{2}}((T) \geq \min(Sc(X_1, x_1), Sc(X_2, x_2)) - \varepsilon.$$

Accordingly $X_i \# Y_{i+1}$ stands for $X_i \#_{\varepsilon_i} Y_{i+1}$ say with $\varepsilon_i = \frac{1}{2^i}$.

Question 2. Let X be a compact smooth Riemannian n-manifold and let X_i be a sequence of Riemannian n-manifolds with $Sc(X_i) \geq \sigma$ and let $U_i \subset X_i$ be domains with smooth boundaries ∂U_i, such that

U_i admit $(1+\varepsilon_i)$-bi-Lipschitz embeddings to X, where $\varepsilon_i \to 0$ for $i \to \infty$.

What bound on the sizes of the boundaries ∂U_i for $i \to \infty$ would imply that $Sc(X) \geq \sigma$?

Partial Answer. If $\sigma = 0$, then the following bound on the diameters of the connected components $comp_{ij} \subset \partial U_i$, which says that the *limit Haussdorf dimension* of of ∂U_i is <1, is sufficient:

$$\sum_j diam(comp_{ij}) \to 0 \text{ for } i \to \infty.$$

In fact, this follows from the ■-*criterion* in section 3.1.

If $n = 3$, this may be close to the necessary condition, but if $n \geq 3$ the sufficiency of the similar strict bound on the *limit Haussdorf dimension* by $n-2$, which says that all ∂U_i can be covered by subsets B_{ij}, such that

$$\sum_j diam(B_{ij})^{n-2} \to 0 \text{ for } i \to \infty,$$

remains problematic.

Remark (a) We make no assumption on geometries of the complements $X_i \setminus U_i$. Thus, the relationship between X andX_i are, unlike any kind of distance, non-symmetric. (If $n = 3$, one imagines $X_i \subset U_i$ as kind of white holes universes emerging from X, where they are seen as black holes.)

Remark (b) The Penrose inequality suggests that if $n = 3$, then requiring that the areas of ∂U_i tend to zero, for $i \to \infty$,h would have little effect (if ar all) on the geometry of the limit space X. But it is unclear what should be the corresponding condition for $n \geq 3$. (Could the areas be replaced by something like 2-waists of ∂U_i?)

I.E. Since the scalar curvature is additive under finite Riemannian products it is tempting to extend the idea to infinite products and iterated fibrations, and to find geometric meaning of the inequality $Sc \geq \sigma$ for infinite dimensional Hilbertian manifolds, such as spaces of maps between Riemannian manifolds. But no plausible conjecture is known in this direction.

II. Instead of the spaces one may focus on analytic techniques used for the study of $Sc \geq \sigma$, in particular the index theory for the Dirac operator and the geometric measure theory and search for generalisations (unification?) of these that would be applicable to singular spaces.

III. One may think of manifolds with $Sc \geq \sigma$ and the methods used for their study as as geometric/analytic embodiment of certain algebraic formulae behind these, such as the GaussTheorema Egregium coupled with the second variation formula and the Schroedinger-Lichnerowicz-Weitzenboeck-(Bochner) formula coupled with the formula(s) involved in the local proof of the index theorem.

Conceivably, there may exist alternative implementations of these formulas in categories which are quite different from those of manifolds and/or of metric spaces, where, e.g. the objects are represented by functors from a category of "decorated graphs" to that of measure spaces as in the last section of [G(billiards) 2014] or

something else, something far removed from the present day idea of what a geometric space is.

7 Metric Invariants Accompanying Scalar Curvature

Many invariants of metric spaces X can be expressed in (quasi)-category theoretical language, e.g. in terms of λ-Lipschitz maps between X and a "measuring rod" space (or spaces) $\underline{X}$. [437]

In fact, the distance function in X is fully encoded by the sets of 1-Lipschitz functions, i.e. distance non-increasing maps $X \to \mathbb{R}$:

(dist$_{cnrt}$) the distance $dist(x_0, x_1)$ is (obviously) equal to the *supremum* of numbers $d \geq 0$, such that X admits a 1-Lipschitz map $f : X \to \mathbb{R}$, such that $f(x_0) = 0$ and $f(x_2 1 = d$.

Alternatively, $dist(x_0, x_1)$ can be defined *covariantly* via maps of two point subsets from $\mathbb{R}$ to X, as follow:

(dist$_{cov}$)the distance $dist(x_0, x_1)$ is equal to the *infimum* of $d \geq 0$, such that $\{0, d\}$ admits a 1-Lipschitz map to X with the *image* $\{x_0, x_1\} \subset X$.

Similarly, one can define the volume of a connected Riemannian n-manifold X as

(vol$_{cov}$) the *infimum* of numbers $v = d^n$, such that that X receives a smooth *locally volume non increasing* map f (i.e. $||\wedge^n df|| \leq 1$) from the cube $[0, d]^n$ *onto* X.

And – this is closer to invariants used in the study of scalar curvature – one can define $vol(X)$ of a closed connected manifold X *contravariantly* as

(vol$_{cntr}$) the supremum of volumes $v = (2n)d^n$ of the boundaries of the cubes $[0, d]^n$, which receive *non-contractible* locally volume non increasing piecewise smooth maps from X.

Exercise. Prove (vol$_{cov}$) and (vol$_{cntr}$).

[437] There are also some questionable, albeit sometimes coming in handy, ad hoc invariants, such as the "injectivity radius", but these are useless as far as the scalar curvature is concerned.

7.1 Multi-Spreads of Riemannianan Manifolds: $\square^\perp$ and $\tilde\square^\perp$

Let $\tilde U$ be a compact n-dimensional manifold possibly with a boundary, let g be Riemannian metric on $\tilde U$ and let $\tilde h \in H_{n-k}(\tilde U)$ be a homology class of codimension k.

The $\square^n$-inequality from section 3.8 for widths of cubes with metrics with $Sc \geq \sigma$ motivates the following.

Definition. The $\square^\perp$*-spread* of a homology class $\tilde h \in H_{n-k}(\tilde U)$, denoted

$$\square^\perp(\tilde h) = \square^\perp_g(\tilde h),$$

is the supremum of the numbers $d \geq 0$, for which there exists

a continuous proper (boundary-to-boundary) map $\psi = (\psi_1, ...\psi_i, ...\psi_k) : U \to [-1,1]^k$, $\psi_i : \tilde U \to [-1,1]$, such that

(a) the homology class of the ψ-pullback of a point[438] is equal to $\tilde h$, symbolically

$$\psi^*[t] = \tilde h,$$

where $[t] \in H_0([-1,1]^k)$, $t \in [-1,1]^k$, is the homology class of a point in $[-1,1]^k$;

(b) the distances between the pullbacks of the opposite faces in the cube $[-1,1]^k$,

$$d_i = dist_g(\psi_i^{-1}(-1), \psi_i^{-1}(1)), i = 1, ..., k,$$

are bounded from below by the following inequality

$$\left(\frac{1}{k}\sum_{i=1}^{k}\frac{1}{d_i^2}\right)^{-\frac{1}{2}} \geq d,$$

that is

$$\frac{1}{k}\sum_{i=1}^{k}\frac{1}{d_i^2} \leq \frac{1}{d^2}.$$

(Equivalently, one could require the maps ψ_i to be d_i^{-1}-Lipschitz.)

Next define $\tilde\square^\perp(h) \geq \square^\perp(h)$ of a homology class of codimension k in a Riemannian n-manifold X, possibly non-compact and with a boundary, denoted $h \in H_{n-k}(X)$, as the supremum of the numbers $d \geq 0$, such there exist

(i) a Riemannian manifold $\tilde U$,

(ii) a homology class $\tilde h \in H_{n-k}(\tilde U)$ with $\square^\perp(\tilde h) = d$,

(iii) a locally isometric map $\phi : \tilde U \to X$, for which the induced homology homomorphism $\phi_* : H_{n-k}(\tilde U) \to H_{n-k}(\tilde X)$ sends $\tilde h$ to h, in writing: $\phi_*(\tilde h) = h$.

