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Four-manifolds, geometries and knots

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Contents

Preface

Every closed surface admits a geometry of constant curvature, and may be classified topologically either by its fundamental group or by its Euler characteristic and orientation character. Closed 3-manifolds have decompositions into geometric pieces, and are determined up to homeomorphism by invariants associated with the fundamental group (whereas the Euler characteristic is always 0). In dimension 4 the Euler characteristic and fundamental group are largely independent, and the class of closed 4-manifolds which admit a geometric decomposition is rather restricted. For instance, there are only 11 such manifolds with finite fundamental group. On the other hand, many complex surfaces admit geometric structures, as do all the manifolds arising from surgery on twist spun simple knots.

The goal of this book is to characterize algebraically the closed 4-manifolds that fibre nontrivially or admit geometries, or which are obtained by surgery on 2-knots, and to provide a reference for the topology of such manifolds and knots. In many cases the Euler characteristic, fundamental group and Stiefel-Whitney classes together form a complete system of invariants for the homotopy type of such manifolds, and the possible values of the invariants can be described explicitly. If the fundamental group is elementary amenable we may use topological surgery to obtain classifications up to homeomorphism. Surgery techniques also work well "stably" in dimension 4 (i.e., modulo connected sums with copies of $S^2 \times S^2$). However, in our situation the fundamental group may have nonabelian free subgroups and the Euler characteristic is usually the minimal possible for the group, and it is not known whether s-cobordisms between such 4-manifolds are always topologically products. Our strongest results are characterizations of infrasolvmanifolds (up to homeomorphism) and aspherical manifolds which fibre over a surface or which admit a geometry of rank > 1 (up to TOP s-cobordism). As a consequence 2-knots whose groups are poly-Z are determined up to Gluck reconstruction and change of orientations by their groups alone.

We shall now outline the chapters in somewhat greater detail. The first chapter is purely algebraic; here we summarize the relevant group theory and present the notions of amenable group, Hirsch length of an elementary amenable group, finiteness conditions, criteria for the vanishing of cohomology of a group with coefficients in a free module, Poincaré duality groups, and Hilbert modules over the von Neumann algebra of a group. The rest of the book may be divided into three parts: general results on homotopy and surgery (Chapters 2-6), geometries and geometric decompositions (Chapters 7-13), and 2-knots (Chapters 14-18).

Some of the later arguments are applied in microcosm to 2-complexes and PD_3-complexes in Chapter 2, which presents equivariant cohomology, L^2-Betti numbers and Poincaré duality. Chapter 3 gives general criteria for two closed 4-manifolds to be homotopy equivalent, and we show that a closed 4-manifold M is aspherical if and only if $\pi_1(M)$ is a PD_4-group of type FF and $\chi(M)=\chi(\pi)$. We show that if the universal cover of a closed 4-manifold is finitely dominated then it is contractible or homotopy equivalent to S^2 or S^3 or the fundamental group is finite. We also consider at length the relationship between fundamental group and Euler characteristic for closed 4-manifolds. In Chapter 4 we show that a closed 4-manifold M fibres homotopically over S^1 with fibre a PD_3-complex if and only if $\chi(M)=0$ and $\pi_1(M)$ is an extension of Z by a finitely presentable normal subgroup. (There remains the problem of recognizing which PD_3-complexes are homotopy equivalent to 3-manifolds). The dual problem of characterizing the total spaces of S^1-bundles over 3-dimensional bases seems more difficult. We give a criterion that applies under some restrictions on the fundamental group. In Chapter 5 we characterize the homotopy types of total spaces of surface bundles. (Our results are incomplete if the base is RP^2). In particular, a closed 4-manifold M is simple homotopy equivalent to the total space of an F-bundle over B (where B and F are closed surfaces and B is aspherical) if and only if $\chi(M)=\chi(B)\chi(F)$ and $\pi_1(M)$ is an extension of $\pi_1(B)$ by a normal subgroup isomorphic to $\pi_1(F)$. (The extension should split if $F=RP^2$). Any such extension is the fundamental group of such a bundle space; the bundle is determined by the extension of groups in the aspherical cases and by the group and Stiefel-Whitney classes if the fibre is S^2 or RP^2. This characterization is improved in Chapter 6, which considers Whitehead groups and obstructions to constructing s-cobordisms via surgery.

The next seven chapters consider geometries and geometric decompositions. Chapter 7 introduces the 4-dimensional geometries and demonstrates the limitations of geometric methods in this dimension. It also gives a brief outline of the connections between geometries, Seifert fibrations and complex surfaces. In Chapter 8 we show that a closed 4-manifold M is homeomorphic to an infrasolvmanifold if and only if $\chi(M)=0$ and $\pi_1(M)$ has a locally nilpotent normal subgroup of Hirsch length at least 3, and two such manifolds are homeomorphic if and only if their fundamental groups are isomorphic. Moreover $\pi_1(M)$ is then a torsion free virtually poly-Z group of Hirsch length 4 and every such group is the fundamental group of an infrasolvmanifold. We also consider in detail the question of when such a manifold is the mapping torus of a self homeomorphism of a 3-manifold, and give a direct and elementary derivation of the fundamental groups of flat 4-manifolds. At the end of this chapter we show that all orientable 4-dimensional infrasolvmanifolds are determined up to diffeomorphism by their fundamental groups. (The corresponding result in other dimensions was known).

Chapters 9-12 consider the remaining 4-dimensional geometries, grouped according to whether the model is homeomorphic to R^4, $S^2 \times R^2$, $S^3 \times R$ or is compact. Aspherical geometric 4-manifolds are determined up to s-cobordism by their homotopy type. However there are only partial characterizations of the groups arising as fundamental groups of $\mathbb{H}^2 \times \mathbb{H}^2$-manifolds, while very little is known about $\mathbb{H}^4$- or $\mathbb{H}^2(\mathbb{C})$-manifolds. We show that the homotopy types of manifolds covered by $S^2 \times R^2$ are determined up to finite ambiguity by their fundamental groups. If the fundamental group is torsion free such a manifold is s-cobordant to the total space of an S^2-bundle over an aspherical surface. The homotopy types of manifolds covered by $S^3 \times R$ are determined by the fundamental group and first nonzero k-invariant; much is known about the possible fundamental groups, but less is known about which k-invariants are realized. Moreover, although the fundamental groups are all "good", so that in principle surgery may be used to give a classification up to homeomorphism, the problem of computing surgery obstructions seems very difficult. We conclude the geometric section of the book in Chapter 13 by considering geometric decompositions of 4-manifolds which are also mapping tori or total spaces of surface bundles, and we characterize the complex surfaces which fibre over S^1 or over a closed orientable 2-manifold.

The final five chapters are on 2-knots. Chapter 14 is an overview of knot theory; in particular it is shown how the classification of higher-dimensional knots may be largely reduced to the classification of knot manifolds. The knot exterior is determined by the knot manifold and the conjugacy class of a normal generator for the knot group, and at most two knots share a given exterior. An essential step is to characterize 2-knot groups. Kervaire gave homological conditions which characterize high dimensional knot groups and which 2-knot groups must satisfy, and showed that any high dimensional knot group with a presentation of deficiency 1 is a 2-knot group. Bridging the gap between the homological and combinatorial conditions appears to be a delicate task. In Chapter 15 we investigate 2-knot groups with infinite normal subgroups which have no noncyclic free subgroups. We show that under mild coherence hypotheses such 2-knot groups usually have nontrivial abelian normal subgroups, and we determine all 2-knot groups with finite commutator subgroup. In Chapter 16 we show that if there is an abelian normal subgroup of rank > 1 then the knot manifold is either

s-cobordant to a $\widetilde{\mathbb{SL}} \times \mathbb{E}^1$-manifold or is homeomorphic to an infrasolvmanifold. In Chapter 17 we characterize the closed 4-manifolds obtained by surgery on certain 2-knots, and show that just eight of the 4-dimensional geometries are realised by knot manifolds. We also consider when the knot manifold admits a complex structure. The final chapter considers when a fibred 2-knot with geometric fibre is determined by its exterior. We settle this question when the monodromy has finite order or when the fibre is R^3/Z^3 or is a coset space of the Lie group Nil^3.

This book arose out of two earlier books of mine, on "*2-Knots and their Groups*" and "*The Algebraic Characterization of Geometric 4-Manifolds*", published by Cambridge University Press for the Australian Mathematical Society and for the London Mathematical Society, respectively. About a quarter of the present text has been taken from these books. [1] However the arguments have been improved in many cases, notably in using Bowditch's homological criterion for virtual surface groups to streamline the results on surface bundles, using L^2-methods instead of localization, completing the characterization of mapping tori, relaxing the hypotheses on torsion or on abelian normal subgroups in the fundamental group and in deriving the results on 2-knot groups from the work on 4-manifolds. The main tools used here beyond what can be found in *Algebraic Topology* [Sp] are cohomology of groups, equivariant Poincaré duality and (to a lesser extent) L^2-(co)homology. Our references for these are the books *Homological Dimension of Discrete Groups* [Bi], *Surgery on Compact Manifolds* [Wl] and *L^2-Invariants: Theory and Applications to Geometry and K-Theory* [Lü], respectively. We also use properties of 3-manifolds (for the construction of examples) and calculations of Whitehead groups and surgery obstructions.

This work has been supported in part by ARC small grants, enabling visits by Steve Plotnick, Mike Dyer, Charles Thomas and Fang Fuquan. I would like to thank them all for their advice, and in particular Steve Plotnick for the collaboration reported in Chapter 18. I would also like to thank Robert Bieri, Robin Cobb, Peter Linnell and Steve Wilson for their collaboration, and Warren Dicks, William Dunbar, Ross Geoghegan, F.T.Farrell, Ian Hambleton, Derek Holt, K.F.Lai, Eamonn O'Brien, Peter Scott and Shmuel Weinberger for their correspondance and advice on aspects of this work.

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[1] See the Acknowledgment following this preface for a summary of the textual borrowings.

Acknowledgment

I wish to thank Cambridge University Press for their permission to use material from my earlier books [H1] and [H2]. The textual borrowings in each Chapter [of the 2002 version] are outlined below.

1. §1, Lemmas 1.7 and 1.10 and Theorem 1.11, §6 (up to the discussion of $\chi(\pi)$), the first paragraph of §7 and Theorem 1.15 are from [H2:Chapter I]. (Lemma 1.1 is from [H1]). §3 is from [H2:Chapter VI].

2. §1, most of §4 and part of §5 are from [H2:Chapter II and Appendix].

3. Lemma 3.1, Theorems 3.2, 3.7-3.9 and Corollaries 3.9.1-3.9.3 of Theorem 3.12 are from [H2:Chapter II]. (Theorem 3.9 has been improved).

4. The statements of Corollaries 4.5.1-4.5.3, Corollary 4.5.4 and most of §7 are from [H2:Chapter III]. (Theorem 11 and the subsequent discussion have been improved).

5. Part of Lemma 5.15 and §4-§5 are from [H2:Chapter IV]. (Theorem 5.19 and Lemmas 5.21 and 5.22 have been improved).

6. §1 (excepting Theorem 6.1), Theorem 6.12 and the proof of Theorem 6.14 are from [H2:Chapter V].

8. Part of Theorem 8.1, §6, most of §7 and §8 are from [H2:Chapter VI].

9. Theorems 9.1, 9.2 and 9.7 are from [H2:Chapter VI], with improvements.

10. Theorems 10.10-10.12 and §6 are largely from [H2:Chapter VII]. (Theorem 10.10 has been improved).

11. Lemma 11.3, §3 and the first three paragraphs of §5 are from [H2:Chapter VIII]. §6 is from [H2:Chapter IV].

12. The introduction, §1-§3, §5, most of §6 (from Lemma 12.5 onwards) and §7 are from [H2:Chapter IX], with improvements (particularly in §7).

14. §1-§5 are from [H1:Chapter I]. §6 and §7 are from [H1:Chapter II].

16. Most of §3 is from [H1:Chapter V].(Theorem 16.4 is new and Theorems 16.5 and 16.6 have been improved).

17. Lemma 2 and Theorem 7 are from [H1:Chapter VIII], while Corollary 17.6.1 is from [H1:Chapter VII]. The first two paragraphs of §8 and Lemma 17.12 are from [H2:Chapter X].

[Added in 2014]

In 2007 some of the material was improved, particularly as regards

(a) *finiteness conditions* (Chapters 3 and 4); and

(b) *(aspherical) Seifert fibred 4-manifolds* (Chapters 7 and 9).

Some results on the equivariant intersection pairing and the notion of (strongly) minimal PD_4-complex were added as new sections §3.5 and (now) §10.8.

Further improvements have been made since, particularly as regards

(c) *non-aspherical geometric 4-manifolds* (Chapter 10–12).

In Chapter 10 we show that every closed 4-manifold with universal cover $\simeq S^2$ is homotopy equivalent to either the total space of an S^2-orbifold bundle over an aspherical 2-orbifold or to the total space of an RP^2-bundle over an aspherical surface. (Most such S^2-orbifold bundle spaces are geometric [Hi13].) Chapter 11 now has a brief sketch of the work of Davis and Weinberger [DW07] on mapping tori of orientation-reversing self homotopy equivalences of lens spaces (quotients of $S^3 \times \mathbb{R}$), and we have added four pages to Chapter 12 on quotients of $S^2 \times S^2$ by $Z/4Z$.

We have cited Perelman's work on geometrization to simplify some statements, since detailed accounts are now available [B-P], and have used the Virtual Fibration Theorem of Agol [Ag13] in Chapters 9 and 13, in connection with the geometry $\mathbb{H}^3 \times \mathbb{E}^1$. We have used the notion of orbifold more widely, improved the discussion of surgery in Chapter 6 and tightened some of the results on 2-knots. In particular, there is a new family of 2-knots with torsion free, solvable groups, which was overlooked before. (See Theorem 16.15.) Six new pages have been added to Chapter 18. In these we settle the questions of reflexivity, amphicheirality and invertibility for the other such knots.

The classification of 4-dimensional infrasolvmanifolds is now essentially known [Hi13d, LT13]. This has been noted, but the details are not included here.

The errors and typos discovered up to 30 June 2014 have been corrected.

I would like to thank J.F. Davis and S.Weinberger for permission to include a summary of [DW07], I.Hambleton, M.Kemp, D.H.Kochloukova and S.K.Roushon for their collaboration in relation to some of the improvements recorded here, and J.G.Ratcliffe for alerting me to some gaps in §4 of Chapter 8.

Part I

Manifolds and PD-complexes

Chapter 1

Group theoretic preliminaries

The key algebraic idea used in this book is to study the homology groups of covering spaces as modules over the group ring of the group of covering transformations. In this chapter we shall summarize the relevant notions from group theory, in particular, the Hirsch-Plotkin radical, amenable groups, Hirsch length, finiteness conditions, the connection between ends and the vanishing of cohomology with coefficients in a free module, Poincaré duality groups and Hilbert modules.

Our principal references for group theory are [Bi], [DD] and [Ro].

1.1 Group theoretic notation and terminology

We shall write $\mathbb{Z}$ for the ring of integers and for the augmentation module of a group, and otherwise write Z for the free (abelian) group of rank 1. Let $F(r)$ be the free group of rank r.

Let G be a group. Then G' and ζG denote the commutator subgroup and centre of G, respectively. The *outer automorphism group* of G is $Out(G) = Aut(G)/Inn(G)$, where $Inn(G) \cong G/\zeta G$ is the subgroup of $Aut(G)$ consisting of conjugations by elements of G. If H is a subgroup of G let $N_G(H)$ and $C_G(H)$ denote the normalizer and centralizer of H in G, respectively. The subgroup H is a *characteristic* subgroup of G if it is preserved under all automorphisms of G. In particular, $I(G) = \{g \in G \mid \exists n > 0,\ g^n \in G'\}$ is a characteristic subgroup of G, and the quotient $G/I(G)$ is a torsion-free abelian group of rank $\beta_1(G)$. A group G is *indicable* if there is an epimorphism $p : G \to Z$, or if $G = 1$. If S is a subset of G then $\langle S\rangle$ and $\langle\langle S\rangle\rangle_G$ (or just $\langle\langle S\rangle\rangle$) are the subgroup generated by S and the *normal closure* of S in G (the intersection of the normal subgroups of G which contain S), respectively.

If P and Q are classes of groups let PQ denote the class of ("P *by* Q") groups G which have a normal subgroup H in P such that the quotient G/H is in Q, and let ℓP denote the class of ("*locally* P") groups such that each finitely generated subgroup is in the class P. In particular, if F is the class of finite groups ℓF is the class of *locally finite* groups. In any group the union of all the locally-finite normal subgroups is the unique maximal locally-finite normal subgroup. Clearly there are no nontrivial homomorphisms from such a group to a torsion-free group. Let *poly-P* be the class of groups with a finite composition series such that each subquotient is in P. Thus if Ab is the class of abelian groups poly-Ab is the class of solvable groups.

Let P be a class of groups which is closed under taking subgroups. A group is *virtually* P if it has a subgroup of finite index in P. Let vP be the class of groups which are virtually P. Thus a *virtually poly-Z* group is one which has a subgroup of finite index with a composition series whose factors are all infinite cyclic. The number of infinite cyclic factors is independent of the choice of finite index subgroup or composition series, and is called the *Hirsch length* of the group. We shall also say that a space virtually has some property if it has a finite regular covering space with that property.

If $p : G \to Q$ is an epimorphism with kernel N we shall say that G is an *extension of $Q = G/N$ by the normal subgroup N*. The action of G on N by conjugation determines a homomorphism from G to $Aut(N)$ with kernel $C_G(N)$ and hence a homomorphism from G/N to $Out(N) = Aut(N)/Inn(N)$. If $G/N \cong Z$ the extension splits: a choice of element t in G which projects to a generator of G/N determines a right inverse to p. Let θ be the automorphism of N determined by conjugation by t in G. Then G is isomorphic to the semidirect product $N \rtimes_\theta Z$. Every automorphism of N arises in this way, and automorphisms whose images in $Out(N)$ are conjugate determine isomorphic semidirect products. In particular, $G \cong N \times Z$ if θ is an inner automorphism.

Lemma 1.1 *Let θ and ϕ automorphisms of a group G such that $H_1(\theta;\mathbb{Q})-1$ and $H_1(\phi;\mathbb{Q})-1$ are automorphisms of $H_1(G;\mathbb{Q}) = (G/G')\otimes\mathbb{Q}$. Then the semidirect products $\pi_\theta = G \rtimes_\theta Z$ and $\pi_\phi = G \rtimes_\phi Z$ are isomorphic if and only if θ is conjugate to ϕ or ϕ^{-1} in $Out(G)$.*

Proof Let t and u be fixed elements of π_θ and π_ϕ, respectively, which map to 1 in Z. Since $H_1(\pi_\theta;\mathbb{Q}) \cong H_1(\pi_\phi;\mathbb{Q}) \cong Q$ the image of G in each group is characteristic. Hence an isomorphism $h : \pi_\theta \to \pi_\phi$ induces an isomorphism $e : Z \to Z$ of the quotients, for some $e = \pm 1$, and so $h(t) = u^e g$ for some g in G. Therefore $h(\theta(h^{-1}(j))) = h(th^{-1}(j)t^{-1}) = u^e gjg^{-1}u^{-e} = \phi^e(gjg^{-1})$ for all j in G. Thus θ is conjugate to ϕ^e in $Out(G)$.

Conversely, if θ and ϕ^e are conjugate in $Out(G)$ there is an f in $Aut(G)$ and a g in G such that $\theta(j) = f^{-1}\phi^e f(gjg^{-1})$ for all j in G. Hence $F(j) = f(j)$ for all j in G and $F(t) = u^e f(g)$ defines an isomorphism $F : \pi_\theta \to \pi_\phi$. □

A subgroup K of a group G is *ascendant* if there is an increasing sequence of subgroups N_α, indexed by ordinals $\leq \beth$, such that $N_0 = K$, N_α is normal in $N_{\alpha+1}$ if $\alpha < \beth$, $N_\beta = \cup_{\alpha<\beta} N_\alpha$ for all limit ordinals $\beta \leq \beth$ and $N_\beth = G$. If $\beth$ is finite K is *subnormal* in G. Such ascendant series are well suited to arguments by transfinite induction.

1.2 Matrix groups

In this section we shall recall some useful facts about matrices over $\mathbb{Z}$.

Lemma 1.2 *Let p be an odd prime. Then the kernel of the reduction modulo (p) homomorphism from $SL(n,\mathbb{Z})$ to $SL(n,\mathbb{F}_p)$ is torsion-free.*

Proof This follows easily from the observation that if A is an integral matrix and $k = p^v q$ with q not divisible by p then $(I + p^r A)^k \equiv I + kp^r A \ mod\ (p^{2r+v})$, and $kp^r \not\equiv 0 \ mod\ (p^{2r+v})$ if $r \geq 1$. □

Similarly, the kernel of reduction *mod* (4) is torsion-free.

Since $SL(n,\mathbb{F}_p)$ has order $(\Pi_{j=0}^{j=n-1}(p^n - p^j))/(p-1)$, it follows that the order of any finite subgroup of $SL(n,\mathbb{Z})$ must divide the highest common factor of these numbers, as p varies over all odd primes. In particular, finite subgroups of $SL(2,\mathbb{Z})$ have order dividing 24, and so are solvable.

Let $A = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, $B = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 1 \end{smallmatrix}\right)$ and $R = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$. Then $A^2 = B^3 = -I$ and $A^4 = B^6 = I$. The matrices A and R generate a dihedral group of order 8, while B and R generate a dihedral group of order 12.

Theorem 1.3 *Let G be a nontrivial finite subgroup of $GL(2,\mathbb{Z})$. Then G is conjugate to one of the cyclic groups generated by A, $A^2 = -I$, B, B^2, R or RA, or to one of the dihedral groups generated by $\{A, R\}$, $\{-I, R\}$, $\{A^2, RA\}$, $\{B, R\}$, $\{B^2, R\}$ or $\{B^2, RB\}$. If $G \neq \langle -I_2 \rangle$ then $N_{GL(2,\mathbb{Z})}(G)$ is finite.*

Proof If $M \in GL(2,\mathbb{Z})$ has finite order then its characteristic polynomial has cyclotomic factors. If the characteristic polynomial is $(X \pm 1)^2$ then $M = \mp I$. (This uses the finite order of M.) If the characteristic polynomial is $X^2 - 1$ then M is conjugate to R or RA. If the characteristic polynomial is $X^2 + 1$, $X^2 - X + 1$ or $X^2 + X + 1$ then it is irreducible, and the corresponding ring of algebraic numbers is a PID. Since any $\mathbb{Z}$-torsion-free module over such a ring is free it follows easily that M is conjugate to A, B or B^2.

The normalizers in $SL(2,\mathbb{Z})$ of the subgroups generated by A, B or B^2 are easily seen to be finite cyclic. Since $G \cap SL(2,\mathbb{Z})$ is solvable it must be cyclic also. As it has index at most 2 in G the rest of the theorem follows easily. □

Although the 12 groups listed in the theorem represent distinct conjugacy classes in $GL(2,\mathbb{Z})$, some of these conjugacy classes coalesce in $GL(2,\mathbb{R})$. (For instance, R and RA are conjugate in $GL(2,\mathbb{Z}[\frac{1}{2}])$.)

Corollary 1.3.1 *Let G be a locally finite subgroup of $GL(2,\mathbb{Q})$. Then G is finite, and is conjugate to one of the above subgroups of $GL(2,\mathbb{Z})$.*

Proof Let L be a finitely generated subgroup of rank 2 in $\mathbb{Q}^2$. If G is finite then $\cup_{g\in G} gL$ is finitely generated, G-invariant and of rank 2, and so G is conjugate to a subgroup of $GL(2,\mathbb{Z})$. In general, as the finite subgroups of G have bounded order G must be finite. $\square$

Theorem 1.3 also follows from the fact that $PSL(2,\mathbb{Z}) = SL(2,\mathbb{Z})/\langle \pm I\rangle$ is a free product $(Z/2Z)*(Z/3Z)$, generated by the images of A and B. (In fact $\langle A, B \mid A^2 = B^3,\ A^4 = 1\rangle$ is a presentation for $SL(2,\mathbb{Z})$.) Moreover, $SL(2,\mathbb{Z})' \cong PSL(2,\mathbb{Z})'$ is freely generated by the images of $ABA^{-1}B^{-1} = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$ and $A^{-1}B^{-1}AB = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 2 \end{smallmatrix}\right)$, while the abelianizations are generated by the images of $AB = \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$. (See §6.2 of [Ro].)

The groups arising as extension of such groups G by Z^2 are the flat 2-orbifold groups, or 2-dimensional crystallographic groups. In three cases $H^2(G;\mathbb{Z}^2) \neq 0$, and there are in fact 17 isomorphism classes of such groups.

Let $\Lambda = \mathbb{Z}[t, t^{-1}]$ be the ring of integral Laurent polynomials. The next theorem is a special case of a classical result of Latimer and MacDuffee.

Theorem 1.4 *There is a 1-1 correspondance between conjugacy classes of matrices in $GL(n,\mathbb{Z})$ with irreducible characteristic polynomial $\Delta(t)$ and isomorphism classes of ideals in $\Lambda/(\Delta(t))$. The set of such ideal classes is finite.*

Proof Let $A \in GL(n,\mathbb{Z})$ have characteristic polynomial $\Delta(t)$ and let $R = \Lambda/(\Delta(t))$. As $\Delta(A) = 0$, by the Cayley-Hamilton Theorem, we may define an R-module M_A with underlying abelian group Z^n by $t.z = A(z)$ for all $z \in Z^n$. As R is a domain and has rank n as an abelian group M_A is torsion-free and of rank 1 as an R-module, and so is isomorphic to an ideal of R. Conversely every R-ideal arises in this way. The isomorphism of abelian groups underlying an R-isomorphism between two such modules M_A and M_B determines a matrix $C \in GL(n,\mathbb{Z})$ such that $CA = BC$. The final assertion follows from the Jordan-Zassenhaus Theorem. $\square$

1.3 The Hirsch-Plotkin radical

The *Hirsch-Plotkin radical* $\sqrt{G}$ of a group G is its maximal locally-nilpotent normal subgroup; in a virtually poly-Z group every subgroup is finitely generated, and so $\sqrt{G}$ is then the maximal nilpotent normal subgroup. If H is normal in G then $\sqrt{H}$ is normal in G also, since it is a characteristic subgroup of H, and in particular it is a subgroup of $\sqrt{G}$.

For each natural number $q \geq 1$ let Γ_q be the group with presentation

$$\langle x, y, z \mid xz = zx,\ yz = zy,\ xy = z^q yx \rangle.$$

Every such group Γ_q is torsion-free and nilpotent of Hirsch length 3.

Theorem 1.5 *Let G be a finitely generated torsion-free nilpotent group of Hirsch length $h(G) \leq 4$. Then either*

(1) *G is free abelian; or*

(2) *$h(G) = 3$ and $G \cong \Gamma_q$ for some $q \geq 1$; or*

(3) *$h(G) = 4$, $\zeta G \cong Z^2$ and $G \cong \Gamma_q \times Z$ for some $q \geq 1$; or*

(4) *$h(G) = 4$, $\zeta G \cong Z$ and $G/\zeta G \cong \Gamma_q$ for some $q \geq 1$.*

In the latter case G has characteristic subgroups which are free abelian of rank 1, 2 and 3. In all cases G is an extension of Z by a free abelian normal subgroup.

Proof The centre ζG is nontrivial and the quotient $G/\zeta G$ is again torsion - abelian, and hence that $G/\zeta G$ is not cyclic. Hence $h(G/\zeta G) \geq 2$, so $h(G) \geq 3$ and $1 \leq h(\zeta G) \leq h(G) - 2$. In all cases ζG is free abelian.

If $h(G) = 3$ then $\zeta G \cong Z$ and $G/\zeta G \cong Z^2$. On choosing elements x and y representing a basis of $G/\zeta G$ and z generating ζG we quickly find that G is isomorphic to one of the groups Γ_q, and thus is an extension of Z by Z^2.

If $h(G) = 4$ and $\zeta G \cong Z^2$ then $G/\zeta G \cong Z^2$, so $G' \subseteq \zeta G$. Since G may be generated by elements x, y, t and u where x and y represent a basis of $G/\zeta G$ and t and u are central it follows easily that G' is infinite cyclic. Therefore ζG is not contained in G' and G has an infinite cyclic direct factor. Hence $G \cong Z \times \Gamma_q$, for some $q \geq 1$, and thus is an extension of Z by Z^3.

The remaining possibility is that $h(G) = 4$ and $\zeta G \cong Z$. In this case $G/\zeta G$ is torsion-free nilpotent of Hirsch length 3. If $G/\zeta G$ were abelian G' would also be infinite cyclic, and the pairing from $G/\zeta G \times G/\zeta G$ into G' defined by the commutator would be nondegenerate and skewsymmetric. But there are no such pairings on free abelian groups of odd rank. Therefore $G/\zeta G \cong \Gamma_q$, for some $q \geq 1$.

Let $\zeta_2 G$ be the preimage in G of $\zeta(G/\zeta G)$. Then $\zeta_2 G \cong Z^2$ and is a characteristic subgroup of G, so $C_G(\zeta_2 G)$ is also characteristic in G. The quotient $G/\zeta_2 G$ acts by conjugation on $\zeta_2 G$. Since $Aut(Z^2) = GL(2,\mathbb{Z})$ is virtually free and $G/\zeta_2 G \cong \Gamma_q/\zeta\Gamma_q \cong Z^2$ and since $\zeta_2 G \neq \zeta G$ it follows that $h(C_G(\zeta_2 G)) = 3$. Since $C_G(\zeta_2 G)$ is nilpotent and has centre of rank ≥ 2 it is abelian, and so $C_G(\zeta_2 G) \cong Z^3$. The preimage in G of the torsion subgroup of $G/C_G(\zeta_2 G)$ is torsion-free, nilpotent of Hirsch length 3 and virtually abelian and hence is abelian. Therefore $G/C_G(\zeta_2 G) \cong Z$. □

Theorem 1.6 *Let π be a torsion-free virtually poly-Z group of Hirsch length 4. Then $h(\sqrt{\pi}) \geq 3$.*

Proof Let S be a solvable normal subgroup of finite index in π. Then the lowest nontrivial term of the derived series of S is an abelian subgroup which is characteristic in S and so normal in π. Hence $\sqrt{\pi} \neq 1$. If $h(\sqrt{\pi}) \leq 2$ then $\sqrt{\pi} \cong Z$ or Z^2. Suppose π has an infinite cyclic normal subgroup A. On replacing π by a normal subgroup σ of finite index we may assume that A is central and that σ/A is poly-Z. Let B be the preimage in σ of a nontrivial abelian normal subgroup of σ/A. Then B is nilpotent (since A is central and B/A is abelian) and $h(B) > 1$ (since $B/A \neq 1$ and σ/A is torsion-free). Hence $h(\sqrt{\pi}) \geq h(\sqrt{\sigma}) > 1$.

If π has a normal subgroup $N \cong Z^2$ then $Aut(N) \cong GL(2,\mathbb{Z})$ is virtually free, and so the kernel of the natural map from π to $Aut(N)$ is nontrivial. Hence $h(C_\pi(N)) \geq 3$. Since $h(\pi/N) = 2$ the quotient π/N is virtually abelian, and so $C_\pi(N)$ is virtually nilpotent.

In all cases we must have $h(\sqrt{\pi}) \geq 3$. □

1.4 Amenable groups

The class of *amenable* groups arose first in connection with the Banach-Tarski paradox. A group is amenable if it admits an invariant mean for bounded $\mathbb{C}$-valued functions [Pi]. There is a more geometric characterization of finitely presentable amenable groups that is more convenient for our purposes. Let X be a finite cell-complex with universal cover $\widetilde{X}$. Then $\widetilde{X}$ is an increasing union of finite subcomplexes $X_j \subseteq X_{j+1} \subseteq \widetilde{X} = \cup_{n\geq 1} X_n$ such that X_j is the union of $N_j < \infty$ translates of some fundamental domain D for $G = \pi_1(X)$. Let N'_j be the number of translates of D which meet the frontier of X_j in $\widetilde{X}$. The sequence $\{X_j\}$ is a *Følner exhaustion* for $\widetilde{X}$ if $\lim(N'_j/N_j) = 0$, and $\pi_1(X)$ is amenable if and only if $\widetilde{X}$ has a Følner exhaustion. This class contains all finite groups and Z, and is closed under the operations of extension, increasing union, and under the formation of sub- and quotient groups. (However nonabelian free groups are not amenable.)

The subclass EG generated from finite groups and Z by the operations of extension and increasing union is the class of *elementary amenable* groups. We may construct this class as follows. Let $U_0 = 1$ and U_1 be the class of finitely generated virtually abelian groups. If U_α has been defined for some ordinal α let $U_{\alpha+1} = (\ell U_\alpha)U_1$ and if U_α has been defined for all ordinals less than some limit ordinal β let $U_\beta = \cup_{\alpha<\beta} U_\alpha$. Let κ be the first uncountable ordinal. Then $EG = \ell U_\kappa$.

This class is well adapted to arguments by transfinite induction on the ordinal $\alpha(G) = \min\{\alpha | G \in U_\alpha\}$. It is closed under extension (in fact $U_\alpha U_\beta \subseteq U_{\alpha+\beta}$) and increasing union, and under the formation of sub- and quotient groups. As U_κ contains every countable elementary amenable group, $U_\lambda = \ell U_\kappa = EG$ if $\lambda > \kappa$. Torsion groups in EG are locally finite and elementary amenable free groups are cyclic. Every locally-finite by virtually solvable group is elementary amenable; however this inclusion is proper.

For example, let Z^∞ be the free abelian group with basis $\{x_i \mid i \in \mathbb{Z}\}$ and let G be the subgroup of $Aut(Z^\infty)$ generated by $\{e_i \mid i \in \mathbb{Z}\}$, where $e_i(x_i) = x_i + x_{i+1}$ and $e_i(x_j) = x_j$ if $j \neq i$. Then G is the increasing union of subgroups isomorphic to groups of upper triangular matrices, and so is locally nilpotent. However it has no nontrivial abelian normal subgroups. If we let ϕ be the automorphism of G defined by $\phi(e_i) = e_{i+1}$ for all i then $G \rtimes_\phi Z$ is a finitely generated torsion-free elementary amenable group which is not virtually solvable.

It can be shown (using the Følner condition) that finitely generated groups of subexponential growth are amenable. The class SG generated from such groups by extensions and increasing unions contains EG (since finite groups and finitely generated abelian groups have polynomial growth), and is the largest class of groups over which topological surgery techniques are known to work in dimension 4 [FT95]. There is a finitely presentable group in SG which is not elementary amenable [Gr98], and a finitely presentable amenable group which is not in SG [BV05].

A group is *restrained* if it has no noncyclic free subgroup. Amenable groups are restrained, but there are finitely presentable restrained groups which are not amenable [OS02, Lo13]. There are also infinite finitely generated torsion groups. (See §14.2 of [Ro].) These are restrained, but are not elementary amenable. No known example is also finitely presentable.

1.5 Hirsch length

In this section we shall use transfinite induction to extend the notion of Hirsch length (as a measure of the size of a solvable group) to elementary amenable groups, and to establish the basic properties of this invariant.

Lemma 1.7 *Let G be a finitely generated infinite elementary amenable group. Then G has normal subgroups $K < H$ such that G/H is finite, H/K is free abelian of positive rank and the action of G/H on H/K by conjugation is effective.*

Proof We may show that G has a normal subgroup K such that G/K is an infinite virtually abelian group, by transfinite induction on $\alpha(G)$. We may assume that G/K has no nontrivial finite normal subgroup. If H is a subgroup of G which contains K and is such that H/K is a maximal abelian normal subgroup of G/K then H and K satisfy the above conditions. □

In particular, finitely generated infinite elementary amenable groups are virtually indicable.

If G is in U_1 let $h(G)$ be the rank of an abelian subgroup of finite index in G. If $h(G)$ has been defined for all G in U_α and H is in ℓU_α let

$$h(H) = \text{l.u.b.}\{h(F) | F \leq H,\ F \in U_\alpha\}.$$

Finally, if G is in $U_{\alpha+1}$, so has a normal subgroup H in ℓU_α with G/H in U_1, let $h(G) = h(H) + h(G/H)$.

Theorem 1.8 *Let G be an elementary amenable group. Then*

(1) *$h(G)$ is well defined;*

(2) *If H is a subgroup of G then $h(H) \leq h(G)$;*

(3) *$h(G) = \text{l.u.b.}\{h(F) \mid F \text{ is a finitely generated subgroup of } G\}$;*

(4) *if H is a normal subgroup of G then $h(G) = h(H) + h(G/H)$.*

Proof We shall prove all four assertions simultaneously by induction on $\alpha(G)$. They are clearly true when $\alpha(G) = 1$. Suppose that they hold for all groups in U_α and that $\alpha(G) = \alpha + 1$. If G is in ℓU_α so is any subgroup, and (1) and (2) are immediate, while (3) follows since it holds for groups in U_α and since each finitely generated subgroup of G is a U_α-subgroup. To prove (4) we may assume that $h(H)$ is finite, for otherwise both $h(G)$ and $h(H) + h(G/H)$ are ∞, by (2). Therefore by (3) there is a finitely generated subgroup $J \leq H$ with $h(J) = h(H)$. Given a finitely generated subgroup Q of G/H we may choose a finitely generated subgroup F of G containing J and whose image in G/H is Q. Since F is finitely generated it is in U_α and so $h(F) = h(H)+h(Q)$. Taking least upper bounds over all such Q we have $h(G) \geq h(H) + h(G/H)$. On the other hand if F is any U_α-subgroup of G then $h(F) = h(F \cap H) + h(FH/H)$, since (4) holds for F, and so $h(G) \leq h(H) + h(G/H)$, Thus (4) holds for G also.

Now suppose that G is not in ℓU_α, but has a normal subgroup K in ℓU_α such that G/K is in U_1. If K_1 is another such subgroup then (4) holds for K and K_1 by the hypothesis of induction and so $h(K) = h(K \cap K_1) + h(KK_1/K)$. Since we also have $h(G/K) = h(G/KK_1)+h(KK_1/K)$ and $h(G/K_1) = h(G/KK_1)+h(KK_1/K_1)$ it follows that $h(K_1)+h(G/K_1) = h(K)+h(G/K)$ and so $h(G)$ is well defined. Property (2) follows easily, as any subgroup of G is an extension of a subgroup of G/K by a subgroup of K. Property (3) holds for K by the hypothesis of induction. Therefore if $h(K)$ is finite K has a finitely generated subgroup J with $h(J) = h(K)$. Since G/K is finitely generated there is a finitely generated subgroup F of G containing J and such that $FK/K = G/K$. Clearly $h(F) = h(G)$. If $h(K)$ is infinite then for every $n \geq 0$ there is a finitely generated subgroup J_n of K with $h(J_n) \geq n$. In either case, (3) also holds for G. If H is a normal subgroup of G then H and G/H are also in $U_{\alpha+1}$, while $H \cap K$ and $KH/H = K/H \cap K$ are in ℓU_α and $HK/K = H/H \cap K$ and G/HK are in U_1. Therefore

$$\begin{aligned} h(H) + h(G/H) &= h(H \cap K) + h(HK/K) + h(HK/H) + h(G/HK) \\ &= h(H \cap K) + h(HK/H) + h(HK/K) + h(G/HK). \end{aligned}$$

Since K is in ℓU_α and G/K is in U_1 this sum gives $h(G) = h(K) + h(G/K)$ and so (4) holds for G. This completes the inductive step. □

Let $\Lambda(G)$ be the maximal locally-finite normal subgroup of G.

Theorem 1.9 *There are functions d and M from $\mathbb{Z}_{\geq 0}$ to $\mathbb{Z}_{\geq 0}$ such that if G is an elementary amenable group of Hirsch length at most h and $\Lambda(G)$ is its* *maximal locally finite normal subgroup then $G/\Lambda(G)$ has a maximal solvable normal subgroup of derived length at most $d(h)$ and index at most $M(h)$.*

Proof We argue by induction on h. Since an elementary amenable group has Hirsch length 0 if and only if it is locally finite we may set $d(0) = 0$ and $M(0) = 1$. Assume that the result is true for all such groups with Hirsch length at most h and that G is an elementary amenable group with $h(G) = h + 1$.

Suppose first that G is finitely generated. Then by Lemma 1.7 there are normal subgroups $K < H$ in G such that G/H is finite, H/K is free abelian of rank $r \geq 1$ and the action of G/H on H/K by conjugation is effective. (Note that $r = h(G/K) \leq h(G) = h + 1$.) Since the kernel of the natural map from $GL(r, \mathbb{Z})$ to $GL(r, \mathbb{F}_3)$ is torsion-free, by Lemma 1.2, we see that G/H embeds in $GL(r, \mathbb{F}_3)$ and so has order at most 3^{r^2}. Since $h(K) = h(G) - r \leq h$ the inductive hypothesis applies for K, so it has a normal subgroup L containing $\Lambda(K)$ and of index at most $M(h)$ such that $L/\Lambda(K)$ has derived length at most $d(h)$ and is the maximal solvable normal subgroup of $K/\Lambda(K)$. As $\Lambda(K)$ and L are characteristic in K they are normal in G. (In particular, $\Lambda(K) = K \cap \Lambda(G)$.) The centralizer of K/L in H/L is a normal solvable subgroup of G/L with index at most $[K : L]![G : H]$ and derived length at most 2. Set $M(h+1) = M(h)!3^{(h+1)^2}$ and $d(h+1) = M(h+1)+2+d(h)$. Then $G.\Lambda(G)$ has a maximal solvable normal subgroup of index at most $M(h + 1)$ and derived length at most $d(h + 1)$ (since it contains the preimage of the centralizer of K/L in H/L).

In general, let $\{G_i \mid i \in I\}$ be the set of finitely generated subgroups of G. By the above argument G_i has a normal subgroup H_i containing $\Lambda(G_i)$ and such that $H_i/\Lambda(G_i)$ is a maximal normal solvable subgroup of $G_i/\Lambda(G_i)$ and has derived length at most $d(h + 1)$ and index at most $M(h + 1)$. Let $N = \max\{[G_i : H_i] \mid i \in I\}$ and choose $\alpha \in I$ such that $[G_\alpha : H_\alpha] = N$. If $G_i \geq G_\alpha$ then $H_i \cap G_\alpha \leq H_\alpha$. Since $[G_\alpha : H_\alpha] \leq [G_\alpha : H_i \cap G_\alpha] = [H_i G_\alpha : H_i] \leq [G_i : H_i]$ we have $[G_i : H_i] = N$ and $H_i \geq H_\alpha$. It follows easily that if $G_\alpha \leq G_i \leq G_j$ then $H_i \leq H_j$.

Set $J = \{i \in I \mid H_\alpha \leq H_i\}$ and $H = \cup_{i \in J} H_i$. If $x, y \in H$ and $g \in G$ then there are indices i, k and $k \in J$ such that $x \in H_i$, $y \in H_j$ and $g \in G_k$. Choose $l \in J$ such that G_l contains $G_i \cup G_j \cup G_k$. Then xy^{-1} and gxg^{-1} are in $H_l \leq H$, and so H is a normal subgroup of G. Moreover if $x_1, \dots, x_N$ is a set of coset representatives for H_α in G_α then it remains a set of coset representatives for H in G, and so $[G; H] = N$.

Let D_i be the $d(h + 1)^{th}$ derived subgroup of H_i. Then D_i is a locally-finite normal subgroup of G_i and so, by an argument similar to that of the above paragraph $\cup_{i\in J}D_i$ is a locally-finite normal subgroup of G. Since it is easily seen that the $d(h+1)^{th}$ derived subgroup of H is contained in $\cup_{i\in J}D_i$ (as each iterated commutator involves only finitely many elements of H) it follows that $H\Lambda(G)/\Lambda(G)\cong H/H\cap\Lambda(G)$ is solvable and of derived length at most $d(h+1)$. □

The above result is from [HL92]. The argument can be simplified to some extent if G is countable and torsion-free. (In fact a virtually solvable group of finite Hirsch length and with no nontrivial locally-finite normal subgroup must be countable, by Lemma 7.9 of [Bi].)

Lemma 1.10 *Let G be an elementary amenable group. If $h(G)=\infty$ then for every $k>0$ there is a subgroup H of G with $k<h(H)<\infty$.*

Proof We shall argue by induction on $\alpha(G)$. The result is vacuously true if $\alpha(G)=1$. Suppose that it is true for all groups in U_α and G is in ℓU_α. Since $h(G)=\text{l.u.b.}\{h(F)|F\leq G,\ F\in U_\alpha\}$ either there is a subgroup F of G in U_α with $h(F)=\infty$, in which case the result is true by the inductive hypothesis, or $h(G)$ is the least upper bound of a set of natural numbers and the result is true. If G is in $U_{\alpha+1}$ then it has a normal subgroup N which is in ℓU_α with quotient G/N in U_1. But then $h(N)=h(G)=\infty$ and so N has such a subgroup. □

Theorem 1.11 *Let G be an elementary amenable group of finite cohomological dimension. Then $h(G)\leq c.d.G$ and G is virtually solvable.*

Proof Since $c.d.G<\infty$ the group G is torsion-free. Let H be a subgroup of finite Hirsch length. Then H is virtually solvable and $c.d.H\leq c.d.G$ so $h(H)\leq c.d.G$. The theorem now follows from Theorem 1.9 and Lemma 1.10. □

1.6 Modules and finiteness conditions

Let G be a group and $w:G\to Z/2Z$ a homomorphism, and let R be a commutative ring. Then $\bar{g}=(-1)^{w(g)}g^{-1}$ defines an anti-involution on $R[G]$. If L is a left $R[G]$-module $\overline{L}$ shall denote the *conjugate* right $R[G]$-module with the same underlying R-module and $R[G]$-action given by $l.g=\bar{g}.l$, for all $l\in L$ and $g\in G$. (We shall also use the overline to denote the conjugate of a right $R[G]$-module.) The conjugate of a free left (right) module is a free right (left) module of the same rank.

We shall also let Z^w denote the G-module with underlying abelian group Z and G-action given by $g.n=(-1)^{w(g)}n$ for all g in G and n in Z.

Lemma 1.12 [Wl65] *Let G and H be groups such that G is finitely presentable and there are homomorphisms $j : H \to G$ and $\rho : G \to H$ with $\rho j = id_H$. Then H is also finitely presentable.*

Proof Since G is finitely presentable there is an epimorphism $p : F \to G$ from a free group $F(X)$ with a finite basis X onto G, with kernel the normal closure of a finite set of relators R. We may choose elements w_x in $F(X)$ such that $j\rho p(x) = p(w_x)$, for all x in X. Then ρ factors through the group K with presentation $\langle X \mid R, x^{-1}w_x, \forall x \in X\rangle$, say $\rho = vu$. Now uj is clearly onto, while $vuj = \rho j = id_H$, and so v and uj are mutually inverse isomomorphisms. Therefore $H \cong K$ is finitely presentable. □

A group G is FP_n if the augmentation $\mathbb{Z}[G]$-module $\mathbb{Z}$ has a projective resolution which is finitely generated in degrees $\leq n$, and it is FP if it has finite cohomological dimension and is FP_n for $n = c.d.G$. It is FF if moreover $\mathbb{Z}$ has a finite resolution consisting of finitely generated free $\mathbb{Z}[G]$-modules. "Finitely generated" is equivalent to FP_1, while "finitely presentable" implies FP_2. Groups which are FP_2 are also said to be *almost finitely presentable.* (There are FP groups which are not finitely presentable [BB97].) An elementary amenable group G is FP_∞ if and only if it is virtually FP, and is then virtually constructible and solvable of finite Hirsch length [Kr93].

If the augmentation $\mathbb{Q}[\pi]$-module $\mathbb{Q}$ has a finite resolution F_* by finitely generated projective modules then $\chi(\pi) = \Sigma(-1)^i dim_{\mathbb{Q}}(\mathbb{Q}\otimes_\pi F_i)$ is independent of the resolution. (If π is the fundamental group of an aspherical finite complex K then $\chi(\pi) = \chi(K)$.) We may extend this definition to groups σ which have a subgroup π of finite index with such a resolution by setting $\chi(\sigma) = \chi(\pi)/[\sigma : \pi]$. (It is not hard to see that this is well defined.)

Let P be a finitely generated projective $\mathbb{Z}[\pi]$-module. Then P is a direct summand of $\mathbb{Z}[\pi]^r$, for some $r \geq 0$, and so is the image of some idempotent $r\times r$-matrix M with entries in $\mathbb{Z}[\pi]$. The *Kaplansky rank* $\kappa(P)$ is the coefficient of $1 \in \pi$ in the trace of M. It depends only on P and is strictly positive if $P \neq 0$. The group π satisfies the *Weak Bass Conjecture* if $\kappa(P) = dim_{\mathbb{Q}}\mathbb{Q}\otimes_\pi P$. This conjecture has been confirmed for linear groups, solvable groups, groups of cohomological dimension ≤ 2 over $\mathbb{Q}$ and PD_3-groups. (See [Ec01] for further details.)