(This definition make sense for an arbitrary metric $dist$ on $\tilde U$.)

Topological Remark. The $\tilde\square^\perp$-spread of h vanishes if and only if none of $\tilde h$ is homologous to any point-pullback, that is the case, for instance, if h has non-zero self intersection $h \frown h \neq 0$.

On the other hand, by a theorem of Serre on cohomotopy groups, If k is odd, or if $h \frown h = 0$, then some non-zero multiple of h, say Nh, has $\square^\perp(Nh) > 0$.

Say that h is $\tilde\square^\perp$*-spread infinite* or that h has infinite $\tilde\square^\perp$-spread if $\tilde\square^\perp(h) = \infty$.

Define $\square^\perp$*-spread and* $\tilde\square^\perp$*-spread* of a compact connected orientable n-dimensional Riemannian manifold X, possibly with a boundary, denoted $\square^\perp(X)$ and $\tilde\square^\perp(X)$,

[438] If ψ is smooth this an actual pullback of a generic point.

as the $\square^\perp$- and $\tilde{\square}^\perp$-spreads of the *zero dimensional homology class* $[x]$ of a single point $x \in X$.

If X *is non-compact* define $\square^\perp(X)$ as $\limsup \square^\perp(X_i)$, $i \in I$, for all compact n-submanifolds X_i exhausting X and let

$\square^\perp(j \cdot X)$ and $\tilde{\square}^\perp(j \cdot X)$ denote these spreads of the j*-multiple* $j \cdot [x] \in H_0(X)$ of the homology class of $x \in X$.

Observe that if X is *compact without boundary* then $\square^\perp(X) = 0$ and that the $\tilde{\square}^\perp$*-spread*, unlike the $\square^\perp$-spread, *of the universal covering* $\tilde{X}$ is *equal* to that of X.

Thus, for instance, the n-torus $\mathbb{T}^n$ is $\tilde{\square}^\perp$-infinite,

$$\tilde{\square}^\perp(\mathbb{T}^n) = \square^\perp(\mathbb{R}^n) = \infty, \text{ while } \square^\perp(\mathbb{T}^n) = 0.$$

And, in general, if a homology class h is representable by a simply connected cycle in X, that is if h is equal to the image of a class $\tilde{h} \in H_{n-k}(\tilde{X})$ under (the homology homomorphism induced by) the universal covering map $\tilde{X} \to X$, then

$$\tilde{\square}^\perp(\tilde{h}) = \tilde{\square}^\perp(h).$$

Say that X is $\tilde{\square}^\perp$*-spread infinite* or that X has infinite $\tilde{\square}^\perp$-spread if $\tilde{\square}^\perp(X) = \infty$, and observe that this property is equivalent to *iso-enlargeability* from [G(inequalities) 2018];

As far as the scalar curvature is concerned, we are interested in *lower bounds* on $\tilde{\square}^\perp(h)$, which are usually easily available, e.g. in the examples (1)-(4) below.

(1) The ball $B^n(R) \subset \mathbb{R}^n$ has

$$\square(B^n(R)) \geq \frac{2R}{\sqrt{n}}.$$

(2) Closed connected surfaces X with *infinite* fundamental groups $\pi_1(X)$ are (obviously) $\tilde{\square}$*-spread infinite*, i.e. $\tilde{\square}(X) = \infty$.

(3) The spread of an n-manifold X with non-empty boundary is (obviously) related to the inradius $inrad(X) = \sup_x(dist, (x, \partial X)$ by the following inequality.

$$\tilde{\square}(X) \leq 2\sqrt{n} \cdot inrad(X),$$

where the equality holds for $X = [0, 2r] \times \mathbb{R}^{n-1}$.

Furthermore, if $n = 2$, then

$$\tilde{\square}(X)(X) \geq \sqrt{2} \cdot inrad(X).$$

This is seen with the universal covering $\tilde{U}$ of X minus the furthest point from the boundary, where $inrad(\tilde{U}) = \frac{1}{2} \cdot inrad(X)$.

In particular, *complete non-compact* surfaces X are $\square$*-infinite*.

(4) Surfaces X *homeomorphic to the 2-sphere* have $\tilde{\square} \perp (X) \geq \sqrt{2} \cdot diam(X)$, that is seen by evaluating $\square$ of the universal cover $\tilde{U}$ of X minus two furthest points in it.

And since the cut loci to all points x in this X contain *conjugate points* to x, the inradii of surfaces $\tilde{U}$, which locally isometrically immerse to X are bounded by $diam(X)$; hence, $\tilde{\square} \perp (X) \leq 2 \cdot diam(X)$.

(All compact simply connected manifolds X have $\tilde{\square}\perp(X) \leq C < \infty$, but no bound on C by the diameter is possible for $n > 2$.

In fact, an arbitrary closed n-manifold X, $n \geq 3$, e.g. $X = S^3$, admits, by geometric surgery argument, Riemannian metrics g_C for all $C > 0$, with $diam_{g_C}(X) = 1$, with $sect.curv(g) \leq \frac{1}{100n^2C^2}$ and, thus, with $\tilde{\square}\perp(X) > C$, where $\tilde{U}$ is the R-ball in the tangent space $T_{x_0}(X)$ for $R = nC$, sent to X by the exponential map and endowed with the Riemannian metric induced from that on X.)

(5) **Product Inequality**. Let $\underline{X}_i$, $i = 1, 2, ..., m$, be Riemannian manifolds of dimensions n_i , possibly with boundaries, non-compact and non-complete and let $f_i : X \to \underline{X}_i$ be proper (infinity-to-infinity boundary-to-boundary) maps and let

$$f = (f_1, ..., f_m) : X \to \underline{X} = \underline{X}_1 \times ... \times \underline{X}_m.$$

Then the $\tilde{\square}^{\perp}$-spread of the homology class $h = f^*[\underline{x}]) \in H_{n-k}(X)$, $k = dim(X) - dim(\underline{X})$ of the point-pullback $f^{-1}(\underline{x})$ of f is bounded from below by the $\tilde{\square}^{\perp}$-spreads of $\underline{X}_i$ as follows.

$$\tilde{\square}^{\perp}(f^*[\underline{x}]) \geq \left(\frac{1}{n}\sum_{i=1}^{m}\frac{n_i}{\tilde{\square}(\underline{X}_i)^2}\right)^{-\frac{1}{2}}$$

that is

$$\frac{1}{\tilde{\square}^{\perp}(f^*[\underline{x}])^2} \leq \frac{1}{n}\sum_{i=1}^{m}\frac{n_i}{\tilde{\square}(\underline{X}_i)^2}.$$

In fact, the $\square^{\perp}$-spread of intersection of cycles h_1 and h_2 of codimensions k_1 and k_2, denoted $d = \square^{\perp}(h_1 \frown h_2)$, satisfies

$$\frac{1}{d^2} \leq \frac{1}{k_1 + k_2}\left(\frac{k_1}{(\square^{\perp}(h_1))^2} + \frac{k_2}{(\square^{\perp}(h_2))^2}\right).$$

Then the the proof for $\square^{\perp}$ follows by induction on m and the corresponding inequality for $\tilde{\square}^{\perp}$ follows.

For instance, the $\square$-spread of the rectangular solid, $d = \square(\times_{i=1}^{n}[0, d_i])$ satisfies

$$\frac{1}{d^2} \leq \frac{1}{n}\sum_i \frac{1}{d_i^2}.$$

(6) Connected sums of compact connected $\tilde{\square}^{\perp}$-*spread infinite* manifolds X with complete manifolds are (obviously) $\tilde{\square}^{\perp}$-*infinite.*

In particular, complete metrics on a $\tilde{\square}^{\perp}$-spread infinite compact manifold X minus a point are $\tilde{\square}^{\perp}$-infinite.

(7) Let X be a complete connected non-compact manifold and $Y \subset X$ be a compact connected submanifold of *codimension 1.*

If the inclusion homomorphism $\pi_1(Y) \to \pi_1(X)$ is *injective*, then

$$\tilde{\square}^{\perp}(X) \geq \tilde{\square}^{\perp}(Y).$$

In particular, if Y is $\tilde{\square}^{\perp}$-*spread infinite* then also Y is $\tilde{\square}^{\perp}$-*spread infinite.*

(8) Let X be a connected complete non-compact manifold and $Y \subset X$ be a compact connected $\square$–*spread infinite* submanifold of *codimension 2*, such that the inclusion homomorphism $\pi_1(Y) \to \pi_1(X)$ is *injective.*

If the real homology class of the ε-circle $S^1_y(\varepsilon) \subset X \setminus Y$ in the normal plane to Y doesn't vanish, then

$$\tilde{\square}^{\perp}(X) \geq \tilde{\square}^{\perp}(Y).$$

Unlike lower bunds, upper bounds on $\tilde{\square}$ find no, at least, no immediate, applications to scalar curvature. What make them amusing is an unexpected complexity of sharp evaluation of $\tilde{\square}$, and even of $\square \leq \tilde{\square}$, in simple examples indicated below, where there are more questions than answers.

(A) The d-cube $[0,d]^n$ satisfies

$$\square[0,d]^n = d.$$

Proof. The inequality $\square[0,d]^n \geq d$ is obvious. (It is the simplest case of the product inequality.)

The lower bound follows from *Besicovich-Derrick* & geometric/arithmetic means inequalities, which shows that $(\square(X))^n \leq vol(X)$, $n = dim(X)$, for all Riemannin manifolds X.

Probably, $\tilde{\square}[0,d]^n$ is equal to d as well, and the following more general property of $\tilde{\square}$ also looks plausible.

(B) *Conjecture.* All *convex* domains $X \subset \mathbb{R}^n$ satisfy $\tilde{\square}(X) = \square(X)$.

(C) The universal covering $\tilde{U}$ of the 2-*ball* $B(r) \subset \mathbb{R}^2$ *minus the center* (obviously) satisfies:

$$\tilde{\square}(\tilde{U}) = \square(\tilde{U}) = \sqrt{2}r,$$

which is equal to the $\square$-spread of the (inscribed) square $\left[-\frac{r}{\sqrt{2}}, \frac{r}{\sqrt{2}}\right] \subset B(r)$.

(D) *Conjecture.* The rectangular solid $\times_{i=1}^n [0,d_i] \subset \mathbb{R}^n$ satisfies

$$\tilde{\square}\left(\bigtimes_{i=1}^{n}([0,d_i]\right) \leq \left(\sum_i \frac{1}{d_i^2}\right)^{-\frac{1}{2}}.$$

(It is obvious that $\tilde{\square}\,(\times_{i=1}^n([0,d_i]) \geq \left(\sum_i \frac{1}{d_i^2}\right)^{-\frac{1}{2}}$.)

(E) *Conjecture.* The $\tilde{\square}$-spread of the ball $B(r) \subset \mathbb{R}^n$ is equal to the (conjectural) $\tilde{\square}$-spread of the inscribed cube (where, obviously, the latter is bounded by the former):

$$\tilde{\square}(B(r)) = \square(B(r)) = \frac{2r}{\sqrt{n}}.$$

Moreover, if a $\tilde{U}$ admits a locally isometric immersion to $B(r)$, if

$$\tilde{\square}(B(r)) = \square(B(r)) = \frac{2r}{\sqrt{n}} = \frac{2r}{\sqrt{n}}$$

and if $n \neq 2$, then $\tilde{U} = B(r)$.

The following is a step toward a weaker, also conjectural, inequality

$\square\,(\times_{i=1}^n([0,d_i]) \leq \left(\sum_i \frac{1}{d_i^2}\right)^{-\frac{1}{2}}$.

(F) ***Proposition.*** Let $X = \times_{i=1}^n[0,d_i] \subset \mathbb{R}^n$ and $X' = \times_{i=1}^n[0,d'_i] \subset \mathbb{R}^n$ be rectangular solids, such that either

(i) *there exists a proper (boundary to boundary) 1-Lipschitz map of odd degree*[439] $X' \to X$,

or

(ii) *there exists a smooth locally expanding (non-decreasing the lengths of smooth curves) embedding* $X \to X'$.

Then

$$\left(\sum_i \frac{1}{d_i^2}\right)^{-\frac{1}{2}} \le \left(\sum_i \frac{1}{(d_i')^2}\right)^{-\frac{1}{2}}.$$

This is an immediate corollary of the following (a) and (b) pointed out to me by Roman Karasev,[440] where (a) depend on the concept of a k-dimensional $\mathbb{Z}_2$-*waist*, denoted $waist_k(X)$, of a Riemannian manifold X (possibly with a boundary). This is a numerical invariant, which is (almost by definition, see next section) is

non-increasing under proper 1-Lipschitz map of odd degree $X' \to X$ *and non-decreasing under locally expanding embedding* $X \to X'$*,i. e.* $waist_k(X') \ge waist.(X)$ *in these cases*

(The □-spread may decrease under locally isometric embedding $X \to X'$. For instance, concentric R-balls in the unit sphere, satisfy: $\Box(B(R)) \ge \Box(B(R'))$ for $\frac{\pi}{2} \le R \le R' \le \pi$.)

(a)*The* $\mathbb{Z}_2$*-waists of the solids* $\times_i [0, d_i]$, $d_1 \le ... \le d_i \le ... \le d_n$, *satisfy*

$$waist_k(\bigtimes_i [0, d_i]) = d_1 \times ... \times d_k \text{ for all } k = 1, ..., n.$$

This is stated, in a slightly different form in corollary 5.3 in [Klartag(waists) 2017] (also see [Akopyan-Karasev(non-radial Gaussian) 2019]).

(b) *If positive numbers* $d_1 \le ... \le d_i \le ... \le d_n$, *and* $d_1' \le ... \le d_i' \le ... \le d_n'$ *satisfy*

$$d_1 \times ... \times d_k \le d_1' \times ... \times d_k'$$

for all $k = 1, ..., n$, *then*

$$\sum_i \frac{1}{d_i^2} \ge \sum_i \frac{1}{(d_i')^2}.$$

Indeed, since the numbers $l_i = -2\log d_i$ dominate $l_i' = -2\log d_i'$, i.e.

$$\sum_i^k l_i \ge \sum_i^k l_i', \ k = 1, ..., n.$$

the *Karamata inequality* applied to to (convex !) function $\exp l$ yields the required inequality:

$$\sum_i \frac{1}{d_i^2} = \sum_i^n \exp l_i \ge \sum_i^n \exp l_i' = \sum_i \frac{1}{(d_i')^2}.$$

(G) *Generalizations.* The above argument yields similar monotonicity of $\Sigma_\alpha = (\sum_i d_i^\alpha)^{\frac{1}{\alpha}}$ for all *negative* α, but it is unclear which (if any) of these Σ_α

[439] *Non-zero* degree should be OK, but I only vaguely see how to prove this.

[440] Besides (a) and (b) Roman has made a few other illuminating remarks, including counter examples to some of my naive suggestions on this subject matter.

is increasing under (globally) *non*-one-to-one locally expanding maps between solids. (This monotonicity may fail for small $|\alpha|$.)

Also, waist evaluation in in [Akopyan-Karasev(tight estimate) 2016] (corollary 4) yields similar monotonicity for maps between ellipsoids with principal axes d_i and d'_i, and between solids and ellipsoids.