The following result from [BS78] shall be useful.

Theorem 1.13 (Bieri-Strebel) *Let G be an FP_2 group with G/G' infinite. Then G is an HNN extension with finitely generated base and associated subgroups.*

Proof (Sketch – We shall assume that G is finitely presentable.) Let $h : F(m) \to G$ be an epimorphism, and let $g_i = h(x_i)$ for $1 \leq i \leq m$. We may assume that g_m has infinite order modulo the normal closure of $\{g_i \mid 1 \leq i < m\}$. Since G is finitely presentable the kernel of h is the normal closure of finitely many relators, of weight 0 in the letter x_m. Each such relator is a product of powers of conjugates of the generators $\{x_i \mid 1 \leq i < m\}$ by powers of x_m. Thus we may assume the relators are contained in the subgroup generated by $\{x_m^j x_i x_m^{-j} \mid 1 \leq i \leq m, -p \leq j \leq p\}$, for some sufficiently large p. Let U be the subgroup of G generated by $\{g_m^j g_i g_m^{-j} \mid 1 \leq i \leq m, -p \leq j < p\}$, and let $V = g_m U g_m^{-1}$. Let B be the subgroup of G generated by $U \cup V$ and let $\tilde{G}$ be the HNN extension with base B and associated subgroups U and V presented by $\tilde{G} = \langle B, s \mid sus^{-1} = \tau(u)\ \forall u \in U\rangle$, where $\tau : U \to V$ is the isomorphism determined by conjugation by g_m in G. There are obvious epimorphisms $\xi : F(m+1) \to \tilde{G}$ and $\psi : \tilde{G} \to G$ with composite h. It is easy to see that $\mathrm{Ker}(h) \leq \mathrm{Ker}(\xi)$ and so $\tilde{G} \cong G$. □

In particular, if G is restrained then it is an ascending HNN extension.

A ring R is *weakly finite* if every onto endomorphism of R^n is an isomorphism, for all $n \geq 0$. (In [H2] the term "SIBN ring" was used instead.) Finitely generated stably free modules over weakly finite rings have well defined ranks, and the rank is strictly positive if the module is nonzero. Skew fields are weakly finite, as are subrings of weakly finite rings. If G is a group its complex group algebra $\mathbb{C}[G]$ is weakly finite, by a result of Kaplansky. (See [Ro84] for a proof.)

A ring R is *(regular) coherent* if every finitely presentable left R-module has a (finite) resolution by finitely generated projective R-modules, and is *(regular) noetherian* if moreover every finitely generated R-module is finitely presentable. A group G is regular coherent or regular noetherian if the group ring $R[G]$ is regular coherent or regular noetherian (respectively) for any regular noetherian ring R. It is coherent as a *group* if all its finitely generated subgroups are finitely presentable.

Lemma 1.14 *If G is a group such that $\mathbb{Z}[G]$ is coherent then every finitely generated subgroup of G is FP_∞.*

Proof Let H be a subgroup of G. Since $\mathbb{Z}[H] \leq \mathbb{Z}[G]$ is a faithfully flat ring extension a left $\mathbb{Z}[H]$-module is finitely generated over $\mathbb{Z}[H]$ if and only if the induced module $\mathbb{Z}[G] \otimes_H M$ is finitely generated over $\mathbb{Z}[G]$. It follows by induction on n that M is FP_n over $\mathbb{Z}[H]$ if and only if $\mathbb{Z}[G] \otimes_H M$ is FP_n over $\mathbb{Z}[G]$.

If H is finitely generated then the augmentation $\mathbb{Z}[H]$-module $\mathbb{Z}$ is finitely presentable over $\mathbb{Z}[H]$. Hence $\mathbb{Z}[G]\otimes_H Z$ is finitely presentable over $\mathbb{Z}[G]$, and so is FP_∞ over $\mathbb{Z}[G]$, since that ring is coherent. Hence $\mathbb{Z}$ is FP_∞ over $\mathbb{Z}[H]$, i.e., H is FP_∞. □

Thus if either G is coherent (as a group) or $\mathbb{Z}[G]$ is coherent (as a ring) every finitely generated subgroup of G is FP_2. As the latter condition shall usually suffice for our purposes below, we shall say that such a group is *almost coherent*. The connection between these notions has not been much studied.

The class of groups whose integral group ring is regular coherent contains the trivial group and is closed under generalised free products and HNN extensions with amalgamation over subgroups whose group rings are regular noetherian, by Theorem 19.1 of [Wd78]. If $[G:H]$ is finite and G is torsion-free then $\mathbb{Z}[G]$ is regular coherent if and only if $\mathbb{Z}[H]$ is. In particular, free groups and surface groups are coherent and their integral group rings are regular coherent, while (torsion-free) virtually poly-Z groups are coherent and their integral group rings are (regular) noetherian.

1.7 Ends and cohomology with free coefficients

A finitely generated group G has 0, 1, 2 or infinitely many ends. It has 0 ends if and only if it is finite, in which case $H^0(G;\mathbb{Z}[G])\cong Z$ and $H^q(G;\mathbb{Z}[G])=0$ for $q>0$. Otherwise $H^0(G;\mathbb{Z}[G])=0$ and $H^1(G;\mathbb{Z}[G])$ is a free abelian group of rank $e(G)-1$, where $e(G)$ is the number of ends of G [Sp49]. The group G has more than one end if and only if it is a nontrivial generalised free product with amalgamation $G\cong A*_C B$ or an HNN extension $A*_C\phi$, where C is a finite group. In particular, it has two ends if and only if it is virtually Z if and only if it has a (maximal) finite normal subgroup F such that $G/F\cong Z$ or D, where $D=(Z/2Z)*(Z/2Z)$ is the *infinite dihedral group* [St] - see also [DD].

If G is a group with a normal subgroup N, and A is a left $\mathbb{Z}[G]$-module there is a *Lyndon-Hochschild-Serre spectral sequence* (LHSSS) for G as an extension of G/N by N and with coefficients A:

$$E_2=H^p(G/N;H^q(N;A))\Rightarrow H^{p+q}(G;A),$$

the r^{th} differential having bidegree $(r,1-r)$. (See Section 10.1 of [Mc].)

Theorem 1.15 [Ro75] *If G has a normal subgroup N which is the union of an increasing sequence of subgroups N_n such that $H^s(N_n;\mathbb{Z}[G])=0$ for $s\le r$ then $H^s(G;\mathbb{Z}[G])=0$ for $s\le r$.*

Proof Let $s \leq r$. Let f be an s-cocycle for N with coefficients $\mathbb{Z}[G]$, and let f_n denote the restriction of f to a cocycle on N_n. Then there is an $(s-1)$-cochain g_n on N_n such that $\delta g_n = f_n$. Since $\delta(g_{n+1}|_{N_n} - g_n) = 0$ and $H^{s-1}(N_n;\mathbb{Z}[G]) = 0$ there is an $(s-2)$-cochain h_n on N_n with $\delta h_n = g_{n+1}|_{N_n} - g_n$. Choose an extension h'_n of h_n to N_{n+1} and let $\hat{g}_{n+1} = g_{n+1} - \delta h'_n$. Then $\hat{g}_{n+1}|_{N_n} = g_n$ and $\delta\hat{g}_{n+1} = f_{n+1}$. In this way we may extend g_0 to an $(s-1)$-cochain g on N such that $f = \delta g$ and so $H^s(N;\mathbb{Z}[G]) = 0$. The LHSSS for G as an extension of G/N by N, with coefficients $\mathbb{Z}[G]$, now gives $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq r$. □

Corollary 1.15.1 *The hypotheses are satisfied if N is the union of an increasing sequence of FP_r subgroups N_n such that $H^s(N_n;\mathbb{Z}[N_n]) = 0$ for $s \leq r$. In particular, if N is the union of an increasing sequence of finitely generated, one-ended subgroups then G has one end.*

Proof We have $H^s(N_n;\mathbb{Z}[G]) = H^s(N_n;\mathbb{Z}[N_n]) \otimes \mathbb{Z}[G/N_n] = 0$, for all $s \leq r$ and all n, since N_n is FP_r. □

If the successive inclusions are finite this corollary may be sharpened further.

Theorem (Gildenhuys-Strebel) *Let $G = \cup_{n\geq 1} G_n$ be the union of an increasing sequence of FP_r subgroups. Suppose that $[G_{n+1} : G_n] < \infty$ and $H^s(G_n;\mathbb{Z}[G_n]) = 0$ for all $s < r$ and $n \geq 1$. If G is not finitely generated then $H^s(G;F) = 0$ for every free $\mathbb{Z}[G]$-module F and all $s \leq r$.* □

The enunciation of this theorem in [GS81] assumes also that $c.d.G_n = r$ for all $n \geq 1$, and concludes that $c.d.G = r$ if and only if G is finitely generated. However the argument establishes the above assertion.

Theorem 1.16 *Let G be a finitely generated group with an infinite restrained normal subgroup N of infinite index. Then $e(G) = 1$.*

Proof Since N is infinite $H^1(G;\mathbb{Z}[G]) \cong H^0(G/N;H^1(N;\mathbb{Z}[G]))$, by the LHSSS. If N is finitely generated $H^1(N;\mathbb{Z}[G]) \cong H^1(N;\mathbb{Z}[N]) \otimes \mathbb{Z}[G/N]$, with the diagonal G/N-action. Since G/N is infinite $H^1(G;\mathbb{Z}[G]) = 0$. If N is locally one-ended or locally virtually Z and not finitely generated $H^1(N;\mathbb{Z}[G]) = 0$, by Theorem 1.15 and the Gildenhuys-Strebel Theorem, respectively. In all of these cases $e(G) = 1$.

There remains the possibility that N is locally finite. If $e(G) > 1$ then $G \cong A *_C B$ or $A *_C \phi$ with C finite, by Stallings' characterization of such groups.

Suppose $G \cong A *_C B$. Since N is infinite there is an $n \in N \setminus C$. We may suppose that $n = gag^{-1}$ for some $a \in A$ and $g \in G$, since elements of finite order in $A *_C B$ are conjugate to elements of A or B, by Theorem 6.4.3 of [Ro]. If $n \notin A$ we may suppose $g = g_1 \dots g_k$ with terms alternately from $A \setminus C$ and $B \setminus C$, and $g_k \in B$. Let $n' = g_0 n g_0^{-1}$, where $g_0 \in A \setminus C$ if k is odd and $g_0 \in B \setminus C$ if k is even (or if $n \in A$). Since N is normal $n' \in N$ also, and since N is restrained $w(n, n') = 1$ in N for some nontrivial word $w \in F(2)$. But this contradicts the "uniqueness of normal form" for such groups. A similar argument shows that G cannot be $A *_C \phi$. Thus G must have one end. □

In particular, a countable restrained group N is either elementary amenable and $h(N) \leq 1$ or is an increasing union of finitely generated, one-ended subgroups.

The second cohomology of a group with free coefficients ($H^2(G; R[G])$, $R = \mathbb{Z}$ or a field) shall play an important role in our investigations.

Theorem (Farrell) *Let G be a finitely presentable group. If G has an element of infinite order and $R = \mathbb{Z}$ or is a field then $H^2(G; R[G])$ is either 0 or R or is not finitely generated.* □

Farrell also showed in [Fa74] that if $H^2(G; \mathbb{F}_2[G]) \cong Z/2Z$ then every finitely generated subgroup of G with one end has finite index in G. Hence if G is also torsion-free then subgroups of infinite index in G are locally free. Bowditch has since shown that such groups are virtually the fundamental groups of aspherical closed surfaces ([Bo04] - see §8 below).

We would also like to know when $H^2(G; \mathbb{Z}[G])$ is 0 (for G finitely presentable). In particular, we expect this to be so if G has an elementary amenable, normal subgroup E such that either $h(E) = 1$ and G/E has one end or $h(E) = 2$ and $[G : E] = \infty$ or $h(E) \geq 3$, or if G is an ascending HNN extension over a finitely generated, one-ended base. Our present arguments for these two cases require stronger finiteness hypotheses, and each use the following result of [BG85].

Theorem (Brown-Geoghegan) *Let G be an HNN extension $B*_\phi$ in which the base B and associated subgroups I and $\phi(I)$ are FP_n. If the homomorphism from $H^q(B; Z[G])$ to $H^q(I; Z[G])$ induced by restriction is injective for some $q \leq n$ then the corresponding homomorphism in the Mayer-Vietoris sequence is injective, so $H^q(G; Z[G])$ is a quotient of $H^{q-1}(I; Z[G])$.* □

We begin with the case of "large" elementary amenable normal subgroups.

Theorem 1.17 *Let G be a finitely presentable group with a locally virtually indicable, restrained normal subgroup E of infinite index. Suppose that either E is abelian of rank 1 and G/E has one end or E is torsion-free, elementary amenable and $h(E) > 1$ or E is almost coherent and has a finitely generated, one-ended subgroup. Then $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq 2$.*

Proof If E is abelian of positive rank and G/E has one end then G is 1-connected at ∞ by Theorem 1 of [Mi87], and so $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq 2$, by [GM86].

Suppose next that E is torsion-free, elementary amenable and $h(E) > 1$. Then G has one end, so $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq 1$. If E is virtually solvable it has a nontrivial characteristic abelian subgroup A. If $h(A) = 1$ then we may assume that $A = \langle\langle a \rangle\rangle_G$, so G/A is finitely presentable. As E/A is infinite G/A has one end, by Theorem 1.16, and so $H^2(G;\mathbb{Z}[G]) = 0$ as before. If $A \cong Z^2$ then $H^2(A;\mathbb{Z}[G])) \cong \mathbb{Z}[G/A]$. Otherwise, A has Z^2 as a subgroup of infinite index and so $H^2(A;\mathbb{Z}[G]) = 0$. If E is not virtually solvable $H^s(E;\mathbb{Z}[G]) = 0$ for all s, by Proposition 3 of [Kr93']. (The argument applies even if E is not finitely generated.) In all cases, an LHSSS argument gives $H^2(G;\mathbb{Z}[G]) = 0$.

We may assume henceforth that E is almost coherent and is an increasing union of finitely generated one-ended subgroups $E_n \subseteq E_{n+1} \cdots \subseteq E = \cup E_n$. Since E is locally virtually indicable there are subgroups $F_n \leq E_n$ such that $[E_n : F_n] < \infty$ and which map onto Z. Since E is almost coherent these subgroups are FP_2. Hence they are HNN extensions over FP_2 bases H_n, by Theorem 1.13, and the extensions are ascending, since E is restrained. Since E_n has one end H_n is infinite and so has one or two ends.

Suppose that H_n has two ends, for all $n \geq 1$. Then E_n is elementary amenable, $h(E_n) = 2$ and $[E_{n+1} : E_n] < \infty$, for all $n \geq 1$. Hence E is elementary amenable and $h(E) = 2$. If E is finitely generated it is FP_2 and so $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq 2$, by an LHSSS argument. This is also the case if E is not finitely generated, for then $H^s(E;\mathbb{Z}[G]) = 0$ for $s \leq 2$, by the Gildenhuys-Strebel Theorem, and we may again apply an LHSSS argument.

Otherwise we may assume that H_n has one end, for all $n \geq 1$. In this case $H^s(F_n;\mathbb{Z}[F_n]) = 0$ for $s \leq 2$, by the Brown-Geoghegan Theorem. Therefore $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq 2$, by Theorem 1.15. □

The theorem applies if E is almost coherent and elementary amenable, since elementary amenable groups are restrained and locally virtually indicable. It also applies if $E = \sqrt{G}$ is large enough, since finitely generated nilpotent groups are virtually poly-Z. Similar arguments show that if $h(\sqrt{G}) \geq r$ then $H^s(G;\mathbb{Z}[G]) = 0$ for $s < r$, and if also $[G:\sqrt{G}] = \infty$ then $H^r(G;\mathbb{Z}[G]) = 0$.

Are the hypotheses that E be almost coherent and locally virtually indicable necessary? Is it sufficient that E be restrained and be an increasing union of finitely generated, one-ended subgroups?

Theorem 1.18 *Let $G = B*_\phi$ be an HNN extension with FP_2 base B and associated subgroups I and $\phi(I) = J$, and which has a restrained normal subgroup $N \leq \langle\langle B \rangle\rangle$. Then $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq 2$ if either*

(1) *the HNN extension is ascending and $B = I \cong J$ has one end; or*

(2) *N is locally virtually Z and G/N has one end; or*

(3) *N has a finitely generated subgroup with one end.*

Proof The first assertion follows immediately from the Brown-Geogeghan Theorem.

Let t be the stable letter, so that $tit^{-1} = \phi(i)$, for all $i \in I$. Suppose that $N \cap J \neq N \cap B$, and let $b \in N \cap B \setminus J$. Then $b^t = t^{-1}bt$ is in N, since N is normal in G. Let a be any element of $N \cap B$. Since N has no noncyclic free subgroup there is a word $w \in F(2)$ such that $w(a, b^t) = 1$ in G. It follows from Britton's Lemma that a must be in I, and so $N \cap B = N \cap I$. In particular, N is the increasing union of copies of $N \cap B$.

Hence G/N is an HNN extension with base $B/N \cap B$ and associated subgroups $I/N \cap I$ and $J/N \cap J$. Therefore if G/N has one end the latter groups are infinite, and so B, I and J each have one end. If N is virtually Z then $H^s(G;\mathbb{Z}[G]) = 0$ for $s \leq 2$, by an LHSSS argument. If N is locally virtually Z but is not finitely generated then it is the increasing union of a sequence of two-ended subgroups and $H^s(N;\mathbb{Z}[G]) = 0$ for $s \leq 1$, by the Gildenhuys-Strebel Theorem. Since $H^2(B;\mathbb{Z}[G]) \cong H^0(B; H^2(N \cap B;\mathbb{Z}[G]))$ and $H^2(I;\mathbb{Z}[G]) \cong H^0(I; H^2(N \cap I;\mathbb{Z}[G]))$, the restriction map from $H^2(B;\mathbb{Z}[G])$ to $H^2(I;\mathbb{Z}[G])$ is injective. If N has a finitely generated, one-ended subgroup N_1, we may assume that $N_1 \leq N \cap B$, and so B, I and J also have one end. Moreover $H^s(N \cap B;\mathbb{Z}[G]) = 0$ for $s \leq 1$, by Theorem 1.15. We again see that the restriction map from $H^2(B;\mathbb{Z}[G])$ to $H^2(I;\mathbb{Z}[G])$ is injective. The result now follows in these cases from the Brown-Geoghegan Theorem. □

The final result of this section is Theorem 8.8 of [Bi].

Theorem (Bieri) *Let G be a nonabelian group with c.d.$G = n$. Then c.d.$\zeta G \leq n-1$, and if ζG has rank $n-1$ then G' is free.* □

1.8 Poincaré duality groups

A group G is a *PD_n-group* if it is FP, $H^p(G;\mathbb{Z}[G]) = 0$ for $p \neq n$ and $H^n(G;\mathbb{Z}[G]) \cong Z$. The "dualizing module" $H^n(G;\mathbb{Z}[G]) = Ext^n_{\mathbb{Z}[G]}(Z,\mathbb{Z}[G])$ is a right $\mathbb{Z}[G]$-module, with G-action determined by a homomorphism $w = w_1(G) : G \to Aut(Z) \cong \mathbb{Z}^\times$. The group is *orientable* (or is a *PD_n^+-group*) if w is trivial, i.e., if $H^n(G;\mathbb{Z}[G])$ is isomorphic to the augmentation module $\mathbb{Z}$. (See [Bi].)

The only PD_1-group is Z. Eckmann, Linnell and Müller showed that every PD_2-group is the fundamental group of a closed aspherical surface. (See Chapter VI of [DD].) Bowditch has since found a much stronger result, which must be close to the optimal characterization of such groups [Bo04].

Theorem (Bowditch) *Let G be an FP_2 group and F a field. Then G is virtually a PD_2-group if and only if $H^2(G;F[G])$ has a 1-dimensional G-invariant subspace.* □

In particular, this theorem applies if $H^2(G;\mathbb{Z}[G]) \cong Z$, for then the image of $H^2(G;\mathbb{Z}[G])$ in $H^2(G;\mathbb{F}_2[G])$ under reduction *mod* (2) is such a subspace.

The following result corresponds to the fact that an infinite covering space of a PL n-manifold is homotopy equivalent to a complex of dimension $< n$ [St77].

Theorem (Strebel) *Let H be a subgroup of infinite index in a PD_n-group G. Then $c.d.H < n$.* □

Let S be a ring. If C is a left S-module and R is a subring of S let $C|_R$ be the left R-module underlying C. If A is a left R-module the abelian group $Hom_R(S|_R, A)$ has a natural left S-module structure given by $((sf)(s') = f(s's)$ for all $f \in Hom_R(S|_R, A)$ and $s, s' \in S$. The groups $Hom_R(C|_R, A)$ and $Hom_S(C, Hom_R(S|_R, A))$ are naturally isomorphic, for the maps I and J defined by $I(f)(c)(s) = f(sc)$ and $J(\theta)(c) = \theta(c)(1)$ for $f : C \to A$ and $\theta : C \to Hom_R(S, A)$ are mutually inverse isomorphisms. When K is a subgroup of π, $R = \mathbb{Z}[K]$ and $S = \mathbb{Z}[\pi]$ we may write $C|_K$ for $C|_R$, and the module $Hom_{\mathbb{Z}[K]}(\mathbb{Z}[\pi]|_K, A)$ is said to be *coinduced from A*. The above isomorphisms give rise to Shapiro's Lemma. In our applications π/K shall usually be infinite cyclic and S is then a twisted Laurent extension of R.

If G is a group and A is a left $\mathbb{Z}[G]$-module let $A|_1$ be the $\mathbb{Z}[G]$-module with the same underlying group and trivial G-action, and let $A^G = Hom_{\mathbb{Z}}(\mathbb{Z}[G], A)$ be the module of functions $\alpha : G \to A$ with G-action given by $(g\alpha)(h) = g.\alpha(hg)$ for all $g, h \in G$. Then ${A|_1}^G$ is coinduced from a module over the trivial group.

Theorem 1.19 *Let π be a PD_n-group with a normal subgroup K such that π/K is a PD_r-group. Then K is a PD_{n-r}-group if and only if it is $FP_{[n/2]}$.*

Proof The condition is clearly necessary. Assume that it holds. After passing to a subgroup of index 2, if necessary, we may assume that $G=\pi/K$ is orientable. It is sufficient to show that the functors $H^s(K;-)$ from left $\mathbb{Z}[K]$-modules to abelian groups commute with direct limit, for all $s\leq n$, for then K is FP_{n-1} [Br75], and the result follows from Theorem 9.11 of [Bi] (and an LHSSS corner argument to identify the dualizing module), Since K is $FP_{[n/2]}$ we may assume $s>n/2$. If A is a $\mathbb{Z}[K]$-module and $W=Hom_{\mathbb{Z}[K]}(\mathbb{Z}[\pi],A)$ then $H^s(K;A)\cong H^s(\pi;W)\cong H_{n-s}(\pi;\overline{W})$, by Shapiro's Lemma and Poincaré duality.

Let A_g be the left $\mathbb{Z}[K]$-module with the same underlying group as A and K-action given by $k.a=\sigma(g)k\sigma(g)^{-1}a$ for all $a\in A$, $g\in G$ and $k\in K$. The $\mathbb{Z}[K]$-epimorphisms $p_g:W\to A_g$ given by $p_g(f)=f(\sigma(g))$ for all $f\in W$ and $g\in G$ determine an isomorphism $W\cong\Pi_{g\in G}A_g$. Hence they induce $\mathbb{Z}$-linear isomorphisms $H_q(K;\overline{W})\cong\Pi_{g\in G}H_q(K;\overline{A_g})$ for $q\leq[n/2]$, since C_* has finite $[n/2]$-skeleton. The $\mathbb{Z}$-linear homomorphisms $t_{q,g}:\overline{A_g}\otimes_{\mathbb{Z}[K]}C_q\to\overline{A}\otimes_{\mathbb{Z}[K]}C_q$ given by $t_{q,g}(a\otimes c)=w(\sigma(g))a\otimes\sigma(g)c$ for all $a\in\overline{A}$ and $c\in C_q$ induce isomorphisms $H_q(K;\overline{A_g})\cong H_q(K;\overline{A})$ for all $q\geq 0$ and $g\in G$. Let $u_{q,g}=t_{q,g}(p_g\otimes id_{C_q})$. Then $u_{q,g}(f\sigma(h)^{-1}\otimes\sigma(h)c)=u_{q,gh}(f\otimes c)$ for all $g,h\in G$, $f\in\overline{W}$, $c\in C_q$ and $q\geq 0$. Hence these composites determine isomorphisms of left $\mathbb{Z}[G]$-modules $H_q(K;\overline{W})\cong A_q^G$, where $A_q=H_q(\overline{A}\otimes_{\mathbb{Z}[K]}C_*)=H_q(K;\overline{A})$ (with trivial G-action) for $q\leq[n/2]$.

Let $D(L)$ denote the conjugate of a left $\mathbb{Z}[G]$-module L with respect to the canonical involution. We shall apply the homology LHSSS

$$E^2_{pq}=H_p(G;D(H_q(K;\overline{W}))\Rightarrow H_{p+q}(\pi;\overline{W}).$$

Poincaré duality for G and another application of Shapiro's Lemma now give $H_p(G;D(A_q^G))\cong H^{r-p}(G;A_q^G)\cong H^{r-p}(1;A_q)$, since A_q^G is coinduced from a module over the trivial group. If $s>[n/2]$ and $p+q=n-s$ then $q\leq[n/2]$ and so $H_p(G;A_q^G)\cong A_q$ if $p=r$ and is 0 otherwise. Thus the spectral sequence collapses to give $H_{n-s}(\pi;\overline{W})\cong H_{n-r-s}(K;\overline{A})$. Since homology commutes with direct limits this proves the theorem. □

The finiteness condition cannot be relaxed further when $r=2$ and $n=4$, for Kapovich has given an example of a pair $\nu<\pi$ with π a PD_4-group, π/ν a PD_2-group and ν finitely generated but not FP_2 [Ka98].

The most useful case of this theorem is when $G \cong Z$. The argument of the first paragraph of the theorem shows that if K is any normal subgroup such that $\pi/K \cong Z$ then $H^n(K;A) \cong H_0(\pi;\overline{W}) = 0$, and so $c.d.K < n$. (This weak version of Strebel's Theorem suffices for some of the applications below.)

Let R be a ring. An R-chain complex has *finite k-skeleton* if it is chain homotopy equivalent to a complex P_* with P_j a finitely generated free R-module for $j \leq k$. If R is a subring of S and C_* is an S-chain complex then C_* is *R-finitely dominated* if $C_*|_R$ is chain homotopy equivalent to a finite projective R-chain complex. The argument of Theorem 1.19 extends easily to the nonaspherical case as follows. (See Chapter 2 for the definition of PD_n-space.)

Theorem 1.19′ *Let M be a PD_n-space, $p : \pi_1(M) \to G$ be an epimorphism with G a PD_r-group and $\nu = \mathrm{Ker}(p)$. If $C_*(\widetilde{M})|_\nu$ has finite $[n/2]$-skeleton $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated and $H^s(M_\nu;\mathbb{Z}[\nu]) \cong H_{n-r-s}(M_\nu;\mathbb{Z}[\nu])$ for all s.* □

If M is aspherical then $M_\nu = K(\nu,1)$ is a PD_{n-r}-space, by Theorem 1.19. In Chapter 4 we shall show that this holds in general.

Corollary 1.19.1 *If either $r = n-1$ or $r = n-2$ and ν is infinite or $r = n-3$ and ν has one end then M is aspherical.* □

1.9 Hilbert modules

Let π be a countable group and let $\ell^2(\pi)$ be the Hilbert space completion of $\mathbb{C}[\pi]$ with respect to the inner product given by $(\Sigma a_g g, \Sigma b_h h) = \Sigma a_g \overline{b_g}$. Left and right multiplication by elements of π determine left and right actions of $\mathbb{C}[\pi]$ as bounded operators on $\ell^2(\pi)$. The (left) von Neumann algebra $\mathcal{N}(\pi)$ is the algebra of bounded operators on $\ell^2(\pi)$ which are $\mathbb{C}[\pi]$-linear with respect to the left action. By the Tomita-Takesaki theorem this is also the bicommutant in $B(\ell^2(\pi))$ of the right action of $\mathbb{C}[\pi]$, i.e., the set of operators which commute with every operator which is right $\mathbb{C}[\pi]$-linear. (See pages 45-52 of [Su].) We may clearly use the canonical involution of $\mathbb{C}[\pi]$ to interchange the roles of left and right in these definitions.

If $e \in \pi$ is the unit element we may define the von Neumann trace on $\mathcal{N}(\pi)$ by the inner product $tr(f) = (f(e), e)$. This extends to square matrices over $\mathcal{N}(\pi)$ by taking the sum of the traces of the diagonal entries. A *Hilbert $\mathcal{N}(\pi)$-module* is a Hilbert space M with a unitary left π-action which embeds isometrically and π-equivariantly into the completed tensor product $H\widehat{\otimes}\ell^2(\pi)$ for some Hilbert space H. It is finitely generated if we may take $H \cong \mathbb{C}^n$ for some integer n. (In this case we do not need to complete the ordinary tensor product over $\mathbb{C}$.) A *morphism* of Hilbert $\mathcal{N}(\pi)$-modules is a π-equivariant bounded linear operator $f : M \to N$. It is a *weak isomorphism* if it is injective and has dense image. A bounded π-linear operator on $\ell^2(\pi)^n = \mathbb{C}^n \otimes \ell^2(\pi)$ is represented by a matrix whose entries are in $\mathcal{N}(\pi)$. The *von Neumann dimension* of a finitely generated Hilbert $\mathcal{N}(\pi)$-module M is the real number $dim_{\mathcal{N}(\pi)}(M) = tr(P) \in [0,\infty)$, where P is any projection operator on $H \otimes \ell^2(\pi)$ with image π-isometric to M. In particular, $dim_{\mathcal{N}(\pi)}(M) = 0$ if and only if $M = 0$. The notions of finitely generated Hilbert $\mathcal{N}(\pi)$-module and finitely generated projective $\mathcal{N}(\pi)$-module are essentially equivalent, and arbitrary $\mathcal{N}(\pi)$-modules have well-defined dimensions in $[0,\infty]$ [Lü].

A sequence of bounded maps between Hilbert $\mathcal{N}(\pi)$-modules

$$M \xrightarrow{\ j\ } N \xrightarrow{\ p\ } P$$

is *weakly exact at* N if $\mathrm{Ker}(p)$ is the closure of $\mathrm{Im}(j)$. If $0 \to M \to N \to P \to 0$ is weakly exact then j is injective, $\mathrm{Ker}(p)$ is the closure of $\mathrm{Im}(j)$ and $\mathrm{Im}(p)$ is dense in P, and $dim_{\mathcal{N}(\pi)}(N) = dim_{\mathcal{N}(\pi)}(M) + dim_{\mathcal{N}(\pi)}(P)$. A finitely generated *Hilbert* $\mathcal{N}(\pi)$*-complex* C_* is a chain complex of finitely generated Hilbert $\mathcal{N}(\pi)$-modules with bounded $\mathbb{C}[\pi]$-linear operators as differentials. The *reduced* L^2-homology is defined to be $\bar{H}_p^{(2)}(C_*) = \mathrm{Ker}(d_p)/\overline{\mathrm{Im}(d_{p+1})}$. The p^{th} L^2-Betti number of C_* is then $dim_{\mathcal{N}(\pi)}\bar{H}_p^{(2)}(C_*)$. (As the images of the differentials need not be closed the *un*reduced L^2-homology modules $H_p^{(2)}(C_*) = \mathrm{Ker}(d_p)/\mathrm{Im}(d_{p+1})$ are not in general Hilbert modules.)

See [Lü] for more on modules over von Neumann algebras and L^2 invariants of complexes and manifolds.

[In this book L^2-Betti number arguments replace the localization arguments used in [H2]. However we shall recall the definition of *safe extension* of a group ring used there. An inclusion of rings $\mathbb{Z}[G] < S$ is a safe extension if it is flat, S is weakly finite and $S \otimes_{\mathbb{Z}[G]} \mathbb{Z} = 0$. If G has a nontrivial elementary amenable normal subgroup whose finite subgroups have bounded order and which has no nontrivial finite normal subgroup then $\mathbb{Z}[G]$ has a safe extension. This is used briefly at the end of Chapter 15 below.]

Chapter 2

2-Complexes and PD_3-complexes

This chapter begins with a review of the notation we use for (co)homology with local coefficients and of the universal coefficient spectral sequence. We then define the L^2-Betti numbers and present some useful vanishing theorems of Lück and Gromov. These invariants are used in §3, where they are used to estimate the Euler characteristics of finite $[\pi, m]$-complexes and to give a converse to the Cheeger-Gromov-Gottlieb Theorem on aspherical finite complexes. Some of the arguments and results here may be regarded as representing in microcosm the bulk of this book; the analogies and connections between 2-complexes and 4-manifolds are well known. We then review Poincaré duality and PD_n-complexes. In §5-§9 we shall summarize briefly what is known about the homotopy types of PD_3-complexes.

2.1 Notation

Let X be a connected cell complex and let $\widetilde{X}$ be its universal covering space. If H is a normal subgroup of $G = \pi_1(X)$ we may lift the cellular decomposition of X to an equivariant cellular decomposition of the corresponding covering space X_H. The cellular chain complex of X_H with coefficients in a commutative ring R is then a complex $C_* = C_*(X_H)$ of left $R[G/H]$-modules, with respect to the action of the covering group G/H. A choice of lifts of the q-cells of X determines a free basis for C_q, for all q, and so C_* is a complex of free modules. If X is a finite complex G is finitely presentable and these modules are finitely generated. If X is finitely dominated, i.e., is a retract of a finite complex, then G is again finitely presentable, by Lemma 1.12. Moreover the chain complex of the universal cover is chain homotopy equivalent over $R[G]$ to a complex of finitely generated projective modules [Wl65]. The Betti numbers of X with coefficients in a field F shall be denoted by $\beta_i(X; F) = dim_F H_i(X; F)$ (or just $\beta_i(X)$, if $F = \mathbb{Q}$).

The i^{th} *equivariant homology* module of X with coefficients $R[G/H]$ is the left module $H_i(X; R[G/H]) = H_i(C_*)$, which is clearly isomorphic to $H_i(X_H; R)$ as an R-module, with the action of the covering group determining its $R[G/H]$-module structure. The i^{th} *equivariant cohomology* module of X with coefficients $R[G/H]$ is the right module $H^i(X; R[G/H]) = H^i(C^*)$, where $C^* =$

$Hom_{R[G/H]}(C_*, R[G/H])$ is the associated cochain complex of right $R[G/H]$-modules. More generally, if A and B are right and left $\mathbb{Z}[G/H]$-modules (respectively) we may define $H_j(X;A) = H_j(A \otimes_{\mathbb{Z}[G/H]} C_*)$ and $H^{n-j}(X;B) = H^{n-j}(Hom_{\mathbb{Z}[G/H]}(C_*, B))$. There is a *Universal Coefficient Spectral Sequence* (UCSS) relating equivariant homology and cohomology:

$$E_2^{pq} = Ext^q_{R[G/H]}(H_p(X; R[G/H]), R[G/H]) \Rightarrow H^{p+q}(X; R[G/H]),$$

with r^{th} differential d_r of bidegree $(1-r, r)$.

If J is a normal subgroup of G which contains H there is also a *Cartan-Leray* spectral sequence relating the homology of X_H and X_J:

$$E^2_{pq} = Tor_p^{R[G/H]}(R[G/J], H_q(X; R[G/H])) \Rightarrow H_{p+q}(X; R[G/J]),$$

with r^{th} differential d^r of bidegree $(-r, r-1)$. (See [Mc] for more details on these spectral sequences.)

If M is a cell complex let $c_M : M \to K(\pi_1(M), 1)$ denote the *classifying map* for the fundamental group and let $f_M : M \to P_2(M)$ denote the second stage of the Postnikov tower for M. (Thus $c_M = c_{P_2(M)} f_M$.) A map $f : X \to K(\pi_1(M), 1)$ lifts to a map from X to $P_2(M)$ if and only if $f^*k_1(M) = 0$, where $k_1(M)$ is the first k-invariant of M in $H^3(\pi_1(M); \pi_2(M))$. In particular, if $k_1(M) = 0$ then $c_{P_2(M)}$ has a cross-section. The *algebraic* 2*-type* of M is the triple $[\pi, \pi_2(M), k_1(M)]$. Two such triples $[\pi, \Pi, \kappa]$ and $[\pi', \Pi', \kappa']$ (corresponding to M and M', respectively) are equivalent if there are isomorphisms $\alpha : \pi \to \pi'$ and $\beta : \Pi \to \Pi'$ such that $\beta(gm) = \alpha(g)\beta(m)$ for all $g \in \pi$ and $m \in \Pi$ and $\beta_*\kappa = \alpha^*\kappa'$ in $H^3(\pi; \alpha^*\Pi')$. Such an equivalence may be realized by a homotopy equivalence of $P_2(M)$ and $P_2(M')$. (The reference [Ba] gives a detailed treatment of Postnikov factorizations of nonsimple maps and spaces.) Throughout this book *closed manifold* shall mean compact, connected TOP manifold without boundary. Every closed manifold has the homotopy type of a finite Poincaré duality complex [KS].

2.2 L^2-Betti numbers

Let X be a finite complex with fundamental group π. The *L^2-Betti numbers* of X are defined by $\beta_i^{(2)}(X) = dim_{\mathcal{N}(\pi)}(\bar{H}_2^{(2)}(\widetilde{X}))$, where the L^2-homology $\bar{H}_i^{(2)}(\widetilde{X}) = \bar{H}_i(C_*^{(2)})$ is the reduced homology of the Hilbert $\mathcal{N}(\pi)$-complex $C_*^{(2)} = \ell^2 \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X})$ of square summable chains on $\widetilde{X}$. They are multiplicative in finite covers, and for $i = 0$ or 1 depend only on π. (In particular, $\beta_0^{(2)}(\pi) = 0$

if π is infinite.) The alternating sum of the L^2-Betti numbers is the Euler characteristic $\chi(X)$. (See [Lü].)

It may be shown that $\beta_i^{(2)}(X) = dim_{\mathcal{N}(\pi)} H_i(\mathcal{N}(\pi) \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{X}))$, and this formulation of the definition applies to arbitrary complexes [CG86, Lü]. In particular, $\beta_i^{(2)}(\pi) = dim_{\mathcal{N}(\pi)} H_i(\pi; \mathcal{N}(\pi))$ is defined for all π. If X is finitely dominated these numbers are finite, and if also π satisfies the Strong Bass Conjecture then the Euler characteristic formula holds [Ec96]. Moreover, $\beta_s^{(2)}(X) = \beta_s^{(2)}(\pi)$ for $s = 0$ or 1, and $\beta_2^{(2)}(X) \geq \beta_2^{(2)}(\pi)$. (See Theorems 1.35 and 6.54 of [Lü].) The argument for Theorem 1.35.5 of [Lü] extends to show that if $\pi \cong A *_C B$ then $\beta_1^{(2)}(\pi) \geq \frac{1}{|C|} - \frac{1}{|A|} - \frac{1}{|B|}$. (Similarly for $A *_C \phi$.) Thus if $\beta_1^{(2)}(\pi) = 0$ then $e(\pi)$ is finite.

Lemma 2.1 *Let $\pi = H*_\phi$ be a finitely presentable group which is an ascending HNN extension with finitely generated base H. Then $\beta_1^{(2)}(\pi) = 0$.*

Proof Let t be the stable letter and let H_n be the subgroup generated by H and t^n, and suppose that H is generated by g elements. Then $[\pi : H_n] = n$, so $\beta_1^{(2)}(H_n) = n\beta_1^{(2)}(\pi)$. But each H_n is also finitely presentable and generated by $g+1$ elements. Hence $\beta_1^{(2)}(H_n) \leq g+1$, and so $\beta_1^{(2)}(\pi) = 0$. □

In particular, this lemma holds if H is normal in π and $\pi/H \cong Z$.

Theorem 2.2 (Lück) *Let π be a group with a finitely generated infinite normal subgroup Δ such that π/Δ has an element of infinite order. Then $\beta_1^{(2)}(\pi) = 0$.*

Proof (Sketch) Let $\rho \leq \pi$ be a subgroup containing Δ such that $\rho/\Delta \cong Z$. The terms in the line $p + q = 1$ of the homology LHSSS for ρ as an extension of Z by Δ with coefficients $\mathcal{N}(\rho)$ have dimension 0, by Lemma 2.1. Since $dim_{\mathcal{N}(\rho)} M = dim_{\mathcal{N}(\pi)}(\mathcal{N}(\pi) \otimes_{\mathcal{N}(\rho)} M)$ for any $\mathcal{N}(\rho)$-module M the corresponding terms for the LHSSS for π as an extension of π/Δ by Δ with coefficients $\mathcal{N}(\pi)$ also have dimension 0 and the theorem follows. □

This is Theorem 7.2.6 of [Lü]. The hypothesis "π/Δ has an element of infinite order" can be relaxed to "π/Δ is infinite" [Ga00]. The next result also derives from [Lü]. (The case $s = 1$ is extended further in [PT11].)

Theorem 2.3 *Let π be a group with an ascendant subgroup N such that $\beta_i^{(2)}(N) = 0$ for all $i \leq s$. Then $\beta_i^{(2)}(\pi) = 0$ for all $i \leq s$.*

Proof Let $N = N_0 < N_1 < ... < N_\beth = \pi$ be an ascendant sequence. Then we may show by transfinite induction on α that $\beta_i^{(2)}(N_\alpha) = 0$ for all $i \leq s$ and $\alpha \leq \beth$, using parts (2) and (3) of Theorem 7.2 of [Lü] for the passages to successor ordinals and to limit ordinals, respectively. $\square$

Corollary 2.3.1 (Gromov) *Let π be a group with an infinite amenable normal subgroup A. Then $\beta_i^{(2)}(\pi) = 0$ for all i.*

Proof If A is an infinite amenable group $\beta_i^{(2)}(A) = 0$ for all i [CG86]. $\square$

Note that the normal closure of an amenable ascendant subgroup is amenable.

2.3 2-Complexes and finitely presentable groups

If a group π has a finite presentation P with g generators and r relators then the *deficiency* of P is $\mathrm{def}(P) = g - r$, and $\mathrm{def}(\pi)$ is the maximal deficiency of all finite presentations of π. Such a presentation determines a finite 2-complex $C(P)$ with one 0-cell, g 1-cells and r 2-cells and with $\pi_1(C(P)) \cong \pi$. Clearly $\mathrm{def}(P) = 1 - \chi(P) = \beta_1(C(P)) - \beta_2(C(P))$ and so $\mathrm{def}(\pi) \leq \beta_1(\pi) - \beta_2(\pi)$. Conversely every finite 2-complex with one 0-cell arises in this way. In general, any connected finite 2-complex X is homotopy equivalent to one with a single 0-cell, obtained by collapsing a maximal tree T in the 1-skeleton $X^{[1]}$.

We shall say that π has *geometric dimension at most* 2, written $g.d.\pi \leq 2$, if it is the fundamental group of a *finite* aspherical 2-complex.

Theorem 2.4 *Let X be a connected finite 2-complex with fundamental group π. Then $\beta_2^{(2)}(X) \geq \beta_2^{(2)}(\pi)$, with equality if and only if X is aspherical.*

Proof Since we may construct $K = K(\pi, 1)$ by adjoining cells of dimension ≥ 3 to X the natural homomorphism $\bar{H}_2(c_X)$ is an epimorphism, and so $\beta_2^{(2)}(X) \geq \beta_2^{(2)}(\pi)$. Since X is 2-dimensional $\pi_2(X) = H_2(\widetilde{X}; \mathbb{Z})$ is a subgroup of $\bar{H}_2^{(2)}(\widetilde{X})$, with trivial image in $\bar{H}_2^{(2)}(\widetilde{K})$. If moreover $\beta_2^{(2)}(X) = \beta_2^{(2)}(\pi)$ then $\bar{H}_2(c_X)$ is an isomorphism, by Lemma 1.13 of [Lü], so $\pi_2(X) = 0$ and X is aspherical. $\square$

Corollary 2.4.1 *Let π be a finitely presentable group. Then $\mathrm{def}(\pi) \leq 1 + \beta_1^{(2)}(\pi) - \beta_2^{(2)}(\pi)$. If $\mathrm{def}(\pi) = 1 + \beta_1^{(2)}(\pi)$ then $g.d.\pi \leq 2$.*

Proof This follows from the theorem and the L^2-Euler characteristic formula, applied to the 2-complex associated to an optimal presentation for π. $\square$

Theorem 2.5 *Let π be a finitely presentable group such that $\beta_1^{(2)}(\pi) = 0$. Then $\mathrm{def}(\pi) \leq 1$, with equality if and only if $g.d.\pi \leq 2$ and $\beta_2(\pi) = \beta_1(\pi) - 1$.*

Proof The upper bound and the necessity of the conditions follow as in Corollary 2.4.1. Conversely, if they hold and X is a finite aspherical 2-complex with $\pi_1(X) \cong \pi$ then $\chi(X) = 1 - \beta_1(\pi) + \beta_2(\pi) = 0$. After collapsing a maximal tree in X we may assume it has a single 0-cell, and then the presentation read off the 1- and 2-cells has deficiency 1. $\square$

This theorem applies if π is finitely presentable and is an ascending HNN extension with finitely generated base H, or has an infinite amenable normal subgroup. In the latter case $\beta_i^{(2)}(\pi) = 0$ for all i, by Theorem 2.3. Thus if X is a finite aspherical 2-complex with $\pi_1(X) \cong \pi$ then $\chi(X) = 0$, and so the condition $\beta_2(\pi) = \beta_1(\pi) - 1$ is redundant.

[Similarly, if $\mathbb{Z}[\pi]$ has a safe extension Ψ and C_* is the equivariant cellular chain complex of the universal cover $\widetilde{X}$ then $\Psi \otimes_{\mathbb{Z}[\pi]} C_*$ is a complex of free left Ψ-modules with bases corresponding to the cells of X. Since Ψ is a safe extension $H_i(X;\Psi) = \Psi \otimes_{\mathbb{Z}[\pi]} H_i(X;\mathbb{Z}[\pi]) = 0$ for all i, and so again $\chi(X) = 0$.]

Corollary 2.5.1 *Let π be a finitely presentable group with an FP_2 normal subgroup N such that $\pi/N \cong Z$. Then $\mathrm{def}(\pi) = 1$ if and only if N is free.*

Proof If $\mathrm{def}(\pi) = 1$ then $g.d.\pi \leq 2$, by Theorem 2.5, and so N is free by Corollary 8.6 of [Bi]. The converse is clear. $\square$

In fact it suffices to assume that N is finitely generated (rather than FP_2) [Ko06]. (See Corollary 4.3.1 below.)

Let $G = F(2) \times F(2)$. Then $g.d.G = 2$ and $\mathrm{def}(G) \leq \beta_1(G) - \beta_2(G) = 0$. Hence $\langle u, v, x, y \mid ux = xu,\ uy = yu,\ vx = xv,\ vy = yv \rangle$ is an optimal presentation, and $\mathrm{def}(G) = 0$. The subgroup N generated by u, vx^{-1} and y is normal in G and $G/N \cong Z$, so $\beta_1^{(2)}(G) = 0$, by Lemma 2.1. However N is not free, since u and y generate a rank two abelian subgroup. It follows from Corollary 2.5.1 that N is not FP_2, and so $F(2) \times F(2)$ is not almost coherent.

The next result is a version of the *Tits alternative* for coherent groups of cohomological dimension 2. For each $m \in Z$ let $Z*_m$ be the group with presentation $\langle a, t \mid tat^{-1} = a^m \rangle$. (Thus $Z*_0 \cong Z$ and $Z*_{-1} \cong Z \rtimes_{-1} Z$.)

Theorem 2.6 *Let π be a finitely generated group such that $c.d.\pi = 2$. Then $\pi \cong Z*_m$ for some $m \neq 0$ if and only if it is almost coherent and restrained and π/π' is infinite.*

Proof The conditions are easily seen to be necessary. Conversely, if π is almost coherent and π/π' is infinite π is an HNN extension with FP_2 base H, by Theorem 1.13. The HNN extension must be ascending as π has no noncyclic free subgroup. Hence $H^2(\pi;\mathbb{Z}[\pi])$ is a quotient of $H^1(H;\mathbb{Z}[\pi]) \cong H^1(H;\mathbb{Z}[H])\otimes\mathbb{Z}[\pi/H]$, by the Brown-Geoghegan Theorem. Now $H^2(\pi;\mathbb{Z}[\pi]) \neq 0$, since $c.d.\pi = 2$, and so $H^1(H;\mathbb{Z}[H]) \neq 0$. Since H is restrained it must have two ends, so $H \cong Z$ and $\pi \cong Z*_m$ for some $m \neq 0$. □

Does this remain true without any such coherence hypothesis?

Corollary 2.6.1 *Let π be a finitely generated group. Then the following are equivalent:*

(1) $\pi \cong Z*_m$ *for some* $m \in Z$;

(2) π *is torsion-free, elementary amenable,* FP_2 *and* $h(\pi) \leq 2$;

(3) π *is elementary amenable and* $c.d.\pi \leq 2$;

(4) π *is elementary amenable and* $\mathrm{def}(\pi) = 1$; *and*

(5) π *is almost coherent and restrained and* $\mathrm{def}(\pi) = 1$.

Proof Condition (1) clearly implies the others. Suppose (2) holds. We may assume that $h(\pi) = 2$ and $h(\sqrt{\pi}) = 1$ (for otherwise $\pi \cong Z$, $Z^2 = Z*_1$ or $Z*_{-1}$). Hence $h(\pi/\sqrt{\pi}) = 1$, and so $\pi/\sqrt{\pi}$ is an extension of Z or D by a finite normal subgroup. If $\pi/\sqrt{\pi}$ maps onto D then $\pi \cong A *_C B$, where $[A : C] = [B : C] = 2$ and $h(A) = h(B) = h(C) = 1$, and so $\pi \cong Z \rtimes_{-1} Z$. But then $h(\sqrt{\pi}) = 2$. Hence we may assume that π maps onto Z, and so π is an ascending HNN extension with finitely generated base H, by Theorem 1.13. Since H is torsion-free, elementary amenable and $h(H) = 1$ it must be infinite cyclic and so (2) implies (1). If (3) holds π is solvable, by Theorems 1.11, and 1.9, and so (1) follows from [Gi79]. If $\mathrm{def}(\pi) = 1$ then π is an ascending HNN extension with finitely generated base, so $\beta_1^{(2)}(\pi) = 0$, by Lemma 2.1. Hence (4) and (5) each imply (1) by Theorems 2.5 and 2.6. □

Note that (3) ⇒ (2) if π is FP_2, so we may then avoid [Gi79]. Are these conditions equivalent to "π is almost coherent and restrained and $c.d.\pi \leq 2$"? Note also that if $\mathrm{def}(\pi) > 1$ then π has noncyclic free subgroups [Ro77].

Let $\mathcal{X}$ be the class of groups of finite graphs of groups, with all edge and vertex groups infinite cyclic. Kropholler has shown that a finitely generated, noncyclic group G is in $\mathcal{X}$ if and only if $c.d.G = 2$ and G has an infinite cyclic subgroup H which meets all its conjugates nontrivially. Moreover G is then coherent, one ended and $g.d.G = 2$ [Kr90'], while $\beta_1^{(2)}(G) = 0$ by Theorem 5.12 of [PT11].

Theorem 2.7 *Let π be a finitely generated group such that $c.d.\pi = 2$. If π has a nontrivial normal subgroup E which is either elementary amenable or almost coherent, locally virtually indicable and restrained then π is in $\mathcal{X}$ and either $E \cong Z$ or π/π' is infinite and π' is abelian.*

Proof If E is elementary amenable it is virtually solvable, by Theorem 1.11, since $c.d.E \leq c.d.\pi$. Otherwise finitely generated subgroups of E are metabelian, by Theorem 2.6 and its Corollary, and so all words in E of the form $[[g,h],[g',h']]$ are trivial. Hence E is metabelian also. Therefore $A = \sqrt{E}$ is nontrivial, and as A is characteristic in E it is normal in π. Since A is the union of its finitely generated subgroups, which are torsion-free nilpotent groups of Hirsch length ≤ 2, it is abelian. If $A \cong Z$ then $[\pi : C_\pi(A)] \leq 2$. Moreover $C_\pi(A)'$ is free, by Bieri's Theorem. If $C_\pi(A)'$ is cyclic then $\pi \cong Z^2$ or $Z \rtimes_{-1} Z$; if $C_\pi(A)'$ is nonabelian then $E = A \cong Z$. Otherwise $c.d.A = c.d.C_\pi(A) = 2$ and so $C_\pi(A) = A$, by Bieri's Theorem. If A has rank 1 then $Aut(A)$ is abelian, so $\pi' \leq C_\pi(A)$ and π is metabelian. If $A \cong Z^2$ then π/A is isomorphic to a subgroup of $GL(2,\mathbb{Z})$, and so is virtually free. As A together with an element $t \in \pi$ of infinite order modulo A would generate a subgroup of cohomological dimension 3, which is impossible, the quotient π/A must be finite. Hence $\pi \cong Z^2$ or $Z \rtimes_{-1} Z$. In all cases π is in $\mathcal{X}$, by Theorem C of [Kr90']. □

If $c.d.\pi = 2$, $\zeta\pi \neq 1$ and π is nonabelian then $\zeta\pi \cong Z$ and π' is free, by Bieri's Theorem. On the evidence of his work on 1-relator groups Murasugi conjectured that if G is a finitely presentable group other than Z^2 and $\mathrm{def}(G) \geq 1$ then $\zeta G \cong Z$ or 1, and is trivial if $\mathrm{def}(G) > 1$, and he verified this for classical link groups [Mu65]. Theorems 2.3, 2.5 and 2.7 together imply that if ζG is infinite then $\mathrm{def}(G) = 1$ and $\zeta G \cong Z$.

It remains an open question whether every finitely presentable group of cohomological dimension 2 has geometric dimension 2. The following partial answer to this question was first obtained by W.Beckmann under the additional assumptions that π is FF and $c.d.\pi \leq 2$ (see [Dy87']).

Theorem 2.8 *Let π be a finitely presentable group. Then $g.d.\pi \leq 2$ if and only if $c.d._{\mathbb{Q}}\pi \leq 2$ and $\mathrm{def}(\pi) = \beta_1(\pi) - \beta_2(\pi)$.*

Proof The necessity of the conditions is clear. Suppose that they hold and that $C(P)$ is the 2-complex corresponding to a presentation for π of maximal deficiency. The cellular chain complex of $\widetilde{C(P)}$ gives an exact sequence

$$0 \to K = \pi_2(C(P)) \to \mathbb{Z}[\pi]^r \to \mathbb{Z}[\pi]^g \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0.$$

Extending coefficients to $\mathbb{Q}$ gives a similar exact sequence, with kernel $\mathbb{Q} \otimes_{\mathbb{Z}} K$ on the left. As $c.d._{\mathbb{Q}}\pi \leq 2$ the image of $\mathbb{Q}[\pi]^r$ in $\mathbb{Q}[\pi]^g$ is projective, by Schanuel's Lemma. Therefore the inclusion of $\mathbb{Q} \otimes_{\mathbb{Z}} K$ into $\mathbb{Q}[\pi]^r$ splits, and $\mathbb{Q} \otimes_{\mathbb{Z}} K$ is projective. Moreover $dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}[\pi]} K) = 0$, and so $\mathbb{Q} \otimes_{\mathbb{Z}} K = 0$, since the Weak Bass Conjecture holds for π [Ec86]. Since K is free as an abelian group it imbeds in $\mathbb{Q} \otimes_{\mathbb{Z}} K$, and so is also 0. Hence $\widetilde{C(P)}$ is contractible, and so $C(P)$ is aspherical. □

The arguments of this section may easily be extended to other highly connected finite complexes. A $[\pi, m]_f$-complex is a finite m-dimensional complex X with $\pi_1(X) \cong \pi$ and with $(m-1)$-connected universal cover $\widetilde{X}$. Such a $[\pi, m]_f$-complex X is aspherical if and only if $\pi_m(X) = 0$. In that case we shall say that π has geometric dimension at most m, written $g.d.\pi \leq m$.

Theorem 2.4′ *Let X be a $[\pi, m]_f$-complex and suppose that $\beta_i^{(2)}(\pi) = 0$ for $i < m$. Then $(-1)^m \chi(X) \geq 0$. If $\chi(X) = 0$ then X is aspherical.* □

In general, the final implication of this theorem cannot be reversed. For $S^1 \vee S^1$ is an aspherical $[F(2), 1]_f$-complex and $\beta_0^{(2)}(F(2)) = 0$, but $\chi(S^1 \vee S^1) \neq 0$.

One of the applications of L^2-cohomology in [CG86] was to show that if X is a finite aspherical complex and $\pi_1(X)$ has an infinite amenable normal subgroup A then $\chi(X) = 0$. (This generalised a theorem of Gottlieb, who assumed that A was a central subgroup [Go65].) We may similarly extend Theorem 2.5 to give a converse to the Cheeger-Gromov extension of Gottlieb's Theorem.

Theorem 2.5′ *Let X be a $[\pi, m]_f$-complex and suppose that π has an infinite amenable normal subgroup. Then X is aspherical if and only if $\chi(X) = 0$.* □

2.4 Poincaré duality

The main reason for studying PD-complexes is that they represent the homotopy theory of manifolds. However they also arise in situations where the geometry does not immediately provide a corresponding manifold. For instance, under suitable finiteness assumptions an infinite cyclic covering space of a closed 4-manifold with Euler characteristic 0 will be a PD_3-complex, but need not be homotopy equivalent to a closed 3-manifold. (See Chapter 11.)

A PD_n*-space* is a space homotopy equivalent to a cell complex which satisfies Poincaré duality of formal dimension n with local coefficients. If X is a PD_n-space with fundamental group π then $C_*(\widetilde{X})$ is $\mathbb{Z}[\pi]$-finitely dominated, so π is FP_2. The PD_n-space X is *finite* if $C_*(\widetilde{X})$ is $\mathbb{Z}[\pi]$-chain homotopy equivalent to a finite free $\mathbb{Z}[\pi]$-complex. It is a PD_n*-complex* if it is finitely dominated. This is so if and only if π is finitely presentable [Br72, Br75]. Finite PD_n-complexes are homotopy equivalent to finite complexes. (Note also that a cell complex X is finitely dominated if and only if $X \times S^1$ is finite. See Proposition 3 of [Rn95].) Although PD_n-complexes are most convenient for our purposes, the broader notion of PD_n-space is occasionally useful. All the PD_n-complexes that we consider shall be connected.

Let P be a PD_n-complex. We may assume that $P = P_o \cup D^n$, where P_o is a complex of dimension $\leq \max\{3, n-1\}$ [Wl67]. If C_* is the cellular chain complex of $\widetilde{P}$ the Poincaré duality isomorphism may be described in terms of a chain homotopy equivalence $\overline{C^*} \cong C_{n-*}$, which induces isomorphisms from $H^j(\overline{C^*})$ to $H_{n-j}(C_*)$, given by cap product with a generator $[P]$ of $H_n(P; Z^{w_1(P)}) = H_n(\bar{Z} \otimes_{\mathbb{Z}[\pi_1(P)]} C_*)$. (Here the first Stiefel-Whitney class $w_1(P)$ is considered as a homomorphism from $\pi_1(P)$ to $Z/2Z$.) From this point of view it is easy to see that Poincaré duality gives rise to ($\mathbb{Z}$-linear) isomorphisms from $H^j(P; B)$ to $H_{n-j}(P; \bar{B})$, where B is any left $\mathbb{Z}[\pi_1(P)]$-module of coefficients. (See [Wl67] or Chapter II of [Wl] for further details.) If P is a Poincaré duality complex then the L^2-Betti numbers also satisfy Poincaré duality. (This does not require that P be finite or orientable!)

A group G is a PD_n-group (as defined in Chapter 1) if and only if $K(G,1)$ is a PD_n-space. For every $n \geq 4$ there are PD_n-groups which are not finitely presentable [Da98].

Dwyer, Stolz and Taylor have extended Strebel's Theorem to show that if H is a subgroup of infinite index in $\pi_1(P)$ then the corresponding covering space P_H has homological dimension $< n$; hence if moreover $n \neq 3$ then P_H is homotopy equivalent to a complex of dimension $< n$ [DST96].

2.5 PD_3-complexes

In this section we shall summarize briefly what is known about PD_n-complexes of dimension at most 3. It is easy to see that a connected PD_1-complex must be homotopy equivalent to S^1. The 2-dimensional case is already quite difficult, but has been settled by Eckmann, Linnell and Müller, who showed that every PD_2-complex is homotopy equivalent to a closed surface. (See Chapter VI of [DD]. This result has been further improved by Bowditch's Theorem.) There are PD_3-complexes with finite fundamental group which are not homotopy equivalent to any closed 3-manifold. On the other hand, Turaev's Theorem below implies that every PD_3-complex with torsion-free fundamental group is homotopy equivalent to a closed 3-manifold if every PD_3-group is a 3-manifold group. The latter is so if the Hirsch-Plotkin radical of the group is nontrivial (see §7 below), but remains open in general.

The *fundamental triple* of a PD_3-complex P is $(\pi_1(P), w_1(P), c_{P*}[P])$. This is a complete homotopy invariant for such complexes. (See also §6 and §9 below.)

Theorem (Hendriks) *Two PD_3-complexes are homotopy equivalent if and only if their fundamental triples are isomorphic.* □

Turaev has characterized the possible triples corresponding to a given finitely presentable group and orientation character, and has used this result to deduce a basic splitting theorem [Tu90].

Theorem (Turaev) *A PD_3-complex is indecomposable with respect to connected sum if and only if its fundamental group is indecomposable with respect to free product.* □

Wall asked whether every orientable PD_3-complex whose fundamental group has infinitely many ends is a proper connected sum [Wl67]. Since the fundamental group of a PD_n-space is FP_2 it is the fundamental group of a finite graph of finitely generated groups in which each vertex group has at most one end and each edge group is finite, by Theorem VI.6.3 of [DD]. Crisp has given a substantial partial answer to Wall's question, based on this observation [Cr00].

Theorem (Crisp) *Let P be an indecomposable orientable PD_3-complex. If $\pi_1(P)$ is not virtually free then it has one end, and so P is aspherical.* □

The arguments of Turaev and Crisp extend to PD_3-spaces in a straightforward manner. In particular, they imply that if P is a PD_3-space then $\pi = \pi_1(P)$ is virtually torsion-free. However, there is an indecomposable orientable PD_3-complex with $\pi \cong S_3 *_{Z/2Z} S_3 \cong F(2) \rtimes S_3$ and double cover homotopy equivalent to $L(3,1)\sharp L(3,1)$. "Most" indecomposable PD_3-complexes with π virtually free have double covers which are homotopy equivalent to connected sums of $\mathbb{S}^3$-manifolds [Hi12].

2.6 The spherical cases

Let P be a PD_3-space with fundamental group π. The Hurewicz Theorem, Poincaré duality and a choice of orientation for P together determine an isomorphism $\pi_2(P) \cong \overline{H^1(\pi;\mathbb{Z}[\pi])}$. In particular, $\pi_2(P)=0$ if and only if π is finite or has one end.

The possible PD_3-complexes with π finite are well understood.

Theorem 2.9 [Wl67] *Let X be a PD_3-complex with finite fundamental group F. Then*

(1) $\widetilde{X}\simeq S^3$, *F has cohomological period dividing 4 and X is orientable;*

(2) *the first nontrivial k-invariant $k(X)$ generates $H^4(F;\mathbb{Z})\cong Z/|F|Z$.*

(3) *the homotopy type of X is determined by F and the orbit of $k(M)$ under $Out(F)\times\{\pm1\}$.*

Proof Since the universal cover $\widetilde{X}$ is also a finite PD_3-complex it is homotopy equivalent to S^3. A standard Gysin sequence argument shows that F has cohomological period dividing 4. Suppose that X is nonorientable, and let C be a cyclic subgroup of F generated by an orientation reversing element. Let $\tilde{Z}$ be the nontrivial infinite cyclic $\mathbb{Z}[C]$-module. Then $H^2(X_C;\tilde{Z})\cong H_1(X_C;\mathbb{Z})\cong C$, by Poincaré duality. But $H^2(X_C;\tilde{Z})\cong H^2(C;\tilde{Z})=0$, since the classifying map from $X_C=\widetilde{X}/C$ to $K(C,1)$ is 3-connected. Therefore X must be orientable and F must act trivially on $\pi_3(X)\cong H_3(\widetilde{X};\mathbb{Z})$.

The image of the orientation class of X generates $H_3(F;\mathbb{Z})\cong Z/|F|Z$. The Bockstein $\beta: H^3(F;\mathbb{Q}/\mathbb{Z})\to H^4(F;\mathbb{Z})$ is an isomorphism, since $H^q(F;\mathbb{Q})=0$ for $q>0$, and the bilinear pairing from $H_3(F;\mathbb{Z})\times H^4(F;\mathbb{Z})$ to $\mathbb{Q}/\mathbb{Z}$ given by $(h,c)\mapsto\beta^{-1}(c)(h)$ is nonsingular. Each generator g of $H_3(F;\mathbb{Z})$ determines an unique $k_g\in H^4(F;\mathbb{Z})$ such that $\beta^{-1}(k_g)(g)=\frac{1}{|F|}$ *mod* $\mathbb{Z}$. The element corresponding to $c_{X*}[X]$ is the first nontrivial k-invariant of X. Inner automorphisms of F act trivially on $H^4(F;\mathbb{Z})$, while changing the orientation of X corresponds to multiplication by -1. Thus the orbit of $k(M)$ under $Out(F)\times\{\pm1\}$ is the significant invariant.

We may construct the third stage of the Postnikov tower for X by adjoining cells of dimension greater than 4 to X. The natural inclusion $j: X\to P_3(X)$ is then 4-connected. If X_1 is another such PD_3-complex and $\theta:\pi_1(X_1)\to F$ is an isomorphism which identifies the k-invariants then there is a 4-connected map $j_1: X_1\to P_3(X)$ inducing θ, which is homotopic to a map with image in the 4-skeleton of $P_3(X)$, and so there is a map $h : X_1 \to X$ such that j_1 is homotopic to jh. The map h induces isomorphisms on π_i for $i \leq 3$, since j and j_1 are 4-connected, and so the lift $\tilde{h} : \widetilde{X}_1 \simeq S^3 \to \widetilde{X} \simeq S^3$ is a homotopy equivalence, by the theorems of Hurewicz and Whitehead. Thus h is itself a homotopy equivalence. □

The list of finite groups with cohomological period dividing 4 is well known. Each such group F and generator $k \in H^4(F;\mathbb{Z})$ is realized by some PD_3^+-complex [Sw60, Wl67]. (See also Chapter 11 below.) In particular, there is an unique homotopy type of PD_3-complexes with fundamental group S_3, but there is no 3-manifold with this fundamental group [Mi57].

The fundamental group of a PD_3-complex P has two ends if and only if $\widetilde{P} \simeq S^2$, and then P is homotopy equivalent to one of the four $\mathbb{S}^2 \times \mathbb{E}^1$-manifolds $S^2 \times S^1$, $S^2 \tilde{\times} S^1$, $RP^2 \times S^1$ or $RP^3 \sharp RP^3$. The following simple lemma leads to an alternative characterization.

Lemma 2.10 *Let X be a finite-dimensional complex with a connected regular covering space $\widehat{X}$ and covering group $C = Aut(\widehat{X}/X)$. If $\widetilde{H}_q(\widehat{X};\mathbb{Z}) = 0$ for $q \neq m$ then $H_{s+m+1}(C;\mathbb{Z}) \cong H_s(C; H_m(\widehat{X};\mathbb{Z}))$, for all $s >> 0$.*

Proof The lemma follows by devissage applied to the homology of $C_*(\widehat{X})$, considered as a chain complex over $\mathbb{Z}[C]$. (In fact $s \geq dim(X) - m$ suffices.) □

Theorem 2.11 *Let P be a PD_3-space whose fundamental group π has a nontrivial finite normal subgroup N. Then either P is homotopy equivalent to $RP^2 \times S^1$ or π is finite.*

Proof We may clearly assume that π is infinite. Then $H_q(\widetilde{P};\mathbb{Z}) = 0$ for $q > 2$, by Poincaré duality. Let $\Pi = \pi_2(P)$. The augmentation sequence

$$0 \to A(\pi) \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0$$

gives rise to a short exact sequence

$$0 \to Hom_{\mathbb{Z}[\pi]}(\mathbb{Z}[\pi], \mathbb{Z}[\pi]) \to Hom_{\mathbb{Z}[\pi]}(A(\pi), \mathbb{Z}[\pi]) \to H^1(\pi;\mathbb{Z}[\pi]) \to 0.$$

Let $f : A(\pi) \to \mathbb{Z}[\pi]$ be a homomorphism and ζ be a central element of π. Then $f.\zeta(i) = f(i)\zeta = \zeta f(i) = f(\zeta i) = f(i\zeta)$ and so $(f.\zeta - f)(i) = f(i(\zeta - 1)) = if(\zeta - 1)$ for all $i \in A(\pi)$. Hence $f.\zeta - f$ is the restriction of a homomorphism from $\mathbb{Z}[\pi]$ to $\mathbb{Z}[\pi]$. Thus central elements of π act trivially on $H^1(\pi;\mathbb{Z}[\pi])$.

If $n \in N$ the centraliser $\gamma = C_\pi(\langle n \rangle)$ has finite index in π, and so the covering space P_γ is again a PD_3-complex with universal covering space $\widetilde{P}$. Therefore $\Pi \cong \overline{H^1(\gamma;\mathbb{Z}[\gamma])}$ as a (left) $\mathbb{Z}[\gamma]$-module. In particular, Π is a free abelian group. Since n is central in γ it acts trivially on $H^1(\gamma;\mathbb{Z}[\gamma])$ and hence via $w(n)$ on Π. Suppose first that $w(n) = 1$. Then Lemma 2.10 (with $X = P$, $\widehat{X} = \widetilde{P}$ and $m = 2$) gives an exact sequence

$$0 \to Z/o(n)Z \to \Pi \to \Pi \to 0,$$

where $o(n)$ is the order of n and the right hand homomorphism is multiplication by $o(n)$, since n acts trivially on Π. As Π is torsion-free we must have $n = 1$.

Therefore if $n \in N$ is nontrivial it has order 2 and $w(n) = -1$. In this case Lemma 2.10 gives an exact sequence

$$0 \to \Pi \to \Pi \to Z/2Z \to 0,$$

where the left hand homomorphism is multiplication by 2. Since Π is a free abelian group it must be infinite cyclic. Hence $\widetilde{P} \simeq S^2$ and $\widetilde{P}/(Z/2Z) \simeq RP^2$. The theorem now follows, since any self homotopy equivalence of RP^2 is homotopic to the identity (compare Theorem 4.4 of [Wl67]). □

If $\pi_1(P)$ has a finitely generated infinite normal subgroup of infinite index then it has one end, and so P is aspherical. We shall discuss this case next.

2.7 PD_3-groups

As a consequence of the work of Turaev and Crisp the study of PD_3-complexes reduces largely to the study of PD_3-groups. It is not yet known whether all such groups are 3-manifold groups, or even whether they must be finitely presentable. The fundamental groups of aspherical 3-manifolds which are Seifert fibred or are finitely covered by surface bundles may be characterized among all PD_3-groups in simple group-theoretic terms.

Theorem 2.12 *Let G be a PD_3-group with a nontrivial FP_2 normal subgroup N of infinite index. Then either*

(1) *$N \cong Z$ and G/N is virtually a PD_2-group; or*

(2) *N is a PD_2-group and G/N has two ends.*

Proof Let e be the number of ends of N. If N is free then $H^3(G;\mathbb{Z}[G]) \cong H^2(G/N;H^1(N;\mathbb{Z}[G]))$. Since N is finitely generated and G/N is FP_2 this is in turn isomorphic to $H^2(G/N;\mathbb{Z}[G/N])^{(e-1)}$. Since G is a PD_3-group we must have $e-1=1$ and so $N\cong Z$. We then have $H^2(G/N;\mathbb{Z}[G/N]) \cong H^3(G;\mathbb{Z}[G])\cong Z$, so G/N is virtually a PD_2-group, by Bowditch's Theorem.

Otherwise $c.d.N=2$ and so $e=1$ or ∞. The LHSSS gives an isomorphism $H^2(G;\mathbb{Z}[G]) \cong H^1(G/N;\mathbb{Z}[G/N])\otimes H^1(N;\mathbb{Z}[N]) \cong H^1(G/N;\mathbb{Z}[G/N])^{e-1}$. Hence either $e=1$ or $H^1(G/N;\mathbb{Z}[G/N])=0$. But in the latter case we have $H^3(G;\mathbb{Z}[G]) \cong H^2(G/N;\mathbb{Z}[G/N])\otimes H^1(N;\mathbb{Z}[N])$ and so $H^3(G;\mathbb{Z}[G])$ is either 0 or infinite dimensional. Therefore $e=1$, and so $H^3(G;\mathbb{Z}[G]) \cong H^1(G/N;\mathbb{Z}[G/N])\otimes H^2(N;\mathbb{Z}[N])$. Hence G/N has two ends and $H^2(N;\mathbb{Z}[N]) \cong Z$, so N is a PD_2-group. □

We shall strengthen this result in Theorem 2.17 below.

Corollary 2.12.1 *A PD_3-space P is homotopy equivalent to the mapping torus of a self homeomorphism of a closed surface if and only if there is an epimorphism $\phi:\pi_1(P)\to Z$ with finitely generated kernel.*

Proof This follows from Theorems 1.19, 2.11 and 2.12. □

If $\pi_1(P)$ is infinite and is a nontrivial direct product then P is homotopy equivalent to the product of S^1 with a closed surface.

Theorem 2.13 *Let G be a PD_3-group. If S is an almost coherent, restrained, locally virtually indicable subgroup then S is virtually solvable. If S has infinite index in G it is virtually abelian.*

Proof Suppose first that S has finite index in G, and so is again a PD_3-group. Since S is virtually indicable we may assume without loss of generality that $\beta_1(S)>0$. Then S is an ascending HNN extension $H*_\phi$ with finitely generated base. Since G is almost coherent H is finitely presentable, and since $H^3(S;\mathbb{Z}[S])\cong Z$ it follows from Lemma 3.4 of [BG85] that H is normal in S and $S/H\cong Z$. Hence H is a PD_2-group, by Theorem 2.12. Since H has no noncyclic free subgroup it is virtually Z^2 and so S and G are virtually poly-Z.

If $[G:S]=\infty$ then $c.d.S\le 2$, by Strebel's Theorem. Let J be a finitely generated subgroup of S. Then J is FP_2 and virtually indicable, and hence is virtually solvable, by Theorem 2.6 and its Corollary. Since J contains a PD_2-group, by Corollary 1.4 of [KK05], it is virtually abelian. Hence S is virtually abelian also. □

As the fundamental groups of virtually Haken 3-manifolds are coherent and locally virtually indicable, this implies the Tits alternative for such groups [EJ73]. A slight modification of the argument gives the following corollary.

Corollary 2.13.1 *A PD_3-group G is virtually poly-Z if and only if it is coherent, restrained and has a subgroup of finite index with infinite abelianization.* □

If $\beta_1(G) \geq 2$ the hypothesis of coherence is redundant, for there is then an epimorphism $p : G \to Z$ with finitely generated kernel, by Theorem D of [BNS87], and the kernel is then FP_2 by Theorem 1.19.

The argument of Theorem 2.13 and its corollary extend to show by induction on m that a PD_m-group is virtually poly-Z if and only if it is restrained and every finitely generated subgroup is FP_{m-1} and virtually indicable.

Theorem 2.14 *Let G be a PD_3-group. Then G is the fundamental group of an aspherical Seifert fibred 3-manifold or a $\mathbb{S}ol^3$-manifold if and only if $\sqrt{G} \neq 1$. Moreover*

(1) $h(\sqrt{G}) = 1$ *if and only if G is the group of an $\mathbb{H}^2 \times \mathbb{E}^1$- or $\widetilde{\mathbb{SL}}$-manifold;*
(2) $h(\sqrt{G}) = 2$ *if and only if G is the group of a $\mathbb{S}ol^3$-manifold;*
(3) $h(\sqrt{G}) = 3$ *if and only if G is the group of an $\mathbb{E}^3$- or $\mathbb{N}il^3$-manifold.*

Proof The necessity of the conditions is clear. (See [Sc83'], or §2 and §3 of Chapter 7 below.) Certainly $h(\sqrt{G}) \leq c.d.\sqrt{G} \leq 3$. Moreover $c.d.\sqrt{G} = 3$ if and only if $[G : \sqrt{G}]$ is finite, by Strebel's Theorem. Hence G is virtually nilpotent if and only if $h(\sqrt{G}) = 3$. If $h(\sqrt{G}) = 2$ then $\sqrt{G}$ is locally abelian, and hence abelian. Moreover $\sqrt{G}$ must be finitely generated, for otherwise $c.d\sqrt{G} = 3$. Thus $\sqrt{G} \cong Z^2$ and case (2) follows from Theorem 2.12.

Suppose now that $h(\sqrt{G}) = 1$ and let $C = C_G(\sqrt{G})$. Then $\sqrt{G}$ is torsion-free abelian of rank 1, so $Aut(\sqrt{G})$ is isomorphic to a subgroup of $\mathbb{Q}^\times$. Therefore G/C is abelian. If G/C is infinite then $c.d.C \leq 2$ by Strebel's Theorem and $\sqrt{G}$ is not finitely generated, so C is abelian, by Bieri's Theorem, and hence G is solvable. But then $h(\sqrt{G}) > 1$, which is contrary to our hypothesis. Therefore G/C is isomorphic to a finite subgroup of $\mathbb{Q}^\times \cong Z^\infty \oplus (Z/2Z)$ and so has order at most 2. In particular, if A is an infinite cyclic subgroup of $\sqrt{G}$ then A is normal in G, and so G/A is virtually a PD_2-group, by Theorem 2.12. If G/A is a PD_2-group then G is the fundamental group of an S^1-bundle over a closed surface. In general, a finite torsion-free extension of the fundamental group of a closed Seifert fibred 3-manifold is again the fundamental group of a closed Seifert fibred 3-manifold, by [Sc83] and Section 63 of [Zi]. □

The heart of this result is the deep theorem of Bowditch. The weaker characterization of fundamental groups of $\mathbb{S}ol^3$-manifolds and aspherical Seifert fibred 3-manifolds as PD_3-groups G such that $\sqrt{G} \neq 1$ and G has a subgroup of finite index with infinite abelianization is much easier to prove [H2]. There is as yet no comparable characterization of the groups of $\mathbb{H}^3$-manifolds, although it may be conjectured that these are exactly the PD_3-groups with no noncyclic abelian subgroups. (It has been recently shown that every closed $\mathbb{H}^3$-manifold is finitely covered by a mapping torus [Ag13].)

$\mathbb{N}il^3$- and $\widetilde{\mathbb{SL}}$-manifolds are orientable, and so their groups are PD_3^+-groups. This can also be seen algebraically, as every such group has a characteristic subgroup H which is a nonsplit central extension of a PD_2^+-group β by Z. An automorphism of such a group H must be orientation preserving.

Theorem 2.14 implies that if a PD_3-group G is not virtually poly-Z then its maximal elementary amenable normal subgroup is Z or 1. For this subgroup is virtually solvable, by Theorem 1.11, and if it is nontrivial then so is $\sqrt{G}$.

Lemma 2.15 *Let G be a group such that $c.d.G = 2$ and let K be an ascendant FP_2 subgroup of G. Then either $[G : K]$ is finite or K is free.*

Proof We may assume that K is not free, and so $c.d.K = c.d.G = 2$. Suppose first that K is normal in G. Then G/K is locally finite, by Corollary 8.6 of [Bi], and so G is the increasing union of a (possibly finite) sequence of FP_2 subgroups $K = U_0 < U_1 < \dots$ such that $[U_{i+1} : U_i]$ is finite, for all $i \geq 0$. It follows from the Kurosh subgroup theorem that if $U \leq V$ are finitely generated groups and $[V : U]$ is finite then V has strictly fewer indecomposable factors than U unless both groups are indecomposable. (See Lemma 1.4 of [Sc76]). Hence if K is a nontrivial free product then $[G : K]$ is finite. Otherwise K has one end, and so $H^s(U_i; \mathbb{Z}[U_i]) = 0$ for $s \leq 1$ and $i \geq 0$. Since K is FP_2, the successive indices are finite and $c.dU_i = 2 = c.d.G$ for all $i \geq 0$ the union is finitely generated, by the Gildenhuys-Strebel Theorem. Hence the sequence terminates and $[G : K]$ is again finite.

If $K = K_0 < K_1 < \dots K_\daleth = G$ is an ascendant chain then $[K_{\alpha+1} : K_\alpha] < \infty$ for all α, by the argument just given. Let ω be the union of the finite ordinals in $\daleth$. Then $\cup_{\alpha<\omega} K_\alpha$ is finitely generated, by the Gildenhuys-Strebel Theorem, and so ω is finite. Hence the chain is finite, and so $[G : K] < \infty$. $\square$

Theorem 2.16 *Let G be a PD_3-group with an ascending sequence of subgroups $K_0 < K_1 < \dots$ such that K_n is normal in K_{n+1} for all $n \geq 0$. If $K = K_0$ is one-ended and FP_2 then the sequence is finite and either $[K_n : K]$ or $[G : K_n]$ is finite, for all $n \geq 0$.*

Proof Suppose that $[K_1 : K]$ and $[G : K_1]$ are both infinite. Since K has one end it is not free and so $c.d.K = c.d.K_1 = 2$, by Strebel's Theorem. Hence there is a free $\mathbb{Z}[K_1]$-module W such that $H^2(K_1; W) \neq 0$, by Proposition 5.1 of [Bi]. Since K is FP_2 and has one end $H^q(K; W) = 0$ for $q = 0$ or 1 and $H^2(K; W)$ is an induced $\mathbb{Z}[K_1/K]$-module. Since $[K_1 : K]$ is infinite $H^0(K_1/K; H^2(K; W)) = 0$, by Lemma 8.1 of [Bi]. The LHSSS for K_1 as an extension of K_1/K by K now gives $H^r(K_1; W) = 0$ for $r \leq 2$, which is a contradiction. A similar argument applies to the other terms of the sequence.

Suppose that $[K_n : K]$ is finite for all $n \geq 0$ and let $\widehat{K} = \cup_{n\geq 0} K_n$. If $c.d.\widehat{K} = 2$ then $[\widehat{K} : K] < \infty$, by Lemma 2.15. Thus the sequence must be finite. □

Corollary 2.16.1 *Let G be a PD_3-group with an FP_2 subgroup H which has one end and is of infinite index in G. Let $H_0 = H$ and $H_{i+1} = N_G(H_i)$ for $i \geq 0$. Then $\widehat{H} = \cup H_i$ is FP_2 and has one end, and either $c.d.\widehat{H} = 2$ and $N_G(\widehat{H}) = \widehat{H}$ or $[G : \widehat{H}] < \infty$ and G is virtually the group of a surface bundle.*

Proof This follows immediately from Theorems 2.12 and 2.16. □

Corollary 2.16.2 *If G has a subgroup H which is a PD_2-group with $\chi(H) = 0$ (respectively, < 0) then either it has such a subgroup which is its own normalizer in G or it is virtually the group of a surface bundle.*

Proof If $c.d.\widehat{H} = 2$ then $[\widehat{H} : H] < \infty$, so $\widehat{H}$ is a PD_2-group, and $\chi(H) = [\widehat{H} : H]\chi(\widehat{H})$. □

It is possible to use the fact that $Out(H)$ is virtually torsion-free instead of appealing to [GS81] to prove this corollary.

Theorem 2.17 *Let G be a PD_3-group with a nontrivial FP_2 subgroup H which is ascendant and of infinite index in G. Then either $H \cong Z$ and H is normal in G or G is virtually poly-Z or H is a PD_2-group, $[G : N_G(H)] < \infty$ and $N_G(H)/H$ has two ends.*

Proof Let $H = H_0 < H_1 < \cdots < H_\beth = G$ be an ascendant sequence and let $\gamma = \min\{\alpha < \beth \mid [H_\alpha : H] = \infty\}$. Let $\widehat{H} = \cup_{\alpha<\gamma} H_\alpha$. Then $h.d.\widehat{H} \leq 2$ and so $[G : \widehat{H}] = \infty$. Hence $c.d.\widehat{H} \leq 2$ also, by Strebel's Theorem, and so either H is free or $[\widehat{H} : H] < \infty$, by Lemma 2.15.

If H is not free then $c.d.\widehat{H} = 2$ and $\widehat{H}$ is FP_2, normal and of infinite index in H_γ. Therefore $[G : H_\gamma] < \infty$ and so H_γ is a PD_3-group, by Theorem 2.16.

Hence $\widehat{H}$ is a PD_2-group and $H_\gamma/\widehat{H}$ has two ends, by Theorem 2.12. Since $[\widehat{H} : H] < \infty$ it follows easily that H is a PD_2-group, $[G : N_G(H)] < \infty$ and $N_G(H)/H$ has two ends.

If $H \cong F(r)$ for some $r > 1$ then γ and $[\widehat{H} : H]$ are finite, since $[H_n : H]$ divides $\chi(H) = 1 - r$ for all $n < \gamma$. A similar argument shows that $H_\gamma/\widehat{H}$ is not locally finite. Let K be a finitely generated subgroup of H_γ which contains $\widehat{H}$ as a subgroup of infinite index. Then $K/\widehat{H}$ is virtually free, by Theorem 8.4 of [Bi], and so K is finitely presentable. In particular, $\chi(K) = \chi(\widehat{H})\chi(K/\widehat{H})$. Now $\chi(K) \leq 0$ (see §9 of [KK05]). Since $\chi(\widehat{H}) < 0$ this is only possible if $\chi(K/\widehat{H}) \geq 0$, and so $K/\widehat{H}$ is virtually Z. Hence we may assume that H_γ is the union of an increasing sequence $N_0 = H < N_1 \leq \dots$ of finitely generated subgroups with N_i/H virtually Z, for $i \geq 1$. For each $i \geq 1$ the group N_i is FP_2, $c.d.N_i = 2$, $H^s(N_i; \mathbb{Z}[N_i]) = 0$ for $s \leq 1$ and $[N_{i+1} : N_i]$ is finite. Therefore H_γ is finitely generated, by the Gildenhuys-Strebel Theorem.

In particular, H_γ is virtually a semidirect product $\widehat{H} \rtimes Z$, and so it is FP_2 and $c.d.H_\gamma = 2$. Hence H_γ is a PD_2-group, by the earlier argument. But PD_2-groups do not have normal subgroups such as $\widehat{H}$. Therefore if H is free it is infinite cyclic: $H \cong Z$. Since $\sqrt{H_\alpha}$ is characteristic in H_α it is normal in $H_{\alpha+1}$, for each $\alpha < \beth$. Transfinite induction now shows that $H \leq \sqrt{G}$. Therefore either $\sqrt{G} \cong Z$, so $H \cong Z$ and is normal in G, or G is virtually poly-Z, by Theorem 2.14. □

If H is a PD_2-group $N_G(H)$ is the fundamental group of a 3-manifold which is double covered by the mapping torus of a surface homeomorphism. There are however $\mathbb{N}il^3$-manifolds whose groups have no normal PD_2-subgroup (although they always have subnormal copies of Z^2).

The original version of this result assumed that H is subnormal in G. (See [BH91] for a proof not using [Bo04] or [KK05].)

2.8 Subgroups of PD_3-groups and 3-manifold groups

The central role played by incompressible surfaces in the geometric study of Haken 3-manifolds suggests strongly the importance of studying subgroups of infinite index in PD_3-groups. Such subgroups have cohomological dimension ≤ 2, by Strebel's Theorem.

There are substantial constraints on 3-manifold groups and their subgroups. Every finitely generated subgroup of a 3-manifold group is the fundamental group of a compact 3-manifold (possibly with boundary), by Scott's Core Theorem [Sc73], and thus is finitely presentable and is either a 3-manifold group or has finite geometric dimension 2 or is a free group. Aspherical closed 3-manifolds are Haken, hyperbolic or Seifert fibred, by the work of Perelman [B-P]. The groups of such 3-manifolds are residually finite [He87], and the centralizer of any element in the group is finitely generated [JS]. Solvable subgroups of such groups are virtually poly-Z [EJ73].

In contrast, any group of finite geometric dimension 2 is the fundamental group of a compact aspherical 4-manifold with boundary, obtained by attaching 1- and 2-handles to D^4. On applying the reflection group trick of Davis [Da83] to the boundary we see that each such group embeds in a PD_4-group. For instance, the product of two nonabelian PD_2^+-groups contains a copy of $F(2)\times F(2)$, and so is a PD_4^+-group which is not almost coherent. No PD_4-group containing a Baumslag-Solitar group $\langle x,t \mid tx^pt^{-1} = x^q\rangle$ is residually finite, since this property is inherited by subgroups. Thus the question of which groups of finite geometric dimension 2 are subgroups of PD_3-groups is critical.

Kapovich and Kleiner have given an algebraic Core Theorem, showing that every one-ended FP_2 subgroup H in a PD_3-group G is the "ambient group" of a PD_3-pair $(H,\mathcal{S})$ [KK05]. Using this the argument of [Kr90a] may be adapted to show that every strictly increasing sequence of centralizers in G has length at most 4 [Hi06]. (The finiteness of such sequences and the fact that centralizers in G are finitely generated or rank 1 abelian are due to Castel [Ca07].) With the earlier work of Kropholler and Roller [KR88, KR89, Kr90, Kr93] it follows that if G has a subgroup $H\cong Z^2$ and $\sqrt{G}=1$ then it splits over a subgroup commensurate with H. It also follows easily from the algebraic Core Theorem that if a subgroup H is an $\mathcal{X}$-group then $H=\pi_1(N)$ for some Seifert fibred 3-manifold N with $\partial N\neq\emptyset$. In particular, no nontrivial Baumslag-Solitar relation holds in G [KK05].

The geometric conclusions of Theorem 2.14 and the coherence of 3-manifold groups suggest that Theorems 2.12 and 2.17 should hold under the weaker hypothesis that N be finitely generated. (Compare Theorem 1.19.) It is known that $F(2)\times F(2)$ is not a subgroup of any PD_3-group [KR89]. This may be regarded as a weak coherence result.

Is there a characterization of virtual PD_3-groups parallel to Bowditch's Theorem? (It may be relevant that homology n-manifolds are manifolds for $n\leq 2$. There is no direct analogue in high dimensions. For every $k\geq 6$ there are FP_k groups G with $H^k(G;\mathbb{Z}[G])\cong Z$ but which are not virtually torsion-free [FS93].)

2.9 $\pi_2(P)$ as a $\mathbb{Z}[\pi]$-module

Let P be a PD_3-space with fundamental group π and orientation character w. If π is finite $\pi_2(P)=0$ and $c_{P*}[P]\in H_3(\pi;Z^w)$ is essentially equivalent to the first nontrivial k-invariant of P, as outlined in Theorem 2.9. Suppose that π is infinite. If N is another PD_3-space and there is an isomorphism $\theta:\nu=\pi_1(N)\to\pi$ such that $w_1(N)=\theta^*w$ then $\pi_2(N)\cong\theta^*\pi_2(P)$ as $\mathbb{Z}[\nu]$-modules. If moreover $k_1(N)=\theta^*k_1(P)$ (modulo automorphisms of the pair $(\nu,\pi_2(N))$) then $P_2(N)\simeq P_2(P)$. Since we may construct these Postnikov 2-stages by adjoining cells of dimension ≥ 4 it follows that there is a map $f:N\to P$ such that $\pi_1(f)=\theta$ and $\pi_2(f)$ is an isomorphism. The homology of the universal covering spaces $\widetilde{N}$ and $\widetilde{P}$ is 0 above degree 2, and so f is a homotopy equivalence, by the Whitehead Theorem. Thus the homotopy type of P is determined by the triple $(\pi,w,k_1(P))$. One may ask how $c_{P*}[P]$ and $k_1(P)$ determine each other.

There is a facile answer: in Turaev's realization theorem for homotopy triples the element of $H_3(\pi;Z^w)$ is used to construct a cell complex X by attaching 2- and 3-cells to the 2-skeleton of $K(\pi,1)$. If C_* is the cellular chain complex of $\widetilde{X}$ then $k_1(X)$ is the class of

$$0\to\pi_2(X)\to C_2/\partial C_3\to C_1\to C_0\to\mathbb{Z}\to 0$$

in $H^3(\pi;\pi_2(X))=Ext^3_{\mathbb{Z}[\pi]}(\mathbb{Z},\pi_2(X))$. Conversely, a class $\kappa\in Ext^3_{\mathbb{Z}[\pi]}(\mathbb{Z},\Pi)$ corresponds to an extension

$$0\to\Pi\to D_2\to D_1\to D_0\to\mathbb{Z}\to 0,$$

with D_1 and D_0 finitely generated free $\mathbb{Z}[\pi]$-modules. Let $\mathcal{D}_*$ be the complex $D_2\to D_1\to D_0$, with augmentation ε to $\mathbb{Z}$. If $\kappa=k_1(P)$ for a PD_3-complex P then $\mathbb{T}or_3^{\mathbb{Z}[\pi]}(\mathbb{Z}^w,\mathcal{D}_*)\cong H_3(P_2(P);\mathbb{Z}^w)\cong Z$ (where $\mathbb{T}or$ denotes hyperhomology), and the augmentation then determines a class in $H_3(\pi;\mathbb{Z}^w)$ (up to sign). Can these connections be made more explicit? Is there a natural homomorphism from $H^3(\pi;\overline{H^1(\pi;\mathbb{Z}[\pi])})$ to $H_3(\pi;Z^w)$?

If P is an orientable 3-manifold which is the connected sum of a 3-manifold whose fundamental group is free of rank r with $s\geq 1$ aspherical 3-manifolds then $\pi_2(P)$ is a finitely generated free $\mathbb{Z}[\pi]$-module of rank $r+s-1$ [Sw73]. We shall give a direct homological argument that applies for PD_3-spaces with torsion-free fundamental group, and we shall also compute $H^2(P;\pi_2(P))$ for such spaces. (This cohomology group arises in studying homotopy classes of self homotopy equivalences of P [HL74].)

Theorem 2.18 *Let P be a PD_3-space with torsion-free fundamental group π and orientation character $w = w_1(P)$. Then*

(1) *if π is a nontrivial free group $\pi_2(P)$ is finitely generated and of projective dimension 1 as a left $\mathbb{Z}[\pi]$-module and $H^2(P;\pi_2(P)) \cong Z$;*

(2) *if π is not free $\pi_2(P)$ is a finitely generated free $\mathbb{Z}[\pi]$-module, c.d.$\pi = 3$, $H_3(c_P;\mathbb{Z}^w)$ is a monomorphism and $H^2(P;\pi_2(P)) = 0$;*

(3) *P is homotopy equivalent to a finite PD_3-complex if and only if π is finitely presentable and FF.*

Proof As observed in §2.6 above, $\pi_2(P) \cong \overline{H^1(\pi;\mathbb{Z}[\pi])}$ as a left $\mathbb{Z}[\pi]$-module. Since π is finitely generated it is a free product of finitely many indecomposable groups, and since π is torsion-free the latter either have one end or are infinite cyclic. If π is free of rank r there is a short exact sequence of left modules

$$0 \to \mathbb{Z}[\pi]^r \to \mathbb{Z}[\pi] \to \mathbb{Z} \to 0.$$

If $r \neq 0$ then $H^0(\pi;\mathbb{Z}[\pi]) = 0$, so dualizing gives an exact sequence of right modules

$$0 \to \mathbb{Z}[\pi] \to \mathbb{Z}[\pi]^r \to H^1(\pi;\mathbb{Z}[\pi]) \to 0.$$

The exact sequence of homology with these coefficients includes the sequence

$$0 \to H_1(P;H^1(\pi;\mathbb{Z}[\pi])) \to H_0(P;\mathbb{Z}[\pi]) \to H_0(P;\mathbb{Z}[\pi]^r)$$

in which the right hand map is 0, and so $H_1(P;H^1(\pi;\mathbb{Z}[\pi])) \cong H_0(P;\mathbb{Z}[\pi]) = Z$. Hence $H^2(P;\pi_2(P)) \cong H_1(P;\overline{\pi_2(P)}) = H_1(P;H^1(\pi;\mathbb{Z}[\pi])) \cong Z$, by Poincaré duality. As π is finitely presentable and projective $\mathbb{Z}[F(r)]$-modules are free [Ba64] P is homotopy equivalent to a finite PD_3-complex.