(H) *Questions.* Is there an "effective" set of inequalities between the numbers d_i and d'_i necessary and sufficient for the existence of an (affine) isometric embedding from solid to solid, $\times_i[0,d_i] \to \times_i[0,d'_i]$, and from ellipsoid to solid or to ellipsoid?

(From a geometric perspective, one would rather have this kind of inequalities for (non-affine) locally injective and/or non-injective locally expanding maps, while from the convexity point of view it is natural to study the convex set of all affine embeddings between convex sets, with a special consideration of affine self-embeddings of such sets.)

(I) *Conjecture.* The manifolds $\underline{X}_i$ from the above product inequality satisfy the equality:

$$\tilde{\square}^{\perp}(f^*[\underline{x}]) = \left(\frac{1}{n}\sum_{i=1}^{m}\frac{n_i}{\tilde{\square}(\underline{X}_i)^2}\right)^{-\frac{1}{2}}.$$

(This generalizes the above conjectural formula $\tilde{\square}\left(\times_{i=1}^{n}([0,d_i]\right) \le \left(\sum_i \frac{1}{d_i^2}\right)^{-\frac{1}{2}}$, and, in fact, may follow from such a formula.)

7.2 Manifolds with Distinguished Side Boundaries and Gauss-Bonnet/Area Inequalities

Let $\partial_{side} \subset \partial X$ be an open subset in the boundary of our Riemannian n-manifold X and let us generalize the definitions of $\square^{\perp}$ and $\tilde{\square}^{\perp}$ for a *relative homology* class $h \in H_{n-k}(X, \partial_{side})$, as earlier but with *side-proper* rather than just proper maps ψ.

Namely:

• the auxiliary n-manifold $\tilde{U}$ also comes with a distinguished *side boundary*, denoted $\tilde{\partial}_{side} \subset \partial\tilde{U}$;

• continuous maps

$$\psi = (\psi_1, ...\psi_i, ...\psi_k) : U \to [-1,1]^k,\ \ \psi_i : \tilde{U} \to [-1,1],$$

must be *side-proper*, which means that they send the complement $\partial X \setminus \partial_{side}$ to the boundary of the cube,

$$\psi(\tilde{U} \setminus \tilde{\partial}_{side}) \subset \partial[-1,1]^k;$$

• locally isometric maps $\phi : \tilde{U} \to X$ must send $\tilde{\partial}_{side} \to \partial_{side} \subset \partial X$.

Theorem: G-B-Inequality. Let X be a Riemannian manifold of dimension n and with a distinguished open subset $\partial_{side} \subset \partial X$ and let $h \in H_2(X, \partial_{side})$ be a relative homology class.

Then h can be represented by an immersed smooth surface $\Sigma \subset X$, the boundary of which is contained in ∂_{side} and such that the integrals of the scalar curvature of X over all connected components S of Σ and of the mean curvature[441]

[441] Our sign convention is such that the boundaries of *convex* domains have *positive* mean curvatures.

of ∂_{side} over $\Theta = \partial S$ are related to the multi-spread $\tilde{\square}$ of the pair (X, ∂_{side}) by the following inequality.
satisfy:

$$\int_S Sc(X,s)ds + 2\int_\Theta mean.curv(\partial_{side},\theta)d\theta \le 4\pi\chi(S) + C_{\tilde{\square}} \cdot area(S),$$

where $\chi(S)$ is the Euler characteristics of S and

👁 $$C_{\tilde{\square}} = \frac{4(n-1)(n-2)\pi^2}{n} \left(\tilde{\square}(X, \partial_{side})\right)^{-2}.$$

The proof of this given in [G-Z(area) 2021]) combines (a version of) the the $\square^n$-inequality for widths of cubes (see sections 3.8 and 5.4) with the argument similar to that in [Zhu(rigidity) for the proof of the sharp equivariant area inequality (see section 2.8), where we consider only the case of $n \ge 7$.

The case $n = 8$, which needs a version of Natan Smale's generic regularity result we postpone till another paper, while $n \ge 8$ needs a generalization of Lohkamp's or of Schoen-Yau's regularization theorems.

Remarks.(a) This theorem, as stated, is non-vacuous *only if* $Sc(X) \ge 0$ and $mean.curv(\partial X) \ge 0$; otherwise, all relative homology classes can be represented by surfaces with *arbitrarily small* integrals, $\int_S Sc(X,s)ds$ or $\int_\Theta mean.curv(\partial_{side},\theta)d\theta$.

(b) If $Sc(X) \ngeq 0$ or $mean.curv(\partial X) \ngeq 0$, a rough, (but meaningful) 👁 kind inequality is possible if, for instance, $Sc(X) \ge -1$, $mean,curv(\partial X) \ge -1$ and

the sectional curvature of X is bounded by $+1$ in the 1-neighbourhood of the region in X, where $Sc(X,x) < 0$ and/or $mean.curv(\partial Xx) < 0$;

the principal curvatures of ∂X are bounded by 1 in the 1-neighbourhood of the region in ∂X, where $mean.curv(\partial Xx) < 0$.

(c) If the boundary of X is mean convex, $mean.curv(\partial Xx) \ge 0$ and $Sc(\ge -1$, then the proof of 👁, which delivers *area minimizing* surface in the class $h \in H_2(X, \partial_{side}$, provide a non-trivial *lower bound* on the area-norm of this class.

But , it is unclear what should be a *correct* version of 👁 for $n \ge 3$, where $Sc(X) \ngeq 0$ and/or $mean.curv(\partial X) \ngeq 0$.

(d) If we allow $C_\square = \frac{4(n-1)(n-2)\pi^2}{n}(\square(X,\partial_{side}))^{-2}$instead of $C_{\tilde{\square}}$ in 👁, then the required surface $\Sigma \hookrightarrow X$ may be assumed *embedded.*

(e) A kind of 👁 (systolic) inequality for metrics with $Sc > 0$ on $S^2 \times S^2$ was established in [Richard(2-systoles) 2020; also a version of Zhu's sharp equivariant area inequality for manifolds with $Sc \ge 0$ and with mean convex boundaries is proven in [Barboza-Conrado](disks) 2019].

Examples of Corollaries. **A**. Let X be a Riemannian manifold diffeomorphic to the product $\pentagon \times \mathbb{R}^{n-2}$, where $\pentagon$ is a planer j-gon, or, more generally, let X be a *manifold with j corners*, which admits a proper (boundary-to-boundary, infinity-to-infinity) map of positive degree $f: X \to \pentagon \times \mathbb{R}^{n-2}$, such that the images of these corners in $\partial\pentagon \times \mathbb{R}^{n-2}$ have non-zero intersection indices with the circles $\partial\pentagon \times t \subset \partial\pentagon \times \mathbb{R}^{n-2}$, $t \in \mathbb{R}^{n-2}$.

If $Sc(X) \ge 0$, if the mean curvature of ∂X away from the corners is ≥ 0 and if the dihedral angles $\angle_i$, $i = 1,...j$, of X at the corners satisfy $\angle_i \le \alpha_i \le \pi$, where

$$\sum_{i=1}^{j} \pi - \alpha_i > 2\pi,$$

then *the map f can't be (globally) Lipschitz.*

Moreover,

there exist sequences of points $x_i, y_i \in X$, such that

$$dist(x_i, y_i) \leq const < \infty, \text{ and } dist(f(x_i), f(y_i)) \to \infty \text{ for } i \to \infty.$$

In fact the inequality 👁 holds for manifolds with *non-smooth* boundaries (here $\partial X = \partial_{side} X$), if the mean curvature understood in a suitable distribution way. But to fully make sense of this one needs additional data on regularity of the boundary $S = \partial\Sigma$.