If π is not free then it is the fundamental group of a finite graph of groups $\mathcal{G}$ in which all the vertex groups are finitely generated and have one end and all the edge groups are trivial. It follows from the Mayer-Vietoris sequences of Theorems 2.10 and 2.12 of [Bi] that $H^1(\pi;\mathbb{Z}[\pi])$ is a free right $\mathbb{Z}[\pi]$-module with basis corresponding to the edges of $\mathcal{G}$. As $H^2(P;\mathbb{Z}[\pi]) = H_1(P;\mathbb{Z}[\pi]) = 0$ and $\pi_2(P)$ is a finitely generated free module it follows that $H^2(P;\pi_2(P)) = 0$.

We may assume that P is 3-dimensional. The cellular chain complex of $\widetilde{P}$ is chain homotopy equivalent to a finitely generated projective $\mathbb{Z}[\pi]$-complex

$$0 \to C_3 \to C_2 \to C_1 \to C_0 \to 0,$$

and we may assume that C_i is free if $i \leq 2$. Then the sequences

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0$$

and

$$0 \to C_3 \to Z_2 \to \pi_2(P) \to 0$$

are exact, where Z_2 is the module of 2-cycles in C_2. Attaching 3-cells to P along a basis for $\pi_2(P)$ gives an aspherical 3-dimensional complex K with fundamental group π. The inclusion of P into K may be identified with c_P, and clearly induces monomorphisms $H_3(P;A) \to H_3(\pi;A)$ for any coefficient module A. Hence $c.d.\pi = 3$.

If π is FF there is a finite free resolution

$$0 \to D_3 \to D_2 \to D_1 \to D_0 \to \mathbb{Z} \to 0.$$

Therefore Z_2 is finitely generated and stably free, by Schanuel's Lemma. Since $\pi_2(P)$ is free $Z_2 \cong \pi_2(P) \oplus C_3$ and so C_3 is also stably free. Hence if moreover π is finitely presentable then P is homotopy equivalent to a finite PD_3-complex. The converse is clear, by the above construction of $K(\pi,1) \simeq K$. $\square$

If π is not torsion-free the projective dimension of $\pi_2(P)$ is infinite. Since π is FP_2 it is accessible, and so $\pi \cong \pi\mathcal{G}$, where $\mathcal{G}$ is a finite graph of groups with all vertex groups finite or one-ended and all edge groups finite. (See Theorem VI.6.3 of [DD].) There is an associated Mayer-Vietoris presentation

$$0 \to \oplus\mathbb{Z}[G_v\backslash\pi] \to \oplus\mathbb{Z}[G_e\backslash\pi] \to H^1(\pi;\mathbb{Z}[\pi]) \to 0,$$

where the sums involve only the *finite* vertex groups G_v (and edge groups G_e). Crisp uses an ingenious combinatorial argument based on such a presentation together with Lemma 2.10 to show that if P is indecomposable, orientable and not aspherical the vertex groups must all be finite, and so π is virtually free. He also extends Theorem 2.11 to show that the centralizer of any orientation-preserving element of finite order is finite [Cr00]. Elementary group theory then leads to the near-determination of the groups of such PD_3-complexes [Hi12]. (It is not yet clear what are the indecomposable *non*-orientable PD_3-complexes.)

Corollary 2.18.1 *Let P be a PD_3-complex. Then $\pi_2(P)$ is finitely presentable as a $\mathbb{Z}[\pi]$-module. Moreover, $H^2(P;\pi_2(P)) \cong H_1(\pi;\pi_2(P))$ is finitely generated of rank 1 if π is virtually free and is finite otherwise.*

Proof Since Crisp's Theorem implies that π is virtually torsion-free, these assertions follow from the theorem, together with an LHSSS argument. $\square$

Chapter 3

Homotopy invariants of PD_4-complexes

The homotopy type of a 4-manifold M is largely determined (through Poincaré duality) by its algebraic 2-type and orientation character. In many cases the formally weaker invariants $\pi_1(M)$, $w_1(M)$ and $\chi(M)$ already suffice. In §1 we give criteria in such terms for a degree-1 map between PD_4-complexes to be a homotopy equivalence, and for a PD_4-complex to be aspherical. We then show in §2 that if the universal covering space of a PD_4-complex is homotopy equivalent to a finite complex then it is either compact, contractible, or homotopy equivalent to S^2 or S^3. In §3 we obtain estimates for the minimal Euler characteristic of PD_4-complexes with fundamental group of cohomological dimension at most 2 and determine the second homotopy groups of PD_4-complexes realizing the minimal value. The class of such groups includes all surface groups and classical link groups, and the groups of many other (bounded) 3-manifolds. The minima are realized by s-parallelizable PL 4-manifolds. In §4 we show that if $\chi(M) = 0$ then $\pi_1(M)$ satisfies some stringent constraints, and in the final section we define the reduced intersection pairing.

3.1 Homotopy equivalence and asphericity

Many of the results of this section depend on the following lemma, in conjunction with use of the Euler characteristic to compute the rank of the surgery kernel. (Lemma 3.1 and Theorem 3.2 derive from Lemmas 2.2 and 2.3 of [Wl].)

Lemma 3.1 *Let R be a ring and C_* be a finite chain complex of projective R-modules. If $H_i(C_*) = 0$ for $i < q$ and $H^{q+1}(Hom_R(C_*, B)) = 0$ for any left R-module B then $H_q(C_*)$ is projective. If moreover $H_i(C_*) = 0$ for $i > q$ then $H_q(C_*) \oplus \bigoplus_{i\equiv q+1\ (2)} C_i \cong \bigoplus_{i\equiv q\ (2)} C_i$.*

Proof We may assume without loss of generality that $q = 0$ and $C_i = 0$ for $i < 0$. We may factor $\partial_1 : C_1 \to C_0$ through $B = \mathrm{Im}\partial_1$ as $\partial_1 = j\beta$, where β is an epimorphism and j is the natural inclusion of the submodule

B. Since $j\beta\partial_2 = \partial_1\partial_2 = 0$ and j is injective $\beta\partial_2 = 0$. Hence β is a 1-cocycle of the complex $Hom_R(C_*, B)$. Since $H^1(Hom_R(C_*, B)) = 0$ there is a homomorphism $\sigma : C_0 \to B$ such that $\beta = \sigma\partial_1 = \sigma j\beta$. Since β is an epimorphism $\sigma j = id_B$ and so B is a direct summand of C_0. This proves the first assertion.

The second assertion follows by an induction on the length of the complex. □

Theorem 3.2 *Let M and N be finite PD_4-complexes. A map $f : M \to N$ is a homotopy equivalence if and only if $\pi_1(f)$ is an isomorphism, $f^*w_1(N) = w_1(M)$, $f_*[M] = \pm[N]$ and $\chi(M) = \chi(N)$.*

Proof The conditions are clearly necessary. Suppose that they hold. Up to homotopy type we may assume that f is a cellular inclusion of finite cell complexes, and so M is a subcomplex of N. We may also identify $\pi_1(M)$ with $\pi = \pi_1(N)$. Let $C_*(M)$, $C_*(N)$ and D_* be the cellular chain complexes of $\widetilde{M}$, $\widetilde{N}$ and $(\widetilde{N}, \widetilde{M})$, respectively. Then the sequence

$$0 \to C_*(M) \to C_*(N) \to D_* \to 0$$

is a short exact sequence of finitely generated free $\mathbb{Z}[\pi]$-chain complexes.

By the projection formula $f_*(f^*a \cap [M]) = a \cap f_*[M] = \pm a \cap [N]$ for any cohomology class $a \in H^*(N; \mathbb{Z}[\pi])$. Since M and N satisfy Poincaré duality it follows that f induces split surjections on homology and split injections on cohomology. Hence $H_q(D_*)$ is the "surgery kernel" in degree $q-1$, and the duality isomorphisms induce isomorphisms from $H^r(Hom_{\mathbb{Z}[\pi]}(D_*, B))$ to $H_{6-r}(\overline{D_*} \otimes B)$, where B is any left $\mathbb{Z}[\pi]$-module. Since f induces isomorphisms on homology and cohomology in degrees ≤ 1, with any coefficients, the hypotheses of Lemma 3.1 are satisfied for the $\mathbb{Z}[\pi]$-chain complex D_*, with $q = 3$, and so $H_3(D_*) = \mathrm{Ker}(\pi_2(f))$ is projective. Moreover $H_3(D_*) \oplus \bigoplus_{i\ odd} D_i \cong \bigoplus_{i\ even} D_i$. Thus $H_3(D_*)$ is a stably free $\mathbb{Z}[\pi]$-module of rank $\chi(E, M) = \chi(M) - \chi(E) = 0$. Hence $H_3(D_*) = 0$, since group rings are weakly finite, and so f is a homotopy equivalence. □

If M and N are merely finitely dominated, rather than finite, then $H_3(D_*)$ is a finitely generated projective $\mathbb{Z}[\pi]$-module such that $H_3(D_*) \otimes_{\mathbb{Z}[\pi]} Z = 0$. If the Wall finiteness obstructions satisfy $f_*\sigma(M) = \sigma(N)$ in $\tilde{K}_0(\mathbb{Z}[\pi])$ then $H_3(D_*)$ is stably free, and the theorem remains true. The theorem holds as stated for arbitrary PD_4-spaces if π satisfies the Weak Bass Conjecture. (Similar comments apply elsewhere in this section.)

We shall see that when N is aspherical and $f = c_M$ we may drop the hypotheses that $f^*w_1(N) = w_1(M)$ and f has degree ± 1.

Corollary 3.2.1 [Ha87] Let N be orientable. Then a map $f : N \to N$ which induces automorphisms of $\pi_1(N)$ and $H_4(N;\mathbb{Z})$ is a homotopy equivalence. $\square$

Any self-map of a geometric manifold of semisimple type (e.g., an $\mathbb{H}^4$-, $\mathbb{H}^2(\mathbb{C})$- or $\mathbb{H}^2\times\mathbb{H}^2$-manifold) with nonzero degree is a homotopy equivalence [Re96].

If X is a cell complex with fundamental group π then $\pi_2(X)\cong H_2(X;\mathbb{Z}[\pi])$, by the Hurewicz Theorem for $\widetilde{X}$, and so there is an *evaluation* homomorphism $ev : H^2(X;\mathbb{Z}[\pi]) \to Hom_{\mathbb{Z}[\pi]}(\pi_2(X),\mathbb{Z}[\pi])$. The latter module may be identified with $H^0(\pi;H^2(\widetilde{X};\mathbb{Z}[\pi]))$, the π-invariant subgroup of the cohomology of $\widetilde{X}$ with coefficients $\mathbb{Z}[\pi]$.

Lemma 3.3 *Let M be a PD_4-space with fundamental group π and let $\Pi = \pi_2(M)$. Then $\Pi\cong\overline{H^2(M;\mathbb{Z}[\pi])}$ and there is an exact sequence*

$$0\to H^2(\pi;\mathbb{Z}[\pi])\to H^2(M;\mathbb{Z}[\pi]) \xrightarrow{ev} Hom_{\mathbb{Z}[\pi]}(\Pi,\mathbb{Z}[\pi])\to H^3(\pi;\mathbb{Z}[\pi])\to 0.$$

Proof This follows from the Hurewicz Theorem, Poincaré duality and the UCSS, since $H^3(M;\mathbb{Z}[\pi])\cong H_1(\widetilde{M};\mathbb{Z})=0$. $\square$

Exactness of much of this sequence can be derived without the UCSS. When π is finite the sequence reduces to the Poincaré duality isomorphism $\pi_2(M)\cong \overline{Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])}$.

Let $ev^{(2)} : H^2_{(2)}(\widetilde{M}) \to Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\ell^2(\pi))$ be the analogous evaluation defined on the *un*reduced L^2-cohomology by $ev^{(2)}(f)(z) = \Sigma f(g^{-1}z)g$ for all square summable 2-cocycles f and all 2-cycles z representing elements of $H_2(X;\mathbb{Z}[\pi])\cong\pi_2(M)$. Part of the next theorem is implicit in [Ec94].

Theorem 3.4 *Let M be a PD_4-complex with fundamental group π. Then*

(1) *if $\beta_1^{(2)}(\pi)=0$ and M is finite or π satisfies the Weak Bass Conjecture then $\chi(M)\geq 0$;*

(2) $\mathrm{Ker}(ev^{(2)})$ *is closed;*

(3) *if $\beta_2^{(2)}(M)=\beta_2^{(2)}(\pi)$ then $H^2(c_M;\mathbb{Z}[\pi]) : H^2(\pi;\mathbb{Z}[\pi])\to H^2(M;\mathbb{Z}[\pi])$ is an isomorphism.*

Proof Since M is a PD_4-complex $\beta_i^{(2)}(M)=\beta_{4-i}^{(2)}(M)$ for all i. If M is finite or π satisfies the Weak Bass Conjecture the alternating sum of the L^2-Betti numbers gives the Euler characteristic [Ec96], and so $\chi(M)=2\beta_0^{(2)}(\pi)-2\beta_1^{(2)}(\pi)+\beta_2^{(2)}(M)$. Hence $\chi(M)\geq\beta_2^{(2)}(M)\geq 0$ if $\beta_1^{(2)}(\pi)=0$.

Let $z \in C_2(\widetilde{M})$ be a 2-cycle and $f \in C_2^{(2)}(\widetilde{M})$ a square-summable 2-cocycle. As $||ev^{(2)}(f)(z)||_2 \leq ||f||_2||z||_2$, the map $f \mapsto ev^{(2)}(f)(z)$ is continuous, for fixed z. Hence if $f = lim f_n$ and $ev^{(2)}(f_n) = 0$ for all n then $ev^{(2)}(f) = 0$.

The inclusion $\mathbb{Z}[\pi] < \ell^2(\pi)$ induces a homomorphism from the exact sequence of Lemma 3.3 to the corresponding sequence with coefficients $\ell^2(\pi)$. (See §1.4 of [Ec94]. Note that we may identify $H^0(\pi; H^2(\widetilde{M}; A))$ with $Hom_{\mathbb{Z}[\pi]}(\pi_2(M), A)$ for $A = \mathbb{Z}[\pi]$ or $\ell^2(\pi)$ since $\widetilde{M}$ is 1-connected.) As $\mathrm{Ker}(ev^{(2)})$ is closed and $ev^{(2)}(\delta g)(z) = ev^{(2)}(g)(\partial z) = 0$ for any square summable 1-chain g, the homomorphism $ev^{(2)}$ factors through the reduced L^2-cohomology $\bar{H}^2_{(2)}(\widetilde{M})$. If $\beta_2^{(2)}(M) = \beta_2^{(2)}(\pi)$ the classifying map $c_M : M \to K(\pi, 1)$ induces weak isomorphisms on reduced L^2-cohomology $\bar{H}^i_{(2)}(\pi) \to \bar{H}^i_{(2)}(\widetilde{M})$ for $i \leq 2$. In particular, the image of $\bar{H}^2_{(2)}(\pi)$ is dense in $\bar{H}^2_{(2)}(\widetilde{M})$. Since $ev^{(2)}$ is trivial on $\bar{H}^2_{(2)}(\pi)$ and $\mathrm{Ker}(ev^{(2)})$ is closed it follows that $ev^{(2)} = 0$. Since the natural homomorphism from $Hom_{\mathbb{Z}[\pi]}(\pi_2(M), \mathbb{Z}[\pi])$ to $Hom_{\mathbb{Z}[\pi]}(\pi_2(M), \ell^2(\pi))$ is a monomorphism it follows that $ev = 0$ also and so $H^2(c_M; \mathbb{Z}[\pi])$ is an isomorphism. □

This gives a complete and natural criterion for asphericity (which we state as a separate theorem to retain the enumeration of the original version of this book).

Theorem 3.5 *Let M be a PD_4-complex with fundamental group π. Then M is aspherical if and only if $H^s(\pi; \mathbb{Z}[\pi]) = 0$ for $s \leq 2$ and $\beta_2^{(2)}(M) = \beta_2^{(2)}(\pi)$.*

Proof The conditions are clearly necessary. If they hold then $H^2(M; \mathbb{Z}[\pi]) \cong H^2(\pi; \mathbb{Z}[\pi]) = 0$ and so M is aspherical, by Poincaré duality. □

Is it possible to replace the hypothesis "$\beta_2^{(2)}(M) = \beta_2^{(2)}(\pi)$" by "$\beta_2(M^+) = \beta_2(\mathrm{Ker}(w_1(M)))$", where $p_+ : M^+ \to M$ is the orientation cover? It is easy to find examples to show that the homological conditions on π cannot be relaxed further.

Corollary 3.5.1 *The PD_4-complex M is finite and aspherical if and only if π is a finitely presentable PD_4-group of type FF and $\chi(M) = \chi(\pi)$.* □

If $\beta_2(\pi) \neq 0$ this follows from Theorem 3.2. For we may assume π and M are orientable, on replacing π by $K = \mathrm{Ker}(w_1(M)) \cap \mathrm{Ker}(w_1(\pi))$ and M by M_K. As $H_2(c_M; \mathbb{Z})$ is onto it is an isomorphism, so c_M has degree ± 1, by Poincaré duality. Is M always aspherical if π is a PD_4-group and $\chi(M) = \chi(\pi)$?

Corollary 3.5.2 *If $\chi(M)=\beta_1^{(2)}(\pi)=0$ and $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$ then M is aspherical and π is a PD_4-group.* $\square$

Corollary 3.5.3 *If $\pi\cong Z^r$ then $\chi(M)\geq 0$, and is 0 only if $r=1,2$ or 4.*

Proof If $r>2$ then $H^s(\pi;\mathbb{Z}[\pi])=0$ for $s\leq 2$. $\square$

Theorem 3.5 implies that if π is a PD_4-group and $\chi(M)=\chi(\pi)$ then $c_{M*}[M]$ is nonzero. If $\chi(M)>\chi(\pi)$ this need not be true. Given any finitely presentable group π there is a finite 2-complex K with $\pi_1(K)\cong\pi$. The boundary of a regular neighbourhood N of some embedding of K in $\mathbb{R}^5$ is a closed orientable 4-manifold M with $\pi_1(M)\cong\pi$. As the inclusion of M into N is 2-connected and K is a deformation retract of N the classifying map c_M factors through c_K and so induces the trivial homomorphism on homology in degrees >2. However if M and π are orientable and $\beta_2(M)<2\beta_2(\pi)$ then c_M must have nonzero degree, for the image of $H^2(\pi;\mathbb{Q})$ in $H^2(M;\mathbb{Q})$ then cannot be self-orthogonal under cup-product.

Theorem 3.6 *Let π be a PD_4-group of type FF. Then* $\mathrm{def}(\pi)<1-\frac{1}{2}\chi(\pi)$.

Proof Suppose that π has a presentation of deficiency $d\geq 1-\frac{1}{2}\chi(\pi)$, and let X be the corresponding finite 2-complex. Then $\beta_2^{(2)}(\pi)-\beta_1^{(2)}(\pi)\leq\beta_2^{(2)}(X)-\beta_1^{(2)}(\pi)=\chi(X)=1-d$. Since we also have $\beta_2^{(2)}(\pi)-2\beta_1^{(2)}(\pi)=\chi(\pi)$ and $\chi(\pi)\geq 2-2d$ it follows that $\beta_1^{(2)}(\pi)\leq d-1$. Hence $\beta_2^{(2)}(X)=0$. Therefore X is aspherical, by Theorem 2.4, and so $c.d.\pi\leq 2$. But this contradicts the hypothesis that π is a PD_4-group. $\square$

Note that if $\chi(\pi)$ is odd the conclusion does not imply that $\mathrm{def}(\pi)\leq-\frac{1}{2}\chi(\pi)$. An old conjecture of H.Hopf asserts that if M is an aspherical smooth $2k$-manifold then $(-1)^k\chi(M)\geq 0$. The first open case is when $k=2$. If Hopf's conjecture is true then $\mathrm{def}(\pi_1(M))\leq 0$. Is $\mathrm{def}(\pi)\leq 0$ for every PD_4-group π? This bound is best possible for groups with $\chi=0$, since the presentation $\langle a,b\mid ba^2=a^3b^2,\ b^2a=a^2b^3\rangle$ gives a Cappell-Shaneson 2-knot group $Z^3\rtimes_A Z$.

The hypothesis on orientation characters in Theorem 3.2 is often redundant.

Theorem 3.7 *Let $f:M\to N$ be a 2-connected map between finite PD_4-complexes with $\chi(M)=\chi(N)$. If $H^2(N;\mathbb{F}_2)\neq 0$ then $f^*w_1(N)=w_1(M)$, and if moreover N is orientable and $H^2(N;\mathbb{Q})\neq 0$ then f is a homotopy equivalence.*

Proof Since f is 2-connected $H^2(f;\mathbb{F}_2)$ is injective, and since $\chi(M)=\chi(N)$ it is an isomorphism. Since $H^2(N;\mathbb{F}_2)\neq 0$, the nondegeneracy of Poincaré duality implies that $H^4(f;\mathbb{F}_2)\neq 0$, and so f is a $\mathbb{F}_2$-(co)homology equivalence. Since $w_1(M)$ is characterized by the Wu formula $x\cup w_1(M)=Sq^1x$ for all x in $H^3(M;\mathbb{F}_2)$, it follows that $f^*w_1(N)=w_1(M)$.

If $H^2(N;\mathbb{Q})\neq 0$ then $H^2(N;\mathbb{Z})$ has positive rank and $H^2(N;\mathbb{F}_2)\neq 0$, so N orientable implies M orientable. We may then repeat the above argument with integral coefficients, to conclude that f has degree ± 1. The result then follows from Theorem 3.2. □

The argument breaks down if, for instance, $M=S^1\tilde{\times}S^3$ is the nonorientable S^3-bundle over S^1, $N=S^1\times S^3$ and f is the composite of the projection of M onto S^1 followed by the inclusion of a factor.

We would like to replace the hypotheses above that there be a map $f:M\to N$ realizing certain isomorphisms by weaker, more algebraic conditions. If M and N are closed 4-manifolds with isomorphic algebraic 2-types then there is a 3-connected map $f:M\to P_2(N)$. The restriction of such a map to $M_o=M\setminus D^4$ is homotopic to a map $f_o:M_o\to N$ which induces isomorphisms on π_i for $i\leq 2$. In particular, $\chi(M)=\chi(N)$. Thus if f_o extends to a map from M to N we may be able to apply Theorem 3.2. However we usually need more information on how the top cell is attached. The characteristic classes and the equivariant intersection pairing on $\pi_2(M)$ are the obvious candidates.

The following criterion arises in studying the homotopy types of circle bundles over 3-manifolds. (See Chapter 4.)

Theorem 3.8 *Let E be a PD_4-complex with fundamental group π and such that $H_4(f_E;Z^{w_1(E)})$ is a monomorphism. A PD_4-complex M is homotopy equivalent to E if and only if there is an isomorphism θ from $\pi_1(M)$ to π such that $w_1(M)=w_1(E)\theta$, there is a lift $\hat{c}:M\to P_2(E)$ of θc_M such that $\hat{c}_*[M]=\pm f_{E*}[E]$ and $\chi(M)=\chi(E)$.*

Proof The conditions are clearly necessary. Conversely, suppose that they hold. We shall adapt to our situation the arguments of Hendriks in analyzing the obstructions to the existence of a degree 1 map between PD_3-complexes realizing a given homomorphism of fundamental groups. For simplicity of notation we shall write $\tilde{Z}$ for $Z^{w_1(E)}$ and also for $Z^{w_1(M)}(=\theta^*\tilde{Z})$, and use θ to identify $\pi_1(M)$ with π and $K(\pi_1(M),1)$ with $K(\pi,1)$. We may suppose the sign of the fundamental class $[M]$ is so chosen that $\hat{c}_*[M]=f_{E*}[E]$.

Let $E_o = E \setminus D^4$. Then $P_2(E_o) = P_2(E)$ and may be constructed as the union of E_o with cells of dimension ≥ 4. Let

$$h : \tilde{Z} \otimes_{\mathbb{Z}[\pi]} \pi_4(P_2(E_o), E_o) \to H_4(P_2(E_o), E_o; \tilde{Z})$$

be the $w_1(E)$-twisted relative Hurewicz homomorphism, and let ∂ be the connecting homomorphism from $\pi_4(P_2(E_o), E_o)$ to $\pi_3(E_o)$ in the exact sequence of homotopy for the pair $(P_2(E_o), E_o)$. Then h and ∂ are isomorphisms since f_{E_o} is 3-connected, and so the homomorphism $\tau_E : H_4(P_2(E); \tilde{Z}) \to \tilde{Z} \otimes_{\mathbb{Z}[\pi]} \pi_3(E_o)$ given by the composite of the inclusion

$$H_4(P_2(E); \tilde{Z}) = H_4(P_2(E_o); \tilde{Z}) \to H_4(P_2(E_o), E_o; \tilde{Z})$$

with h^{-1} and $1 \otimes_{\mathbb{Z}[\pi]} \partial$ is a monomorphism. Similarly $M_o = M \setminus D^4$ may be viewed as a subspace of $P_2(M_o)$ and there is a monomorphism τ_M from $H_4(P_2(M); \tilde{Z})$ to $\tilde{Z} \otimes_{\mathbb{Z}[\pi]} \pi_3(M_o)$. These monomorphisms are natural with respect to maps defined on the 3-skeleta (i.e., E_o and M_o).

The classes $\tau_E(f_{E*}[E])$ and $\tau_M(f_{M*}[M])$ are the images of the primary obstructions to retracting E onto E_o and M onto M_o, under the Poincaré duality isomorphisms from $H^4(E, E_o; \pi_3(E_o))$ to $H_0(E \setminus E_o; \tilde{Z} \otimes_{\mathbb{Z}[\pi]} \pi_3(E_o)) = \tilde{Z} \otimes_{\mathbb{Z}[\pi]} \pi_3(E_o)$ and $H^4(M, M_o; \pi_3(M_o))$ to $\tilde{Z} \otimes_{\mathbb{Z}[\pi]} \pi_3(M_o)$, respectively. Since M_o is homotopy equivalent to a cell complex of dimension ≤ 3 the restriction of $\hat{c}$ to M_o is homotopic to a map from M_o to E_o. Let $\hat{c}_\sharp$ be the homomorphism from $\pi_3(M_o)$ to $\pi_3(E_o)$ induced by $\hat{c}|M_o$. Then $(1 \otimes_{\mathbb{Z}[\pi]} \hat{c}_\sharp)\tau_M(f_{M*}[M]) = \tau_E(f_{E*}[E])$. It follows as in [Hn77] that the obstruction to extending $\hat{c}|M_o : M_o \to E_o$ to a map d from M to E is trivial.

Since $f_{E*}d_*[M] = \hat{c}_*[M] = f_{E*}[E]$ and f_{E*} is a monomorphism in degree 4 the map d has degree 1, and so is a homotopy equivalence, by Theorem 3.2. □

If there is such a lift $\hat{c}$ then $c_M^*\theta^* k_1(E) = 0$ and $\theta_* c_{M*}[M] = c_{E*}[E]$.

3.2 Finitely dominated covering spaces

In this section we shall show that if a PD_4-complex M has a finitely dominated, infinite regular covering space then either M is aspherical or its universal covering space is homotopy equivalent to S^2 or S^3. In Chapters 4 and 5 we shall see that such manifolds are close to being total spaces of fibre bundles.

Theorem 3.9 *Let M be a PD_4-complex with fundamental group π, and let M_ν be the covering space associated to $\nu = \mathrm{Ker}(p)$, where $p : \pi \to G$ is an epimorphism. Suppose that M_ν is finitely dominated. Then*

(1) *G has finitely many ends;*

(2) *if M_ν is acyclic then it is contractible and M is aspherical;*

(3) *if G has one end and ν is infinite and FP_3 then M is aspherical and M_ν is homotopy equivalent to an aspherical closed surface or to S^1;*

(4) *if G has one end and ν is finite but M_ν is not acyclic then $M_\nu \simeq S^2$ or RP^2;*

(5) *G has two ends if and only if M_ν is a PD_3-complex.*

Proof We may clearly assume that G is infinite. As $\mathbb{Z}[G]$ has no nonzero left ideal (i.e., submodule) which is finitely generated as an abelian group $Hom_{\mathbb{Z}[G]}(H_q(M_\nu;\mathbb{Z}),\mathbb{Z}[G]) = 0$ for all $q \geq 0$, and so the bottom row of the UCSS for the covering p is 0. From Poincaré duality and the UCSS we find that $H_4(M_\nu;\mathbb{Z}) = H^0(G;\mathbb{Z}[G]) = 0$ and $H^1(G;\mathbb{Z}[G]) \cong \overline{H_3(M_\nu;\mathbb{Z})}$. As this group is finitely generated, and as G is infinite, G has one or two ends. Similarly, $H^2(G;\mathbb{Z}[G])$ is finitely generated and so $H^2(G;\mathbb{Z}[G]) \cong Z$ or 0.

If M_ν is acyclic $D_* = \mathbb{Z}[G] \otimes_{\mathbb{Z}[\pi]} C_*(\widetilde{M})$ is a resolution of the augmentation $\mathbb{Z}[G]$-module $\mathbb{Z}$ and $H^q(D_*) \cong \overline{H_{4-q}(M_\nu;\mathbb{Z})}$. Hence G is a PD_4-group, and so $H_s(\widetilde{M};\mathbb{Z}) = H_s(M_\nu;\mathbb{Z}[\nu]) = H^{-s}(M_\nu;\mathbb{Z}[\nu]) = 0$ for $s > 0$, by Theorem 1.19$'$. Thus M_ν is contractible and so M is aspherical.

Suppose that G has one end. If $H^2(G;\mathbb{Z}[G]) \cong Z$ then G is virtually a PD_2-group, by Bowditch's Theorem, and so M_ν is a PD_2-complex, by [Go79]. In general, $C_*(\widetilde{M})|_\nu$ is chain homotopy equivalent to a finitely generated projective $\mathbb{Z}[\nu]$-chain complex P_* and $H_3(M_\nu;\mathbb{Z}) = H_4(M_\nu;\mathbb{Z}) = 0$. If ν is FP_3 then the augmentation $\mathbb{Z}[\nu]$-module Z has a free resolution F_* which is finitely generated in degrees ≤ 3. On applying Schanuel's Lemma to the exact sequences

$$0 \to Z_2 \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

and

$$0 \to \partial F_3 \to F_2 \to F_1 \to F_0 \to \mathbb{Z} \to 0$$

derived from these two chain complexes we find that Z_2 is finitely generated as a $\mathbb{Z}[\nu]$-module. Hence $\Pi = \pi_2(M) = \pi_2(M_\nu)$ is also finitely generated as a $\mathbb{Z}[\nu]$-module and so $Hom_\pi(\Pi,\mathbb{Z}[\pi]) = 0$. If moreover ν is infinite then $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, so $\Pi = 0$, by Lemma 3.3, and M is aspherical. If $H^2(G;\mathbb{Z}[G]) = 0$ a spectral sequence corner argument then shows that $H^3(G;\mathbb{Z}[G]) \cong Z$ and $M_\nu \simeq S^1$. (See the following theorem.)

If ν is finite but M_ν is not acyclic then the universal covering space $\widetilde{M}$ is also finitely dominated but not contractible, and $\Pi = H_2(\widetilde{M};\mathbb{Z})$ is a nontrivial finitely generated abelian group, while $H_3(\widetilde{M};\mathbb{Z}) = H_4(\widetilde{M};\mathbb{Z}) = 0$. If C is a finite cyclic subgroup of π there are isomorphisms $H_{n+3}(C;\mathbb{Z}) \cong H_n(C;\Pi)$, for all $n \geq 4$, by Lemma 2.10. Suppose that C acts trivially on Π. Then if n is odd this isomorphism reduces to $0 = \Pi/|C|\Pi$. Since Π is finitely generated, this implies that multiplication by $|C|$ is an isomorphism. On the other hand, if n is even we have $Z/|C|Z \cong \{a \in \Pi \mid |C|a = 0\}$. Hence we must have $C = 1$. Now since Π is finitely generated any torsion subgroup of $Aut(\Pi)$ is finite. (Let T be the torsion subgroup of Π and suppose that $\Pi/T \cong Z^r$. Then the natural homomorphism from $Aut(\Pi)$ to $Aut(\Pi/T)$ has finite kernel, and its image is isomorphic to a subgroup of $GL(r,\mathbb{Z})$, which is virtually torsion-free.) Hence as π is infinite it must have elements of infinite order. Since $H^2(\pi;\mathbb{Z}[\pi]) \cong \bar{\Pi}$, by Lemma 3.3, it is a finitely generated abelian group. Therefore it must be infinite cyclic, by Corollary 5.2 of [Fa74]. Hence $\widetilde{M} \simeq S^2$ and ν has order at most 2, so $M_\nu \simeq S^2$ or RP^2.

Suppose now that M_ν is a PD_3-complex. After passing to a finite covering of M, if necessary, we may assume that M_ν is orientable. Then $H^1(G;\mathbb{Z}[G]) \cong \overline{H_3(M_\nu;\mathbb{Z})}$, and so G has two ends. Conversely, if G has two ends we may assume that $G \cong Z$, after passing to a finite covering of M, if necessary. Hence M_ν is a PD_3-complex, by [Go79]. $\square$

The hypotheses that M be a PD_4-complex and M_ν be finitely dominated can be relaxed to requiring that M be a PD_4-space and $C_*(\widetilde{M})$ be $\mathbb{Z}[\nu]$-finitely dominated, and the appeal to [Go79] can be avoided. (See Theorem 4.1.) It can be shown that the hypothesis in (3) that ν be FP_3 is redundant if M is a *finite* PD_4-space. (See [Hi13b].)

Corollary 3.9.1 *The covering space M_ν is homotopy equivalent to a closed surface if and only if it is finitely dominated and $H^2(G;\mathbb{Z}[G]) \cong Z$.* $\square$

In this case M has a finite covering space which is homotopy equivalent to the total space of a surface bundle over an aspherical closed surface. (See Chapter 5.)

Corollary 3.9.2 *The covering space M_ν is homotopy equivalent to S^1 if and only if it is finitely dominated, G has one end, $H^2(G;\mathbb{Z}[G]) = 0$ and ν is a nontrivial finitely generated free group.*

Proof If $M_\nu \simeq S^1$ then it is finitely dominated and M is aspherical, and the conditions on G follow from the LHSSS. The converse follows from part (3) of Theorem 3.9, since ν is infinite and FP. $\square$

In fact any finitely generated free normal subgroup F of a PD_n-group π must be infinite cyclic. For $\pi/FC_\pi(F)$ embeds in $Out(F)$, so $v.c.d.\pi/FC_\pi(F) \leq v.c.d.Out(F(r)) < \infty$. If F is nonabelian then $C_\pi(F) \cap F = 1$ and so π/F is an extension of $\pi/FC_\pi(F)$ by $C_\pi(F)$. Hence $v.c.d.\pi/F < \infty$. Since F is finitely generated π/F is FP_∞. Hence we may apply Theorem 9.11 of [Bi], and an LHSSS corner argument gives a contradiction.

In the simply connected case "finitely dominated", "homotopy equivalent to a finite complex" and "having finitely generated homology" are all equivalent.

Corollary 3.9.3 *If $H_*(\widetilde{M};\mathbb{Z})$ is finitely generated then either M is aspherical or $\widetilde{M}$ is homotopy equivalent to S^2 or S^3 or $\pi_1(M)$ is finite.* □

This was first stated (for $\pi_1(M)$ satisfying a homological finiteness condition) in [Ku78]. We shall examine the spherical cases more closely in Chapters 10 and 11. (The arguments in these chapters may apply also to PD_n-complexes with universal covering space homotopy equivalent to S^{n-1} or S^{n-2}. The analogues in higher codimensions appear to be less accessible.)

The following variation on the aspherical case shall be used in Theorem 4.8, but belongs naturally here.

Theorem 3.10 *Let ν be a nontrivial FP_3 normal subgroup of infinite index in a PD_4-group π, and let $G = \pi/\nu$. Then either*

(1) *ν is a PD_3-group and G has two ends;*

(2) *ν is a PD_2-group and G is virtually a PD_2-group; or*

(3) *$\nu \cong Z$, $H^s(G;\mathbb{Z}[G]) = 0$ for $s \neq 3$ and $H^3(G;\mathbb{Z}[G]) \cong Z$.*

Proof Since $c.d.\nu < 4$, by Strebel's Theorem, ν is FP and hence G is FP_∞. The E_2 terms of the LHSSS with coefficients $\mathbb{Q}[\pi]$ can then be expressed as $E_2^{pq} = H^p(G;\mathbb{Q}[G]) \otimes H^q(\nu;\mathbb{Q}[\nu])$. If $H^j(G;\mathbb{Q}[G])$ and $H^k(\nu;\mathbb{Q}[\nu])$ are the first nonzero such cohomology groups then E_2^{jk} persists to E_∞ and hence $j+k=4$. Therefore $H^j(G;\mathbb{Q}[G]) \otimes H^{4-j}(\nu;\mathbb{Q}[\nu]) \cong Q$, and so $H^j(G;\mathbb{Q}[G]) \cong H^{4-j}(\nu;\mathbb{Q}[\nu]) \cong Q$. If G has two ends it is virtually Z, and then ν is a PD_3-group, by Theorem 9.11 of [Bi]. If $H^2(\nu;\mathbb{Q}[\nu]) \cong H^2(G;\mathbb{Q}[G]) \cong Q$ then ν and G are virtually PD_2-groups, by Bowditch's Theorem. Since ν is torsion-free it is then a PD_2-group. The only remaining possibility is (3). □

In case (1) π has a subgroup of index ≤ 2 which is a semidirect product $H \rtimes_\theta Z$ with $\nu \leq H$ and $[H:\nu] < \infty$. Is it sufficient that ν be FP_2? Must the quotient π/ν be virtually a PD_3-group in case (3)?

Corollary 3.10.1 *If K is FP_2 and is ascendant in ν where ν is an FP_3 normal subgroup of infinite index in the PD_4-group π then K is a PD_k-group for some $k < 4$.*

Proof This follows from Theorem 3.10 together with Theorem 2.17. □

What happens if we drop the hypothesis that the covering be regular? It follows easily from Theorem 2.18 that a PD_3-complex has a finitely dominated infinite covering space if and only if its fundamental group has one or two ends [Hi08]. We might conjecture that if a PD_4-complex M has a finitely dominated infinite covering space $\widehat{M}$ then either M is aspherical or $\widetilde{M}$ is homotopy equivalent to S^2 or S^3 or M has a finite covering space which is homotopy equivalent to the mapping torus of a self homotopy equivalence of a PD_3-complex. (In particular, $\pi_1(M)$ has one or two ends.) In [Hi08] we extend the arguments of Theorem 3.9 to show that if $\pi_1(\widehat{M})$ is FP_3 and ascendant in π the only other possibility is that $\pi_1(\widehat{M})$ has two ends, $h(\sqrt{\pi}) = 1$ and $H^2(\pi;\mathbb{Z}[\pi])$ is not finitely generated. This paper also considers in more detail FP ascendant subgroups of PD_4-groups, corresponding to the aspherical case.

3.3 Minimizing the Euler characteristic

It is well known that every finitely presentable group is the fundamental group of some closed orientable 4-manifold. Such manifolds are far from unique, for the Euler characteristic may be made arbitrarily large by taking connected sums with simply connected manifolds. Following Hausmann and Weinberger [HW85], we may define an invariant $q(\pi)$ for any finitely presentable group π by

$$q(\pi) = \min\{\chi(M) | M \textit{ is a } PD_4 \textit{ complex with } \pi_1(M) \cong \pi\}.$$

We may also define related invariants q^X where the minimum is taken over the class of PD_4-complexes whose normal fibration has an X-reduction. There are the following basic estimates for q^{SG}, which is defined in terms of PD_4^+-complexes.

Lemma 3.11 *Let π be a finitely presentable group with a subgroup H of finite index and let F be a field. Then*

(1) $1 - \beta_1(H;F) + \beta_2(H;F) \leq [\pi : H](1 - \mathrm{def}\pi)$;

(2) $2 - 2\beta_1(H;F) + \beta_2(H;F) \leq [\pi : H]q^{SG}(\pi)$;

(3) $q^{SG}(\pi) \leq 2(1 - \mathrm{def}(\pi))$;

(4) *if* $H^4(\pi; F) = 0$ *then* $q^{SG}(\pi) \geq 2(1-\beta_1(\pi; F)+\beta_2(\pi; F))$, *and if moreover* $H^4(\pi; \mathbb{F}_2) = 0$ *then* $q(\pi) \geq 2(1 - \beta_1(\pi; \mathbb{F}_2) + \beta_2(\pi; \mathbb{F}_2))$ *also.*

Proof Let C be the 2-complex corresponding to a presentation for π of maximal deficiency and let C_H be the covering space associated to the subgroup H. Then $\chi(C) = 1 - \mathrm{def}\pi$ and $\chi(C_H) = [\pi : H]\chi(\pi)$. Condition (1) follows since $\beta_1(H; F) = \beta_1(C_H; F)$ and $\beta_2(H; F) \leq \beta_2(C_H; F)$.

Condition (2) follows similarly on considering the Euler characteristics of a PD_4^+-complex M with $\pi_1(M) \cong \pi$ and of the associated covering space M_H.

The boundary of a regular neighbourhood of a PL embedding of C in R^5 is a closed orientable 4-manifold realizing the upper bound in (3).

The image of $H^2(\pi; F)$ in $H^2(M; F)$ has dimension $\beta_2(\pi; F)$, and is self-annihilating under cup-product if $H^4(\pi; F) = 0$. In that case $\beta_2(M; F) \geq 2\beta_2(\pi; F)$, which implies the first part of (4). The final observation follows since all PD_n-complexes are orientable over $\mathbb{F}_2$. □

Condition (2) was used in [HW85] to give examples of finitely presentable superperfect groups which are not fundamental groups of homology 4-spheres. (See Chapter 14 below.)

If π is a finitely presentable, orientable PD_4-group we see immediately that $q^{SG}(\pi) \geq \chi(\pi)$. Multiplicativity then implies that $q(\pi) = \chi(\pi)$ if $K(\pi, 1)$ is a finite PD_4-complex.

For groups of cohomological dimension at most 2 we can say more.

Theorem 3.12 *Let* X *be a* PD_4*-complex with fundamental group* π *such that c.d.*$\pi \leq 2$, *and let* $C_* = C_*(X; \mathbb{Z}[\pi])$. *Then*

(1) C_* *is* $\mathbb{Z}[\pi]$*-chain homotopy equivalent to* $D_* \oplus L[2] \oplus D^{4-*}$, *where* D_* *is a projective resolution of* $\mathbb{Z}$, $L[2]$ *is a finitely generated projective module* L *concentrated in degree* 2 *and* D^{4-*} *is the conjugate dual of* D_*, *shifted to terminate in degree* 2;

(2) $\pi_2(X) \cong L \oplus \overline{H^2(\pi; \mathbb{Z}[\pi])}$;

(3) $\chi(X) \geq 2\chi(\pi)$, *with equality if and only if* $L = 0$;

(4) $Hom_{\mathbb{Z}[\pi]}(\overline{H^2(\pi; \mathbb{Z}[\pi])}, \mathbb{Z}[\pi]) = 0$.

Proof The chain complex C_* gives a resolution of the augmentation module

$$0 \to \mathrm{Im}(\partial_2^C) \to C_1 \to C_0 \to \mathbb{Z} \to 0.$$

Let D_* be the corresponding chain complex with $D_0 = C_0$, $D_1 = C_1$ and $D_2 = \mathrm{Im}(\partial_2^C)$. Since $c.d.\pi \leq 2$ and D_0 and D_1 are projective modules D_2 is projective, by Schanuel's Lemma. Therefore the epimorphism from C_2 to D_2 splits, and so C_* is a direct sum $C_* \cong D_* \oplus (C/D)_*$. Since X is a PD_4-complex C_* is chain homotopy equivalent to C^{4-*}. The first two assertions follow easily.

On taking homology with simple coefficients $\mathbb{Q}$, we see that $\chi(X) = 2\chi(\pi) + \dim_{\mathbb{Q}} \mathbb{Q} \otimes_\pi L$. Hence $\chi(X) \geq 2\chi(\pi)$. Since π satisfies the Weak Bass Conjecture [Ec86] and L is projective, $L = 0$ if and only if $\dim_{\mathbb{Q}} \mathbb{Q} \otimes_\pi L = 0$.

Let $\delta : D_2 \to D_1$ be the inclusion. Then $\overline{\overline{H^2(\pi;\mathbb{Z}[\pi])} = \mathrm{Cok}(\delta^\dagger)}$, where $\delta^\dagger$ is the conjugate transpose of δ. Hence $\overline{Hom_{\mathbb{Z}[\pi]}(\overline{H^2(\pi;\mathbb{Z}[\pi])}, \mathbb{Z}[\pi])} = \mathrm{Ker}(\delta^{\dagger\dagger})$. But $\delta^{\dagger\dagger} = \delta$, which is injective, and so $Hom_{\mathbb{Z}[\pi]}(\overline{H^2(\pi;\mathbb{Z}[\pi])}, \mathbb{Z}[\pi]) = 0$. □

The appeal to the Weak Bass Conjecture may be avoided if X and $K(\pi,1)$ are homotopy equivalent to finite complexes. For then L is stably free, and so is 0 if and only if $\mathbb{Z} \otimes_{\mathbb{Z}[\pi]} L = 0$, since group rings are weakly finite.

Similar arguments may be used to prove the following variation.

Addendum *Suppose that $c.d._R\pi \leq 2$ for some ring R. Then $R \otimes \pi_2(M) \cong P \oplus \overline{H^2(\pi;R[\pi])}$, where P is a projective $R[\pi]$-module, and $\chi(M) \geq 2\chi(\pi;R) = 2(1 - \beta_1(\pi;R) + \beta_2(\pi;R))$. If R is a subring of $\mathbb{Q}$ then $\chi(M) = 2\chi(\pi;R)$ if and only if $\pi_2(M) \cong \overline{H^2(\pi;\mathbb{Z}[\pi])}$.* □

There are many natural examples of 4-manifolds with $\pi_1(M) = \pi$ having nontrivial torsion and such that $c.d._{\mathbb{Q}}\pi \leq 2$ and $\chi(M) = 2\chi(\pi)$. (See Chapters 10 and 11.) However all the known examples satisfy $v.c.d.\pi \leq 2$.

Corollary 3.12.1 *If $H_2(\pi;\mathbb{F}_2) \neq 0$ the Hurewicz homomorphism from $\pi_2(M)$ to $H_2(M;\mathbb{F}_2)$ is nonzero.*

Proof By the addendum to the theorem, $H_2(M;\mathbb{F}_2)$ has dimension at least $2\beta_2(\pi)$, and so cannot be isomorphic to $H_2(\pi;\mathbb{F}_2)$ unless both are 0. □

Corollary 3.12.2 *If $\pi = \pi_1(P)$ where P is an aspherical finite 2-complex then $q(\pi) = 2\chi(P)$. The minimum is realized by an s-parallelizable PL 4-manifold.*

Proof If we choose a PL embedding $j : P \to \mathbb{R}^5$, the boundary of a regular neighbourhood N of $j(P)$ is an s-parallelizable PL 4-manifold with fundamental group π and with Euler characteristic $2\chi(P)$. □

By Theorem 2.8 a finitely presentable group is the fundamental group of an aspherical finite 2-complex if and only if it has cohomological dimension ≤ 2 and is efficient, i.e. has a presentation of deficiency $\beta_1(\pi;\mathbb{Q}) - \beta_2(\pi;\mathbb{Q})$. It is not known whether every finitely presentable group of cohomological dimension 2 is efficient.

In Chapter 5 we shall see that if P is an aspherical closed surface and M is a closed 4-manifold with $\pi_1(M) \cong \pi$ then $\chi(M) = q(\pi)$ if and only if M is homotopy equivalent to the total space of an S^2-bundle over P. The homotopy types of such minimal 4-manifolds for π may be distinguished by their Stiefel-Whitney classes. Note that if π is orientable then $S^2 \times P$ is a minimal 4-manifold for π which is both s-parallelizable and also a projective algebraic complex surface. Note also that the conjugation of the module structure in the theorem involves the orientation character of M which may differ from that of the PD_2-group π.

Corollary 3.12.3 *If π is the group of an unsplittable μ-component 1-link then $q(\pi) = 0$.* □

If π is the group of a μ-component n-link with $n \geq 2$ then $H_2(\pi;\mathbb{Q}) = 0$ and so $q(\pi) \geq 2(1-\mu)$, with equality if and only if π is the group of a 2-link. (See Chapter 14.)

Corollary 3.12.4 *If π is an extension of Z by a finitely generated free normal subgroup then $q(\pi) = 0$.* □

In Chapter 4 we shall see that if M is a closed 4-manifold with $\pi_1(M)$ such an extension then $\chi(M) = q(\pi)$ if and only if M is homotopy equivalent to a manifold which fibres over S^1 with fibre a closed 3-manifold with free fundamental group, and then π and $w_1(M)$ determine the homotopy type.