However, for just keeping track of the inequality $\sum_{i=1}^{j} \pi - \alpha_i > 2\pi$ in the integral $\int_\Theta mean.curv(\partial_{side}, \theta) d\theta$, one can simply smooth the boundary ∂X in an obvious manner and thus approximate X by domains $X_\varepsilon \subset X$ with smooth boundaries. Then 👁, applied to these X_ε, yields the corollary for $\varepsilon \to 0$. (We suggests the reader would fill in the details of this argument.)

B. Let $\underline{S}$ be compact connected surface with a boundary, $\bigcirc$ be a planar k-gon, X be a Riemannian n-manifold and let

$$f : X \to \underline{X} = \underline{S} \times \bigcirc \times \mathbb{R}^{n-4}$$

be a diffeomorphism. (A continuous proper map of degree one will do).

Define the side boundary of X as the one corresponding to the boundary $\partial \underline{S}$,

$$\partial_{side}(X) = f^{-1}(\partial \underline{S} \times \bigcirc \times \mathbb{R}^{n-4}),$$

(this ∂_{side} is smooth) and let $\partial_\angle X \subset \partial X$ be the "cornered part" of the boundary of X, that is

$$\partial_\angle X = f^{-1}(\underline{S} \times \partial\bigcirc \times \mathbb{R}^{n-4}),$$

where the faces and the "corners" of $\partial_\angle X$ correspond to the edges and the vertices of $\bigcirc$.

Let the following four conditions be satisfied.

(•) *The map f is roughly asymptotically Lipschitz-like:*

$$dist(f(x)f(y)) \leq \mathcal{L}(dist(x,y))$$

for some continuous function $\mathcal{L}(d) = \mathcal{L}_f(d)$, $d \geq 0$, and all $x, y \in X$ with $dist(x,y) \geq 1$, e.g. $||df|| \leq const < \infty$.

(••) *The faces of $\partial_\angle X$ are mean convex, i.e have positive mean curvatures.*

(• • •) *The dihedral angles $\angle_i$, $i = 1, 2, ...k$, between the faces of $\partial_\angle X$ at all points in the "corners" are all bounded as follows.*

$$\angle_i \leq \frac{2\pi}{l}$$

where l is a positive integer, such that

if $k = 3$, then $l \geq 6$ i.e. $\angle_i \leq \frac{\pi}{3}$,

if $k = 4, 5$, then $l \geq 4$ i.e. $\angle_i \leq \frac{\pi}{2}$,

if $k \geq 6$, then $l \geq 3$, i.e. $\angle_i \leq \frac{\pi}{2}$.

(• • ••) *Either l is even or let, for every pair of adjacent $(n-1)$-faces in $\partial_\angle$, say ∂_i and ∂_{i+1} there exist an isometric, i.e. preserving the induced Riemannian* *metric, involution of $\partial_\angle$, which interchanges these faces, $\partial_i \leftrightarrow \partial_{i+1}$, and fixes the corner $\partial_i \cup \partial_{i+1}$ between them.*[442]

Then X contains a surface Σ as in the above theorem. In fact, there exists a smooth compact connected oriented surface $S \subset X$ with $\partial_{side}(X)$ which represent a non-zero *homology class in $H_2(X, \partial_{side}(X)$ and such that*

$$\int_S Sc(X,s)ds + 2\int_\Theta mean.curv(\partial_{side},\theta)d\theta \le 4\pi\chi(\underline{S}).$$

About the Proof. Develop X by reflections in the faces, divide the resulting manifold $\tilde{X}$ (diffeomorphic to $S\times\mathbb{R}^2\times\mathbb{R}^{n-4}$) by a non-torsion subgroup Γ_0 of finite index in the reflection group Γ (that isometrically acts on $\tilde{X}$ with $\tilde{X}/\Gamma = X$) and smooth the (natural continuous) Riemannian metric on X/Γ_0 with almost no decrease of its scalar curvature.

This reduce the problem to the case, where ⬠ is replaced by a closed surface of positive genus and where G-B-inequality applies.

Exercises (a) Fill in the details in this argument.

(b) Extend the proof to the case of higher dimensional "reflection polyhedra"instead of ⬠,e.g. for m-cubes $[0,1]^m$.

(c) Apply (b) to $\underline{X} = \underline{S}\times[0,1]^{n-2}$ and work out yet another criterion for $Sc \ge 0$ additionally to these in section 3.1.

(d) Formulate and proof the hyperbolic version (i.e. for $Sc \ge \sigma < 0$) of this criterion in the spirit $(2_{\le 0})$ in section 3.1.1.

(e) Formulate and proof the version of this for $Sc \ge \sigma < 0$ by taking into account the area of $S \subset X$.

7.3 Width, Waist and other Slicing Invariants

Given numerical invariant INV of k-dimensional spaces Y, one defines the "slicing version " of INV for n-dimensional X, $n \ge k$, as the infimum of the numbers I, such that X can be "sliced" into k-dimensional subspaces $Y = Y_{\underline{x}} \subset X$, parametrized by an $(n-k)$-dimensional space $\underline{X} \ni \underline{x}$, such that $INV \le I$.

Example 1: *Uryson's Width.* If INV stands for "diameter" then the corresponding slicing invariant of a, say locally compact metric space X, called k-width is defined, via slicings of X by, where pullbacks of points under continuous maps $f: X \to \underline{X}$ for polyhedral (triangulated) spaces $\underline{X}$ of dimensions $dim(\underline{X}) = m = dim(X) - k$..

If the dimension of X is unspecified of if X is infinite dimensional, we speak of *codimension* m width.)

Exercises.[443] (a) Evaluate the widths of balls, ellipsoids simplices and rectangular solids in Euclidean spaces.

(b) Decide whether the k-width is (essentially) non-increasing under proper 1-Lipschitz maps of non-zero degrees between Riemannian n-manifolds for all

[442] Although, the existence of this involution is probbaly unneeded in the present case, it suggests a generalisation of the ⬠-problem from section 3.1.1 by adding to the structure of the manifold V an action of a compact group G_∂ on its boundary, where the metric g in this problem must be required to be G_∂-invariant.

In fact, the persistence of $\mathbb{T}^\rtimes$-stabilization also suggests an addition of an action of a compact group G on V with requirement of g being G-invariant as well.

[443] I haven't done these exercises.

$k \le n$: the existence of such a map $X_1 \to X_2$ should(?) imply that $width_k(X_2) \le width_k(X_1)$, or at least, that $width_k(X_2) \le const_n \cdot width_k(X_1)$.

Below, as a matter of instance, we formulate a quantified version of the classical bound on Lebesgue covering dimension by the Hausdorff dimension conjectured in [Guth(volumes of balls-width) 2011], proved in [Lio-Li-Na-Ro(filling) 2019] and refined in [Papasoglu(width) 2019], where also a (relatively) direct proof was found.

Theorem A. There exists a universal constant $\epsilon = \epsilon_n > 0$, such that all *proper* (closed bounded subsets are compact) metric spaces X admit the following bound on the codimension $n-1$ Uryson width.

Let, for some $R = R_X > 0$, all pairs of concentric balls R-balls,

$$B_x(R) \subset B_x(10R) \subset X, \ x \in X,$$

admit closed subsets S pinched between the boundaries of these balls,

$$S \subset B_x(10R) \smallsetminus B_x(R),$$

such that

$\bullet_1$ S separates the ball $B_x(R)$ from the complement $X \smallsetminus B_x(10R)$, i.e. no connected component of this complement intersects both $B_x(R)$ and $X \smallsetminus B_x(10R)$;

$\bullet_2$ S can be covered by countably many balls

$$S \subset \bigcup_i B_{x_i}(r_i),$$

such that

$$\sum_i r_i^{n-1} \le \epsilon R.$$

Then X admits a continuous map into an $(n-1)$-dimensional polyhedral space, $f : X \to \underline{X}$, such that

$$diam(f^{-1}(\underline{x})) \le R, \ for\ all\ \underline{x} \in \underline{X}.$$

(A significant instance is
of this, proven by Guth, is that of Riemannian n-manifolds X, where the inequality $vol_x(B(1)) \le \epsilon_n$ for sufficiently small $\epsilon = \epsilon_n > 0$, implies that $width_1(X) \le const_n \varepsilon$.

For example,

all Riemannian n-manifolds X satisfy: $width_1(X) \le const_n vol(X)^{\frac{1}{n}}$.)

Another basic property of Uryson width, now in relation to curvature, is the following.

Theorem B. [Perelman(width) 1995] *The the volumes of all Riemannian n-manifolds (and singular Alexandrov spaces) X with non-negative sectional curvatures are bounded by their Uryson width (essentially) the same way as it is for rectangular solids*

$$\frac{1}{const_n} \prod_{k=1}^{n} width_k(X) \le vol(X) \le const_n \prod_{k=1}^{n} width_k(X).$$

(*Probably*, there are similar bounds for the waists of these manifolds:

$$\frac{1}{const_n}\prod_{k=1}^{l} width_k(X) \le waist_l(X) \le const_n \prod_{k=1}^{l} width_k(X), l = 1, ..., n.)$$

Example 2: *From Volumes to Waists.* If *INV* represents the k-volume of k-dimensional Riemannian manifolds, then the corresponding slicing invariant of Riemannian n-manifolds is called the *k-wast*, denoted $waist_k(X)$, which, in the simplest case, can be defined with slicings of X by pullbacks of points under continuous maps $f : X \to \underline{X} = \mathbb{R}^{n-k}$ with $vol_k(f^{-1}(\underline{x}))$ understood as *k-dimensional Hausdorff measure.*

It is known (see section 1.3 in [G(singularities) 2009]) that all Riemannian n-manifolds have strictly positive k-waists for $k \le n$:

Every continuous map $f : X \to \mathbb{R}^{n-k}$ *admits a point* $\underline{x} \in \mathbb{R}^{n-k}$*, such that*

$$Hau_k(f^{-1}(\underline{x})) \ge \delta = \delta_X > 0.$$

However, (non-trivial) sharp bounds on the waist, such as $waist_k(S^n) = vol_k(S^k)$ for unit spheres, have been proved only under annoying, probably unnecessary, assumptions on f, such, e.g. as being smooth *generic* or *piece-wise real analytic.*[444]

$\mathbb{Z}_2$*-Waist and the Even Degree Problem.* The known proof of the lower bounds on waists of manifolds X, such as rectangular solids, for example, which depend on a Borsuk-Ulam topological lemma, apply to the $\mathbb{Z}_2$*-waists* defined in terms of the *Morse spectrum of the k-volume function* on the space of $\mathbb{Z}_2$-cycles of dimension k (see [Guth(Steenrod) 2007], [G(Morse Spectra) 2017]), where this waist is monotone decreasing under smooth maps $f : X_1 \to X_2$ of *odd degree*:

if the map f is k-volume non-increasing, $||\wedge^k df|| \le 1$*, and* $deg(f)$ *is odd, then* $\mathbb{Z}_2$-$waist_k(X_2) \le \mathbb{Z}_2$-$waist_k(X_1)$.

But it is *unclear* if such monotonicity holds for all k-volume non-increasing maps with *non-zero degree.*

Almgen's Min-Max Theorem. There is an alternative proof of the sharp lower bound on $waist_k(S^n)$, that relies on Almgen's min-max theorem, which delivers minimal subvarifolds of volume $\le v$ in Riemannian manifolds sliced into cycles of volumes $\le v$ (see [Guth (waist) 2014]).

this Although proof, doesn't (seem to) apply to rectangular solids, it does yield

sharp lower bounds for the k-waists of compact manifolds with *sectional curvatures* $\ge \kappa > 0$ (see G(singularities) 2009]).

However, the following remains unsettled.

Problems. A. Extend Almgen's method to *singular* Alexandrov spaces with $sect.curv \ge \kappa$.

B. Develop a unified method that would yield, for instance, sharp inequalities for products of spaces X_i with $sect.curv(X_i) \ge \kappa_i > 0$.

[444]See [G(filling) 1983], [G(waist) 2003], [Guth (waist) 2014], [Akopyan-Karasev(tight estimate) 2016], [Akopyan-Karasev(non-radial Gaussian) 2019], [Klartag(waists) 2017].

Spherical Waists with the the Dirac operator. The sharp parametric area contraction theorem from section 3.4.3 implies the sharp lower bound on the *spherical waists* of N-spheres:

the space of smooth *strictly area decreasing* maps $f : S^2 \to S^N$ is contractible in the space of all continuous maps $S^2 \to S^N$ for all $N \geq 2$.

Moreover,

Let $\underline{X} = (\underline{X}g)$ be a compact Riemannian N-manifold with *positive curvature operator*, e.g. a convex hypersurface in $\mathbb{R}^{N+1}$ and let $\underline{g}_\circ = \underline{g}_\circ(\underline{x}) = \frac{1}{N(N-1)} Sc(\underline{X}, \underline{x}) \underline{g}_\circ(\underline{x})$. Then the argument used in the proof of the sharp parametric area contraction theorem yields the bound on the spherical waist of $\underline{X}_\circ = (\underline{X}, \underline{g}_\circ)$ from below:

$Sc(\underline{X}) > 0$, then the space $\mathcal{F}_\circ$ of smooth *strictly area decreasing* maps $f : S^2 \to \underline{X}_\circ$ is *contractible in the space of all continuous maps* $S^2 \to \underline{X}_\circ$ for all $N \geq 2$.

Questions. (a) Does the space $\mathcal{F}_\circ$ is contractible?

(b) Is the ordinary 2-waist of $\underline{X}_\circ$ is similarly bounded from below as

$$waist_2(\underline{X}_\circ) \geq 4\pi?$$

In particular, is the space of maps $f : \Sigma \to \underline{X}_\circ$, where Σ is surface of genus>0 and where $area(f(\Sigma)) < 4\pi$, also contractible in the space of all continous maps $\Sigma \to \underline{X}_\circ$?

(c) Is there a counterpart of the above for $n > 2$, e.g. for maps $S^n \to (X, g_?)$, $n > 2$, in the spirit of Almgren's style proof of the lower waist bound for manifolds with $sect.curv > 0$?

7.4 Hyperspherical Radii, their Parametric and k-Volume Multi-contracting Versions

From a category/homotopy theoretic point of view the main role of Riemannian metrics on manifolds X and Y is a definition of a "norm" on smooth maps $f : X \to Y$, where we distinguish the following.

$\bullet^k_{sup}$ The sup-norm on the kth exterior power of the differential of f, denoted

$$|| \wedge^k df|| = \sup_{x \in X} || \wedge^k df(x)||^{\frac{1}{k}}.$$

For instance, the inequality $|| \wedge^k df|| < 1$ means that f strictly decreases the k-volumes of smooth k-submanifolds in X.

$\bullet^k_{trace}$ The normalized trace norm on $\wedge^k df(x)$,

$$|| \wedge^k df||_{trace} = \sup_{x \in X} \frac{1}{\binom{n}{k}} (trace \wedge^k df(x))^{\frac{1}{k}},$$

(In terms of an orthonormal frame $e_1, ... e_n \in T_x(X)$, for which the vectors $df(e_i) \in T_y(Y)$, $y = f(x)$ are orthogonal

$$trace \wedge^k df(x) = \sum_{1 \leq i_1 < i_2 < ... < i_k \leq n} \lambda_{i_1} \cdot \lambda_{i_2} \cdot ... \cdot \lambda_{i_k}.$$

for $\lambda_i = ||df(e_i)||$.)