Finite generation of the normal subgroup is essential; $F(2)$ is an extension of Z by $F(\infty)$, and $q(F(2)) = 2\chi(F(2)) = -2$.

Let π be the fundamental group of a closed orientable 3-manifold. Then $\pi \cong F * \nu$ where F is free of rank r and ν has no infinite cyclic free factors. Moreover $\nu = \pi_1(N)$ for some closed orientable 3-manifold N. If M_0 is the closed 4-manifold obtained by surgery on $\{n\} \times S^1$ in $N \times S^1$ then $M = M_0 \sharp(\sharp^r(S^1 \times S^3)$

is a smooth s-parallelisable 4-manifold with $\pi_1(M) \cong \pi$ and $\chi(M) = 2(1-r)$. Hence $q^{SG}(\pi) = 2(1-r)$, by part (4) of Lemma 3.11.

The arguments of Theorem 3.12 give stronger results in this case also.

Theorem 3.13 *Let π be a finitely presentable PD_3-group, and let M be a PD_4-complex with fundamental group π and $w_1(\pi) = w_1(M)$. Then $q(\pi) = 2$, and there are finitely generated projective $\mathbb{Z}[\pi]$-modules P and P' such that $\pi_2(M) \oplus P \cong A(\pi) \oplus P'$, where $A(\pi)$ is the augmentation ideal of $\mathbb{Z}[\pi]$.*

Proof Let N be a PD_3-complex with fundamental group π. We may suppose that $N = N_o \cup D^3$, where $N_o \cap D^3 = S^2$. Let $M = N_o \times S^1 \cup S^2 \times D^2$. Then M is a PD_4-complex, $\chi(M) = 2$ and $\pi_1(M) \cong \pi$. Hence $q(\pi) \leq 2$. On the other hand, $q(\pi) \geq 2$ by part (4) of Lemma 3.11, and so $q(\pi) = 2$.

For any left $\mathbb{Z}[\pi]$-module N let $e^i N = Ext^i_{\mathbb{Z}[\pi]}(N, \mathbb{Z}[\pi])$, to simplify the notation. The cellular chain complex for the universal covering space of M gives exact sequences

$$0 \to C_4 \to C_3 \to Z_2 \to H_2 \to 0 \tag{3.1}$$

and

$$0 \to Z_2 \to C_2 \to C_1 \to C_0 \to \mathbb{Z} \to 0. \tag{3.2}$$

Since π is a PD_3-group the augmentation module $\mathbb{Z}$ has a finite projective resolution of length 3. On comparing sequence 3.2 with such a resolution and applying Schanuel's lemma we find that Z_2 is a finitely generated projective $\mathbb{Z}[\pi]$-module. Since π has one end, the UCSS reduces to an exact sequence

$$0 \to H^2 \to e^0 H_2 \to e^3\mathbb{Z} \to H^3 \to e^1 H_2 \to 0 \tag{3.3}$$

and isomorphisms $H^4 \cong e^2 H_2$ and $e^3 H_2 = e^4 H_2 = 0$. Poincaré duality implies that $H^3 = 0$ and $H^4 \cong \overline{\mathbb{Z}}$. Hence sequence 3.3 reduces to

$$0 \to H^2 \to e^0 H_2 \to e^3\mathbb{Z} \to 0 \tag{3.4}$$

and $e^1 H_2 = 0$. On dualizing the sequence 3.1 and conjugating we get an exact sequence of left modules

$$0 \to \overline{e^0 H_2} \to \overline{e^0 Z_2} \to \overline{e^0 C_3} \to \overline{e^0 C_4} \to \overline{e^2 H_2} \cong \mathbb{Z} \to 0. \tag{3.5}$$

Schanuel's lemma again implies that $\overline{e^0 H_2}$ is a finitely generated projective module. Now $\pi_2(M) \cong \overline{H^2}$, by Poincaré duality, and $\overline{e^3\mathbb{Z}} \cong \mathbb{Z}$, since π is a PD_3-group and $w_1(M) = w_1(\pi)$. Hence the final assertion follows from sequence 3.4 and Schanuel's Lemma. ∎

Can Theorem 3.13 be extended to all torsion-free 3-manifold groups, or more generally to all free products of PD_3-groups?

There has been some related work estimating the difference $\chi(M) - |\sigma(M)|$ where M is a closed orientable 4-manifold M with $\pi_1(M) \cong \pi$ and where $\sigma(M)$ is the signature of M. In particular, this difference is always ≥ 0 if $\beta_1^{(2)}(\pi) = 0$. (See [JK93] and §3 of Chapter 7 of [Lü].) The minimum value of this difference ($p(\pi) = \min\{\chi(M) - |\sigma(M)|\}$) is another numerical invariant of π, which is studied in [Ko94].

3.4 Euler Characteristic 0

In this section we shall consider the interaction of the fundamental group and Euler characteristic from another point of view. We shall assume that $\chi(M) = 0$ and show that if π is an ascending HNN extension then it satisfies some very stringent conditions. The groups $Z*_m$ shall play an important role. We shall approach our main result via several lemmas.

We begin with a simple observation relating Euler characteristic and fundamental group which shall be invoked in several of the later chapters. Recall that if G is a group then $I(G)$ is the minimal normal subgroup such that $G/I(G)$ is free abelian.

Lemma 3.14 *Let M be a PD_4-space with $\chi(M) \leq 0$. If M is orientable then $H^1(M;\mathbb{Z}) \neq 0$ and so $\pi = \pi_1(M)$ maps onto Z. If $H^1(M;\mathbb{Z}) = 0$ then π maps onto D.*

Proof The covering space M_W corresponding to $W = \mathrm{Ker}(w_1(M))$ is orientable and $\chi(M_W) = 2 - 2\beta_1(M_W) + \beta_2(M_W) = [\pi : W]\chi(M) \leq 0$. Therefore $\beta_1(W) = \beta_1(M_W) > 0$ and so $W/I(W) \cong Z^r$ for some $r > 0$. Since $I(W)$ is characteristic in W it is normal in π. As $[\pi : W] \leq 2$ it follows easily that $\pi/I(W)$ maps onto Z or D. □

Note that if $M = RP^4 \sharp RP^4$, then $\chi(M) = 0$ and $\pi_1(M) \cong D$, but $\pi_1(M)$ does not map onto Z.

Lemma 3.15 *Let M be a PD_4^+-complex such that $\chi(M) = 0$ and $\pi = \pi_1(M)$ is an extension of $Z*_m$ by a finite normal subgroup F, for some $m \neq 0$. Then the abelian subgroups of F are cyclic. If $F \neq 1$ then π has a subgroup of finite index which is a central extension of $Z*_n$ by a nontrivial finite cyclic group, where n is a power of m.*

Proof Let $\widehat{M}$ be the infinite cyclic covering space corresponding to the subgroup $I(\pi)$. Since M is compact and $\Lambda = \mathbb{Z}[Z]$ is noetherian the groups $H_i(\widehat{M};\mathbb{Z}) = H_i(M;\Lambda)$ are finitely generated as Λ-modules. Since M is orientable, $\chi(M) = 0$ and $H_1(M;\mathbb{Z})$ has rank 1 they are Λ-torsion modules, by the Wang sequence for the projection of $\widehat{M}$ onto M. Now $H_2(\widehat{M};\mathbb{Z}) \cong \overline{Ext^1_\Lambda(I(\pi)/I(\pi)',\Lambda)}$, by Poincaré duality. There is an exact sequence

$$0 \to T \to I(\pi)/I(\pi)' \to I(Z*_m) \cong \Lambda/(t-m) \to 0,$$

where T is a finite Λ-module. Therefore $Ext^1_\Lambda(I(\pi)/I(\pi)',\Lambda) \cong \Lambda/(t-m)$ and so $H_2(I(\pi);\mathbb{Z})$ is a quotient of $\Lambda/(mt-1)$, which is isomorphic to $Z[\frac{1}{m}]$ as an abelian group. Now $I(\pi)/\mathrm{Ker}(f) \cong Z[\frac{1}{m}]$ also, and $H_2(Z[\frac{1}{m}];\mathbb{Z}) \cong Z[\frac{1}{m}] \wedge Z[\frac{1}{m}] = 0$. (See page 334 of [Ro].) Hence $H_2(I(\pi);\mathbb{Z})$ is finite, by an LHSSS argument, and so is cyclic, of order relatively prime to m.

Let t in π generate $\pi/I(\pi) \cong Z$. Let A be a maximal abelian subgroup of F and let $C = C_\pi(A)$. Then $q = [\pi : C]$ is finite, since F is finite and normal in π. In particular, t^q is in C and C maps onto Z, with kernel J, say. Since J is an extension of $Z[\frac{1}{m}]$ by a finite normal subgroup its centre ζJ has finite index in J. Therefore the subgroup G generated by ζJ and t^q has finite index in π, and there is an epimorphism f from G onto $Z*_{m^q}$, with kernel A. Moreover $I(G) = f^{-1}(I(Z*_{m^q}))$ is abelian, and is an extension of $Z[\frac{1}{m}]$ by the finite abelian group A. Hence it is isomorphic to $A \oplus Z[\frac{1}{m}]$. (See page 106 of [Ro].) Now $H_2(I(G);\mathbb{Z})$ is cyclic of order prime to m. On the other hand $H_2(I(G);\mathbb{Z}) \cong (A \wedge A) \oplus (A \otimes Z[\frac{1}{m}])$ and so A must be cyclic.

If $F \neq 1$ then A is cyclic, nontrivial, central in G and $G/A \cong Z*_{m^q}$. □

Lemma 3.16 *Let M be a finite PD_4-complex with fundamental group π. Suppose that π has a nontrivial finite cyclic central subgroup F with quotient $G = \pi/F$ such that $g.d.G = 2$, $e(G) = 1$ and $\mathrm{def}(G) = 1$. Then $\chi(M) \geq 0$. If $\chi(M) = 0$ and $\Xi = \mathbb{F}_p[G]$ is a weakly finite ring for some prime p dividing $|F|$ then π is virtually Z^2.*

Proof Let $\widehat{M}$ be the covering space of M with group F, and let c_q be the number of q-cells of M, for $q \geq 0$. Let $C_* = C_*(M;\Xi) = \mathbb{F}_p \otimes C_*(M)$ be the equivariant cellular chain complex of $\widehat{M}$ with coefficients $\mathbb{F}_p$ and let $H_p = H_p(M;\Xi) = H_p(\widehat{M};\mathbb{F}_p)$. For any left Ξ-module H let $e^qH = \overline{Ext^q_\Xi(H,\Xi)}$.

Since $\widehat{M}$ is connected and F is cyclic $H_0 \cong H_1 \cong \mathbb{F}_p$ and since G has one end Poincaré duality and the UCSS give $H_3 = H_4 = 0$, an exact sequence

$$0 \to e^2\mathbb{F}_p \to H_2 \to e^0H_2 \to e^2H_1 \to H_1 \to e^1H_2 \to 0$$

and an isomorphism $e^2H_2 \cong \mathbb{F}_p$. Since $g.d.G = 2$ and $\mathrm{def}(G) = 1$ the augmentation module has a resolution

$$0 \to \Xi^r \to \Xi^{r+1} \to \Xi \to \mathbb{F}_p \to 0.$$

The chain complex C_* gives four exact sequences

$$0 \to Z_1 \to C_1 \to C_0 \to \mathbb{F}_p \to 0,$$
$$0 \to Z_2 \to C_2 \to Z_1 \to \mathbb{F}_p \to 0,$$
$$0 \to B_2 \to Z_2 \to H_2 \to 0$$
$$\text{and} \qquad 0 \to C_4 \to C_3 \to B_2 \to 0.$$

Using Schanuel's Lemma several times we find that the cycle submodules Z_1 and Z_2 are stably free, of stable ranks $c_1 - c_0$ and $c_2 - c_1 + c_0$, respectively. Dualizing the last two sequences gives two new sequences

$$0 \to e^0B_2 \to e^0C_3 \to e^0C_4 \to e^1B_2 \to 0$$
$$\text{and} \qquad 0 \to e^0H_2 \to e^0Z_2 \to e^0B_2 \to e^1H_2 \to 0,$$

and an isomorphism $e^1B_2 \cong e^2H_2 \cong \mathbb{F}_p$. Further applications of Schanuel's Lemma show that e^0B_2 is stably free of rank $c_3 - c_4$, and hence that e^0H_2 is stably free of rank $c_2 - c_1 + c_0 - (c_3 - c_4) = \chi(M)$. Since Ξ maps onto the field $\mathbb{F}_p$ the rank must be non-negative, and so $\chi(M) \geq 0$.

If $\chi(M) = 0$ and $\Xi = \mathbb{F}_p[G]$ is a weakly finite ring then $e^0H_2 = 0$ and so $e^2\mathbb{F}_p = e^2H_1$ is a submodule of $\mathbb{F}_p \cong H_1$. Moreover it cannot be 0, for otherwise the UCSS would give $H_2 = 0$ and then $H_1 = 0$, which is impossible. Therefore $e^2\mathbb{F}_p \cong \mathbb{F}_p$.

Since G is torsion-free and indicable it must be a PD_2-group, by Theorem V.12.2 of [DD]. Since $\mathrm{def}(G) = 1$ it follows that $G \cong Z^2$ or $Z \rtimes_{-1} Z$, and hence that π is also virtually Z^2. □

The hypothesis on Ξ is satisfied if G is an extension of an amenable group by a free normal subgroup [AO'M02]. In particular, this is so if G' is finitely generated, by Corollary 4.3.1.

We may now give the main result of this section.

Theorem 3.17 *Let M be a finite PD_4-complex whose fundamental group π is an ascending HNN extension with finitely generated base B. Then $\chi(M) \geq 0$, and hence $q(\pi) \geq 0$. If $\chi(M) = 0$ and B is FP_2 and finitely ended then either π has two ends or $\pi \cong Z*_m$ or $Z *_m \rtimes (Z/2Z)$ for some $m \neq 0$ or ± 1 or π is virtually Z^2 or M is aspherical.*

Proof The L^2 Euler characteristic formula gives $\chi(M) = \beta_2^{(2)}(M) \geq 0$, since $\beta_i^{(2)}(M) = \beta_i^{(2)}(\pi) = 0$ for $i = 0$ or 1, by Lemma 2.1.

Let $\phi : B \to B$ be the monomorphism determining $\pi \cong B*_\phi$. If B is finite then ϕ is an automorphism and so π has two ends. If B is FP_2 and has one end then $H^s(\pi;\mathbb{Z}[\pi]) = 0$ for $s \leq 2$, by the Brown-Geoghegan Theorem. If moreover $\chi(M) = 0$ then M is aspherical, by Corollary 3.5.2.

If B has two ends then it is an extension of Z or D by a finite normal subgroup F. As ϕ must map F isomorphically to itself, F is normal in π, and is the maximal finite normal subgroup of π. Moreover $\pi/F \cong Z*_m$, for some $m \neq 0$, if $B/F \cong Z$, and is a semidirect product $Z *_m \rtimes (Z/2Z)$, with a presentation $\langle a, t, u \mid tat^{-1} = a^m,\ tut^{-1} = ua^r,\ u^2 = 1,\ uau = a^{-1}\rangle$, for some $m \neq 0$ and some $r \in Z$, if $B/F \cong D$. (On replacing t by $a^{[r/2]}t$, if necessary, we may assume that $r = 0$ or 1.)

Suppose first that M is orientable, and that $F \neq 1$. Then π has a subgroup σ of finite index which is a central extension of $Z*_{m^q}$ by a finite cyclic group, for some $q \geq 1$, by Lemma 3.15. Let p be a prime dividing q. Since $Z*_{m^q}$ is a torsion-free solvable group the ring $\Xi = \mathbb{F}_p[Z*_{m^q}]$ has a skew field of fractions L, which as a right Ξ-module is the direct limit of the system $\{\Xi_\theta \mid 0 \neq \theta \in \Xi\}$, where each $\Xi_\theta = \Xi$, the index set is ordered by right divisibility ($\theta \leq \phi\theta$) and the map from Ξ_θ to $\Xi_{\phi\theta}$ sends ξ to $\phi\xi$ [KLM88]. In particular, Ξ is a weakly finite ring and so π is virtually Z^2, by Lemma 3.16.

If M is nonorientable then either $w_1(M)|_F$ is injective, so $\pi \cong Z *_m \rtimes (Z/2Z)$, or π is virtually Z^2. $\square$

Is M still aspherical if B is assumed only finitely generated and one ended?

Corollary 3.17.1 *Let M be a finite PD_4-complex such that $\chi(M) = 0$ and $\pi = \pi_1(M)$ is almost coherent and restrained. Then either π has two ends or $\pi \cong Z*_m$ or $Z*_m \rtimes (Z/2Z)$ for some $m \neq 0$ or ± 1 or π is virtually Z^2 or M is aspherical.*

Proof Let $\pi^+ = \mathrm{Ker}(w_1(M))$. Then π^+ maps onto Z, by Lemma 3.14, and so is an ascending HNN extension $\pi^+ \cong B*_\phi$ with finitely generated base B. Since π is almost coherent B is FP_2, and since π has no nonabelian free subgroup B has at most two ends. Hence Lemma 3.16 and Theorem 3.17 apply, so either π has two ends or M is aspherical or $\pi^+ \cong Z*_m$ or $Z *_m \rtimes (Z/2Z)$ for some $m \neq 0$ or ± 1. In the latter case $\sqrt{\pi}$ is isomorphic to a subgroup of the additive rationals Q, and $\sqrt{\pi} = C_\pi(\sqrt{\pi})$. Hence the image of π in $Aut(\sqrt{\pi}) \leq Q^\times$ is infinite. Therefore π maps onto Z and so is an ascending HNN extension $B*_\phi$, and we may again use Theorem 3.17. □

Does this corollary remain true without the hypothesis that π be almost coherent?

There are nine groups which are virtually Z^2 and are fundamental groups of PD_4-complexes with Euler characteristic 0. (See Chapter 11.) Are any of the groups $Z *_m \rtimes(Z/2Z)$ with $|m| > 1$ realized by PD_4-complexes with $\chi = 0$? If π is restrained and M is aspherical must π be virtually poly-Z? (Aspherical 4-manifolds with virtually poly-Z fundamental groups are characterized in Chapter 8.)

Let G is a group with a presentation of deficiency d and $w : G \to \{\pm 1\}$ be a homomorphism, and let $\langle x_i,\ 1 \leq i \leq m \mid r_j,\ 1 \leq j \leq n\rangle$ be a presentation for G with $m - n = d$. We may assume that $w(x_i) = +1$ for $i \leq m-1$. Let $X = \natural^m(S^1 \times D^3)$ if $w = 1$ and $X = (\natural^{m-1}(S^1 \times D^3))\natural(S^1 \tilde{\times} D^3)$ otherwise. The relators r_j may be represented by disjoint orientation preserving embeddings of S^1 in ∂X, and so we may attach 2-handles along product neighbourhoods, to get a bounded 4-manifold Y with $\pi_1(Y) = G$, $w_1(Y) = w$ and $\chi(Y) = 1 - d$. Doubling Y gives a closed 4-manifold M with $\chi(M) = 2(1 - d)$ and $(\pi_1(M), w_1(M))$ isomorphic to (G, w).

Since the groups $Z*_m$ have deficiency 1 it follows that any homomorphism $w : Z*_m \to \{\pm 1\}$ may be realized as the orientation character of a closed 4-manifold with fundamental group $Z*_m$ and Euler characteristic 0. What other invariants are needed to determine the homotopy type of such a manifold?

3.5 The intersection pairing

Let X be a PD_4-complex with fundamental group π and let $w = w_1(X)$. In this section it shall be convenient to work with left modules. Thus if L is a left $\mathbb{Z}[\pi]$-module we shall let $L^\dagger = \overline{Hom_{\mathbb{Z}[\pi]}(L, \mathbb{Z}[\pi])}$ be the conjugate dual module. If L is free, stably free or projective so is $L^\dagger$.

Let $H = \overline{H^2(X; \mathbb{Z}[\pi])}$ and $\Pi = \pi_2(X)$, and let $D : H \to \Pi$ and $ev : H \to \Pi^\dagger$ be the Poincaré duality isomorphism and the evaluation homomorphism, respectively. The *cohomology intersection pairing* $\lambda : H \times H \to \mathbb{Z}[\pi]$ is defined by $\lambda(u, v) = ev(v)(D(u))$, for all $u, v \in H$. This pairing is w-hermitian: $\lambda(gu, hv) = g\lambda(u, v)\bar{h}$ and $\lambda(v, u) = \overline{\lambda(u, v)}$ for all $u, v \in H$ and $g, h \in \pi$. Since $\lambda(u, e) = 0$ for all $u \in H$ and $e \in E = \overline{H^2(\pi; \mathbb{Z}[\pi])}$ the pairing λ induces a pairing $\lambda_X : H/E \times H/E \to \mathbb{Z}[\pi]$, which we shall call the *reduced* intersection pairing. The adjoint homomorphism $\tilde{\lambda}_X : H/E \to (H/E)^\dagger$ is given by $\tilde{\lambda}_X([v])([u]) = \lambda(u,v) = ev(v)(D(u))$, for all $u, v \in H$. It is a monomorphism, and λ_X is *nonsingular* if $\tilde{\lambda}_X$ is an isomorphism.

Lemma 3.18 *Let X be a PD_4-complex with fundamental group π, and let $E = \overline{H^2(\pi;\mathbb{Z}[\pi])}$.*

(1) *If λ_X is nonsingular then $\overline{H^3(\pi;\mathbb{Z}[\pi])}$ embeds as a submodule of $E^\dagger$;*
(2) *if λ_X is nonsingular and $H^2(c_X;\mathbb{Z}[\pi])$ splits then $E^\dagger \cong \overline{H^3(\pi;\mathbb{Z}[\pi])}$;*
(3) *if $H^3(\pi;\mathbb{Z}[\pi]) = 0$ then λ_X is nonsingular;*
(4) *if $H^3(\pi;\mathbb{Z}[\pi]) = 0$ and Π is a finitely generated projective $\mathbb{Z}[\pi]$-module then $E = 0$;*
(5) *if $H^1(\pi;\mathbb{Z}[\pi])$ and Π are projective then $c.d.\pi = 4$.*

Proof Let $p : \Pi \to \Pi/D(E)$ and $q : H \to H/E$ be the canonical epimorphisms. Poincaré duality induces an isomorphism $\gamma : H/E \cong \Pi/D(E)$. It is straightforward to verify that $p^\dagger(\gamma^\dagger)^{-1}\tilde{\lambda}_X q = ev$. If λ_X is nonsingular then $\tilde{\lambda}_X$ is an isomorphism, and so $\mathrm{Coker}(p^\dagger) = \mathrm{Coker}(ev)$. The first assertion follows easily, since $\mathrm{Coker}(p^\dagger) \leq E^\dagger$.

If moreover $H^2(c_X;\mathbb{Z}[\pi])$ splits then so does p, and so $E^\dagger \cong \mathrm{Coker}(p^\dagger)$.

If $H^3(\pi;\mathbb{Z}[\pi]) = 0$ then ev is an epimorphism and so $p^\dagger$ is an epimorphism. Since $p^\dagger$ is also a monomorphism it is an isomorphism. Since ev and q are epimorphisms with the same kernel it folows that $\tilde{\lambda}_X = \gamma^\dagger(p^\dagger)^{-1}$, and so $\tilde{\lambda}_X$ is also an isomorphism.

If Π is finitely generated and projective then so is $\Pi^\dagger$, and $\Pi \cong \Pi^{\dagger\dagger}$. If moreover $H^3(\pi;\mathbb{Z}[\pi]) = 0$ then $\Pi \cong H \cong E \oplus \Pi^\dagger$. Hence E is also finitely generated and projective, and $E \cong E^{\dagger\dagger} = 0$.

If $H^1(\pi;\mathbb{Z}[\pi])$ and Π are projective then we may obtain a projective resolution of $\mathbb{Z}$ of length 4 from $C_* = C_*(\widetilde{X})$ by replacing C_3 and C_4 by $C_3 \oplus \Pi$ and $C_4 \oplus \overline{H^1(\pi;\mathbb{Z}[\pi])}$, respectively, and modifying ∂_3 and ∂_4 appropriately. Since $H_3(X;\mathbb{Z}[\pi]) \cong \overline{H^1(\pi;\mathbb{Z}[\pi])}$ it is also projective. It follows from the UCSS that $H^4(\pi;\mathbb{Z}[\pi]) \neq 0$. Hence $c.d.\pi = 4$. $\square$

In particular, the cohomology intersection pairing is nonsingular if and only if $H^2(\pi;\mathbb{Z}[\pi]) = H^3(\pi;\mathbb{Z}[\pi]) = 0$. If X is a 4-manifold counting intersections of generic immersions of S^2 in $\widetilde{X}$ gives an equivalent pairing on Π.

We do not know whether the hypotheses in this lemma can be simplified. For instance, is $H^2(\pi;\mathbb{Z}[\pi])^\dagger$ always 0? Does "Π projective" imply that $H^3(\pi;\mathbb{Z}[\pi]) = 0$? Projectivity of $\Pi^\dagger$ and $H^2(\pi;\mathbb{Z}[\pi]) = 0$ together do not imply this. For if π is a PD_3^+-group and $w = w_1(\pi)$ there are finitely generated projective $\mathbb{Z}[\pi]$-modules P and P' such that $\Pi \oplus P \cong A(\pi) \oplus P'$, where $A(\pi)$ is the augmentation ideal of $\mathbb{Z}[\pi]$, by Theorem 3.13, and so $\Pi^\dagger$ is projective. However $H^3(\pi;\mathbb{Z}[\pi]) \cong \mathbb{Z} \neq 0$.

The module Π is finitely generated if and only if π is of type FP_3. As observed in the proof of Theorem 2.18, if π is a free product of infinite cyclic groups and groups with one end and is not a free group then $H^1(\pi;\mathbb{Z}[\pi])$ is a free $\mathbb{Z}[\pi]$-module. An argument similar to that for part(5) of the lemma shows that $c.d.\pi \leq 5$ if and only if π is torsion-free and $p.d._{\mathbb{Z}[\pi]}\Pi \leq 2$.

If Y is a second PD_4-complex we write $\lambda_X \cong \lambda_Y$ if there is an isomorphism $\theta : \pi \cong \pi_1(Y)$ such that $w_1(X) = w_1(Y)\theta$ and a $\mathbb{Z}[\pi]$-module isomorphism $\Theta : \pi_2(X) \cong \theta^*\pi_2(Y)$ inducing an isometry of cohomology intersection pairings. If $f : X \to Y$ is a 2-connected degree-1 map the "surgery kernel" $K_2(f) = \mathrm{Ker}(\pi_2(f))$ and "surgery cokernel" $K^2(f) = \overline{\mathrm{Cok}(H^2(f;\mathbb{Z}[\pi]))}$ are finitely generated and projective, and are stably free if X and Y are finite complexes, by Lemma 2.2 of [Wl]. (See also Theorem 3.2 above.) Moreover cap product with $[X]$ induces an isomorphism from $K^2(f)$ to $K_2(f)$. The pairing $\lambda_f = \lambda|_{K^2(f)\times K^2(f)}$ is nonsingular, by Theorem 5.2 of [Wl].

Chapter 4

Mapping tori and circle bundles

Stallings showed that if M is a 3-manifold and $f : M \to S^1$ a map which induces an epimorphism $f_* : \pi_1(M) \to Z$ with infinite kernel K then f is homotopic to a bundle projection if and only if M is irreducible and K is finitely generated. Farrell gave an analogous characterization in dimensions ≥ 6, with the hypotheses that the homotopy fibre of f is finitely dominated and a torsion invariant $\tau(f) \in Wh(\pi_1(M))$ is 0. The corresponding results in dimensions 4 and 5 are constrained by the present limitations of geometric topology in these dimensions. (In fact there are counter-examples to the most natural 4-dimensional analogue of Farrell's theorem [We87].)

Quinn showed that if the base B and homotopy fibre F of a fibration $p : M \to B$ are finitely dominated then the total space M is a Poincaré duality complex if and only if both the base and fibre are Poincaré duality complexes. (The paper [Go79] gives an elegant proof for the case when M, B and F are finite complexes. The general case follows on taking products with copies of S^1 to reduce to the finite case and using the Künneth theorem.)

We shall begin by giving a purely homological proof of a version of this result, for the case when M and B are PD-spaces and $B = K(G, 1)$ is aspherical. The homotopy fibre F is then the covering space associated to the kernel of the induced epimorphism from $\pi_1(M)$ to G. Our algebraic approach requires only that the equivariant chain complex of F have finite $[n/2]$-skeleton. In the next two sections we use the finiteness criterion of Ranicki and the fact that Novikov rings associated to finitely generated groups are weakly finite to sharpen this finiteness hypotheses when $B = S^1$, corresponding to infinite cyclic covers of M. The main result of §4.4 is a 4-dimensional homotopy fibration theorem with hypotheses similar to those of Stallings and a conclusion similar to that of Gottlieb and Quinn. The next two sections consider products of 3-manifolds with S^1 and covers associated to ascendant subgroups.

We shall treat fibrations of PD_4-complexes over surfaces in Chapter 5, by a different, more direct method. In the final section of this chapter we consider instead bundles with fibre S^1. We give conditions for a PD_4-complex to fibre over a PD_3-complex with homotopy fibre S^1, and show that these conditions are sufficient if the fundamental group of the base is torsion-free but not free.

4.1 PD_r-covers of PD_n-spaces

Let M be a PD_n-space and $p:\pi=\pi_1(M)\to G$ an epimorphism with G a PD_r-group, and let M_ν be the covering space corresponding to $\nu=\mathrm{Ker}(p)$. If M is aspherical and ν is $FP_{[n/2]}$ then ν is a PD_{n-r}-group and $M_\nu=K(\nu,1)$ is a PD_{n-r}-space, by Theorem 9.11 of [Bi]. In general, there are isomorphisms $H^q(M_\nu;\mathbb{Z}[\nu])\cong H_{n-r-q}(M_\nu;\mathbb{Z}[\nu])$, by Theorem 1.19$'$. However in the nonaspherical case it is not clear that there are such isomorphisms induced by cap product with a class in $H_{n-r}(M_\nu;\mathbb{Z}[\nu])$. If M is a PD_n-complex and ν is finitely presentable M_ν is finitely dominated, and we could apply the Gottlieb-Quinn Theorem to conclude that M_ν is a PD_{n-r}-complex. We shall give instead a purely homological argument which does not require π or ν to be finitely presentable, and so applies under weaker finiteness hypotheses.

A group G is a *weak* PD_r-group if $H^q(G;\mathbb{Z}[G])\cong Z$ if $q=r$ and is 0 otherwise [Ba80]. If $r\leq 2$ an FP_2 group is a weak PD_r-group if and only if it is virtually a PD_r-group. This is easy for $r\leq 1$ and is due to Bowditch when $r=2$ [Bo04]. Barge has given a simple homological argument to show that if G is a weak PD_r-group, M is a PD_n-space and $\eta_G\in H^r(M;\mathbb{Z}[G])$ is the image of a generator of $H^r(G;\mathbb{Z}[G])$ then cap product with $[M_\nu]=\eta_G\cap[M]$ induces isomorphisms with simple coefficients [Ba80]. We shall extend his argument to the case of arbitrary local coefficients, using coinduced modules to transfer arguments about subgroups and covering spaces to contexts where Poincaré duality applies,

All tensor products $N\otimes P$ in the following theorem are taken over $\mathbb{Z}$.

Theorem 4.1 *Let M be a PD_n-space and $p:\pi=\pi_1(M)\to G$ an epimorphism with G a weak PD_r-group, and let $\nu=\mathrm{Ker}(p)$. If $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated then M_ν is a PD_{n-r}-space.*

Proof Let C_* be a finitely generated projective $\mathbb{Z}[\pi]$-chain complex which is chain homotopy equivalent to $C_*(\widetilde{M})$. Since $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated there is a finitely generated projective $\mathbb{Z}[\nu]$-chain complex E_* and a pair of $\mathbb{Z}[\nu]$-linear chain homomorphisms $\theta:E_*\to C_*|_\nu$ and $\phi:C_*|_\nu\to E_*$ such that $\theta\phi\sim I_{C_*}$ and $\phi\theta\sim I_{E_*}$. Let $C^q=Hom_{\mathbb{Z}[\pi]}(C_q,\mathbb{Z}[\pi])$ and $E^q=Hom_{\mathbb{Z}[\nu]}(E_q,\mathbb{Z}[\nu])$, and let $\widehat{\mathbb{Z}[\pi]}=Hom_{\mathbb{Z}[\nu]}(\mathbb{Z}[\pi]|_\nu,\mathbb{Z}[\nu])$ be the module coinduced from $\mathbb{Z}[\nu]$. Then there are isomorphisms $\Psi:H^q(E^*)\cong H^q(C_*;\widehat{\mathbb{Z}[\pi]})$, determined by θ and Shapiro's Lemma.

The complex $\mathbb{Z}[G]\otimes_{\mathbb{Z}[\pi]}C_*$ is an augmented complex of finitely generated projective $\mathbb{Z}[G]$-modules with finitely generated integral homology. Therefore G

is of type FP_∞, by Theorem 3.1 of [St96]. Hence the augmentation $\mathbb{Z}[G]$-module $\mathbb{Z}$ has a resolution A_* by finitely generated projective $\mathbb{Z}[G]$-modules. Let $A^q = Hom_{\mathbb{Z}[G]}(A_q, \mathbb{Z}[G])$ and let $\eta \in H^r(A^*) = H^r(G; \mathbb{Z}[G])$ be a generator. Let $\varepsilon_C : C_* \to A_*$ be a chain map corresponding to the projection of p onto G, and let $\eta_G = \varepsilon_C^* \eta \in H^r(C_*; \mathbb{Z}[G])$. The augmentation $A_* \to \mathbb{Z}$ determines a chain homotopy equivalence $p : C_* \otimes A_* \to C_* \otimes \mathbb{Z} = C_*$. Let $\sigma : G \to \pi$ be a set-theoretic section.

We may define cup-products relating the cohomology of M_ν and M as follows. Let $e : \widehat{\mathbb{Z}[\pi]} \otimes \mathbb{Z}[G] \to \mathbb{Z}[\pi]$ be the pairing given by $e(\alpha \otimes g) = \sigma(g).\alpha(\sigma(g)^{-1})$ for all $\alpha : \mathbb{Z}[\pi] \to \mathbb{Z}[\nu]$ and $g \in G$. Then e is independent of the choice of section σ and is $\mathbb{Z}[\pi]$-linear with respect to the diagonal left π-action on $\widehat{\mathbb{Z}[\pi]} \otimes \mathbb{Z}[G]$. Let $d : C_* \to C_* \otimes C_*$ be a π-equivariant diagonal, with respect to the diagonal left π-action on $C_* \otimes C_*$, and let $j = (1 \otimes \varepsilon_C)d : C_* \to C_* \otimes A_*$. Then $pj = Id_{C_*}$ and so j is a chain homotopy equivalence. We define the cup-product $[f] \cup \eta_G$ in $H^{p+r}(C^*) = H^{p+r}(M; \mathbb{Z}[\pi])$ by $[f] \cup \eta_G = e_\# d^*(\Psi([f]) \times \eta_G) = e_\# j^*(\Psi([f]) \times \eta)$ for all $[f] \in H^p(E^*) = H^p(M_\nu; \mathbb{Z}[\nu])$.

If C is a left $\mathbb{Z}[\pi]$-module let $D = Hom_{\mathbb{Z}[\nu]}(C|_\nu, \mathbb{Z}[\pi])$ have the left G-action determined by $(g\lambda)(c) = \sigma(g)\lambda(\sigma(g)^{-1}c)$ for all $c \in C$ and $g \in G$. If C is free with basis $\{c_i | 1 \leq i \leq n\}$ there is an isomorphism of left $\mathbb{Z}[G]$-modules $\Theta : D \cong (|\mathbb{Z}[\pi]|^G)^n$ given by $\Theta(\lambda)(g) = (\sigma(g).\lambda(\sigma(g)^{-1}c_1), \dots, \sigma(g).\lambda(\sigma(g)^{-1}c_n))$ for all $\lambda \in D$ and $g \in G$, and so D is coinduced from a module over the trivial group.

Let $D^q = Hom_{\mathbb{Z}[\nu]}(C_q|_\nu, \mathbb{Z}[\pi])$ and let $\rho : E^* \otimes \mathbb{Z}[G] \to D^*$ be the $\mathbb{Z}$-linear cochain homomorphism defined by $\rho(f \otimes g)(c) = \sigma(g) f \phi(\sigma(g)^{-1}c)$ for all $c \in C_q$, $\lambda \in D^q$, $f \in E^q$, $g \in G$ and all q. Then the G-action on D^q and ρ are independent of the choice of section σ, and ρ is $\mathbb{Z}[G]$-linear if $E^q \otimes \mathbb{Z}[G]$ has the left G-action given by $g(f \otimes g') = f \otimes gg'$ for all $g, g' \in G$ and $f \in E^q$.

If $\lambda \in D^q$ then $\lambda\theta_q(E_q)$ is a finitely generated $\mathbb{Z}[\nu]$-submodule of $\mathbb{Z}[\pi]$. Hence there is a family of homomorphisms $\{f_g \in E^q | g \in F\}$, where F is a finite subset of G, such that $\lambda\theta_q(e) = \Sigma_{g \in F} f_g(e)\sigma(g)$ for all $e \in E_q$. Let $\lambda_g(e) = \sigma(g)^{-1} f_g(\phi\sigma(g)\theta(e))\sigma(g)$ for all $e \in E_q$ and $g \in F$. Let $\Phi(\lambda) = \Sigma_{g \in F} \lambda_g \otimes g \in E^q \otimes \mathbb{Z}[G]$. Then Φ is a $\mathbb{Z}$-linear cochain homomorphism. Moreover $[\rho\Phi(\lambda)] = [\lambda]$ for all $[\lambda] \in H^q(D^*)$ and $[\Phi\rho(f \otimes g)] = [f \otimes g]$ for all $[f \otimes g] \in H^q(E^* \otimes \mathbb{Z}[G])$, and so ρ is a chain homotopy equivalence. (It is not clear that Φ is $\mathbb{Z}[G]$-linear on the cochain level, but we shall not need to know this).

We now compare the hypercohomology of G with coefficients in the cochain complexes $E^* \otimes \mathbb{Z}[G]$ and D^*. On one side we have $\mathbb{H}^n(G; E^* \otimes \mathbb{Z}[G]) =$

$H^n_{tot}(Hom_{\mathbb{Z}[G]}(A_*, E^* \otimes \mathbb{Z}[G]))$, which may be identified with $H^n_{tot}(E^* \otimes A^*)$ since A_q is finitely generated for all $q \geq 0$. This is in turn isomorphic to $H^{n-r}(E^*) \otimes H^r(G; \mathbb{Z}[G]) \cong H^{n-r}(E^*)$, since G acts trivially on E^* and is a weak PD_r-group.

On the other side we have $\mathbb{H}^n(G; D^*) = H^n_{tot}(Hom_{\mathbb{Z}[G]}(A_*, D^*))$. The cochain homomorphism ρ induces a morphism of double complexes from $E^* \otimes A^*$ to $Hom_{\mathbb{Z}[G]}(A_*, D^*)$ by $\rho^{pq}(f \otimes \alpha)(a) = \rho(f \otimes \alpha(a)) \in D^p$ for all $f \in E^p$, $\alpha \in A^q$ and $a \in A_q$ and all $p, q \geq 0$. Let $\hat{\rho}^p([f]) = [\rho^{pr}(f \times \eta)] \in \mathbb{H}^{p+r}(G; D^*)$ for all $[f] \in H^p(E^*)$. Then $\hat{\rho}^p : H^p(E^*) \to \mathbb{H}^{p+r}(G; D^*)$ is an isomorphism, since $[f] \mapsto [f \times \eta]$ is an isomorphism and ρ is a chain homotopy equivalence. Since C_p is a finitely generated projective $\mathbb{Z}[\pi]$-module D^p is a direct summand of a coinduced module. Therefore $H^i(G; D^p) = 0$ for all $i > 0$, while $H^0(G; D^p) = Hom_{\mathbb{Z}[\pi]}(C_p, \mathbb{Z}[\pi])$, for all $p \geq 0$. Hence $\mathbb{H}^n(G; D^*) \cong H^n(C^*)$ for all n.

Let $f \in E^p$, $a \in A_r$ and $c \in C_p$, and suppose that $\eta(a) = \Sigma n_g g$. Since $\hat{\rho}^p([f])(a)(c) = \rho(f \otimes \eta(a))(c) = \Sigma n_g \sigma(g) f \phi(\sigma(g)^{-1}c) = ([f] \cup \eta)(c, a)$ it follows that the homomorphisms from $H^p(E^*)$ to $H^{p+r}(C^*)$ given by cup-product with η_G are isomorphisms for all p.

Let $[M] \in H_n(M; \mathbb{Z}^w)$ be a fundamental class for M, and let $[M_\nu] = \eta_G \cap [M] \in H_{n-r}(M; \mathbb{Z}^w \otimes \mathbb{Z}[G]) = H_{n-r}(M_\nu; \mathbb{Z}^{w|_\nu})$. Then cap product with $[M_\nu]$ induces isomorphisms $H^p(M_\nu; \mathbb{Z}[\nu]) \cong H_{n-r-p}(M_\nu; \mathbb{Z}[\nu])$ for all p, since $c \cap [M_\nu] = (c \cup \eta_G) \cap [M]$ in $H_{n-r-p}(M; \mathbb{Z}[\pi]) = H_{n-r-p}(M_\nu; \mathbb{Z}[\nu]) = H_{n-r-p}(\widetilde{M}; \mathbb{Z})$ for $c \in H^p(M_\nu; \mathbb{Z}[\nu])$. Thus M_ν is a PD_{n-r}-space. □

Theorems 1.19′ and 4.1 together give the following version of the Gottlieb-Quinn Theorem for covering spaces.

Corollary 4.1.1 *Let M be a PD_n-space and $p : \pi = \pi_1(M) \to G$ an epimorphism with G a PD_r-group, and let $\nu = \mathrm{Ker}(p)$. Then M_ν is a PD_{n-r}-space if and only if $C_*(\widetilde{M})|_\nu$ has finite $[n/2]$-skeleton.*

Proof The conditions are clearly necessary. Conversely, if M_ν has finite $[n/2]$-skeleton then C_* is $\mathbb{Z}[\nu]$-finitely dominated, by Theorem 1.19′, and hence is a PD_{n-r}-space, by Theorem 4.1. □

Corollary 4.1.2 *The space M_ν is a PD_{n-r}-complex if and only if it is homotopy equivalent to a complex with finite $[n/2]$-skeleton and ν is finitely presentable.* □

Corollary 4.1.3 *If π is a PD_r-group $\widetilde{M}$ is a PD_{n-r}-complex if and only if $H_q(\widetilde{M};\mathbb{Z})$ is finitely generated for all $q \leq [n/2]$.* $\square$

Stark used Theorem 3.1 of [St96] with the Gottlieb-Quinn Theorem to deduce that if M is a PD_n-complex and $v.c.d.\pi/\nu < \infty$ then π/ν is of type vFP, and therefore is virtually a PD-group. Is there a purely algebraic argument to show that if M is a PD_n-space, ν is a normal subgroup of π and $C_*(\widetilde{M})$ is $\mathbb{Z}[\nu]$-finitely dominated then π/ν must be a weak PD-group?

4.2 Novikov rings and Ranicki's criterion

The results of the above section apply in particular when $G = Z$. In this case however we may use an alternative finiteness criterion of Ranicki to get a slightly stronger result, which can be shown to be best possible. The results of this section are based on joint work with Kochloukova (in [HK07]).

Let π be a group, $\rho : \pi \to Z$ an epimorphism with kernel ν and $t \in \pi$ an element such that $\rho(t) = 1$. Let $\alpha : \nu \to \nu$ be the automorphism determined by $\alpha(h) = tht^{-1}$ for all h in ν. This automorphism extends to a ring automorphism (also denoted by α) of the group ring $R = \mathbb{Z}[\nu]$, and the ring $S = \mathbb{Z}[\pi]$ may then be viewed as a twisted Laurent extension, $\mathbb{Z}[\pi] = \mathbb{Z}[\nu]_\alpha[t, t^{-1}]$. The *Novikov ring* $\widehat{\mathbb{Z}[\pi]}_\rho$ associated to π and ρ is the ring of (twisted) Laurent series $\Sigma_{j\geq a}\kappa_j t^j$, for some $a \in \mathbb{Z}$, with coefficients κ_j in $\mathbb{Z}[\nu]$. Multiplication of such series is determined by conjugation in π: if $g \in \nu$ then $tg = (tgt^{-1})t$. If π is finitely generated the Novikov rings $\widehat{\mathbb{Z}[\pi]}_\rho$ are weakly finite [Ko06]. Let $\widehat{S}_+ = \widehat{\mathbb{Z}[\pi]}_\rho$ and $\widehat{S}_- = \widehat{\mathbb{Z}[\pi]}_{-\rho}$.

An α-*twisted* endomorphism of an R-module E is an additive function $h : E \to E$ such that $h(re) = \alpha(r)h(e)$ for all $e \in E$ and $r \in R$, and h is an α-twisted automorphism if it is bijective. Such an endomorphism h extends to α-twisted endomorphisms of the modules $S \otimes_R E$, $\widehat{E}_+ = \widehat{S}_+ \otimes_R E$ and $\widehat{E}_- = \widehat{S}_- \otimes_R E$ by $h(s \otimes e) = tst^{-1} \otimes h(e)$ for all $e \in E$ and $s \in S$, $\widehat{S}_+$ or $\widehat{S}_-$, respectively. In particular, left multiplication by t determines α-twisted automorphisms of $S \otimes_R E$, $\widehat{E}_+$ and $\widehat{E}_-$ which commute with h.

If E is finitely generated then $1 - t^{-1}h$ is an automorphism of $\widehat{E}_-$, with inverse given by a geometric series: $(1 - t^{-1}h)^{-1} = \Sigma_{k\geq 0}t^{-k}h^k$. (If E is not finitely generated this series may not give a function with values in $\widehat{E}_-$, and $t - h = t(1 - t^{-1}h)$ may not be surjective). Similarly, if k is an α^{-1}-twisted endomorphism of E then $1 - tk$ is an automorphism of $\widehat{E}_+$.

If P_* is a chain complex with an endomorphism $\beta : P_* \to P_*$ let $P_*[1]$ be the suspension and $\mathcal{C}(\beta)_*$ be the mapping cone. Thus $\mathcal{C}(\beta)_q = P_{q-1} \oplus P_q$, and $\partial_q(p, p') = (-\partial p, \beta(p) + \partial p')$, and there is a short exact sequence

$$0 \to P_* \to \mathcal{C}(\beta)_* \to P_*[1] \to 0.$$

The connecting homomorphisms in the associated long exact sequence of homology are induced by β. The *algebraic mapping torus* of an α-twisted self chain homotopy equivalence h of an R-chain complex E_* is the mapping cone $\mathcal{C}(1 - t^{-1}h)$ of the endomorphism $1 - t^{-1}h$ of the S-chain complex $S \otimes_R E_*$.

Lemma 4.2 *Let E_* be a projective chain complex over R which is finitely generated in degrees $\leq d$ and let $h : E_* \to E_*$ be an α-twisted chain homotopy equivalence. Then $H_q(\widehat{S}_- \otimes_S \mathcal{C}(1 - t^{-1}h)_*) = 0$ for $q \leq d$.*

Proof There is a short exact sequence

$$0 \to S \otimes_R E_* \to \mathcal{C}(1 - z^{-1}h)_* \to S \otimes_R E_*[1] \to 0.$$

Since E_* is a complex of projective R-modules the sequence

$$0 \to \widehat{E}_{*-} \to \widehat{S}_- \otimes_S \mathcal{C}(1 - t^{-1}h)_* \to \widehat{E}_{*-}[1] \to 0$$

obtained by extending coefficients is exact. Since $1-t^{-1}h$ induces isomorphisms on $\widehat{E}_{q-}$ for $q \leq d$ it induces isomorphisms on homology in degrees $< d$ and an epimorphism on homology in degree d. Therefore $H_q(\widehat{S}_- \otimes_S \mathcal{C}(1 - t^{-1}h)_*) = 0$ for $q \leq d$, by the long exact sequence of homology. □

The next theorem is our refinement of Ranicki's finiteness criterion [HK07].