Such a "norm" defines a "norm" on homotopy and/or other classes $[f]$ of maps f, by

$$\text{"norm"}[f] = \inf_{f \in [f]} \text{"norm"}(f),$$

where a relevant example is where $[f] = [f]_{h,\underline{h}}$ consists of the maps that send a given homology class $h \in H_*(X)$ to a given set $\{\underline{h}\}$ or a set of classes $\underline{h} \in H_*(Y)$.

For instance if $Y = S^n$ and the set $\{\underline{h}\}$ consists of non-zero multiples of the fundamental class $[S^n] \in H_n(S^n)$, we define various *hyperspherical radii* of h as the reciprocals of such norms,

$$Rad^{\mathsf{norm}}_{S^n}(h) = \frac{1}{\text{"}\mathsf{norm}\text{"}[f]},$$

where "norm" may stand for $\| \wedge^k df\|$ and $\| \wedge^k df\|_{trace}$.

And if the class $[f]$ consists of the maps $f : X \to S^n$ with *non-zero homology homomorphism* $: H_n(X) \to H_n(S^n) = \mathbb{Z}$, we write

$$Rad^{\mathsf{norm}}_{S^n}(X) = \frac{1}{\text{"}\mathsf{norm}\text{"}[f]} = \sup_{0 \neq h \in H_n(X)} Rad^{\mathsf{norm}}_{S^n}(h).$$

In particular, if X is a connected orientable n-manifold and $[f]$ is the class of *locally constant at infinity maps* $f : X \to S^n$ *of non-zero degrees*, i.e. which *dominate* non-zero-multiples of the fundamental class $[S^n] \in H_n(S^n)$, we speak of *hyperspherical radii of* X,

$$Rad^{\mathsf{norm}}_{S^n}(X) = Rad^{\mathsf{norm}}_{S^n}[X] = \frac{1}{\text{"}\mathsf{norm}\text{"}[f]},$$

with an emphasis on the norms" $Lip = \|df\|$, $\| \wedge^k df\|$ and $\| \wedge^k df\|_{trace}$, $k = 1, 2$, where non-trivial bounds on these radii for manifolds X with $Sc(X) \geq \sigma > 0$, are given (in different terms) in section 3.4.1.

Exercise (a) Show that the hyperspherical radius of the R-sphere $S^n(R)$ defined with any of the norms $\bullet^k_{sup}$ and $\bullet^k_{trace}$ is equal to R.

(b) Evaluate these radii for the (open) Euclidean ball $B^n(R)$ and the cube $[0, R]^n$.[445]

(c) Show that if X is a (Riemannian product) cylinder, $X = X_0 \times \mathbb{R}^1$, and $[f]_{\underline{h}}$ is the class of maps $f : X \to Y$, which send the fundamental class of X to $\underline{h} \in H_n(Y)$, $n = dim(X)$, then the "norms" of the multiples of $\underline{h}$ are bounded by the corresponding "norms" of $\underline{h}$,

$$\text{"}\mathsf{norm}\text{"}(j \cdot \underline{h}) \leq \text{"}\mathsf{norm}\text{"}(\underline{h}),\ j = 0, \pm 1, \pm 2, \ldots .$$

Spaces of Maps and Parametric Radii. A norm on maps $f : X \to Y$ can be regarded as a function on the space $\mathcal{F}$ of maps $X \to Y$ (not only on the set of homotopy classes of maps).

Call such a function $\Psi : \mathcal{F} \to [0, \infty)$ and define a (Morse-kind) filtration on the homology $H_*(\mathcal{F})$, by the *images of the homology homomorphisms* induced by the sublevels of Ψ to $\mathcal{F}$,

$$H_*(\Psi^{-1}[0, \lambda]) \to H_*(\mathcal{F}),\ 0 \leq \lambda < \infty,$$

where these images are denoted

$$H_*(\mathcal{F}|_{\underset{\Psi}{\leq}} \lambda) \subset H_*(\mathcal{F}).$$

[445] I haven't done this exercise for the cube.

Equivalently, Ψ defines a function on $H_*(\mathcal{F})$, call it

$$\Psi_* : H_*(\mathcal{F}) \to [0,\infty),$$

where $\Psi_*(h)$ is the infimum of $\lambda|geq0$ for which $h \in H_*(\mathcal{F} \underset{\Psi}{\leq} \lambda)$.

In other words,

the inequality $\Psi_*(h) \leq \lambda$ for $h \in H_i\mathcal{F})$ signifies that

h is representable by a family P of maps $f_p : X \to Y$, $p \in P$, where P is an oriented i-pseudomanifold and $\Psi(f_p) \leq \lambda$ for all $p \in P$.

Example: Stabilized Radii. Let X be an orientable n-manifold and Y be the unit sphere S^{n+m}. Then the homology of the space $\mathcal{F}$ of continuous maps $f : X \to S^{n+m}$ vanishes for $0 < k < m$ and $H_m(\mathcal{F}) = \mathbb{Z}$. Define the (stabilized) spherical radii of X, by

$$Rad^{\mathsf{norm}}_{S^{n+m}}(X) = Rad^{\mathsf{norm}}_{S^{n+m}}(X \times \mathbb{R}^m),$$

and observe that such a radius is equal to the infimum of the "norms" of the non-zero classes in $H_m(\mathcal{F})$.

Remark. If the above "norm" is associated with $\| \wedge^n df\|$, $n = dim(X)$, (where the maps with $"norm"(f) = \| \wedge^n df\| \leq 1$ are volume non-increasing), then, according the the sharp waist inequality from the previous section, the stabilized radii are equal to the basic one:

$$Rad^{\wedge^n}_{S^{n+m}}(X) = Rad^{\wedge^n}_{S^n} \text{ for all } m = 1,2,\dots\ .$$

Stabilization Conjecture . If $k < n$ then the stabilized radii satisfy:

$$Rad^{\wedge^k}_{S^n}(X) \geq Rad^{\wedge^k}_{S^n} \geq c_{n,m,k} Rad^{\wedge^k}_{S^k}(X),$$

for all $k = 1,\dots,n-1$ and universal constants $c_{n,m,k}$, such that

$$1 > c_{n,1,k} > c_{n,2,k} > \dots > c_{n,m,k} > \dots \geq c_n > 0.$$

Admission. I haven't proved that either $c_{2,3,1} > 0$ or that $c_{2,3,1} = 1$, that is a possible decrease (if any) of minimal Lipschitz constants for maps $X \times \mathbb{R}^1 \to S^3$ with non-zero degrees compared to such maps $X \to S^2$ of oriented surfaces X.

Diagrams and Multiple Norms. All of the above definitions can be generalized by replacing single maps between Riemannian manifolds by diagrams $\mathcal{D} = f_I$ of maps f_i with homotopy commutativity relations imposed on some sub-diagrams in $\mathcal{D}$.

We have met simple instances of such diagrams for distance and area multi-contracting maps to products,

$$f = (f_1, f_2, \dots f_k) : \underline{X} \to \underline{X}_1 \times \underline{X}_2 \times \dots \times \underline{X}_k$$

(see section 3.4.4), where a "total norm" of such an f related to scalar curvature is

$$\left(\frac{1}{k}\sum_{i=1}^{k}\frac{1}{("norm"(f_i))^2}\right)^{-\frac{1}{2}}.$$

Problem Find constraints on norms $Lip(f_i) = \|df_i\|$ and on $\| \wedge^2 df_i\|$ for more complicated diagrams $f_I = \{f_i\}$ of maps between manifolds with Sc-normalized and/or $\mathbb{T}^{\rtimes}$-stabilized (see section 2.4) manifolds with positive scalar curvatures.

7.5 m-Radii of Uniformly Contractible Spaces

Define the m-radius with an above "norm" of a Riemannian manifold X as the supremum of the hyperspherical radii of all m-cycles in X, or, more formally as

$$Rad_m^{norm}(X) = \sup_{V \subset X} Rad_{S^m}^{norm}(V).$$

where the supremum is taken over all relatively compact open (not to worry about pathologies) subsets V in X.

Exercise. Show that if X is an n-dimensional Riemannian manifold with $H_{n-1}(X) = 0$, then

$$Rad_{n-1}^{Lip}(X) \le const_n Rad_{S^n}^{Lip}$$

for $const_n < 10n$.

If X is uniformly contractible (see section 3.10.3) then – it is (almost) obvious – that $Rad_1^{Lip}(X) = \infty$. But it is unclear, in general, if this true for Rad_m^{Lip}, for $m \ge 2$.

Below is a partial result in this direction slightly generalizing that in §9$\frac{3}{11}$ [G(positive) 1996].

Lipschitz Suspension Lemma. Let a Riemannian manifold X contain a double sequence of triples of disjoint $(m-1)$-cycles, i.e. of oriented $(m-1)$-dimensional sub-pseudomanifolds, $A_{ij}, B_{ij}, C_{ij} \subset X$, and let

$$[0, A_{ij}], [A_{ij}, B_{ij}], [B_{ij}, C_{ij}] \subset X$$

be m-chains represented by oriented m-sub-pseudomanifolds with boundaries $-A_{ij}$, $A_i \cup -B_{ij}$ and $B_{ij} \cup -C_{ij}$.

Let

$\bullet_1$ $R_{ij} = Rad_{S^{m-1}}(A_{ij}) \ge R_i \to \infty$ *for* $i \to \infty$;

$\bullet_2$ *the diameters* $diam[0, A_{ij}] \le d_j$ *for some constants* d_j *and all* i;

$\bullet_3$ $[A_{ij}, B_{ij}]$ *is contained in the* $\delta \cdot R_i$*-neighbourhood of* $B_{i,j}$, *i.e.* $dist(x, A_{ij})$, $x \in [A_{ij}, B_{ij}]$, *is bounded by* $n\ \delta \cdot R_i$, *for* $\delta \le \frac{1}{10n}$, $n = dim(X)$;

$\bullet_4$ $dist([B_{ij}, C_{ij}], [0, A_{ij}]) \ge r_i \to \infty$ *for* $i \to \infty$.

$\bullet_5$ $dist(C_{i,j}, [0, A_{i,j}] \ge d_j^+ \to \infty$ *for* $j \to \infty$.

Then, if X is uniformly contractible (or uniformly rationally acyclic) and $n = dim(X) > m$, then

$$Rad_m^{Lip}(X) = \infty.$$

Proof. To keep track of these $\bullet_1$-$\bullet_5$, visualize $A_{i,j}$, B_{ij} and C_{ij} as concentric circles of radii i, $\frac{41}{40}i$ and $2i + j$ in $\mathbb{R}^2 \subset X = \mathbb{R}^3$, where the chains $[0, A_{ij}], [A_{ij}, B_{ij}], [B_{ij}, C_{ij}] \subset X$ are the annuli between these circles.

Then proceed with the proof by observing that uniform contractibility of X implies that the cycle C_{ij} for $j >> i$ (much greater) bounds chain, call it $[C_{ij}, \varnothing_{ij}]$ with the the support far from $[0, A_{i,j}]$, say

$$dist([C_{ij}, \varnothing_{ij}], [0, A_{i,j}]) \ge 10R_i.$$

Then the union D_{ij}^m of these four chains

$$D_{ij}^m = [0, A_{ij}] \cup [A_{ij}, B_{ij}] \cup [B_{ij}, C_{ij}] \cup [C_{ij}, \varnothing_{ij}]$$

makes a m-cycle, such that

$$Rad^{Lip}_{S^m}(D^m_{ij}) \geq \varepsilon R_i,$$

say, for $\varepsilon = \frac{1}{1000n^3}$.

This is shown by constructing a λ-Lipschitz map $D^m_{ij} \to S^m(R_i)$ with non-zero degree and with $\lambda < 100n^2$, such that $[0, A_{ij}] \subset D^m_{ij}$ goes to the south pole of $S^m(R_i)$ and $[B_{ij}, C_{ij}] \cup [C_{ij}, \varnothing_{ij}]$ to the north pole and where the two main ingredients of this construction are the following:

(i) a λ_1-Lipschitz extension of the 1-Lipshitz map $A_{ij} \to S^{m-1}(R_i)$ to the δR_i-neighbourhood of $A_{ij} \subset X$, for $\delta = \frac{1}{10n}$;

(ii) distance function $x \to dist(x, A_{ij})$, $x \in X$.

(Fitting all this together is left to the reader.)

Large Scale Lipschitz Uniform Embeddings. A map between metric spaces, $\phi : Z \to X$ is LSL if there exist positive constants λ and e, such that

$$dist(\phi(z_1 z_), f(z_2)) \leq \lambda \cdot dist(z_1, z_2) + e.$$

A map $\phi : Z \to X$ is LSUE if there exists a function $\Delta(D) = \Delta_\phi(D)$, such that $\Delta(D) \to \infty$ for $D \to \infty$ and

$$dist(\phi(z_1), f(z_2)) \geq \Delta(dist(z_1, z_2)).$$

A map $f : Y \to X$ is LSLU embedding if it is LSL as well as LSU.

LSLUE-Lemma. Let X and Z be Riemannian manifolds, $A_{ij}, B_{ij}, C_{ij} \subset Z$ and $[0, A_{ij}], [A_{ij}, B_{ij}], [B_{ij}, C_{ij}] \subset Z$ be $(m-1)$-cycles and m-chains satisfying the above conditions $\bullet_1$-$\bullet_5$ and let $Z \to X$ be an LSLU embedding.

If X is uniformly contractible (uniformly rationally acyclic will do) then X also contains $(m-1)$-cycles and m-chains cycles, which satisfy $\bullet_1$-$\bullet_5$.

Thus, for instance,

[$\star$] *if $dim(Z) = m$, if $Rad^{Lip}_{S^m}(Z) = \infty$, if X is uniformly contractible and if $dim(X) > m$, then*

$$Rad^{Lip}_m(X) = \infty.$$

[$\star\star$] ***Example of Corollary.*** Let X be a compact aspherical manifold of dimension six.

If the fundamental group $\pi_1(X)$ contains a surface group Γ (e.g. $\Gamma = \mathbb{Z}^2$) as a subgroup, then X admits no metric with $Sc > 0$.

Proof. The inclusion $\Gamma \subset \pi_1(X)$ implies that the universal covering Z of the surface with the fundamental group Γ admits an LSLU embedding to the universal covering $\tilde{X}$ of X. Hence, $Rad^{Lip}_2(\tilde{X}) = \infty$.

On the other hand, an easy argument (see §9$\frac{3}{11}$ [G(positive) 1996] and [G(aspherical) 2020] shows that if a uniformly contractible n-manifold $\tilde{X}$) satisfies $Rad^{Lip}_m = \infty$, then it contains compact submanifolds Y of dimension $n - m - 1$, which have arbitrarily large filling radii, while, if $Sc(X) \geq \sigma$, then $\mathbb{T}^\rtimes$-stabilizations $Y_\rtimes$ of Y have their scalar curvatures bounded from below by $\sigma/2 > 0$.

This, in the present 6d-case, contradicts to the bound $fillrad(Y) \leq const \cdot \sigma$ for $dim(Y) = 3$. QED.

Exercise. Extend all $5d$-results from section 3.10.3 to $(5+m-1)$-manifolds X, which admit maps of non-zero degree to uniformly contractible (and uniformly rationally acyclic) manifolds $\underline{X}$ (and pseudomanifolds with at most 2-dimensional singularities), where the fundamental groups $\pi_1(\underline{X})$ contains subgroups Γ, which serve as fundamental groups of compact m-pseudomanifolds the universal coverings Z of which have $Rad^{Lip}_{S^m}(Z)=\infty$.

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