Theorem 4.3 *Let C_* be a finitely generated projective S-chain complex. Then $i^!C_*$ has finite d-skeleton if and only if $H_q(\widehat{S}_\pm \otimes_S C_*) = 0$ for $q \leq d$.*

Proof We may assume without loss of generality that C_q is a finitely generated free S-module for all $q \leq d+1$, with basis $X_i = \{c_{q,i}\}_{i \in I(q)}$. We may also assume that $0 \notin \partial_i(X_i)$ for $i \leq d+1$, where $\partial_i : C_i \to C_{i-1}$ is the differential of the complex. Let $h_\pm$ be the $\alpha^{\pm 1}$-twisted automorphisms of $i^!C_*$ induced by multiplication by $z^{\pm 1}$ in C_*. Let $f_q(z^k r c_{q,i}) = (0, z^k \otimes r c_{q,i}) \in (S \otimes_R C_{q-1}) \oplus (S \otimes_R C_q)$. Then f_* defines S-chain homotopy equivalences from C_* to each of $\mathcal{C}(1 - z^{-1}h_+)$ and $\mathcal{C}(1 - zh_-)$.

Suppose first that $k_* : i^!C_* \to E_*$ and $g_* : E_* \to i^!C_*$ are chain homotopy equivalences, where E_* is a projective R-chain complex which is finitely generated in degrees $\leq d$. Then $\theta_\pm = k_* h_\pm g_*$ are $\alpha^{\pm 1}$-twisted self homotopy equivalences of E_*, and $\mathcal{C}(1-z^{-1}h_+)$ and $\mathcal{C}(1-zh_-)$ are chain homotopy equivalent to $\mathcal{C}(1-z^{-1}\theta_+)$ and $\mathcal{C}(1-z\theta_-)$, respectively. Therefore $H_q(\widehat{S}_- \otimes_S C_*) = H_q(\widehat{S}_- \otimes_S \mathcal{C}(1-z^{-1}\theta_+)) = 0$ and $H_q(\widehat{S}_+ \otimes_S C_*) = H_q(\widehat{S}_+ \otimes_S \mathcal{C}(1-z\theta_-)) = 0$ for $q \leq d$, by Lemma 5, applied twice.

Conversely, suppose that $H_i(\widehat{S}_\pm \otimes_S C_*) = 0$ for all $i \leq k$. Adapting an idea from [BR88], we shall define inductively a support function $supp_X$ for $\lambda \in \cup_{i\leq d+1} C_i$ with values finite subsets of $\{z^j\}_{j\in\mathbb{Z}}$ so that

(1) $supp_X(0) = \emptyset$;
(2) if $x \in X_0$ then $supp_X(z^j x) = z^j$;
(3) if $x \in X_i$ for $1 \leq i \leq d+1$ then $supp_X(z^j x) = z^j.supp_X(\partial_i(x))$;
(4) if $s = \sum_j r_j z^j \in S$, where $r_j \in R$, $supp_X(sx) = \cup_{r_j\neq 0} supp_X(z^j x)$;
(5) if $0 \leq i \leq d+1$ and $\lambda = \sum_{s_x\in S, x\in X_i} s_x x$ then $supp_X(\lambda) = \cup_{s_x\neq 0, x\in X_i} supp_X(s_x x)$.

Then $supp_X(\partial_i(\lambda)) \subseteq supp_X(\lambda)$ for all $\lambda \in C_i$ and all $1 \leq i \leq d+1$. Since $X = \cup_{i\leq d+1} X_i$ is finite there is a positive integer b such that

$$\cup_{x\in X_i, i\leq d+1} supp_X(x) \subseteq \{z^j\}_{-b\leq j\leq b}.$$

Define two subcomplexes C^+ and C^- of C which are 0 in degrees $i \geq d+2$ as follows:

(1) if $i \leq d+1$ an element $\lambda \in C_i$ is in C^+ if and only if $supp_X(\lambda) \subseteq \{z^j\}_{j\geq -b}$; and
(2) if $i \leq d+1$ an element $\lambda \in C_i$ is in C^- if and only if $supp_X(\lambda) \subseteq \{z^j\}_{j\leq b}$.

Then $\cup_{i\leq d+1} X_i \subseteq (C^+)^{[d+1]} \cap (C^-)^{[d+1]}$ and so $(C^+)^{[d+1]} \cup (C^-)^{[d+1]} = C^{[d+1]}$, where the upper index $*$ denotes the $*$-skeleton. Moreover $(C^+)^{[d+1]}$ is a complex of free finitely generated $R_\alpha[z]$-modules, $(C^-)^{[d+1]}$ is a complex of free finitely generated $R_\alpha[z^{-1}]$-modules, $(C^+)^{[d+1]} \cap (C^-)^{[d+1]}$ is a complex of free finitely generated R-modules and

$$C^{[d+1]} = S \otimes_{R_\alpha[z]} (C^+)^{[d+1]} = S \otimes_{R_\alpha[z^{-1}]} (C^-)^{[d+1]}.$$

Furthermore there is a Mayer-Vietoris exact sequence

$$0 \to (C^+)^{[d+1]} \cap (C^-)^{[d+1]} \to (C^+)^{[d+1]} \oplus (C^-)^{[d+1]} \to C^{[d+1]} \to 0.$$

Thus the $(d+1)$-skeletons of C, C^+ and C^- satisfy "algebraic transversality" in the sense of [Rn95].

Then to prove the theorem it suffices to show that C^+ and C^- are each chain homotopy equivalent over R to a complex of projective R-modules which is finitely generated in degrees $\leq d$. As in [Rn95] there is an exact sequence of $R_\alpha[z^{-1}]$-module chain complexes

$$0 \to (C^-)^{[d+1]} \to C^{[d+1]} \oplus R_\alpha[[z^{-1}]] \otimes_{R_\alpha[z^{-1}]} (C^-)^{[d+1]} \to \widehat{S}_- \otimes_S C^{[d+1]} \to 0.$$

Let $\tilde{i}$ denot the inclusion of $(C^-)^{[d+1]}$ into the central term. Inclusions on each component define a chain homomorphism

$$\tilde{j} : (C^+)^{[d+1]} \cap (C^-)^{[d+1]} \to (C^+)^{[d+1]} \oplus R_\alpha[[z^{-1}]] \otimes_{R_\alpha[z^{-1}]} (C^-)^{[d+1]}$$

such that the mapping cones of $\tilde{i}$ and $\tilde{j}$ are chain equivalent R-module chain complexes. The map induced by $\tilde{i}$ in homology is an epimorphism in degree d and an isomorphism in degree $< d$, since $H_i(\widehat{S}_- \otimes_S C^{[d+1]}) = 0$ for $i \leq d$. In particular all homologies in degrees $\leq d$ of the mapping cone of $\tilde{i}$ are 0. Hence all homologies of the mapping cone of $\tilde{j}$ are 0 in degrees $\leq d$. Then $(C^+)^{[d+1]}$ is homotopy equivalent over R to a chain complex of projectives over R whose k-skeleton is a summand of $(C^+)^{[d]} \cap (C^-)^{[d]}$. This completes the proof. $\square$

The argument for the converse is entirely due to Kochloukova.

As an application we shall give a quick proof of Kochloukova's improvement of Corollary 2.5.1.

Corollary 4.3.1 [Ko06] Let π be a finitely presentable group with a finitely generated normal subgroup N such that $\pi/N \cong Z$. Then $\mathrm{def}(\pi) = 1$ if and only if N is free.

Proof Let X be the finite 2-complex corresponding to an optimal presentation of π. If $\mathrm{def}(G) = 1$ then $\chi(X) = 0$ and X is aspherical, by Theorem 2.5. Hence $C_* = C_*(\widetilde{X})$ is a finite free resolution of the augmentation module $\mathbb{Z}$. Let $A_\pm$ be the two Novikov rings corresponding to the two epimorphisms $\pm p : \pi \to Z$ with kernel N. Then $H_j(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = 0$ for $j \leq 1$, by Theorem 4.3. But then $H_2(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*)$ is stably free, by Lemma 3.1. Since $\chi(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = \chi(C_*) = \chi(X) = 0$ and the rings $A_\pm$ are weakly finite [Ko06] these modules are 0. Thus $H_j(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = 0$ for all j, and so $C_*|_\nu$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\nu]$-complex, by Theorem 2 of [Rn95]. In particular, N is FP_2 and hence is free, by Corollary 8.6 of [Bi].

The converse is clear. $\square$

4.3 Infinite cyclic covers

The *mapping torus* of a self homotopy equivalence $f : X \to X$ is the space $M(f) = X \times [0,1]/\sim$, where $(x,0) \sim (f(x),1)$ for all $x \in X$. The function $p([x,t]) = e^{2\pi it}$ defines a map $p : M(f) \to S^1$ with homotopy fibre X, and the induced homomorphism $p_* : \pi_1(M(f)) \to Z$ is an epimorphism if X is path-connected. Conversely, let E be a connected cell complex and let $f : E \to S^1$ be a map which induces an epimorphism $f_* : \pi_1(E) \to Z$, with kernel ν. Then $E_\nu = E \times_{S^1} R = \{(x,y) \in E \times R \mid f(x) = e^{2\pi iy}\}$, and $E \simeq M(\phi)$, where $\phi : E_\nu \to E_\nu$ is the generator of the covering group given by $\phi(x,y) = (x, y+1)$ for all (x,y) in E_ν.

Theorem 4.4 *Let M be a finite PD_n-space with fundamental group π and let $p : \pi \to Z$ be an epimorphism with kernel ν. Then M_ν is a PD_{n-1}-space if and only if $\chi(M) = 0$ and $C_*(\widetilde{M_\nu}) = C_*(\widetilde{M})|_\nu$ has finite $[(n-1)/2]$-skeleton.*

Proof If M_ν is a PD_{n-1}-space then $C_*(\widetilde{M_\nu})$ is $\mathbb{Z}[\nu]$-finitely dominated [Br72]. In particular, $H_*(M;\Lambda) = H_*(M_\nu;\mathbb{Z})$ is finitely generated. The augmentation Λ-module $\mathbb{Z}$ has a short free resolution $0 \to \Lambda \to \Lambda \to \mathbb{Z} \to 0$, and it follows easily from the exact sequence of homology for this coefficient sequence that $\chi(M) = 0$ [Mi68]. Thus the conditions are necessary.

Suppose that they hold. Let $A_\pm$ be the two Novikov rings corresponding to the two epimorphisms $\pm p : \pi \to Z$ with kernel ν. Then $H_j(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = 0$ for $j \leq [(n-1)/2]$, by Theorem 4.3. Hence $H_j(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = 0$ for $j \geq n - [(n-1)/2]$, by duality. If n is even there is one possible nonzero module, in degree $m = n/2$. But then $H_m(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*)$ is stably free, by the finiteness of M and Lemma 3.1. Since $\chi(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = \chi(C_*) = \chi(M) = 0$ and the rings $A_\pm$ are weakly finite [Ko06] these modules are 0. Thus $H_j(A_\pm \otimes_{\mathbb{Z}[\pi]} C_*) = 0$ for all j, and so $C_*|_\nu$ is chain homotopy equivalent to a finite projective $\mathbb{Z}[\nu]$-complex, by Theorem 4.4. Thus the result follows from Theorem 4.1. □

When n is odd $[n/2] = [(n-1)/2]$, so the finiteness condition on M_ν agrees with that of Corollary 4.1.1 (for $G = Z$), but it is slightly weaker if n is even. Examples constructed by elementary surgery on simple n-knots show that the $FP_{[(n-1)/2]}$ condition is best possible, even when $\pi \cong Z$ and $\nu = 1$.

Corollary 4.4.1 *Under the same hypotheses on M and π, if $n \neq 4$ then M_ν is a PD_{n-1}-complex if and only if it is homotopy equivalent to a complex with finite $[(n-1)/2]$-skeleton.*

Proof If $n \leq 3$ every PD_{n-1}-space is a PD_{n-1}-complex, while if $n \geq 5$ then $[(n-1)/2] \geq 2$ and so ν is finitely presentable. $\square$

If $n \leq 3$ we need only assume that M is a PD_n-space and ν is finitely *generated*.

It remains an open question whether every PD_3-space is finitely dominated. The arguments of [Tu90] and [Cr00] on the factorization of PD_3-complexes into connected sums are essentially homological, and so every PD_3-space is a connected sum of aspherical PD_3-spaces and a PD_3-complex with virtually free fundamental group. Thus the question of whether every PD_3-space is finitely dominated reduces to whether every PD_3-group is finitely presentable.

4.4 The case $n = 4$

If $M(f)$ is the mapping torus of a self homotopy equivalence of a PD_3-space then $\chi(M) = 0$ and $\pi_1(M)$ is an extension of Z by a finitely generated normal subgroup. These conditions characterize such mapping tori, by Theorem 4.4. We shall summarize various related results in the following theorem.

Theorem 4.5 *Let M be a finite PD_4-space whose fundamental group π is an extension of Z by a finitely generated normal subgroup ν. Then*

(1) $\chi(M) \geq 0$, *with equality if and only if $H_2(M_\nu; \mathbb{Q})$ is finitely generated;*

(2) $\chi(M) = 0$ *if and only if M_ν is a PD_3-space;*

(3) *if $\chi(M) = 0$ then M is aspherical if and only if ν is a PD_3-group if and only if ν has one end;*

(4) *if M is aspherical then $\chi(M) = 0$ if and only if ν is a PD_3-group if and only ν is FP_2.*

Proof Since $C_*(\widetilde{M})$ is finitely dominated and $\mathbb{Q}\Lambda = \mathbb{Q}[t, t^{-1}]$ is noetherian the homology groups $H_q(M_\nu; \mathbb{Q})$ are finitely generated as $\mathbb{Q}\Lambda$-modules. Since ν is finitely generated they are finite dimensional as $\mathbb{Q}$-vector spaces if $q < 2$, and hence also if $q > 2$, by Poincaré duality. Now $H_2(M_\nu; \mathbb{Q}) \cong \mathbb{Q}^r \oplus (\mathbb{Q}\Lambda)^s$ for some $r, s \geq 0$, by the Structure Theorem for modules over a PID. It follows easily from the Wang sequence for the covering projection from M_ν to M, that $\chi(M) = s \geq 0$.

The space M_ν is a PD_3-space if and only if $\chi(M) = 0$, by Theorem 4.4.

Since M is aspherical if and only if M_ν is aspherical, (3) follows from (2) and the facts that PD_3-groups have one end and a PD_3-space is aspherical if and only if its fundamental group has one end.

If M is aspherical and $\chi(M) = 0$ then ν is a PD_3-group. If ν is a PD_3-group it is FP_2. If M is aspherical and ν is FP_2 then ν is a PD_3-group, by Theorem 1.19 (or Theorem 4.4), and so $\chi(M) = 0$. □

In particular, if $\chi(M) = 0$ then $q(\pi) = 0$. This observation and the bound $\chi(M) \geq 0$ were given in Theorem 3.17. (They also follow on counting bases for the cellular chain complex of M_ν and extending coefficients to $\mathbb{Q}(t)$.)

If $\chi(M) = 0$ and ν is finitely presentable then M_ν is a PD_3-complex. However M_ν need not be homotopy equivalent to a *finite* complex. If M is a *simple* PD_4-complex and a generator of $Aut(M_\nu/M) \cong \pi/\nu$ has finite order in the group of self homotopy equivalences of M_ν then M is finitely covered by a simple PD_4-complex homotopy equivalent to $M_\nu \times S^1$. In this case M_ν must be homotopy finite by [Rn86].

If $\pi \cong \nu \rtimes Z$ is a PD_4-group with ν finitely generated then $\chi(\pi) = 0$ if and only if ν is FP_2, by Theorem 4.5. However the latter conditions need not hold. Let F be the orientable surface of genus 2. Then $G = \pi_1(F)$ has a presentation $\langle a_1, a_2, b_1, b_2 \mid [a_1, b_1] = [a_2, b_2]\rangle$. The group $\pi = G \times G$ is a PD_4-group, and the subgroup $\nu \leq \pi$ generated by the images of (a_1, a_1) and the six elements $(x, 1)$ and $(1, x)$, for $x = a_2$, b_1 or b_2, is normal in π, with quotient $\pi/\nu \cong Z$. However $\chi(\pi) = 4 \neq 0$ and so ν cannot be FP_2.

It can be shown that the finitely generated subgroup N of $F(2) \times F(2)$ defined after Theorem 2.4 has one end. However $H^2(F(2) \times F(2); \mathbb{Z}[F(2) \times F(2)]) \neq 0$. (Note that $q(F(2) \times F(2)) = 2$, by Corollary 3.12.2.)

Corollary 4.5.1 *Let M be a finite PD_4-space with $\chi(M) = 0$ and whose fundamental group π is an extension of Z by a normal subgroup ν. If π has an infinite cyclic normal subgroup C which is not contained in ν then the covering space M_ν with fundamental group ν is a PD_3-complex.*

Proof We may assume without loss of generality that M is orientable and that C is central in π. Since π/ν is torsion-free $C \cap \nu = 1$, and so $C\nu \cong C \times \nu$ has finite index in π. Thus by passing to a finite cover we may assume that $\pi = C \times \nu$. Hence ν is finitely presentable and so Theorem 4.5 applies. □

Since ν has one or two ends if it has an infinite cyclic normal subgroup, Corollary 4.5.1 remains true if $C \leq \nu$ and ν is finitely presentable. In this case ν is the fundamental group of a Seifert fibred 3-manifold, by Theorem 2.14.

Corollary 4.5.2 *Let M be a finite PD_4-space with $\chi(M)=0$ and whose fundamental group π is an extension of Z by a finitely generated normal subgroup ν. If ν is finite then it has cohomological period dividing 4. If ν has one end then M is aspherical and so π is a PD_4-group. If ν has two ends then $\nu\cong Z$, $Z\oplus(Z/2Z)$ or $D=(Z/2Z)*(Z/2Z)$. If moreover ν is finitely presentable the covering space M_ν with fundamental group ν is a PD_3-complex.*

Proof The final hypothesis is only needed if ν is one-ended, as finite groups and groups with two ends are finitely presentable. If ν is finite then $\widetilde{M}\simeq S^3$ and so the first assertion holds. (See Chapter 11 for more details.) If ν has one end we may use Theorem 4.5. If ν has two ends and its maximal finite normal subgroup is nontrivial then $\nu\cong Z\oplus(Z/2Z)$, by Theorem 2.11 (applied to the PD_3-complex M_ν). Otherwise $\nu\cong Z$ or D. □

In Chapter 6 we shall strengthen this Corollary to obtain a fibration theorem for 4-manifolds with torsion-free elementary amenable fundamental group.

Corollary 4.5.3 *Let M be a finite PD_4-space with $\chi(M)=0$ and whose fundamental group π is an extension of Z by a normal subgroup $\nu\cong F(r)$. Then M is homotopy equivalent to a closed PL 4-manifold which fibres over the circle, with fibre $\sharp^r S^1\times S^2$ if $w_1(M)|_\nu$ is trivial, and $\sharp^r S^1\tilde{\times}S^2$ otherwise. The bundle is determined by the homotopy type of M.*

Proof Since M_ν is a PD_3-complex with free fundamental group it is homotopy equivalent to $N=\sharp^r S^1\times S^2$ if $w_1(M)|_\nu$ is trivial and to $\sharp^r S^1\tilde{\times}S^2$ otherwise. Every self homotopy equivalence of a connected sum of S^2-bundles over S^1 is homotopic to a self-homeomorphism, and homotopy implies isotopy for such manifolds [La]. Thus M is homotopy equivalent to such a fibred 4-manifold, and the bundle is determined by the homotopy type of M. □

It is easy to see that the natural map from $Homeo(N)$ to $Out(F(r))$ is onto. If a self homeomorphism f of $N=\sharp^r S^1\times S^2$ induces the trivial outer automorphism of $F(r)$ then f is homotopic to a product of twists about nonseparating 2-spheres [Hn]. How is this manifest in the topology of the mapping torus?

Corollary 4.5.4 *Let M be a finite PD_4-space with $\chi(M)=0$ and whose fundamental group π is an extension of Z by a torsion-free normal subgroup ν which is the fundamental group of a closed 3-manifold N. Then M is homotopy equivalent to the mapping torus of a self homeomorphism of N.*

Proof There is a homotopy equivalence $f : N \to M_\nu$, by Turaev's Theorem. (See §5 of Chapter 2.) The indecomposable factors of N are either Haken, hyperbolic or Seifert fibred 3-manifolds, by the Geometrization Conjecture (see [B-P]). Let $t : M_\nu \to M_\nu$ be the generator of the covering transformations. Then there is a self homotopy equivalence $u : N \to N$ such that $fu \sim tf$. As each aspherical factor of N has the property that self homotopy equivalences are homotopic to PL homeomorphisms (by [Hm], Mostow rigidity or [Sc83]), and a similar result holds for $\sharp^r(S^1 \times S^2)$ (by [La]), u is homotopic to a homeomorphism [HL74], and so M is homotopy equivalent to the mapping torus of this homeomorphism. □

The hypothesis that M be finite is redundant in each of the last two corollaries, since $\tilde{K}_0(\mathbb{Z}[\pi]) = 0$. (See Theorem 6.3.) All known PD_3-complexes with torsion-free fundamental group are homotopy equivalent to 3-manifolds.

If the irreducible connected summands of the closed 3-manifold $N = \sharp_i N_i$ are P^2-irreducible and sufficiently large or have fundamental group Z then every self homotopy equivalence of N is realized by an unique isotopy class of homeomorphisms [HL74]. However if N is not aspherical then it admits nontrivial self-homeomorphisms ("rotations about 2-spheres") which induce the identity on ν, and so such bundles are not determined by the group alone.

Let $f : M \to E$ be a homotopy equivalence, where E is a finite PD_4-complex with $\chi(E) = 0$ and fundamental group $\pi = \nu \rtimes Z$, where ν is finitely presentable. Then $w_1(M) = f^*w_1(E)$ and $c_{E*}f_*[M] = \pm c_{E*}[E]$ in $H_4(\pi; Z^{w_1(E)})$. Conversely, if $\chi(M) = 0$ and there is an isomorphism $\theta : \pi_1(M) \cong \pi$ such that $w_1(M) = \theta_i^* w$ and $\theta_{1*}c_{M*}[M] = c_{E*}[E]$ then E_ν and M_ν are PD_3-complexes, by Theorem 4.5. A Wang sequence argument as in the next theorem shows that the fundamental triples of E_ν and M_ν are isomorphic, and so they are homotopy equivalent, by Hendrik's Theorem. What additional conditions are needed to determine the homotopy type of such mapping tori? Our next result is a partial step in this direction.

Theorem 4.6 *Let E be a finite PD_4-complex with $\chi(E) = 0$ and whose fundamental group π is an extension of Z by a finitely presentable normal subgroup ν which is not virtually free. Let $\Pi = \overline{H^2(\pi; \mathbb{Z}[\pi])}$. A PD_4-complex M is homotopy equivalent to E if and only if $\chi(M) = 0$, there is an isomorphism θ from $\pi_1(M)$ to π such that $w_1(M) = w_1(E)\theta$, $\theta^{*-1}k_1(M)$ and $k_1(E)$ generate the same subgroup of $H^3(\pi; \Pi)$ under the action of $Out(\pi) \times Aut_{\mathbb{Z}[\pi]}(\Pi)$, and there is a lift $\hat{c} : M \to P_2(E)$ of θc_M such that $\hat{c}_*[M] = \pm f_{E*}[E]$ in $H_4(P_2(E); Z^{w_1(E)})$.*

Proof The conditions are clearly necessary. Suppose that they hold. The infinite cyclic covering spaces $N = E_\nu$ and M_ν are PD_3-complexes, by Theorem 4.5, and $\pi_2(E) \cong \Pi$ and $\pi_2(M) \cong \theta^*\Pi$, by Theorem 3.4. The maps c_N and c_E induce a homomorphism between the Wang sequence for the fibration of E over S^1 and the corresponding Wang sequence for $K(\pi,1)$. Since ν is not virtually free $H_3(c_N; Z^{w_1(E)})$ is a monomorphism. Hence $H_4(c_E; Z^{w_1(E)})$ and *a fortiori* $H_4(f_E; Z^{w_1(E)})$ are monomorphisms, and so Theorem 3.8 applies. □

As observed in the first paragraph of §9 of Chapter 2, the conditions on θ and the k-invariants also imply that $M_\nu \simeq E_\nu$.

The original version of this book gave an exposition of the extension of Barge's argument to local coefficients for the case when $G \cong Z$, instead of the present Theorem 4.1, and used this together with an L^2-argument, instead of the present Theorem 4.3, to establish the results corresponding to Theorem 4.5 for the case when ν was FP_2.

4.5 Products

If $M = N \times S^1$, where N is a closed 3-manifold, then $\chi(M) = 0$, Z is a direct factor of $\pi_1(M)$, $w_1(M)$ is trivial on this factor and the Pin^--condition $w_2 = w_1^2$ holds. These conditions almost characterize such products up to homotopy equivalence. We need also a constraint on the other direct factor of the fundamental group.

Theorem 4.7 *Let M be a finite PD_4-complex whose fundamental group π has no 2-torsion. Then M is homotopy equivalent to a product $N \times S^1$, where N is a closed 3-manifold, if and only if $\chi(M) = 0$, $w_2(M) = w_1(M)^2$ and there is an isomorphism $\theta : \pi \to \nu \times Z$ such that $w_1(M)\theta^{-1}|_Z = 0$, where ν is a (2-torsion-free) 3-manifold group.*

Proof The conditions are clearly necessary, since the Pin^--condition holds for 3-manifolds.

If these conditions hold then the covering space M_ν with fundamental group ν is a PD_3-complex, by Theorem 4.5 above. Since ν is a 3-manifold group and has no 2-torsion it is a free product of cyclic groups and groups of aspherical closed 3-manifolds. Hence there is a homotopy equivalence $h : M_\nu \to N$, where N is a connected sum of lens spaces and aspherical closed 3-manifolds, by Turaev's Theorem. (See §5 of Chapter 2.) Let ϕ generate the covering group

$Aut(M/M_\nu) \cong Z$. Then there is a self homotopy equivalence $\psi : N \to N$ such that $\psi h \sim h\phi$, and M is homotopy equivalent to the mapping torus $M(\psi)$. We may assume that ψ fixes a basepoint and induces the identity on $\pi_1(N)$, since $\pi_1(M) \cong \nu \times Z$. Moreover ψ preserves the local orientation, since $w_1(M)\theta^{-1}|_Z = 0$. Since ν has no element of order 2 N has no two-sided projective planes and so ψ is homotopic to a rotation about a 2-sphere [Hn]. Since $w_2(M) = w_1(M)^2$ the rotation is homotopic to the identity and so M is homotopy equivalent to $N \times S^1$. □

Let ρ be an essential map from S^1 to $SO(3)$, and let $M = M(\tau)$, where $\tau : S^2 \times S^1 \to S^2 \times S^1$ is the twist map, given by $\tau(x,y) = (\rho(y)(x), y)$ for all (x,y) in $S^2 \times S^1$. Then $\pi_1(M) \cong Z \times Z$, $\chi(M) = 0$, and $w_1(M) = 0$, but $w_2(M) \neq w_1(M)^2 = 0$, so M is not homotopy equivalent to a product. (Clearly however $M(\tau^2) = S^2 \times S^1 \times S^1$.)

To what extent are the constraints on ν necessary? There are orientable 4-manifolds which are homotopy equivalent to products $N \times S^1$ where $\nu = \pi_1(N)$ is finite and is *not* a 3-manifold group. (See Chapter 11.) Theorem 4.1 implies that M is homotopy equivalent to a product of an aspherical PD_3-complex with S^1 if and only if $\chi(M) = 0$ and $\pi_1(M) \cong \nu \times Z$ where ν has one end.

There are 4-manifolds which are simple homotopy equivalent to $S^1 \times RP^3$ (and thus satisfy the hypotheses of our theorem) but which are not homeomorphic to mapping tori [We87].

4.6 Ascendant subgroups

In this brief section we shall give another characterization of aspherical PD_4-complexes with finite covering spaces which are homotopy equivalent to mapping tori.

Theorem 4.8 *Let M be a PD_4-complex. Then M is aspherical and has a finite cover which is homotopy equivalent to a mapping torus if and only if $\chi(M) = 0$ and $\pi = \pi_1(M)$ has an ascendant FP_3 subgroup G of infinite index and such that $H^s(G; \mathbb{Z}[G]) = 0$ for $s \leq 2$. In that case G is a PD_3-group, $[\pi : N_\pi(G)] < \infty$ and $e(N_\pi(G)/G) = 2$.*

Proof The conditions are clearly necessary. Suppose that they hold and that $G = G_0 < G_1 < ... < G_\beth = \pi$ is an ascendant sequence. Let $\gamma = \min\{\alpha \mid [G_\alpha : G] = \infty\}$. Transfinite induction using the LHSSS with coefficients

$\mathbb{Z}[\pi]$ and Theorem 1.15 shows that $H^s(\pi; Z[\pi]) = 0$ for $s \leq 2$. If γ is finite then $\beta_1^{(2)}(G_\gamma) = 0$, since it has a finitely generated normal subgroup of infinite index [Ga00]. Otherwise γ is the first infinite ordinal, and $[G_{j+1} : G_j] < \infty$ for all $j < \gamma$. In this case $\beta_1^{(2)}(G_n) = \beta_1^{(2)}(G)/[G_n : G]$ and so $\lim_{n\to\infty} \beta_1^{(2)}(G_n) = 0$. It then follows from Theorems 6.13 and 6.54(7) of [Lü] that $\beta_1^{(2)}(G_\gamma) = 0$. In either case it then follows that $\beta_1^{(2)}(G_\alpha) = 0$ for all $\gamma \leq \alpha \leq \daleth$ by Theorem 2.3 (which is part of Theorem 7.2 of [Lü]). Hence M is aspherical, by Theorem 3.5.

On the other hand $H^s(G_\gamma; W) = 0$ for $s \leq 3$ and any free $\mathbb{Z}[G_\gamma]$-module W, so $c.d.G_\gamma = 4$. Hence $[\pi : G_\gamma] < \infty$, by Strebel's Theorem. Therefore G_γ is a PD_4-group. In particular, it is finitely generated and so $\gamma < \infty$. If $\gamma = \beta + 1$ then $[G_\beta : G] < \infty$. It follows easily that $[\pi : N_\pi(G)] < \infty$. Hence G is a PD_3-group and $N_\pi(G)/G$ has two ends, by Theorem 3.10. $\square$

The hypotheses on G could be replaced by "*G is a PD_3-group*", for then $[\pi : G] = \infty$, by Theorem 3.12.

We shall establish an analogous result for PD_4-complexes M such that $\chi(M) = 0$ and $\pi_1(M)$ has an ascendant subgroup of infinite index which is a PD_2-group in Chapter 5.

4.7 Circle bundles

In this section we shall consider the "dual" situation, of PD_4-complexes which are homotopy equivalent to the total space of a S^1-bundle over a 3-dimensional base N. Lemma 4.9 presents a number of conditions satisfied by such spaces. (These conditions are not all independent.) Bundles $c_N^*\xi$ induced from S^1-bundles over $K(\pi_1(N), 1)$ are given equivalent characterizations in Lemma 4.10. In Theorem 4.11 we shall show that the conditions of Lemmas 4.9 and 4.10 characterize the homotopy types of such bundle spaces $E(c_N^*\xi)$, provided $\pi_1(N)$ is torsion-free but not free.

Since $BS^1 \simeq K(Z, 2)$ any S^1-bundle over a connected base B is induced from some bundle over $P_2(B)$. For each epimorphism $\gamma : \mu \to \nu$ with cyclic kernel and such that the action of μ by conjugation on $\mathrm{Ker}(\gamma)$ factors through multiplication by ± 1 there is an S^1-bundle $p(\gamma) : X(\gamma) \to Y(\gamma)$ whose fundamental group sequence realizes γ and which is universal for such bundles; the total space $E(p(\gamma))$ is a $K(\mu, 1)$ space (cf. Proposition 11.4 of [Wl]).

Lemma 4.9 *Let $p : E \to B$ be the projection of an S^1-bundle ξ over a connected cell complex B. Then*

(1) $\chi(E) = 0$;

(2) *the natural map $p_* : \pi = \pi_1(E) \to \nu = \pi_1(B)$ is an epimorphism with cyclic kernel, and the action of ν on $\mathrm{Ker}(p_*)$ induced by conjugation in π is given by $w = w_1(\xi) : \pi_1(B) \to Z/2Z \cong \{\pm 1\} \leq Aut(\mathrm{Ker}(p_*))$;*

(3) *if B is a PD-complex $w_1(E) = p^*(w_1(B) + w)$;*

(4) *if B is a PD_3-complex there are maps $\hat{c} : E \to P_2(B)$ and $y : P_2(B) \to Y(p_*)$ such that $c_{P_2(B)} = c_{Y(p_*)}y$, $y\hat{c} = p(p_*)c_E$ and $(\hat{c}, c_E)_*[E] = \pm G(f_{B*}[B])$ where G is the Gysin homomorphism from $H_3(P_2(B); Z^{w_1(B)})$ to $H_4(P_2(E); Z^{w_1(E)})$;*

(5) *If B is a PD_3-complex $c_{E*}[E] = \pm G(c_{B*}[B])$, where G is the Gysin homomorphism from $H_3(\nu; Z^{w_B})$ to $H_4(\pi; Z^{w_E})$;*

(6) $\mathrm{Ker}(p_*)$ *acts trivially on $\pi_2(E)$.*

Proof Condition(1) follows from the multiplicativity of the Euler characteristic in a fibration. If α is any loop in B the total space of the induced bundle $\alpha^*\xi$ is the torus if $w(\alpha) = 0$ and the Klein bottle if $w(\alpha) = 1$ in $Z/2Z$; hence $gzg^{-1} = z^{\epsilon(g)}$ where $\epsilon(g) = (-1)^{w(p_*(g))}$ for g in $\pi_1(E)$ and z in $\mathrm{Ker}(p_*)$. Conditions (2) and (6) then follow from the exact homotopy sequence. If the base B is a PD-complex then so is E, and we may use naturality and the Whitney sum formula (applied to the Spivak normal bundles) to show that $w_1(E) = p^*(w_1(B) + w_1(\xi))$. (As $p^* : H^1(B; \mathbb{F}_2) \to H^1(E; \mathbb{F}_2)$ is a monomorphism this equation determines $w_1(\xi)$.)

Condition (4) implies (5), and follows from the observations in the paragraph preceding the lemma. (Note that the Gysin homomorphisms G in (4) and (5) are well defined, since $H_1(\mathrm{Ker}(\gamma); Z^{w_E})$ is isomorphic to Z^{w_B}, by (3).) □

Bundles with $\mathrm{Ker}(p_*) \cong Z$ have the following equivalent characterizations.

Lemma 4.10 *Let $p : E \to B$ be the projection of an S^1-bundle ξ over a connected cell complex B. Then the following conditions are equivalent:*

(1) *ξ is induced from an S^1-bundle over $K(\pi_1(B), 1)$ via c_B;*

(2) *for each map $\beta : S^2 \to B$ the induced bundle $\beta^*\xi$ is trivial;*

(3) *the induced epimorphism $p_* : \pi_1(E) \to \pi_1(B)$ has infinite cyclic kernel.*

If these conditions hold then $c(\xi) = c_B^*\Xi$, where $c(\xi)$ is the characteristic class of ξ in $H^2(B;Z^w)$ and Ξ is the class of the extension of fundamental groups in $H^2(\pi_1(B);Z^w) = H^2(K(\pi_1(B),1);Z^w)$, where $w = w_1(\xi)$.

Proof Condition (1) implies condition (2) as for any such map β the composite $c_B\beta$ is nullhomotopic. Conversely, as we may construct $K(\pi_1(B),1)$ by adjoining cells of dimension ≥ 3 to B condition (2) implies that we may extend ξ over the 3-cells, and as S^1-bundles over S^n are trivial for all $n > 2$ we may then extend ξ over the whole of $K(\pi_1(B),1)$, so that (2) implies (1). The equivalence of (2) and (3) follows on observing that (3) holds if and only if $\partial\beta = 0$ for all such β, where ∂ is the connecting map from $\pi_2(B)$ to $\pi_1(S^1)$ in the exact sequence of homotopy for ξ, and on comparing this with the corresponding sequence for $\beta^*\xi$.

As the natural map from the set of S^1-bundles over $K(\pi,1)$ with $w_1 = w$ (which are classified by $H^2(K(\pi,1);Z^w)$) to the set of extensions of π by Z with π acting via w (which are classified by $H^2(\pi;Z^w)$) which sends a bundle to the extension of fundamental groups is an isomorphism we have $c(\xi) = c_B^*(\Xi)$. □

If N is a closed 3-manifold which has no summands of type $S^1 \times S^2$ or $S^1\tilde{\times}S^2$ (i.e., if $\pi_1(N)$ has no infinite cyclic free factor) then every S^1-bundle over N with $w = 0$ restricts to a trivial bundle over any map from S^2 to N. For if ξ is such a bundle, with characteristic class $c(\chi)$ in $H^2(N;\mathbb{Z})$, and $\beta : S^2 \to N$ is any map then $\beta_*(c(\beta^*\xi) \cap [S^2]) = \beta_*(\beta^*c(\xi) \cap [S^2]) = c(\xi) \cap \beta_*[S^2] = 0$, as the Hurewicz homomorphism is trivial for such N. Since β_* is an isomorphism in degree 0 it follows that $c(\beta^*\xi) = 0$ and so $\beta^*\xi$ is trivial. (A similar argument applies for bundles with $w \neq 0$, provided the induced 2-fold covering space N^w has no summands of type $S^1 \times S^2$ or $S^1\tilde{\times}S^2$.)

On the other hand, if η is the Hopf fibration the bundle with total space $S^1 \times S^3$, base $S^1 \times S^2$ and projection $id_{S^1} \times \eta$ has nontrivial pullback over any essential map from S^2 to $S^1 \times S^2$, and is not induced from any bundle over $K(Z,1)$. Moreover, $S^1 \times S^2$ is a 2-fold covering space of $RP^3\sharp RP^3$, and so the above hypothesis on summands of N is not stable under passage to 2-fold coverings (corresponding to a homomorphism w from $\pi_1(N)$ to $Z/2Z$).

Theorem 4.11 *Let M be a PD_4-complex and N a PD_3-complex whose fundamental group is torsion-free but not free. Then M is homotopy equivalent to the total space of an S^1-bundle over N which satisfies the conditions of Lemma* 4.10 *if and only if*

(1) $\chi(M) = 0$;

(2) *there is an epimorphism* $\gamma : \pi = \pi_1(M) \to \nu = \pi_1(N)$ *with* $\mathrm{Ker}(\gamma) \cong Z$;

(3) $w_1(M) = (w_1(N) + w)\gamma$, *where* $w : \nu \to Z/2Z \cong Aut(\mathrm{Ker}(\gamma))$ *is determined by the action of* ν *on* $\mathrm{Ker}(\gamma)$ *induced by conjugation in* π;

(4) $k_1(M) = \gamma^* k_1(N)$ *(and so* $P_2(M) \simeq P_2(N) \times_{K(\nu,1)} K(\pi, 1)$*)*;

(5) $f_{M*}[M] = \pm G(f_{N*}[N])$ *in* $H_4(P_2(M); Z^{w_1(M)})$, *where* G *is the Gysin homomorphism in degree* 3.

If these conditions hold then M *has minimal Euler characteristic for its fundamental group, i.e.,* $q(\pi) = 0$.

Remark The first three conditions and Poincaré duality imply that $\pi_2(M) \cong \gamma^*\pi_2(N)$, the $\mathbb{Z}[\pi]$-module with the same underlying group as $\pi_2(N)$ and with $\mathbb{Z}[\pi]$-action determined by the homomorphism γ.

Proof Since these conditions are homotopy invariant and hold if M is the total space of such a bundle, they are necessary. Suppose conversely that they hold. As ν is torsion-free N is the connected sum of a 3-manifold with free fundamental group and some aspherical PD_3-complexes [Tu90]. As ν is not free there is at least one aspherical summand. Hence $c.d.\nu = 3$ and $H_3(c_N; Z^{w_1(N)})$ is a monomorphism.

Let $p(\gamma) : K(\pi, 1) \to K(\nu, 1)$ be the S^1-bundle corresponding to γ and let $E = N \times_{K(\nu,1)} K(\pi, 1)$ be the total space of the S^1-bundle over N induced by the classifying map $c_N : N \to K(\nu, 1)$. The bundle map covering c_N is the classifying map c_E. Then $\pi_1(E) \cong \pi = \pi_1(M)$, $w_1(E) = (w_1(N) + w)\gamma = w_1(M)$, as maps from π to $Z/2Z$, and $\chi(E) = 0 = \chi(M)$, by conditions (1) and (3). The maps c_N and c_E induce a homomorphism between the Gysin sequences of the S^1-bundles. Since N and ν have cohomological dimension 3 the Gysin homomorphisms in degree 3 are isomorphisms. Hence $H_4(c_E; Z^{w_1(E)})$ is a monomorphism, and so *a fortiori* $H_4(f_E; Z^{w_1(E)})$ is also a monomorphism.

Since $\chi(M) = 0$ and $\beta_1^{(2)}(\pi) = 0$, by Theorem 2.3, part (3) of Theorem 3.4 implies that $\pi_2(M) \cong \overline{H^2(\pi; \mathbb{Z}[\pi])}$. It follows from conditions (2) and (3) and the LHSSS that $\pi_2(M) \cong \pi_2(E) \cong \gamma^*\pi_2(N)$ as $\mathbb{Z}[\pi]$-modules. Conditions (4) and (5) then give us a map $(\hat{c}, c_M)$ from M to $P_2(E) = P_2(N) \times_{K(\nu,1)} K(\pi, 1)$ such that $(\hat{c}, c_M)_*[M] = \pm f_{E*}[E]$. Hence M is homotopy equivalent to E, by Theorem 3.8.

The final assertion now follows from part (1) of Theorem 3.4. □

As $\pi_2(N)$ is a projective $\mathbb{Z}[\nu]$-module, by Theorem 2.18, it is homologically trivial and so $H_q(\pi;\gamma^*\pi_2(N)\otimes Z^{w_1(M)}) = 0$ if $q \geq 2$. Hence it follows from the spectral sequence for $c_{P_2(M)}$ that $H_4(P_2(M);Z^{w_1(M)})$ maps onto $H_4(\pi;Z^{w_1(M)})$, with kernel isomorphic to $H_0(\pi;\Gamma(\pi_2(M)))\otimes Z^{w_1(M)})$, where $\Gamma(\pi_2(M)) = H_4(K(\pi_2(M),2);\mathbb{Z})$ is Whitehead's universal quadratic construction on $\pi_2(M)$. (See Chapter I of [Ba'].) This suggests that there may be another formulation of the theorem in terms of conditions (1-3), together with some information on $k_1(M)$ and the intersection pairing on $\pi_2(M)$. If N is aspherical conditions (4) and (5) are vacuous or redundant.

Condition (4) is vacuous if ν is a free group, for then $c.d.\pi \leq 2$. In this case the Hurewicz homomorphism from $\pi_3(N)$ to $H_3(N;Z^{w_1(N)})$ is 0, and so $H_3(f_N;Z^{w_1(N)})$ is a monomorphism. The argument of the theorem would then extend if the Gysin map in degree 3 for the bundle $P_2(E)\to P_2(N)$ were a monomorphism. If $\nu = 1$ then M is orientable, $\pi\cong Z$ and $\chi(M) = 0$, so $M\simeq S^3\times S^1$. In general, if the restriction on ν is removed it is not clear that there should be a degree 1 map from M to such a bundle space E.

It would be of interest to have a theorem with hypotheses involving only M, without reference to a model N. There is such a result in the aspherical case.

Theorem 4.12 *A finite PD_4-complex M is homotopy equivalent to the total space of an S^1-bundle over an aspherical PD_3-complex if and only if $\chi(M) = 0$ and $\pi = \pi_1(M)$ has an infinite cyclic normal subgroup A such that π/A has one end and finite cohomological dimension.*

Proof The conditions are clearly necessary. Conversely, suppose that they hold. Since π/A has one end $H^s(\pi/A;\mathbb{Z}[\pi/A]) = 0$ for $s\leq 1$ and so an LHSSS calculation gives $H^t(\pi;\mathbb{Z}[\pi]) = 0$ for $t\leq 2$. Moreover $\beta_1^{(2)}(\pi) = 0$, by Theorem 2.3. Hence M is aspherical and π is a PD_4-group, by Corollary 3.5.2. Since A is FP_∞ and $c.d.\pi/A<\infty$ the quotient π/A is a PD_3-group, by Theorem 9.11 of [Bi]. Therefore M is homotopy equivalent to the total space of an S^1-bundle over the PD_3-complex $K(\pi/A,1)$. □

Note that a finitely generated torsion-free group has one end if and only if it is indecomposable as a free product and is neither infinite cyclic nor trivial.

In general, if M is homotopy equivalent to the total space of an S^1-bundle over some 3-manifold then $\chi(M) = 0$ and $\pi_1(M)$ has an infinite cyclic normal subgroup A such that $\pi_1(M)/A$ is virtually of finite cohomological dimension. Do these conditions characterize such homotopy types?

Chapter 5

Surface bundles

In this chapter we shall show that a closed 4-manifold M is homotopy equivalent to the total space of a fibre bundle with base and fibre closed surfaces if and only if the obviously necessary conditions on the Euler characteristic and fundamental group hold. When the base is S^2 we need also conditions on the characteristic classes of M, and when the base is RP^2 our results are incomplete. We shall defer consideration of bundles over RP^2 with fibre T or Kb and $\partial \neq 0$ to Chapter 11, and those with fibre S^2 or RP^2 to Chapter 12.

5.1 Some general results

If B, E and F are connected finite complexes and $p : E \to B$ is a Hurewicz fibration with fibre homotopy equivalent to F then $\chi(E) = \chi(B)\chi(F)$ and the long exact sequence of homotopy gives an exact sequence

$$\pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 1$$

in which the image of $\pi_2(B)$ under the *connecting homomorphism* ∂ is in the centre of $\pi_1(F)$. (See page 51 of [Go68].) These conditions are clearly homotopy invariant.

Hurewicz fibrations with base B and fibre X are classified by homotopy classes of maps from B to the Milgram classifying space $BE(X)$, where $E(X)$ is the monoid of all self homotopy equivalences of X, with the compact-open topology [Mi67]. If X has been given a base point the evaluation map from $E(X)$ to X is a Hurewicz fibration with fibre the subspace (and submonoid) $E_0(X)$ of base point preserving self homotopy equivalences [Go68].

Let T and Kb denote the torus and Klein bottle, respectively.

Lemma 5.1 *Let F be an aspherical closed surface and B a closed smooth manifold. There are natural bijections from the set of isomorphism classes of smooth F-bundles over B to the set of fibre homotopy equivalence classes of Hurewicz fibrations with fibre F over B and to the set $\coprod_{[\xi]} H^2(B;\zeta\pi_1(F)^\xi)$, where the union is over conjugacy classes of homomorphisms $\xi : \pi_1(B) \to Out(\pi_1(F))$ and $\zeta\pi_1(F)^\xi$ is the $\mathbb{Z}[\pi_1(F)]$-module determined by ξ.*

Proof If $\zeta\pi_1(F) = 1$ the identity components of $Diff(F)$ and $E(F)$ are contractible [EE69]. Now every automorphism of $\pi_1(F)$ is realizable by a diffeomorphism and homotopy implies isotopy for self diffeomorphisms of surfaces. (See Chapter V of [ZVC].) Therefore $\pi_0(Diff(F)) \cong \pi_0(E(F)) \cong Out(\pi_1(F))$, and the inclusion of $Diff(F)$ into $E(F)$ is a homotopy equivalence. Hence $BDiff(F) \simeq BE(F) \simeq K(Out(\pi_1(F),1)$, so smooth F-bundles over B and Hurewicz fibrations with fibre F over B are classified by the (unbased) homotopy set

$$[B, K(Out(\pi_1(F),1))] = Hom(\pi_1(B), Out(\pi_1(F)))/\backsim,$$

where $\xi \backsim \xi'$ if there is an $\alpha \in Out(\pi_1(F))$ such that $\xi'(b) = \alpha\xi(b)\alpha^{-1}$ for all $b \in \pi_1(B)$.

If $\zeta\pi_1(F) \neq 1$ then $F = T$ or Kb. Left multiplication by T on itself induces homotopy equivalences from T to the identity components of $Diff(T)$ and $E(T)$. (Similarly, the standard action of S^1 on Kb induces homotopy equivalences from S^1 to the identity components of $Diff(Kb)$ and $E(Kb)$. See Theorem III.2 of [Go65].) Let $\alpha : GL(2,\mathbb{Z}) \to Aut(T) \leq Diff(T)$ be the standard linear action. Then the natural maps from the semidirect product $T \rtimes_\alpha GL(2,\mathbb{Z})$ to $Diff(T)$ and to $E(T)$ are homotopy equivalences. Therefore $BDiff(T)$ is a $K(Z^2,2)$-fibration over $K(GL(2,\mathbb{Z}),1)$. It follows that T-bundles over B are classified by two invariants: a conjugacy class of homomorphisms $\xi : \pi_1(B) \to GL(2,\mathbb{Z})$ together with a cohomology class in $H^2(B;(Z^2)^\xi)$. A similar argument applies if $F = Kb$. □

Theorem 5.2 *Let M be a PD_4-complex and B and F aspherical closed surfaces. Then M is homotopy equivalent to the total space of an F-bundle over B if and only if $\chi(M) = \chi(B)\chi(F)$ and $\pi = \pi_1(M)$ is an extension of $\pi_1(B)$ by $\pi_1(F)$. Moreover every extension of $\pi_1(B)$ by $\pi_1(F)$ is realized by some surface bundle, which is determined up to isomorphism by the extension.*

Proof The conditions are clearly necessary. Suppose that they hold. If $\zeta\pi_1(F) = 1$ each homomorphism $\xi : \pi_1(B) \to Out(\pi_1(F))$ corresponds to an unique equivalence class of extensions of $\pi_1(B)$ by $\pi_1(F)$, by Proposition 11.4.21 of [Ro]. Hence there is an F-bundle $p : E \to B$ with $\pi_1(E) \cong \pi$ realizing the extension, and p is unique up to bundle isomorphism. If $F = T$ then every homomorphism $\xi : \pi_1(B) \to GL(2,\mathbb{Z})$ is realizable by an extension (for instance, the semidirect product $Z^2 \rtimes_\xi \pi_1(B)$) and the extensions realizing ξ are classified up to equivalence by $H^2(\pi_1(B);(Z^2)^\xi)$. As B is aspherical the natural map from bundles to group extensions is a bijection. Similar arguments apply if $F = Kb$. In all cases the bundle space E is aspherical, and so π is an FF PD_4-group. Such extensions satisfy the Weak Bass Conjecture, by Theorem 5.7 of [Co95]. Hence $M \simeq E$, by Corollary 3.5.1. □

Such extensions (with $\chi(F) < 0$) were shown to be realizable by bundles in [Jo79].

5.2 Bundles with base and fibre aspherical surfaces

In many cases the group $\pi_1(M)$ determines the bundle up to diffeomorphism of its base. Lemma 5.3 and Theorems 5.4 and 5.5 are based on [Jo94].

Lemma 5.3 *Let G_1 and G_2 be groups with no nontrivial abelian normal subgroup. If H is a normal subgroup of $G = G_1 \times G_2$ which contains no nontrivial direct product then either $H \leq G_1 \times \{1\}$ or $H \leq \{1\} \times G_2$.*

Proof Let P_i be the projection of H onto G_i, for $i = 1, 2$. If $(h, h') \in H$, $g_1 \in G_1$ and $g_2 \in G_2$ then $([h, g_1], 1) = [(h, h'), (g_1, 1)]$ and $(1, [h', g_2])$ are in H. Hence $[P_1, P_1] \times [P_2, P_2] \leq H$. Therefore either P_1 or P_2 is abelian, and so is trivial, since P_i is normal in G_i, for $i = 1, 2$. □

Theorem 5.4 *Let π be a group with a normal subgroup K such that K and π/K are PD_2-groups with trivial centres.*

1. *If $C_\pi(K) = 1$ and K_1 is a finitely generated normal subgroup of π then $C_\pi(K_1) = 1$ also.*
2. *The index $[\pi : KC_\pi(K)]$ is finite if and only if π is virtually a direct product of PD_2-groups.*

Proof (1) Let $z \in C_\pi(K_1)$. If $K_1 \leq K$ then $[K : K_1] < \infty$ and $\zeta K_1 = 1$. Let $M = [K : K_1]!$. Then $f(k) = k^{-1}z^M kz^{-M}$ is in K_1 for all k in K. Now $f(kk_1) = k_1^{-1}f(k)k_1$ and also $f(kk_1) = f(kk_1k^{-1}k) = f(k)$ (since K_1 is a normal subgroup centralized by z), for all k in K and k_1 in K_1. Hence $f(k)$ is central in K_1, and so $f(k) = 1$ for all k in K. Thus z^M centralizes K. Since π is torsion-free we must have $z = 1$. Otherwise the image of K_1 under the projection $p : \pi \to \pi/K$ is a nontrivial finitely generated normal subgroup of π/K, and so has trivial centralizer. Hence $p(z) = 1$. Now $[K, K_1] \leq K \cap K_1$ and so $K \cap K_1 \neq 1$, for otherwise $K_1 \leq C_\pi(K)$. Since z centralizes the nontrivial normal subgroup $K \cap K_1$ in K we must again have $z = 1$.

(2) Since K has trivial centre $KC_\pi(K) \cong K \times C_\pi(K)$ and so the condition is necessary. Suppose that $f : G_1 \times G_2 \to \pi$ is an isomorphism onto a subgroup of finite index, where G_1 and G_2 are PD_2-groups. Let $H = K \cap f(G_1 \times G_2)$. Then $[K : H] < \infty$ and so H is also a PD_2-group, and is normal in $f(G_1 \times G_2)$. We may assume that $H \leq f(G_1)$, by Lemma 5.3. Then $f(G_1)/H$ is finite and is isomorphic to a subgroup of $f(G_1 \times G_2)/K \leq \pi/K$, so $H = f(G_1)$. Now $f(G_2)$ normalizes K and centralizes H, and $[K : H] < \infty$. Hence $f(G_2)$ has a subgroup of finite index which centralizes K, as in part (1). Hence $[\pi : KC_\pi(K)] < \infty$. □

It follows immediately that if π and K are as in the theorem whether

(1) $C_\pi(K) \neq 1$ and $[\pi : KC_\pi(K)] = \infty$;

(2) $[\pi : KC_\pi(K)] < \infty$; or

(3) $C_\pi(K) = 1$

depends only on π and not on the subgroup K. In [Jo94] these cases are labeled as types I, II and III, respectively. (In terms of the action $\xi : \pi/K \to Out(K)$: if $\mathrm{Im}(\xi)$ is infinite and $\mathrm{Ker}(\xi) \neq 1$ then π is of type I, if $\mathrm{Im}(\xi)$ is finite then π is of type II, and if ξ is injective then π is of type III.)

Theorem 5.5 *Let π be a group with a normal subgroup K such that K and π/K are virtually PD_2-groups with no non-trivial finite normal subgroup. If $\sqrt{\pi} = 1$ and $C_\pi(K) \neq 1$ then π has at most one other nontrivial finitely generated normal subgroup $K_1 \neq K$ which contains no nontrivial direct product and is such that π/K_1 has no non-trivial finite normal subgroup. In that case $K_1 \cap K = 1$ and $[\pi : KC_\pi(K)] < \infty$.*

Proof Let $p : \pi \to \pi/K$ be the quotient epimorphism. Then $p(C_\pi(K))$ is a nontrivial normal subgroup of π/K, since $K \cap C_\pi(K) = \zeta K = 1$. Suppose that $K_1 < \pi$ is a nontrivial finitely generated normal subgroup which contains no nontrivial direct product and is such that π/K_1 has no non-trivial finite normal subgroup. Let $\Sigma = K_1 \cap (KC_\pi(K))$. Since Σ is normal in $KC_\pi(K) \cong K \times C_\pi(K)$ and $\Sigma \leq K_1$ we must have either $\Sigma \leq K$ or $\Sigma \leq C_\pi(K)$, by Lemma 5.3.

If $\Sigma \leq K$ then $p(K_1) \cap p(C_\pi(K)) = 1$, and so $p(K_1)$ centralizes the nontrivial normal subgroup $p(C_\pi(K))$ in π/K. Therefore $K_1 \leq K$ and so $[K : K_1] < \infty$. Since π/K_1 has no non-trivial finite normal subgroup we find $K_1 = K$.

If $\Sigma \leq C_\pi(K)$ then $K_1 \cap K = 1$. Hence $[K, K_1] = 1$, since each subgroup is normal in π, and so $K_1 \leq C_\pi(K)$. Moreover $[\pi/K : p(K_1)] < \infty$ since $p(K_1)$ is a nontrivial finitely generated normal subgroup of π/K, and so K_1 and $C_\pi(K)$ are PD_2-groups and $[\pi : KC_\pi(K)] =]\pi/K : p(C_\pi(K))] \leq [\pi/K : p(K_1)] < \infty$.

If $K_1 \neq K$ and K_2 is another such subgroup of π then K_2 also has finite index in $C_\pi(K)$, by the same argument. Since π/K_1 and π/K_2 have no non-trivial finite normal subgroup it follows that $K_1 = K_2$. □

Corollary 5.5.1 [Jo93] *Let α and β be automorphisms of π, and suppose that $\alpha(K) \cap K = 1$. Then $\beta(K) = K$ or $\alpha(K)$. In particular, $Aut(K \times K) \cong Aut(K)^2 \rtimes (Z/2Z)$.* □

Groups of type I have an unique such normal subgroup K, while groups of type II have at most two such subgroups, by Theorem 5.5. We shall obtain a somewhat weaker result for groups of type III as a corollary of Theorem 5.6.

We shall use the following corollary in Chapter 9.

Corollary 5.5.2 *Let π be a PD_4-group such that $\sqrt{\pi} = 1$. Then the following conditions are equivalent:*

(1) *π has a subgroup $\rho \cong \alpha \times \beta$ where α and β are PD_2-groups;*

(2) *π has a normal subgroup $\sigma \cong K \times L$ of finite index where K and L are PD_2-groups and $[\pi : N_\pi(K)] \leq 2$;*

(3) *π has a subgroup τ such that $[\pi : \tau] \leq 2$ and $\tau \leq G \times H$ where G and H are virtually PD_2-groups.*

Proof Suppose that (1) holds. Then $[\pi : \rho] < \infty$, by Strebel's Theorem. Let N be the intersection of the conjugates of ρ in π. Then N is normal in π and $[\pi : N] < \infty$. We shall identify $\alpha \cong \alpha \times \{1\}$ and $\beta \cong \{1\} \times \beta$ with subgroups of π. Let $K = \alpha \cap N$ and $L = \beta \cap N$. Then K and L are PD_2-groups, $K \cap L = 1$ and $\sigma = K.L \cong K \times L$ is normal in N and has finite index in π. Moreover N/K and N/L are isomorphic to subgroups of finite index in β and α, respectively, and so are also PD_2-groups. If $\sqrt{\pi} = 1$ all these groups have trivial centre, and so any automorphism of N must either fix K and L or interchange them, by Theorem 5.5. Hence σ is normal in π and $[\pi : N_\pi(K)] \leq 2$.

If (2) holds then $N_\pi(K) = N_\pi(L)$. Let $\tau = N_\pi(K)$ and let $p_G : \tau \to G = \tau/C_\pi(K)$ and $p_H : \tau \to H = \tau/C_\pi(L)$ be the natural epimorphisms. Then $p_G|_K$, $p_H|_L$ and (p_G, p_H) are injective and have images of finite index in G, H and $G \times H$ respectively. In particular, G and H are virtually PD_2-groups.

If (3) holds let $\alpha = \tau \cap (G \times \{1\})$ and $\beta = \tau \cap (\{1\} \times H)$. Then α and β have finite index in G and H, respectively, and are torsion-free. Hence they are PD_2-groups and clearly $\alpha \cap \beta = 1$. Therefore $\rho = \alpha.\beta \cong \alpha \times \beta$. □

It can be shown that these three conditions remain equivalent under the weaker hypothesis that π be a PD_4-group which is not virtually abelian (using Lemma 9.4 for the implication (1) $\Rightarrow$ (3)).

Theorem 5.6 *Let π be a group with normal subgroups K and K_1 such that K, K_1 and π/K are PD_2-groups, π/K_1 is torsion-free and $\chi(\pi/K)<0$. Then either $K_1=K$ or $K_1\cap K=1$ and $\pi\cong K\times K_1$ or $\chi(K_1)<\chi(\pi/K)$.*

Proof Let $p:\pi\to\pi/K$ be the quotient epimorphism. If $K_1\leq K$ then $K_1=K$, as in Theorem 5.5. Otherwise $p(K_1)$ has finite index in π/K and so $p(K_1)$ is also a PD_2-group. As the minimum number of generators of a PD_2-group G is $\beta_1(G;\mathbb{F}_2)$, we have $\chi(K_1)\leq\chi(p(K_1))\leq\chi(\pi/K)$. We may assume that $\chi(K_1)\geq\chi(\pi/K)$. Hence $\chi(K_1)=\chi(\pi/K)$ and so $p|_{K_1}$ is an epimorphism. Therefore K_1 and π/K have the same orientation type, by the nondegeneracy of Poincaré duality with coefficients $\mathbb{F}_2$ and the Wu relation $w_1\cup x=x^2$ for all $x\in H^1(G;\mathbb{F}_2)$ and PD_2-groups G. Hence $K_1\cong\pi/K$. Since PD_2-groups are hopfian $p|_{K_1}$ is an isomorphism. Hence $[K,K_1]\leq K\cap K_1=1$ and so $\pi=K.K_1\cong K\times\pi/K$. □

Corollary 5.6.1 [Jo99] *There are only finitely many such subgroups $K<\pi$.*

Proof We may assume that $\zeta K=1$ and π is of type III. There is an epimorphism $\rho:\pi\to Z/\chi(\pi)Z$ such that $\rho(K)=0$. Then $\chi(\mathrm{Ker}(\rho))=\chi(\pi)^2$. Since π is not virtually a product K is the only normal PD_2-subgroup of $\mathrm{Ker}(\rho)$ with quotient a PD_2-group and such that $\chi(K)^2\leq\chi(\mathrm{Ker}(\rho))$. The corollary follows since there are only finitely many such epimorphisms ρ. □

See §14 of Chapter V of [BHPV] for examples of type III admitting at least two such normal subgroups. The next corollary follows by elementary arithmetic.

Corollary 5.6.2 *If $K_1\neq K$ and $\chi(K_1)=-1$ then $\pi\cong K\times K_1$.* □

Corollary 5.6.3 *Let M and M' be the total spaces of bundles ξ and ξ' with the same base B and fibre F, where B and F are aspherical closed surfaces such that $\chi(B)<\chi(F)$. Then M' is diffeomorphic to M via a fibre-preserving diffeomorphism if and only if $\pi_1(M')\cong\pi_1(M)$.* □

Compare the statement of Melvin's Theorem on total spaces of S^2-bundles (Theorem 5.13 below.)

We can often recognise total spaces of aspherical surface bundles under weaker hypotheses on the fundamental group.

Theorem 5.7 *Let M be a PD_4-complex with fundamental group π. Then the following conditions are equivalent:*

(1) *M is homotopy equivalent to the total space of a bundle with base and fibre aspherical closed surfaces:*

(2) *π has an FP_2 normal subgroup K such that π/K is a PD_2-group and $\pi_2(M) = 0$;*

(3) *π has a normal subgroup N which is a PD_2-group, π/N is torsion-free and $\pi_2(M) = 0$.*

Proof Clearly (1) implies (2) and (3). Conversely they each imply that π has one end and so M is aspherical. If K is an FP_3 normal subgroup in π and π/K is a PD_2-group then K is a PD_2-group, by Theorem 1.19. If N is a normal subgroup which is a PD_2-group then π/N is virtually a PD_2-group, by Theorem 3.10. Since it is torsion-free it is a PD_2-group and so the theorem follows from Theorem 5.2. □

If $\zeta N = 1$ then π/N is an extension of $C_\pi(N)$ by a subgroup of $Out(N)$. Thus we may argue instead that $v.c.d.\pi/N < \infty$ and π/N is FP_∞, so π/N is virtually a PD_2-group, by Theorem 9.11 of [Bi].

Corollary 5.7.1 *The PD_4-complex M is homotopy equivalent to the total space of a T- or Kb-bundle over an aspherical closed surface, if and only if $\chi(M) = 0$ and π has a normal subgroup $A \cong Z^2$ or $Z \rtimes_{-1} Z$ such that π/A is torsion free.*

Proof The conditions are clearly necessary. If they hold then M is aspherical, by Theorem 2.2 and Corollary 3.5.2, and so this corollary follows from part (3) of Theorem 5.7 □

Kapovich has given examples of aspherical closed 4-manifolds M such that $\pi_1(M)$ is an extension of a PD_2-group by a finitely generated normal subgroup which is not FP_2 [Ka13].

Theorem 5.8 *Let M be a PD_4-complex whose fundamental group π has an ascendant FP_2 subgroup G of infinite index with one end and such that $\chi(M) = 0$. Then M is aspherical. If moreover $c.d.G = 2$ and $\chi(G) \neq 0$ then G is a PD_2-group and either $[\pi : N_\pi(G)] < \infty$ or there is a subnormal chain $G < J < K \leq \pi$ such that $[\pi : K] < \infty$ and $K/J \cong J/G \cong Z$.*

Proof The argument of the first paragraph of the proof of Theorem 4.8 applies equally well here to show that M is aspherical.

Assume henceforth that $c.d.G = 2$ and $\chi(G) < 0$. If $G < \tilde{G} < G_\gamma$ and $c.d.\tilde{G} = 2$ then $[\tilde{G} : G] < \infty$, by Lemma 2.15. Hence $\tilde{G}$ is FP and $[\tilde{G} : G] \leq |\chi(G)|$, since $\chi(G) = [\tilde{G} : G]\chi(\tilde{G})$. We may assume that $\tilde{G}$ is maximal among all groups of cohomological dimension 2 in an ascendant chain from G to π. Let $G = G_0 < G_1 < ... < G_\daleth = \pi$ be such an ascendant chain, with $\tilde{G} = G_n$ for some finite ordinal n. Then $[G_{n+1} : G] = \infty$ and $c.d.G_{n+1} \geq 3$.

If $\tilde{G}$ is normal in π then $\tilde{G}$ is a PD_2-group and $\pi/\tilde{G}$ is virtually a PD_2-group, by Theorem 3.10. Moreover $[\pi : N_\pi(G)] < \infty$, since $\tilde{G}$ has only finitely many subgroups of index $[\tilde{G} : G]$. Therefore π has a normal subgroup $K \leq N_\pi(G)$ such that $[\pi : K] < \infty$ and K/G is a PD_2^+-group.

Otherwise, replacing G_{n+1} by the union of the terms G_α which normalize $\tilde{G}$ and reindexing, if necessary, we may assume that $\tilde{G}$ is not normal in G_{n+2}. Let h be an element of G_{n+2} such that $h\tilde{G}h^{-1} \neq \tilde{G}$, and let $H = \tilde{G}.h\tilde{G}h^{-1}$. Then $\tilde{G}$ is normal in H and H is normal in G_{n+1}, so $[H : \tilde{G}] = \infty$ and $c.d.H = 3$. Moreover H is FP, by Proposition 8.3 of [Bi], and $H^s(H; \mathbb{Z}[H]) = 0$ for $s \leq 2$, by an LHSSS argument.

If $c.d.G_{n+1} = 3$ then G_{n+1}/H is locally finite, by Theorem 8.2 of [Bi]. Hence it is finite, by the Gildenhuys-Strebel Theorem. Therefore G_{n+1} is FP and $H^s(G_{n+1}; \mathbb{Z}[G_{n+1}]) = 0$ for $s \leq 2$. Since G_{n+1} is also ascendant in π it is a PD_3-group, $[\pi : N_\pi(G_{n+1})] < \infty$ and $N_\pi(G_{n+1})/G_{n+1}$ has two ends, by Theorem 4.8. Hence $G_{n+1}/\tilde{G}$ has two ends also, and $\tilde{G}$ is a PD_2-group, by Theorem 2.12. We may easily find subgroups $J \leq G_{n+1}$ and $K \leq N_\pi(G_{n+1})$ such that $G < J < K$, $J/G \cong K/J \cong Z$ and $[\pi : K] < \infty$.

If $c.d.G_{n+1} = 4$ then $[\pi : G_{n+1}]$ is again finite and G_{n+1} is a PD_4-group. Hence the result follows as for the case when $\tilde{G}$ is normal in π. □

Corollary 5.8.1 *If $\chi(M) = 0$, G is a PD_2-group, $\chi(G) \neq 0$ and G is normal in π then M has a finite covering space which is homotopy equivalent to the total space of a surface bundle over T.*

Proof Since G is normal in π and M is aspherical M has a finite covering which is homotopy equivalent to a $K(G,1)$-bundle over an aspherical orientable surface, as in Theorem 5.7. Since $\chi(M) = 0$ the base must be T. □

If π/G is virtually Z^2 then it has a subgroup of index at most 6 which maps onto Z^2 or $Z \rtimes_{-1} Z$.

Let G be a PD_2-group such that $\zeta G = 1$. Let θ be an automorphism of G whose class in $Out(G)$ has infinite order and let $\lambda : G \to Z$ be an epimorphism. Let $\pi = (G \times Z) \rtimes_\phi Z$ where $\phi(g,n) = (\theta(g), \lambda(g)+n)$ for all $g \in G$ and $n \in Z$. Then G is subnormal in π but this group is not virtually the group of a surface bundle over a surface.

If π has an ascendant subgroup G which is a PD_2-group with $\chi(G) = 0$ then $\sqrt{G} \cong Z^2$ is ascendant in π and hence contained in $\sqrt{\pi}$. In this case $h(\sqrt{\pi}) \geq 2$ and so either Theorem 8.1 or Theorem 9.2 applies, to show that M has a finite covering space which is homotopy equivalent to the total space of a T-bundle over an aspherical closed surface.

5.3 Bundles with aspherical base and fibre S^2 or RP^2

Let $E^+(S^2)$ denote the connected component of id_{S^2} in $E(S^2)$, i.e., the submonoid of degree 1 maps. The connected component of id_{S^2} in $E_0(S^2)$ may be identified with the double loop space $\Omega^2 S^2$.

Lemma 5.9 *Let X be a finite 2-complex. Then there are natural bijections $[X; BO(3)] \cong [X; BE(S^2)] \cong H^1(X; \mathbb{F}_2) \times H^2(X; \mathbb{F}_2)$.*

Proof As a self homotopy equivalence of a sphere is homotopic to the identity if and only if it has degree $+1$ the inclusion of $O(3)$ into $E(S^2)$ is bijective on components. Evaluation of a self map of S^2 at the basepoint determines fibrations of $SO(3)$ and $E^+(S^2)$ over S^2, with fibre $SO(2)$ and $\Omega^2 S^2$, respectively, and the map of fibres induces an isomorphism on π_1. On comparing the exact sequences of homotopy for these fibrations we see that the inclusion of $SO(3)$ in $E^+(S^2)$ also induces an isomorphism on π_1. Since the Stiefel-Whitney classes are defined for any spherical fibration and w_1 and w_2 are nontrivial on suitable S^2-bundles over S^1 and S^2, respectively, the inclusion of $BO(3)$ into $BE(S^2)$ and the map $(w_1, w_2) : BE(S^2) \to K(Z/2Z, 1) \times K(Z/2Z, 2)$ induces isomorphisms on π_i for $i \leq 2$. The lemma follows easily. □

Thus there is a natural 1-1 correspondance between S^2-bundles and spherical fibrations over such complexes, and any such bundle ξ is determined up to isomorphism over X by its total Stiefel-Whitney class $w(\xi) = 1 + w_1(\xi) + w_2(\xi)$. (From another point of view: if $w_1(\xi) = w_1(\xi')$ there is an isomorphism of the restrictions of ξ and ξ' over the 1-skeleton $X^{[1]}$. The difference $w_2(\xi) - w_2(\xi')$ is the obstruction to extending any such isomorphism over the 2-skeleton.)

Theorem 5.10 *Let M be a PD_4-complex and B an aspherical closed surface. Then the following conditions are equivalent:*

(1) $\pi_1(M) \cong \pi_1(B)$ *and* $\chi(M) = 2\chi(B)$*;*

(2) $\pi_1(M) \cong \pi_1(B)$ *and* $\widetilde{M} \simeq S^2$*;*

(3) *M is homotopy equivalent to the total space of an S^2-bundle over B.*

Proof If (1) holds then $H_3(\widetilde{M};\mathbb{Z}) = H_4(\widetilde{M};\mathbb{Z}) = 0$, as $\pi_1(M)$ has one end, and $\pi_2(M) \cong \overline{H^2(\pi;\mathbb{Z}[\pi])} \cong Z$, by Theorem 3.12. Hence $\widetilde{M}$ is homotopy equivalent to S^2. If (2) holds we may assume that there is a Hurewicz fibration $h : M \to B$ which induces an isomorphism of fundamental groups. As the homotopy fibre of h is $\widetilde{M}$, Lemma 5.9 implies that h is fibre homotopy equivalent to the projection of an S^2-bundle over B. Clearly (3) implies the other conditions. □

We shall summarize some of the key properties of the Stiefel-Whitney classes of such bundles in the following lemma.

Lemma 5.11 *Let ξ be an S^2-bundle over a closed surface B, with total space M and projection $p : M \to B$. Then*

(1) *ξ is trivial if and only if $w(M) = p^*w(B)$;*

(2) $\pi_1(M) \cong \pi_1(B)$ *acts on* $\pi_2(M)$ *by multiplication by* $w_1(\xi)$*;*

(3) *the intersection form on $H_2(M;\mathbb{F}_2)$ is even if and only if $w_2(\xi) = 0$;*

(4) *if $q : B' \to B$ is a 2-fold covering map with connected domain B' then $w_2(q^*\xi) = 0$.*

Proof (1) Applying the Whitney sum formula and naturality to the tangent bundle of the B^3-bundle associated to ξ gives $w(M) = p^*w(B) \cup p^*w(\xi)$. Since p is a 2-connected map the induced homomorphism p^* is injective in degrees ≤ 2 and so $w(M) = p^*w(B)$ if and only if $w(\xi) = 1$. By Lemma 5.9 this is so if and only if ξ is trivial, since B is 2-dimensional.

(2) It is sufficient to consider the restriction of ξ over loops in B, where the result is clear.

(3) By Poincaré duality, the intersection form is even if and only if the Wu class $v_2(M) = w_2(M) + w_1(M)^2$ is 0. Now

$$\begin{aligned} v_2(M) &= p^*(w_1(B) + w_1(\xi))^2 + p^*(w_2(B) + w_1(B) \cup w_1(\xi) + w_2(\xi)) \\ &= p^*(w_2(B) + w_1(B) \cup w_1(\xi) + w_2(\xi) + w_1(B)^2 + w_1(\xi)^2) \\ &= p^*(w_2(\xi)), \end{aligned}$$

since $w_1(B) \cup \eta = \eta^2$ and $w_1(B)^2 = w_2(B)$, by the Wu relations for B. Hence $v_2(M) = 0$ if and only if $w_2(\xi) = 0$, as p^* is injective in degree 2.

(4) We have $q_*(w_2(q^*\xi) \cap [B']) = q_*((q^*w_2(\xi)) \cap [B']) = w_2(\xi) \cap q_*[B']$, by the projection formula. Since q has degree 2 this is 0, and since q_* is an isomorphism in degree 0 we find $w_2(q^*\xi) \cap [B'] = 0$. Therefore $w_2(q^*\xi) = 0$, by Poincaré duality for B'. □

Melvin has determined criteria for the total spaces of S^2-bundles over a compact surface to be diffeomorphic, in terms of their Stiefel-Whitney classes. We shall give an alternative argument for the cases with aspherical base.

Lemma 5.12 *Let B be a closed surface and w be the Poincaré dual of $w_1(B)$. If u_1 and u_2 are elements of $H_1(B;\mathbb{F}_2) \setminus \{0, w\}$ such that $u_1.u_1 = u_2.u_2$ then there is a diffeomorphism $f : B \to B$ which is a composite of Dehn twists about two-sided essential simple closed curves and such that $f_*(u_1) = u_2$.*

Proof For simplicity of notation, we shall use the same symbol for a simple closed curve u on B and its homology class in $H_1(B;\mathbb{F}_2)$. The curve u is two-sided if and only if $u.u = 0$. In that case we shall let c_u denote the automorphism of $H_1(B;\mathbb{F}_2)$ induced by a Dehn twist about u. Note also that $u.u = u.w$ and $c_v(u) = u + (u.v)v$ for all u and two-sided v in $H_1(B;\mathbb{F}_2)$.

If B is orientable it is well known that the group of isometries of the intersection form acts transitively on $H_1(B;\mathbb{F}_2)$, and is generated by the automorphisms c_u. Thus the claim is true in this case.

If $w_1(B)^2 \neq 0$ then $B \cong RP^2\sharp T_g$, where T_g is orientable. If $u_1.u_1 = u_2.u_2 = 0$ then u_1 and u_2 are represented by simple closed curves in T_g, and so are related by a diffeomorphism which is the identity on the RP^2 summand. If $u_1.u_1 = u_2.u_2 = 1$ let $v_i = u_i + w$. Then $v_i.v_i = 0$ and this case follows from the earlier one.

Suppose finally that $w_1(B) \neq 0$ but $w_1(B)^2 = 0$; equivalently, that $B \cong Kb\sharp T_g$, where T_g is orientable. Let $\{w, z\}$ be a basis for the homology of the Kb summand. In this case w is represented by a 2-sided curve. If $u_1.u_1 = u_2.u_2 = 0$ and $u_1.z = u_2.z = 0$ then u_1 and u_2 are represented by simple closed curves in T_g, and so are related by a diffeomorphism which is the identity on the Kb summand. The claim then follows if $u.z = 1$ for $u = u_1$ or u_2, since we then have $c_w(u).c_w(u) = c_w(u).z = 0$. If $u.u \neq 0$ and $u.z = 0$ then $(u+z).(u+z) = 0$ and $c_{u+z}(u) = z$. If $u.u \neq 0$, $u.z \neq 0$ and $u \neq z$ then $c_{u+z+w}c_w(u) = z$. Thus if $u_1.u_1 = u_2.u_2 = 1$ both u_1 and u_2 are related to z. Thus in all cases the claim is true. □

Theorem 5.13 (Melvin) *Let ξ and ξ' be two S^2-bundles over an aspherical closed surface B. Then the following conditions are equivalent:*

(1) *there is a diffeomorphism $f : B \to B$ such that $\xi = f^*\xi'$;*

(2) *the total spaces $E(\xi)$ and $E(\xi')$ are diffeomorphic; and*

(3) *$w_1(\xi) = w_1(\xi')$ if $w_1(\xi) = 0$ or $w_1(B)$, $w_1(\xi) \cup w_1(B) = w_1(\xi') \cup w_1(B)$ and $w_2(\xi) = w_2(\xi')$.*

Proof Clearly (1) implies (2). A diffeomorphism $h : E \to E'$ induces an isomorphism on fundamental groups; hence there is a diffeomorphism $f : B \to B$ such that fp is homotopic to $p'h$. Now $h^*w(E') = w(E)$ and $f^*w(B) = w(B)$. Hence $p^*f^*w(\xi') = p^*w(\xi)$ and so $w(f^*\xi') = f^*w(\xi') = w(\xi)$. Thus $f^*\xi' = \xi$, by Lemma 5.9, and so (2) implies (1).

If (1) holds then $f^*w(\xi') = w(\xi)$. Since $w_1(B) = v_1(B)$ is the characteristic element for the cup product pairing from $H^1(B;\mathbb{F}_2)$ to $H^2(B;\mathbb{F}_2)$ and $H^2(f;\mathbb{F}_2)$ is the identity $f^*w_1(B) = w_1(B)$, $w_1(\xi) \cup w_1(B) = w_1(\xi') \cup w_1(B)$ and $w_2(\xi) = w_2(\xi')$. Hence(1) implies (3).

If $w_1(\xi) \cup w_1(B) = w_1(\xi') \cup w_1(B)$ and $w_1(\xi)$ and $w_1(\xi')$ are neither 0 nor $w_1(B)$ then there is a diffeomorphism $f : B \to B$ such that $f^*w_1(\xi') = w_1(\xi)$, by Lemma 5.12 (applied to the Poincaré dual homology classes). Hence (3) implies (1). □

Corollary 5.13.1 *There are 4 diffeomorphism classes of S^2-bundle spaces if B is orientable and $\chi(B) \leq 0$, 6 if $B = Kb$ and 8 if B is nonorientable and $\chi(B) < 0$.* □

See [Me84] for a more geometric argument, which applies also to S^2-bundles over surfaces with nonempty boundary. The theorem holds also when $B = S^2$ or RP^2; there are 2 such bundles over S^2 and 4 over RP^2. (See Chapter 12.)

Theorem 5.14 *Let M be a PD_4-complex with fundamental group π. The following are equivalent:*

(1) *$\pi \neq 1$ and $\pi_2(M) \cong Z$.*

(2) *$\widetilde{M} \simeq S^2$;*

(3) *M has a covering space of degree ≤ 2 which is homotopy equivalent to the total space of an S^2-bundle over an aspherical closed surface;*

(4) *π is virtually a PD_2-group and $\chi(M) = 2\chi(\pi)$.*

If these conditions hold the kernel K of the natural action of π on $\pi_2(M)$ is a PD_2-group.

Proof Suppose that (1) holds. If π is finite and $\pi_2(M) \cong Z$ then $\widetilde{M} \simeq CP^2$, and so admits no nontrivial free group actions, by the Lefshetz fixed point theorem. Hence π must be infinite. Then $H_0(\widetilde{M};\mathbb{Z}) = Z$, $H_1(\widetilde{M};\mathbb{Z}) = 0$ and $H_2(\widetilde{M};\mathbb{Z}) = \pi_2(M)$, while $H_3(\widetilde{M};\mathbb{Z}) \cong \overline{H^1(\pi;\mathbb{Z}[\pi])}$ and $H_4(\widetilde{M};\mathbb{Z}) = 0$. Now $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi]) = 0$, since π is infinite and $\pi_2(M) \cong Z$. Therefore $H^2(\pi;\mathbb{Z}[\pi])$ is infinite cyclic, by Lemma 3.3, and so π is virtually a PD_2-group, by Bowditch's Theorem. Hence $H_3(\widetilde{M};\mathbb{Z}) = 0$ and so $\widetilde{M} \simeq S^2$. If C is a finite cyclic subgroup of K then $H_{n+3}(C;\mathbb{Z}) \cong H_n(C;H_2(\widetilde{M};\mathbb{Z}))$ for all $n \geq 2$, by Lemma 2.10. Therefore C must be trivial, so K is torsion-free. Hence K is a PD_2-group and (3) now follows from Theorem 5.10. Clearly (3) implies (2) and (2) implies (1). The equivalence of (3) and (4) follows from Theorem 5.10. □

A straightfoward Mayer-Vietoris argument may be used to show directly that if $H^2(\pi;\mathbb{Z}[\pi]) \cong Z$ then π has one end.

Lemma 5.15 *Let X be a finite 2-complex. Then there are natural bijections $[X;BSO(3)] \cong [X;BE(RP^2)] \cong H^2(X;\mathbb{F}_2)$.*

Proof Let $(1,0,0)$ and $[1:0:0]$ be the base points for S^2 and RP^2 respectively. A based self homotopy equivalence f of RP^2 lifts to a based self homotopy equivalence f^+ of S^2. If f is based homotopic to the identity then $deg(f^+) = 1$. Conversely, any based self homotopy equivalence is based homotopic to a map which is the identity on RP^1; if moreover $deg(f^+) = 1$ then this map is the identity on the normal bundle and it quickly follows that f is based homotopic to the identity. Thus $E_0(RP^2)$ has two components. The diffeomorphism g defined by $g([x:y:z]) = [x:y:-z]$ is isotopic to the identity (rotate in the (x,y)-coordinates). However $deg(g^+) = -1$. It follows that $E(RP^2)$ is connected. As every self homotopy equivalence of RP^2 is covered by a degree 1 self map of S^2, there is a natural map from $E(RP^2)$ to $E^+(S^2)$.

We may use obstruction theory to show that $\pi_1(E_0(RP^2))$ has order 2. Hence $\pi_1(E(RP^2))$ has order at most 4. Suppose that there were a homotopy f_t through self maps of RP^2 with $f_0 = f_1 = id_{RP^2}$ and such that the loop $f_t(*)$ is essential, where $*$ is a basepoint. Let F be the map from $RP^2 \times S^1$ to RP^2 determined by $F(p,t) = f_t(p)$, and let α and β be the generators of $H^1(RP^2;\mathbb{F}_2)$ and $H^1(S^1;\mathbb{F}_2)$, respectively. Then $F^*\alpha = \alpha\otimes 1 + 1\otimes\beta$ and so $(F^*\alpha)^3 = \alpha^2\otimes\beta$ which is nonzero, contradicting $\alpha^3 = 0$. Thus there can be no such homotopy, and so the homomorphism from $\pi_1(E(RP^2))$ to $\pi_1(RP^2)$ induced by the evaluation map must be trivial. It then follows from the exact sequence of homotopy for this evaluation map that the order of $\pi_1(E(RP^2))$ is at most 2. The group $SO(3) \cong O(3)/(\pm I)$ acts isometrically on RP^2. As the composite of the maps on π_1 induced by the inclusions $SO(3) \subset E(RP^2) \subset E^+(S^2)$ is an isomorphism of groups of order 2 the first map also induces an isomorphism. It follows as in Lemma 5.9 that there are natural bijections $[X; BSO(3)] \cong [X; BE(RP^2)] \cong H^2(X; \mathbb{F}_2)$. □

Thus there is a natural 1-1 correspondance between RP^2-bundles and orientable spherical fibrations over such complexes. The RP^2-bundle corresponding to an orientable S^2-bundle is the quotient by the fibrewise antipodal involution. In particular, there are two RP^2-bundles over each closed aspherical surface.

Theorem 5.16 *Let M be a PD_4-complex and B an aspherical closed surface. Then the following conditions are equivalent:*

(1) $\pi_1(M) \cong \pi_1(B) \times (Z/2Z)$ *and* $\chi(M) = \chi(B)$*;*

(2) $\pi_1(M) \cong \pi_1(B) \times (Z/2Z)$ *and* $\widetilde{M} \simeq S^2$*;*

(3) *M is homotopy equivalent to the total space of an RP^2-bundle over B.*

Proof Suppose that (1) holds, and let $w : \pi_1(M) \to Z/2Z$ be the projection onto the $Z/2Z$ factor. Then the covering space associated with the kernel of w satisfies the hypotheses of Theorem 5.10 and so $\widetilde{M} \simeq S^2$.

If (2) holds the homotopy fibre of the map h from M to B inducing the projection of $\pi_1(M)$ onto $\pi_1(B)$ is homotopy equivalent to RP^2. The map h is fibre homotopy equivalent to the projection of an RP^2-bundle over B, by Lemma 5.15.

If E is the total space of an RP^2-bundle over B, with projection p, then $\chi(E) = \chi(B)$ and the long exact sequence of homotopy gives a short exact sequence $1 \to Z/2Z \to \pi_1(E) \to \pi_1(B) \to 1$. Since the fibre has a product neighbourhood, $j^*w_1(E) = w_1(RP^2)$, where $j : RP^2 \to E$ is the inclusion of the fibre over the basepoint of B, and so $w_1(E)$ considered as a homomorphism from $\pi_1(E)$ to $Z/2Z$ splits the injection j_*. Therefore $\pi_1(E) \cong \pi_1(B) \times (Z/2Z)$ and so (1) holds, as these conditions are clearly invariant under homotopy. □

We may use the above results to refine some of the conclusions of Theorem 3.9 on PD_4-complexes with finitely dominated covering spaces.

Theorem 5.17 *Let M be a PD_4-complex with fundamental group π, and let $p:\pi\to G$ be an epimorphism with FP_2 kernel ν. Suppose that $H^2(G;\mathbb{Z}[G])\cong Z$. Then the following conditions are equivalent:*

(1) $Hom_{\mathbb{Z}[\pi]}(\pi_2(M),\mathbb{Z}[\pi])=0$;

(2) $C_*(\widetilde{M})|_\nu$ *has finite 2-skeleton;*

(3) *the associated covering space M_ν is homotopy equivalent to a closed surface;*

(4) *M has a finite covering space which is homotopy equivalent to the total space of a surface bundle over an aspherical closed surface.*

Proof By Bowditch's Theorem G is virtually a PD_2-group. Hence π has one end and $H^2(\pi;\mathbb{Z}[\pi])\cong Z$, if ν is finite, and is 0 otherwise, by an LHSSS argument.

If (1) holds $\pi_2(M)\cong\overline{H^2(\pi;\mathbb{Z}[\pi])}$, by Lemma 3.3. If (2) holds $\pi_2(M)\cong H_2(M_\nu;\mathbb{Z}[\nu])\cong H^0(M_\nu;\mathbb{Z}[\nu])$, by Theorem 1.19′. In either case, if ν is finite $\pi_2(M)\cong Z$, while if ν is infinite $\pi_2(M)=0$ and M is aspherical. Condition (3) now follows from Theorems 5.10, 5.16 and 1.19, and (4) follows easily.

If (4) holds then π is infinite and $\pi_2(M)=\pi_2(M_\nu)\cong Z$ or is 0, and so (1) holds. □

The total spaces of such bundles with base an aspherical surface have minimal Euler characteristic for their fundamental groups (i.e. $\chi(M)=q(\pi)$), by Theorem 3.12 and the remarks in the paragraph preceding it.

The FP_2 hypothesis is in general necessary, as observed after Theorem 5.7. (See [Ka98].) However it may be relaxed when G is virtually Z^2 and $\chi(M)=0$.

Theorem 5.18 *Let M be a finite PD_4-complex with fundamental group π. Then M is homotopy equivalent to the total space of a surface bundle over T or Kb if and only if π is an extension of Z^2 or $Z\rtimes_{-1}Z$ (respectively) by a finitely generated normal subgroup ν and $\chi(M)=0$.*

Proof The conditions are clearly necessary. If they hold the covering space M_ν associated to the subgroup ν is homotopy equivalent to a closed surface, by Corollaries 4.5.2 and 2.12.1. The result then follows from Theorems 5.2, 5.10 and 5.16. □

In particular, if π is the nontrivial extension of Z^2 by $Z/2Z$ then $q(\pi)>0$.

5.4 Bundles over S^2

Since S^2 is the union of two discs along a circle, an F-bundle over S^2 is determined by the homotopy class of the clutching function in $\pi_1(Diff(F))$. (This group is isomorphic to $\zeta\pi_1(F)$ and hence to $H^2(S^2;\zeta\pi_1(F))$.) On the other hand, if M is a PD^4-complex then cellular approximation gives bijections $H^2(M;\mathbb{Z})=[M;CP^\infty]=[M;CP^2]$, and a map $f:M\to CP^2$ factors through $CP^2\setminus D^4\sim S^2$ if and only if $\deg(f)=0$. Thus if $u\in H^2(M;\mathbb{Z})$ and i_2 generates $H^2(S^2;\mathbb{Z})$ then $u=f^*i_2$ for some $f:M\to S^2$ if and only if $u^2=0$. The map is uniquely determined by u, by Theorem 8.4.11 of [Sp].

Theorem 5.19 *Let M be a PD_4-complex with fundamental group π and F a closed surface. Then M is homotopy equivalent to the total space of an F-bundle over S^2 if and only if $\chi(M)=2\chi(F)$ and*

1. *(when $\chi(F)<0$) $\pi\cong\pi_1(F)$, $w_1(M)=c_M^*w_1(F)$ and $w_2(M)=w_1(M)^2=(c_M^*w_1(F))^2$; or*
2. *(when $F=T$) $\pi\cong Z^2$ and $w_1(M)=w_2(M)=0$, or $\pi\cong Z\oplus(Z/nZ)$ for some $n>0$ and, if $n=1$ or 2, $w_1(M)=0$; or*
3. *(when $F=Kb$) $\pi\cong Z\rtimes_{-1}Z$, $w_1(M)\neq 0$ and $w_2(M)=w_1(M)^2=0$, or π has a presentation $\langle x,y\mid yxy^{-1}=x^{-1},\ y^{2n}=1\rangle$ for some $n>0$, where $w_1(M)(x)=0$ and $w_1(M)(y)=1$; or*
4. *(when $F=S^2$) $\pi=1$ and the index $\sigma(M)=0$; or*
5. *(when $F=RP^2$) $\pi=Z/2Z$, $w_1(M)\neq 0$ and there is a class u of infinite order in $H^2(M;\mathbb{Z})$ and such that $u^2=0$.*

Proof Let $p_E:E\to S^2$ be such a bundle. Then $\chi(E)=2\chi(F)$ and $\pi_1(E)\cong\pi_1(F)/\partial\pi_2(S^2)$, where $\mathrm{Im}(\partial)\leq\zeta\pi_1(F)$ [Go68]. The characteristic classes of E restrict to the characteristic classes of the fibre, as it has a product neighbourhood. As the base is 1-connected E is orientable if and only if the fibre is orientable. Thus the conditions on χ, π and w_1 are all necessary. We shall treat the other assertions case by case.

(1) If $\chi(F)<0$ any F-bundle over S^2 is trivial, by Lemma 5.1. Thus the conditions are necessary. Conversely, if they hold then c_M is fibre homotopy equivalent to the projection of an S^2-bundle ξ with base F, by Theorem 5.10. The conditions on the Stiefel-Whitney classes then imply that $w(\xi)=1$ and hence that the bundle is trivial, by Lemma 5.11. Therefore M is homotopy equivalent to $S^2\times F$.

(2) If $\partial = 0$ there is a map $q : E \to T$ which induces an isomorphism of fundamental groups, and the map $(p_E, q) : E \to S^2 \times T$ is clearly a homotopy equivalence, so $w(E) = 1$. Conversely, if $\chi(M) = 0$, $\pi \cong Z^2$ and $w(M) = 1$ then M is homotopy equivalent to $S^2 \times T$, by Theorem 5.10 and Lemma 5.11.

If $\chi(M) = 0$ and $\pi \cong Z \oplus (Z/nZ)$ for some $n > 0$ then the covering space $M_{Z/nZ}$ corresponding to the torsion subgroup Z/nZ is homotopy equivalent to a lens space L, by Corollary 4.5.2. As observed in Chapter 4 the manifold M is homotopy equivalent to the mapping torus of a generator of the group of covering transformations $Aut(M_{Z/nZ}/M) \cong Z$. Since the generator induces the identity on $\pi_1(L) \cong Z/nZ$ it is homotopic to id_L, if $n > 2$. This is also true if $n = 1$ or 2 and M is orientable. (See Section 29 of [Co].) Therefore M is homotopy equivalent to $L \times S^1$, which fibres over S^2 via the composition of the projection to L with the Hopf fibration of L over S^2. (Hence $w(M) = 1$ in these cases also.)

(3) As in part (2), if $\pi_1(E) \cong Z \rtimes_{-1} Z = \pi_1(Kb)$ then E is homotopy equivalent to $S^2 \times Kb$ and so $w_1(E) \neq 0$, while $w_2(E) = 0$. Conversely, if $\chi(M) = 0$, $\pi \cong \pi_1(Kb)$, M is nonorientable and $w_1(M)^2 = w_2(M) = 0$ then M is homotopy equivalent to $S^2 \times Kb$. Suppose now that π and w_1 satisfy the second alternative (corresponding to bundles with $\partial \neq 0$). Let $q : M^+ \to M$ be the orientation double cover. Then M^+ satisfies the hypotheses of part (3), and so there is a map $p^+ : M^+ \to S^2$ with homotopy fibre T. Now $H^2(q; \mathbb{Z})$ is an epimorphism, since $H^3(Z/2Z; \mathbb{Z}) = H^2(Z/2Z; H^1(M^+; \mathbb{Z})) = 0$. Therefore $p^+ = pq$ for some map $p : M \to S^2$. Comparison of the exact sequences of homotopy for p^+ and p shows that the homotopy fibre of p must be Kb. As in Theorem 5.2 above p is fibre homotopy equivalent to a bundle projection.

(4) There are just two S^2-bundles over S^2, with total spaces $S^2 \times S^2$ and $S^2 \tilde{\times} S^2 = CP^2 \sharp -CP^2$, respectively. Thus the conditions are necessary. If M satisfies these conditions then $H^2(M; \mathbb{Z}) \cong Z^2$ and there is an element u in $H^2(M; \mathbb{Z})$ which generates an infinite cyclic direct summand and has square $u \cup u = 0$. Thus $u = f^* i_2$ for some map $f : M \to S^2$. Since u generates a direct summand there is a homology class z in $H_2(M; \mathbb{Z})$ such that $u \cap z = 1$, and therefore (by the Hurewicz theorem) there is a map $z : S^2 \to M$ such that fz is homotopic to id_{S^2}. The homotopy fibre of f is 1-connected and has $\pi_2 \cong Z$, by the long exact sequence of homotopy. It then follows easily from the spectral sequence for f that the homotopy fibre has the homology of S^2. Therefore f is fibre homotopy equivalent to the projection of an S^2-bundle over S^2.

(5) Since $\pi_1(Diff(RP^2)) = Z/2Z$ (see page 21 of [EE69]) there are two RP^2-bundles over S^2. Again the conditions are clearly necessary. If they hold we

may assume that u generates an infinite cyclic direct summand of $H^2(M;\mathbb{Z})$ and that $u = g^* i_2$ for some map $g : M \to S^2$. Let $q : M^+ \to M$ be the orientation double cover and $g^+ = gq$. Since $H_2(Z/2Z;\mathbb{Z}) = 0$ the second homology of M is spherical. Thus there is a map $z = qz^+ : S^2 \to M$ such that $gz = g^+ z^+$ is homotopic to id_{S^2}. Hence the homotopy fibre of g^+ is S^2, by case (5). Since the homotopy fibre of g has fundamental group $Z/2Z$ and is double covered by the homotopy fibre of g^+ it is homotopy equivalent to RP^2. It follows as in Theorem 5.16 that g is fibre homotopy equivalent to the projection of an RP^2-bundle over S^2. □

Theorems 5.2, 5.10 and 5.16 may each be rephrased as giving criteria for maps from M to B to be fibre homotopy equivalent to fibre bundle projections. With the hypotheses of Theorem 5.19 (and assuming also that $\partial = 0$ if $\chi(M) = 0$) we may conclude that a map $f : M \to S^2$ is fibre homotopy equivalent to a fibre bundle projection if and only if $f^* i_2$ generates an infinite cyclic direct summand of $H^2(M;\mathbb{Z})$.

It follows from Theorem 5.10 that the conditions on the Stiefel-Whitney classes are independent of the other conditions when $\pi \cong \pi_1(F)$. Note also that the nonorientable S^3- and RP^3-bundles over S^1 are not T-bundles over S^2, while if $M = CP^2 \sharp CP^2$ then $\pi = 1$ and $\chi(M) = 4$ but $\sigma(M) \neq 0$. See Chapter 12 for further information on parts (4) and (5).

5.5 Bundles over RP^2

Since $RP^2 = Mb \cup D^2$ is the union of a Möbius band Mb and a disc D^2, a bundle $p : E \to RP^2$ with fibre F is determined by a bundle over Mb which restricts to a trivial bundle over ∂Mb, i.e. by a conjugacy class of elements of order dividing 2 in $\pi_0(Homeo(F))$, together with the class of a gluing map over $\partial Mb = \partial D^2$ modulo those which extend across D^2 or Mb, i.e. an element of a quotient of $\pi_1(Homeo(F))$. If F is aspherical $\pi_0(Homeo(F)) \cong Out(\pi_1(F))$, while $\pi_1(Homeo(F)) \cong \zeta\pi_1(F)$ [Go65].

We may summarize the key properties of the algebraic invariants of such bundles with F an aspherical closed surface in the following lemma. Let $\tilde{Z}$ be the nontrivial infinite cyclic $Z/2Z$-module. The groups $H^1(Z/2Z;\tilde{Z})$, $H^1(Z/2Z;\mathbb{F}_2)$ and $H^1(RP^2;\tilde{Z})$ are canonically isomorphic to $Z/2Z$.

Lemma 5.20 *Let $p : E \to RP^2$ be the projection of an F-bundle, where F is an aspherical closed surface, and let x be the generator of $H^1(RP^2;\tilde{Z})$. Then*

(1) $\chi(E) = \chi(F)$;

(2) $\partial(\pi_2(RP^2)) \leq \zeta\pi_1(F)$ *and there is an exact sequence of groups*

$$0 \to \pi_2(E) \to Z \xrightarrow{\partial} \pi_1(F) \to \pi_1(E) \to Z/2Z \to 1;$$

(3) *if* $\partial = 0$ *then* $\pi_1(E)$ *acts nontrivially on* $\pi_2(E) \cong Z$ *and the covering space* E_F *with fundamental group* $\pi_1(F)$ *is homeomorphic to* $S^2 \times F$, *so* $w_1(E)|_{\pi_1(F)} = w_1(E_F) = w_1(F)$ *(as homomorphisms from* $\pi_1(F)$ *to* $Z/2Z$*) and* $w_2(E_F) = w_1(E_F)^2$;

(4) *if* $\partial \neq 0$ *then* $\chi(F) = 0$, $\pi_1(E)$ *has two ends,* $\pi_2(E) = 0$ *and* $Z/2Z$ *acts by inversion on* $\partial(Z)$;

(5) $p^*x^3 = 0 \in H^3(E; p^*\tilde{Z})$.

Proof Condition (1) holds since the Euler characteristic is multiplicative in fibrations, while (2) is part of the long exact sequence of homotopy for p. The image of ∂ is central by [Go68], and is therefore trivial unless $\chi(F) = 0$. Conditions (3) and (4) then follow as the homomorphisms in this sequence are compatible with the actions of the fundamental groups, and E_F is the total space of an F-bundle over S^2, which is a trivial bundle if $\partial = 0$, by Theorem 5.19. Condition (5) holds since $H^3(RP^2; \tilde{Z}) = 0$. □

Let π be a group which is an extension of $Z/2Z$ by a normal subgroup G, and let $t \in \pi$ be an element which maps nontrivially to $\pi/G = Z/2Z$. Then $u = t^2$ is in G and conjugation by t determines an automorphism α of G such that $\alpha(u) = u$ and α^2 is the inner automorphism given by conjugation by u.

Conversely, let α be an automorphism of G whose square is inner, say $\alpha^2(g) = ugu^{-1}$ for all $g \in G$. Let $v = \alpha(u)$. Then $\alpha^3(g) = \alpha^2(\alpha(g)) = u\alpha(g)c^{-1} = \alpha(\alpha^2(g)) = v\alpha(g)v^{-1}$ for all $g \in G$. Therefore vu^{-1} is central. In particular, if the centre of G is trivial α fixes u, and we may define an extension

$$\xi_\alpha : 1 \to G \to \Pi_\alpha \to Z/2Z \to 1$$

in which Π_α has the presentation $\langle G, t_\alpha \mid t_\alpha g t_\alpha^{-1} = \alpha(g),\ t_\alpha^2 = u\rangle$. If β is another automorphism in the same outer automorphism class then ξ_α and ξ_β are equivalent extensions. (Note that if $\beta = \alpha.c_h$, where c_h is conjugation by h, then $\beta(\alpha(h)uh) = \alpha(h)uh$ and $\beta^2(g) = \alpha(h)uh.g.(\alpha(h)uh)^{-1}$ for all $g \in G$.)

Lemma 5.21 *If* $\chi(F) < 0$ *or* $\chi(F) = 0$ *and* $\partial = 0$ *then an* F*-bundle over* RP^2 *is determined up to isomorphism by the corresponding extension of fundamental groups.*

Proof If $\chi(F) < 0$ such bundles and extensions are each determined by an element ξ of order 2 in $Out(\pi_1(F))$. If $\chi(F) = 0$ bundles with $\partial = 0$ are the restrictions of bundles over $RP^\infty = K(Z/2Z, 1)$ (compare Lemma 4.10). Such bundles are determined by an element ξ of order 2 in $Out(\pi_1(F))$ and a cohomology class in $H^2(Z/2Z; \zeta\pi_1(F)^\xi)$, by Lemma 5.1, and so correspond bijectively to extensions also. $\square$

Lemma 5.22 *Let M be a PD_4-complex with fundamental group π. A map $f : M \to RP^2$ is fibre homotopy equivalent to the projection of a bundle over RP^2 with fibre an aspherical closed surface if $\pi_1(f)$ is an epimorphism and either*

(1) $\chi(M) \leq 0$ *and $\pi_2(f)$ is an isomorphism; or*

(2) $\chi(M) = 0$, *π has two ends and $\pi_3(f)$ is an isomorphism.*

Proof In each case π is infinite, by Lemma 3.14. In case (1) $H^2(\pi; \mathbb{Z}[\pi]) \cong Z$ (by Lemma 3.3) and so π has one end, by Bowditch's Theorem. Hence $\widetilde{M} \simeq S^2$. Moreover the homotopy fibre of f is aspherical, and its fundamental group is a surface group. (See Chapter X for details.) In case (2) $\widetilde{M} \simeq S^3$, by Corollary 4.5.2. Hence the lift $\tilde{f} : \widetilde{M} \to S^2$ is fibre homotopy equivalent to the Hopf map, and so induces isomorphisms on all higher homotopy groups. Therefore the homotopy fibre of f is aspherical. As $\pi_2(M) = 0$ the fundamental group of the homotopy fibre is a (torsion-free) infinite cyclic extension of π and so must be either Z^2 or $Z \rtimes_{-1} Z$. Thus the homotopy fibre of f is homotopy equivalent to T or Kb. In both cases the argument of Theorem 5.2 now shows that f is fibre homotopy equivalent to a surface bundle projection. $\square$

5.6 Bundles over RP^2 with $\partial = 0$

Let F be a closed aspherical surface and $p : M \to RP^2$ be a bundle with fibre F, and such that $\pi_2(M) \cong Z$. (This condition is automatic if $\chi(F) < 0$.) Then $\pi = \pi_1(M)$ acts nontrivially on $\pi_2(M)$. The covering space M_κ associated to the kernel κ of the action is an F-bundle over S^2, and so $M_\kappa \cong S^2 \times F$, since all such bundles are trivial. In particular, $v_2(M) \in H^2(\pi; \mathbb{F}_2)$, and $v_2(M)|_\kappa = 0$. The projection admits a section if and only if $\pi \cong \kappa \rtimes Z/2Z$.

Our attempt (in the original version of this book) to characterize more general surface bundles over RP^2 had an error (in the claim that restriction from $H^2(RP^2; Z^u)$ to $H^2(S^2; Z)$ is an isomorphism). We provide instead several partial results. Further progress might follow from a better understanding of maps from 4-complexes to RP^2. The reference [Si67] cited in the former (flawed) theorem of this section remains potentially useful here.

The product $RP^2 \times F$ is easily characterized.

Theorem 5.23 *Let M be a closed 4-manifold with fundamental group π, and let F be an aspherical closed surface. Then the following are equivalent.*

(1) $M \simeq RP^2 \times F$;

(2) $\pi \cong Z/2Z \times \pi_1(F)$, $\chi(M) = \chi(F)$ *and* $v_2(M) = 0$;

(3) $\pi \cong Z/2Z \times \pi_1(F)$, $\chi(M) = \chi(F)$ *and* $M \simeq E$, *where E is the total space of an F-bundle over RP^2.*

Proof Clearly (1) $\Rightarrow$ (2) and (3). If (2) holds then M is homotopy equivalent to the total space of an RP^2-bundle over F, by Theorem 5.16. This bundle must be trivial since $v_2(M) = 0$. If (3) holds then there are maps $q : M \to F$ and $p : M \to RP^2$ such that $\pi_1(p)$ and $\pi_1(q)$ are the projections of π onto its factors and $\pi_2(p)$ is surjective. The map $(p, q) : M \to RP^2 \times F$ is then a homotopy equivalence. □

The implication (3) $\Rightarrow$ (1) fails if $F = RP^2$ or S^2.

The characterization of bundles with sections is based on a study of S^2-orbifold bundles. (See Chapter 10 below and [Hi13].)

Theorem 5.24 *Let F be an aspherical closed surface. A closed orientable 4-manifold M is homotopy equivalent to the total space of an F-bundle over RP^2 with a section if and only if $\pi = \pi_1(M)$ has an element of order 2, $\pi_2(M) \cong Z$ and $\kappa = \mathrm{Ker}(u) \cong \pi_1(F)$, where u is the natural action of π on $\pi_2(M)$.*

Proof The conditions are clearly necessary. Suppose that they hold. We may assume that π is not a direct product $\kappa \times Z/2Z$. Therefore M is not homotopy equivalent to an RP^2-bundle space. Hence it is homotopy equivalent to the total space E of an S^2-orbifold bundle over a 2-orbifold B. (See Corollary 10.8.1 below.) The involution ζ of F corresponding to the orbifold covering has non-empty fixed point set, since π has torsion. Let $M_{st} = S^2 \times F/\sim$, where $(s, f) \sim (-s, \zeta(f))$. Then M_{st} is the total space of an F-bundle over RP^2, and th e fixed points of ζ determine sections of this bundle.

The double cover of E corresponding to κ is an S^2-bundle over F. Since M is orientable and κ acts trivially on $\pi_2(M)$, F must also be orientable and the covering involution of F over B must be orientation-reversing. Since π has torsion ΣB is a non-empty union of reflector curves, and since F is orientable these are "untwisted". Therefore $M \simeq M_{st}$, by Corollary 4.8 of [Hi13]. $\square$

Orientability is used here mainly to ensure that B has a reflector curve.

When π is torsion-free M is homotopy equivalent to the total space of an S^2-bundle over a surface B, with $\pi = \pi_1(B)$ acting nontrivially on the fibre. Inspection of the geometric models for such bundle spaces given in Chapter 10 below shows that if also $v_2(M) \neq 0$ then the bundle space fibres over RP^2. Is the condition $v_2(M) \neq 0$ necessary?

5.7 Sections of surface bundles

If a bundle $p : E \to B$ with base and fibre aspherical surfaces has a section then its fundamental group sequence splits. The converse holds if the action ξ can be realized by a group of based self homeomorphisms of the fibre F. (This is so if $F = T$ or Kb.) The sequence splits if and only if the action factors through $Aut(\pi_1(F))$ and the class of the extension in $H^2(\pi_1(B); \zeta\pi_1(F))$ is 0. This cohomology group is trivial if $\chi(F) < 0$, and the class is easily computed if $\chi(F) = 0$. In particular, if B is orientable and $F = T$ then p has a section if and only if $H_1(E; \mathbb{Z}) \cong H_1(B; \mathbb{Z}) \oplus H_0(B; \pi_1(F))$. (The T-bundles over T which are coset spaces of the nilpotent Lie groups $Nil^3 \times \mathbb{R}$ and Nil^4 do not satisfy this criterion, and so do not have sections.)

If p_* splits and s and s' are two sections determining the same lift $\widetilde{\xi} : \pi_1(B) \to Aut(\pi_1(F))$ then $s'(g)s(g)^{-1}$ is in $\zeta\pi_1(F)$, for all $g \in \pi_1(B)$. Hence the sections are parametrized (up to conjugation by an element of $\pi_1(F)$) by $H^1(\pi_1(B); \zeta\pi_1(F))$. In particular, if $\chi(F) < 0$ and p_* has a section then the section is unique up to conjugation by an element of $\pi_1(F)$.

It follows easily from Theorem 5.19 that nontrivial bundles over S^2 with aspherical fibre do not admit sections.

See also [Hi13e].

Chapter 6

Simple homotopy type and surgery

The problem of determining the high-dimensional manifolds within a given homotopy type has been successfully reduced to the determination of normal invariants and surgery obstructions. This strategy applies also in dimension 4, provided that the fundamental group is in the class SA generated from groups with subexponential growth by extensions and increasing unions [FT95]. (Essentially all the groups in this class that we shall discuss in this book are in fact virtually solvable.) We may often avoid this hypothesis by using 5-dimensional surgery to construct s-cobordisms.

We begin by showing that the Whitehead group of the fundamental group is trivial for surface bundles over surfaces, most circle bundles over geometric 3-manifolds and for many mapping tori. In §2 we define the modified surgery structure set, parametrizing s-cobordism classes of simply homotopy equivalences of closed 4-manifolds. This notion allows partial extensions of surgery arguments to situations where the fundamental group is not elementary amenable. Although many papers on surgery do not explicitly consider the 4-dimensional cases, their results may often be adapted to these cases. In §3 we comment briefly on approaches to the s-cobordism theorem and classification using stabilization by connected sum with copies of $S^2 \times S^2$ or by cartesian product with R.

In §4 we show that 4-manifolds M such that $\pi = \pi_1(M)$ is torsion-free virtually poly-Z and $\chi(M) = 0$ are determined up to homeomorphism by their fundamental group (and Stiefel-Whitney classes, if $h(\pi) < 4$). We also characterize 4-dimensional mapping tori with torsion-free, elementary amenable fundamental group and show that the structure sets for total spaces of RP^2-bundles over T or Kb are finite. In §5 we extend this finiteness to RP^2-bundle spaces over closed hyperbolic surfaces and show that total spaces of bundles with fibre S^2 or an aspherical closed surface over aspherical bases are determined up to s-cobordism by their homotopy type. (We shall consider bundles with base or fibre geometric 3-manifolds in Chapter 13.)

6.1 The Whitehead group

In this section we shall rely heavily upon the work of Waldhausen in [Wd78]. The class of groups *Cl* is the smallest class of groups containing the trivial group and which is closed under generalised free products and HNN extensions with amalgamation over regular coherent subgroups and under filtering direct limit. This class is also closed under taking subgroups, by Proposition 19.3 of [Wd78]. If G is in *Cl* then so is $G\times Z^n$, and $Wh(G)=\tilde{K}(\mathbb{Z}[G])=0$, by Theorem 19.4 of [Wd78]. The argument for this theorem actually shows that if $G\cong A*_C B$ and C is regular coherent then there are *Mayer-Vietoris sequences*:

$$Wh(A)\oplus Wh(B)\to Wh(G)\to\tilde{K}_0(\mathbb{Z}[C])\to\tilde{K}_0(\mathbb{Z}[A])\oplus\tilde{K}_0(\mathbb{Z}[B])\to\tilde{K}_0(\mathbb{Z}[G])\to 0$$

and similarly if $G\cong A*_C$. (See Sections 17.1.3 and 17.2.3 of [Wd78].)

The class *Cl* contains all free groups and poly-Z groups and the class $\mathcal{X}$ of Chapter 2. (In particular, all the groups $Z*_m$ are in *Cl*.) Since every PD_2-group is either poly-Z or is the generalised free product of two free groups with amalgamation over infinite cyclic subgroups it is regular coherent, and is in *Cl*. Hence homotopy equivalences between S^2-bundles over aspherical surfaces are simple. The following extension implies the corresponding result for quotients of such bundle spaces by free involutions.

Theorem 6.1 *Let π be a semidirect product $\rho\rtimes(Z/2Z)$ where ρ is a surface group. Then $Wh(\pi)=0$.*

Proof Assume first that $\pi\cong\rho\times(Z/2Z)$. Let $\Gamma=\mathbb{Z}[\rho]$. There is a cartesian square expressing $\Gamma[Z/2Z]=\mathbb{Z}[\rho\times(Z/2Z)]$ as the pullback of the reduction of coefficients map from Γ to $\Gamma_2=\Gamma/2\Gamma=\mathbb{Z}/2\mathbb{Z}[\rho]$ over itself. (The two maps from $\Gamma[Z/2Z]$ to Γ send the generator of $Z/2Z$ to $+1$ and -1, respectively.) The Mayer-Vietoris sequence for algebraic K-theory traps $K_1(\Gamma[Z/2Z])$ between $K_2(\Gamma_2)$ and $K_1(\Gamma)^2$. (See Theorem 6.4 of [Mi].) Now since $c.d.\rho=2$ the higher K-theory of $R[\rho]$ can be computed in terms of the homology of ρ with coefficients in the K-theory of R (cf. the Corollary to Theorem 5 of the introduction of [Wd78]). In particular, the map from $K_2(\Gamma)$ to $K_2(\Gamma_2)$ is onto, while $K_1(\Gamma)=K_1(\mathbb{Z})\oplus(\rho/\rho')$ and $K_1(\Gamma_2)=\rho/\rho'$. It now follows easily that $K_1(\Gamma[Z/2Z])$ is generated by the images of $K_1(\mathbb{Z})=\{\pm1\}$ and $\rho\times(Z/2Z)$, and so $Wh(\rho\times(Z/2Z))=0$.

If $\pi=\rho\rtimes(Z/2Z)$ is not such a direct product it is isomorphic to a discrete subgroup of $Isom(\mathbb{X})$ which acts properly discontinuously on X, where $\mathbb{X}=\mathbb{E}^2$ or $\mathbb{H}^2$. (See [EM82, Zi].) The singularities of the corresponding 2-orbifold $\pi\backslash X$

are either cone points of order 2 or reflector curves; there are no corner points and no cone points of higher order. Let $|\pi\backslash X|$ be the surface obtained by forgetting the orbifold structure of $\pi\backslash X$, and let m be the number of cone points. Then $\chi(|\pi\backslash X|)-(m/2)=\chi_{orb}(\pi\backslash X)\leq 0$, by the Riemann-Hurwitz formula [Sc83'], so either $\chi(|\pi\backslash X|)\leq 0$ or $\chi(|\pi\backslash X|)=1$ and $m\geq 2$ or $|\pi\backslash X|\cong S^2$ and $m\geq 4$.

We may separate $\pi\backslash X$ along embedded circles (avoiding the singularities) into pieces which are either (i) discs with at least two cone points; (ii) annuli with one cone point; (iii) annuli with one boundary a reflector curve; or (iv) surfaces other than D^2 with nonempty boundary. In each case the inclusions of the separating circles induce monomorphisms on orbifold fundamental groups, and so π is a generalized free product with amalgamation over copies of Z of groups of the form (i) $*^m(Z/2Z)$ (with $m\geq 2$); (ii) $Z*(Z/2Z)$; (iii) $Z\oplus(Z/2Z)$; or (iv) $*^mZ$, by the Van Kampen theorem for orbifolds [Sc83]. The Mayer-Vietoris sequences for algebraic K-theory now give $Wh(\pi)=0$. □

The argument for the direct product case is based on one for showing that $Wh(Z\oplus(Z/2Z))=0$ from [Kw86].

Not all such orbifold groups arise in this way. For instance, the orbifold fundamental group of a torus with one cone point of order 2 has the presentation $\langle x,y\mid [x,y]^2=1\rangle$. Hence it has torsion-free abelianization, and so cannot be a semidirect product as above.

The orbifold fundamental groups of flat 2-orbifolds are the 2-dimensional crystallographic groups. Their finite subgroups are cyclic or dihedral, of order properly dividing 24, and have trivial Whitehead group. In fact $Wh(\pi)=0$ for π any such 2-dimensional crystallographic group [Pe98]. (If π is the fundamental group of an orientable hyperbolic 2-orbifold with k cone points of orders $\{n_1,\dots n_k\}$ then $Wh(\pi)\cong\oplus_{i=1}^k Wh(Z/n_iZ)$ [LS00].)

The argument for the next result is essentially due to F.T.Farrell.

Theorem 6.2 *If π is an extension of $\pi_1(B)$ by $\pi_1(F)$ where B and F are aspherical closed surfaces then $Wh(\pi)=\tilde{K}_0(\mathbb{Z}[\pi])=0$.*

Proof If $\chi(B)<0$ then B admits a complete riemannian metric of constant negative curvature -1. Moreover the only virtually poly-Z subgroups of $\pi_1(B)$ are 1 and Z. If G is the preimage in π of such a subgroup then G is either $\pi_1(F)$ or is the group of a Haken 3-manifold. It follows easily that for any $n\geq 0$ the group $G\times Z^n$ is in Cl and so $Wh(G\times Z^n)=0$. Therefore any such G

is K-flat and so the bundle is admissible, in the terminology of [FJ86]. Hence $Wh(\pi) = \tilde{K}_0(\mathbb{Z}[\pi]) = 0$ by the main result of that paper.

If $\chi(B) = 0$ then this argument does not work, although if moreover $\chi(F) = 0$ then π is poly-Z, so $Wh(\pi) = \tilde{K}_0(\mathbb{Z}[\pi]) = 0$ by Theorem 2.13 of [FJ]. We shall sketch an argument of Farrell for the general case. Lemma 1.4.2 and Theorem 2.1 of [FJ93] together yield a spectral sequence (with coefficients in a simplicial cosheaf) whose E^2 term is $H_i(X/\pi_1(B); Wh'_j(p^{-1}(\pi_1(B)^x)))$ and which converges to $Wh'_{i+j}(\pi)$. Here $p : \pi \to \pi_1(B)$ is the epimorphism of the extension and X is a certain universal $\pi_1(B)$-complex which is contractible and such that all the nontrivial isotropy subgroups $\pi_1(B)^x$ are infinite cyclic and the fixed point set of each infinite cyclic subgroup is a contractible (nonempty) subcomplex. The Whitehead groups with negative indices are the lower K-theory of $\mathbb{Z}[G]$ (i.e., $Wh'_n(G) = K_n(\mathbb{Z}[G])$ for all $n \leq -1$), while $Wh'_0(G) = \tilde{K}_0(\mathbb{Z}[G])$ and $Wh'_1(G) = Wh(G)$. Note that $Wh'_{-n}(G)$ is a direct summand of $Wh(G \times Z^{n+1})$. If $i + j > 1$ then $Wh'_{i+j}(\pi)$ agrees rationally with the higher Whitehead group $Wh_{i+j}(\pi)$. Since the isotropy subgroups $\pi_1(B)^x$ are infinite cyclic or trivial $Wh(p^{-1}(\pi_1(B)^x) \times Z^n) = 0$ for all $n \geq 0$, by the argument of the above paragraph, and so $Wh'_j(p^{-1}(\pi_1(B)^x)) = 0$ if $j \leq 1$. Hence the spectral sequence gives $Wh(\pi) = \tilde{K}_0(\mathbb{Z}[\pi]) = 0$. □

A closed 3-manifold is a *Haken manifold* if it is irreducible and contains an incompressible 2-sided surface. Every aspherical closed 3-manifold N is either Haken, hyperbolic or Seifert-fibred, by the work of Perelman [B-P], and so either has an infinite solvable fundamental group or it has a JSJ decomposition along a finite family of disjoint incompressible tori and Klein bottles so that the complementary components are Seifert fibred or hyperbolic. Every closed 3-manifold with a metric of non-positive curvature is *virtually fibred* (i.e., finitely covered by a mapping torus), and so every aspherical closed 3-manifold is virtually Haken [Ag13, PW12].

If an aspherical closed 3-manifold has a JSJ decomposition with at least one hyperbolic component then it has a metric of non-positive curvature [Lb95]. Otherwise it is a *graph manifold*: either it has solvable fundamental group or it has a JSJ decomposition into Seifert fibred pieces. It is a *proper* graph manifold if the minimal such JSJ decomposition is non-trivial. A criterion for a proper graph manifold to be virtually fibred is given in [Ne97].

Theorem 6.3 *Let N be a connected sum of aspherical graph manifolds, and let $\nu = \pi_1(N)$ and $\pi = \nu \rtimes_\theta Z$, where $\theta \in Aut(\nu)$. Then $\nu \times Z^n$ is regular coherent, and $Wh(\pi \times Z^n) = \tilde{K}_0(\mathbb{Z}[\pi \times Z^n]) = 0$, for all $n \geq 0$.*

Proof The group ν is either polycyclic or is a generalized free product with amalgamation along poly-Z subgroups (1, Z^2 or $Z\rtimes_{-1}Z$) of fundamental groups of Seifert fibred 3-manifolds (possibly with boundary). The group rings of torsion-free polycyclic groups are regular noetherian, and hence regular coherent. If G is the fundamental group of a Seifert fibred 3-manifold then it has a subgroup G_o of finite index which is a central extension of the fundamental group of a surface B (possibly with boundary) by Z. We may assume that G is not solvable and hence that $\chi(B)<0$. If ∂B is nonempty then $G_o\cong Z\times F$ and so is an iterated generalized free product of copies of Z^2, with amalgamation along infinite cyclic subgroups. Otherwise we may split B along an essential curve and represent G_o as the generalised free product of two such groups, with amalgamation along a copy of Z^2. In both cases G_o is regular coherent, and therefore so is G, since $[G:G_o]<\infty$ and $c.d.G<\infty$.

Since ν is the generalised free product with amalgamation of regular coherent groups, with amalgamation along poly-Z subgroups, it is also regular coherent. Hence so is $\nu\times Z^n$. Let N_i be an irreducible summand of N and let $\nu_i=\pi_1(N_i)$. If N_i is Haken then ν_i is in Cl and so $Wh(\nu_i\times Z^n)=0$, for all $n\geq 0$. Otherwise N_i is a Seifert fibred 3-manifold which is not sufficiently large, and the argument of [Pl80] extends easily to prove this. Since $\tilde{K}_0(\mathbb{Z}[\sigma])$ is a direct summand of $Wh(\sigma\times Z)$, for any group σ, we have $\tilde{K}_0(\mathbb{Z}[\nu_i\times Z^n])=0$, for all $n\geq 0$. The Mayer-Vietoris sequences for algebraic K-theory now give, firstly, $Wh(\nu\times Z^n)=\tilde{K}_0(\mathbb{Z}[\nu\times Z^n])=0$, and then $Wh(\pi\times Z^n)=\tilde{K}_0(\mathbb{Z}[\pi\times Z^n])=0$ also. □

All 3-manifold groups are coherent as *groups* [Hm]. If we knew that their group *rings* were regular coherent then we could use [Wd78] instead of [FJ86] to give a purely algebraic proof of Theorem 6.2, for as surface groups are free products of free groups with amalgamation over an infinite cyclic subgroup, an extension of one surface group by another is a free product of groups with $Wh=0$, amalgamated over the group of a surface bundle over S^1. Similarly, we could deduce from [Wd78] and the work of Perelman [B-P] that $Wh(\nu\rtimes_\theta Z)=0$ for any torsion-free 3-manifold group $\nu=\pi_1(N)$ where N is a closed 3-manifold.

Theorem 6.4 *Let N be a closed 3-manifold such that $\nu=\pi_1(N)$ is torsion-free, and let μ be a group with an infinite cyclic normal subgroup A such that $\mu/A\cong\nu$. Then $Wh(\mu)=Wh(\nu)=0$.*

Proof Let $N=\sharp_{1\leq i\leq n}N_i$ be the factorization of N into irreducibles, and let $\nu\cong *_{1\leq i\leq n}\nu_i$, where $\nu_i=\pi_1(N_i)$. The irreducible factors are either Haken, hyperbolic or Seifert fibred, by the work of Perelman [B-P]. Let μ_i be the preimage of ν_i in μ, for $1 \leq i \leq n$. Then μ is the generalized free product of the μ_i's, amalgamated over infinite cyclic subgroups. For all $1 \leq i \leq n$ we have $Wh(\mu_i) = 0$, by Lemma 1.1 of [St84] if $K(\nu_i, 1)$ is Haken, by the main result of [FJ86] if it is hyperbolic, by an easy extension of the argument of [Pl80] if it is Seifert fibred but not Haken and by Theorem 19.5 of [Wd78] if ν_i is infinite cyclic. The Mayer-Vietoris sequences for algebraic K-theory now give $Wh(\mu) = Wh(\nu) = 0$ also. □

Theorem 6.4 may be used to strengthen Theorem 4.11 to give criteria for a closed 4-manifold M to be *simple* homotopy equivalent to the total space of an S^1-bundle, if $\pi_1(M)$ is torsion-free.

6.2 The s-cobordism structure set

The TOP structure set for a closed 4-manifold M with fundamental group π and orientation character $w : \pi \to \{\pm 1\}$ is

$$S_{TOP}(M) = \{f : N \to M \mid N \ a \ TOP \ 4{-}manifold, \ f \ a \ simple \ h.e.\}/\sim,$$

where $f_1 \sim f_2$ if $f_1 = f_2 h$ for some homeomomorphism $h : N_1 \to N_2$. If π is "good" (e.g., if it is in SA) then $L_5^s(\pi, w)$ acts on the structure set $S_{TOP}(M)$, and the orbits of the action ω correspond to the normal invariants $\eta(f)$ of simple homotopy equivalences [FQ, FT95]. The surgery sequence

$$[SM; G/TOP] \xrightarrow{\sigma_5} L_5^s(\pi, w) \xrightarrow{\omega} S_{TOP}(M) \xrightarrow{\eta} [M; G/TOP] \xrightarrow{\sigma_4} L_4^s(\pi, w)$$

may then be identified with the algebraic surgery sequence of [Rn]. The additions on the homotopy sets $[X, G/TOP]$ derive from an H-space structure on G/TOP. (In low dimensions this is unambiguous, as G/TOP has Postnikov 5-stage $K(Z/2Z, 2) \times K(Z, 4)$, which has an unique H-space structure.) We shall not need to specify the addition on $S_{TOP}(M)$.

As it is not yet known whether 5-dimensional s-cobordisms over other fundamental groups are products, we shall redefine the structure set by setting

$$S^s_{TOP}(M) = \{f : N \to M \mid N \ a \ TOP \ 4{-}manifold, \ f \ a \ simple \ h.e.\}/\approx,$$

where $f_1 \approx f_2$ if there is a map $F : W \to M$ with domain W an s-cobordism with $\partial W = N_1 \cup N_2$ and $F|_{N_i} = f_i$ for $i = 1, 2$. If the s-cobordism theorem holds over π this is the usual TOP structure set for M. We shall usually write $L_n(\pi, w)$ for $L^s_n(\pi, w)$ if $Wh(\pi) = 0$ and $L_n(\pi)$ if moreover w is trivial. When the orientation character is nontrivial and otherwise clear from the context we shall write $L_n(\pi,-)$. We shall say that a closed 4-manifold is *s-rigid* if it is determined up to s-cobordism by its homotopy type. The homotopy set $[M;G/TOP]$ may be identified with the set of normal maps (f,b), where $f:N\to M$ is a degree 1 map and b is a stable framing of $T_N\oplus f^*\xi$, for some TOP R^n-bundle ξ over M. If $f:N\to M$ is a homotopy equivalence, with homotopy inverse h, let $\xi=h^*\nu_N$ and b be the framing determined by a homotopy from hf to id_N. Let $\hat{f}\in[M,G/TOP]$ be the homotopy class corresponding to (f,b). Let k_2 generate $H^2(G/TOP;\mathbb{F}_2)\cong Z/2Z$ and l_4 generate $H^4(G/TOP;\mathbb{Z})\cong Z$, with image $[l_4]$ in $H^4(G/TOP;\mathbb{F}_2)$. The function from $[M;G/TOP]$ to $H^2(M;\mathbb{F}_2)\oplus H^4(M;\mathbb{Z})$ which sends $\hat{f}$ to $(\hat{f}^*(k_2),\hat{f}^*(l_4))$ is an isomorphism. Let $KS(M)\in H^4(M;\mathbb{F}_2)$ be the Kirby-Siebenmann obstruction to lifting the TOP normal fibration of M to a vector bundle. If $\hat{f}$ is a normal map then

$$KS(M)-(f^*)^{-1}KS(N)=\hat{f}^*(k_2^2+[l_4]),$$

and $\hat{f}$ factors through G/PL if and only if this difference is 0 [KT98]. If M is orientable then $\hat{f}^*(l_4)([M])=(\sigma(M)-\sigma(N))/8$, where $\sigma(M)$ is the signature of the intersection pairing on $H_2(M;\mathbb{Z})$, and so

$$(KS(M)-(f^*)^{-1}KS(N)-\hat{f}^*(k_2)^2)([M])\equiv(\sigma(M)-\sigma(N))/8 \quad mod\ (2).$$

The *Kervaire-Arf invariant* of a normal map $\hat{g}:N^{2q}\to G/TOP$ is the image of the surgery obstruction in $L_{2q}(Z/2Z,-)=Z/2Z$ under the homomorphism induced by the orientation character, $c(\hat{g})=L_{2q}(w_1(N))(\sigma_{2q}(\hat{g}))$. The argument of Theorem 13.B.5 of [Wl] may be adapted to show that there are universal classes K_{4i+2} in $H^{4i+2}(G/TOP;\mathbb{F}_2)$ (for $i\geq 0$) such that

$$c(\hat{g})=(w(M)\cup\hat{g}^*((1+Sq^2+Sq^2Sq^2)\Sigma K_{4i+2}))\cap[M].$$

Moreover $K_2=k_2$, since c induces the isomorphism $\pi_2(G/TOP)=Z/2Z$. In the 4-dimensional case this expression simplifies to

$$c(\hat{g})=(w_2(M)\cup\hat{g}^*(k_2)+\hat{g}^*(Sq^2k_2))([M])=(w_1(M)^2\cup\hat{g}^*(k_2))([M]).$$

The *codimension-2 Kervaire invariant* of a 4-dimensional normal map $\hat{g}$ is $kerv(\hat{g})=\hat{g}^*(k_2)$. Its value on a 2-dimensional homology class represented by an immersion $y:Y\to M$ is the Kervaire-Arf invariant of the normal map induced over the surface Y.

The structure set may overestimate the number of homeomorphism types within the homotopy type of M, if M has self homotopy equivalences which are not homotopic to homeomorphisms. Such "exotic" self homotopy equivalences may often be constructed as follows. Given $\alpha:S^2\to M$, let $\beta:S^4\to M$ be the composition $\alpha\eta S\eta$, where η is the Hopf map, and let $s:M\to M\vee S^4$ be the pinch map obtained by shrinking the boundary of a 4-disc in M. Then the composite $f_\alpha=(id_M\vee\beta)s$ is a self homotopy equivalence of M.

Lemma 6.5 [No64] *Let M be a closed 4-manifold and let $\alpha : S^2 \to M$ be a map such that $\alpha_*[S^2] \neq 0$ in $H_2(M;\mathbb{F}_2)$ and $\alpha^* w_2(M) = 0$. Then $kerv(\widehat{f_\alpha}) \neq 0$ and so f_α is not normally cobordant to a homeomorphism.*

Proof Since $\alpha_*[S^2] \neq 0$ there is a $u \in H_2(M;\mathbb{F}_2)$ such that $\alpha_*[S^2].u = 1$. This class may be realized as $u = g_*[Y]$ where Y is a closed surface and $g : Y \to M$ is transverse to f_α. Then $g^* kerv(\widehat{f_\alpha})[Y]$ is the Kervaire-Arf invariant of the normal map induced over Y and is nontrivial. (See Theorem 5.1 of [CH90] for details.) □

The family of surgery obstruction maps may be identified with a natural transformation from $\mathbb{L}_0$-homology to L-theory. (In the nonorientable case we must use w-twisted $\mathbb{L}_0$-homology.) In dimension 4 the cobordism invariance of surgery obstructions (as in §13B of [Wl]) leads to the following formula.

Theorem 6.6 [Da05] *There are homomorphisms $I_0 : H_0(\pi; Z^w) \to L_4(\pi, w)$ and $\kappa_2 : H_2(\pi;\mathbb{F}_2) \to L_4(\pi, w)$ such that for any $\hat{f} : M \to G/TOP$ the surgery obstruction is $\sigma_4(\hat{f}) = I_0(c_{M*}(\hat{f}^*(l_4) \cap [M])) + \kappa_2(c_{M*}(kerv(\hat{f}) \cap [M]))$.* □

In the orientable case the signature homomorphism from $L_4(\pi)$ to Z is a left inverse for $I_0 : Z \to L_4(\pi)$, but in general I_0 is not injective. This formula can be made somewhat more explicit as follows.

Theorem 6.6′ [Da05] *If $\hat{f} = (f, b)$ where $f : N \to M$ is a degree 1 map then the surgery obstructions are given by*

$$\sigma_4(\hat{f}) = I_0((\sigma(N) - \sigma(M))/8) + \kappa_2(c_{M*}(kerv(\hat{f}) \cap [M])), \qquad \text{if } w = 1, \quad \text{and}$$

$$\sigma_4(\hat{f}) = I_0(KS(N) - KS(M) + kerv(\hat{f})^2) + \kappa_2(c_{M*}(kerv(\hat{f}) \cap [M])), \text{ if } w \neq 1.$$

(In the latter case we identify $H^4(M;\mathbb{Z})$, $H^4(N;\mathbb{Z})$ and $H^4(M;\mathbb{F}_2)$ with $H_0(\pi; Z^w) = Z/2Z$.) □

The homomorphism σ_4 is trivial on the image of η, but in general we do not know whether a 4-dimensional normal map with trivial surgery obstruction must be normally cobordant to a simple homotopy equivalence. (See however [Kh07] and [Ym07].) In our applications we shall always have a simple homotopy equivalence in hand.

A more serious problem is that it is not clear how to define the action ω in general. We shall be able to circumvent this problem by *ad hoc* arguments in some cases. (There is always an action on the homological structure set, defined in terms of $\mathbb{Z}[\pi]$-homology equivalences [FQ].)

If we fix an isomorphism $i_Z : Z \to L_5(Z)$ we may define a function $I_\pi : \pi \to L_5^s(\pi)$ for any group π by $I_\pi(g) = g_*(i_Z(1))$, where $g_* : Z = L_5(Z) \to L_5^s(\pi)$ is induced by the homomorphism sending 1 in Z to g in π. Then $I_Z = i_Z$ and I_π is natural in the sense that if $f : \pi \to H$ is a homomorphism then $L_5(f)I_\pi = I_H f$. As abelianization and projection to the summands of Z^2 induce an isomorphism from $L_5(Z * Z)$ to $L_5(Z)^2$ [Ca73], it follows easily from naturality that I_π is a homomorphism (and so factors through π/π') [We83]. We shall extend this to the nonorientable case by defining $I_\pi^+ : \mathrm{Ker}(w) \to L_5^s(\pi; w)$ as the composite of $I_{\mathrm{Ker}(w)}$ with the homomorphism induced by inclusion.

Theorem 6.7 *Let M be a closed 4-manifold with fundamental group π and let $w = w_1(M)$. Given any $\gamma \in \mathrm{Ker}(w)$ there is a normal cobordism from id_M to itself with surgery obstruction $I_\pi^+(\gamma) \in L_5^s(\pi, w)$.*

Proof We may assume that γ is represented by a simple closed curve with a product neighbourhood $U \cong S^1 \times D^3$. Let P be the E_8 manifold [FQ] and delete the interior of a submanifold homeomorphic to $D^3 \times [0,1]$ to obtain P_o. There is a normal map $p : P_o \to D^3 \times [0,1]$ (*rel* boundary). The surgery obstruction for $p \times id_{S^1}$ in $L_5(Z) \cong L_4(1)$ is given by a codimension-1 signature (see §12B of [Wl]), and generates $L_5(Z)$. Let $Y = (M \setminus intU) \times [0,1] \cup P_o \times S^1$, where we identify $(\partial U) \times [0,1] = S^1 \times S^2 \times [0,1]$ with $S^2 \times [0,1] \times S^1$ in $\partial P_o \times S^1$. Matching together $id|_{(M \setminus intU) \times [0,1]}$ and $p \times id_{S^1}$ gives a normal cobordism Q from id_M to itself. The theorem now follows by the additivity of surgery obstructions and naturality of the homomorphisms I_π^+. □

In particular, if π is in SA then the image of I_π^+ acts trivially on $S_{TOP}(M)$.

Corollary 6.7.1 *Let $\lambda_* : L_5^s(\pi) \to L_5(Z)^d = Z^d$ be the homomorphism induced by a basis $\{\lambda_1, ..., \lambda_d\}$ for $Hom(\pi, Z)$. If M is orientable, $f : M_1 \to M$ is a simple homotopy equivalence and $\theta \in L_5(Z)^d$ there is a normal cobordism from f to itself whose surgery obstruction in $L_5(\pi)$ has image θ under λ_*.*

Proof If $\{\gamma_1, ..., \gamma_d\} \in \pi$ represents a "dual basis" for $H_1(\pi; \mathbb{Z})$ modulo torsion (so that $\lambda_i(\gamma_j) = \delta_{ij}$ for $1 \leq i, j \leq d$), then $\{\lambda_*(I_\pi(\gamma_1)), ..., \lambda_*(I_\pi(\gamma_d))\}$ is a basis for $L_5(Z)^d$. □

If π is free or is a PD_2^+-group the homomorphism λ_* is an isomorphism [Ca73]. In most of the other cases of interest to us the following corollary applies.

Corollary 6.7.2 *If M is orientable and* $\mathrm{Ker}(\lambda_*)$ *is finite then* $S^s_{TOP}(M)$ *is finite. In particular, this is so if* $\mathrm{Coker}(\sigma_5)$ *is finite.*

Proof The signature difference maps $[M;G/TOP] = H^4(M;\mathbb{Z}) \oplus H^2(M;\mathbb{F}_2)$ onto $L_4(1) = Z$ and so there are only finitely many normal cobordism classes of simple homotopy equivalences $f : M_1 \to M$. Moreover, $\mathrm{Ker}(\lambda_*)$ is finite if σ_5 has finite cokernel, since $[SM;G/TOP] \cong Z^d \oplus (Z/2Z)^d$. Suppose that $F : N \to M \times I$ is a normal cobordism between two simple homotopy equivalences $F_- = F|\partial_- N$ and $F_+ = F|\partial_+ N$. By Theorem 6.7 there is another normal cobordism $F' : N' \to M \times I$ from F_+ to itself with $\lambda_*(\sigma_5(F')) = \lambda_*(-\sigma_5(F))$. The union of these two normal cobordisms along $\partial_+ N = \partial_- N'$ is a normal cobordism from F_- to F_+ with surgery obstruction in $\mathrm{Ker}(\lambda_*)$. If this obstruction is 0 we may obtain an s-cobordism W by 5-dimensional surgery (rel ∂). □

The surgery obstruction groups for a semidirect product $\pi \cong G \rtimes_\theta Z$, may be related to those of the (finitely presentable) normal subgroup G by means of Theorem 12.6 of [Wl]. If $Wh(\pi) = Wh(G) = 0$ this theorem asserts that there is an exact sequence

$$\dots L_m(G, w|_G) \xrightarrow{1-w(t)\theta_*} L_m(G, w|_G) \to L_m(\pi, w) \to L_{m-1}(G, w|_G) \dots,$$

where t generates π modulo G and $\theta_* = L_m(\theta, w|_G)$. The following result is based on Theorem 15.B.1 of [Wl].

Theorem 6.8 *Let M be a 4-manifold which is homotopy equivalent to a mapping torus $M(\theta)$, where θ is a self-homeomorphism of an aspherical closed 3-manifold N. If $Wh(\pi_1(M)) = Wh(\pi_1(M) \times Z) = 0$ then M is s-cobordant to $M(\theta)$ and $\widetilde{M}$ is homeomorphic to R^4.*

Proof The surgery obstruction homomorphisms σ_i^N are isomorphisms for all large i [Ro11]. Comparison of the Mayer-Vietoris sequences for $\mathbb{L}_0$-homology and L-theory (as in Proposition 2.6 of [St84]) shows that σ_i^M and $\sigma_i^{M\times S^1}$ are also isomorphisms for all large i, and so $S_{TOP}(M(\Theta) \times S^1)$ has just one element. If $h : M \to M(\Theta)$ is a homotopy equivalence then $h \times id$ is homotopic to a homeomorphism $M \times S^1 \cong M(\Theta) \times S^1$, and so $M \times \mathbb{R} \cong M(\Theta) \times \mathbb{R}$. This product contains s-cobordisms bounded by disjoint copies of M and $M(\Theta)$.

The final assertion follows from Corollary 7.3B of [FQ] since M is aspherical and π is 1-connected at ∞ [Ho77]. □

It remains an open question whether aspherical closed manifolds with isomorphic fundamental groups must be homeomorphic. This has been verified in higher dimensions in many cases, in particular under geometric assumptions [FJ], and under assumptions on the combinatorial structure of the group [Ca73, St84, NS85]. We shall see that many aspherical 4-manifolds are determined up to s-cobordism by their groups.

There are more general "Mayer-Vietoris" sequences which lead to calculations of the surgery obstruction groups for certain generalized free products and HNN extensions in terms of those of their building blocks [Ca73, St87].

A subgroup H of a group G is *square-root closed* in G if $g^2 \in H$ implies $g \in H$, for $g \in G$. A group π is *square-root closed accessible* if it can be obtained from the trivial group by iterated HNN extensions with associated subgroups square-root closed in the base group and amalgamated products over square-root closed subgroups. In particular, finitely generated free groups and poly-Z groups are square-root closed accessible. A geometric argument implies that cuspidal subgroups of the fundamental group Γ of a complete hyperbolic manifold of finite volume are maximal parabolic subgroups, and hence are square root closed in Γ. If S is a closed surface with $\chi(S) < 0$ it may be decomposed as the union of two subsurfaces with connected boundary and hyperbolic interior. Therefore all PD_2-groups are square-root closed accessible.

Lemma 6.9 *Let π be either the group of a finite graph of groups, all of whose vertex groups are infinite cyclic, or a square root closed accessible group of cohomological dimension 2. Then I_π^+ is an epimorphism. If M is a closed 4-manifold with fundamental group π the surgery obstruction maps $\sigma_4(M)$ and $\sigma_5(M)$ are epimorphisms.*

Proof Since π is in Cl we have $Wh(\pi) = 0$ and a comparison of Mayer-Vietoris sequences shows that the assembly map from $H_*(\pi; \mathbb{L}_0^w)$ to $L_*(\pi, w)$ is an isomorphism [Ca73, St87]. Since $c.d.\pi \leq 2$ and $H_1(\mathrm{Ker}(w); \mathbb{Z})$ maps onto $H_1(\pi; Z^w)$ the component of this map in degree 1 may be identified with I_π^+. In general, the surgery obstruction maps factor through the assembly map. Since $c.d.\pi \leq 2$ the homomorphism $c_{M*} : H_*(M; D) \to H_*(\pi; D)$ is onto for any local coefficient module D, and so the lemma follows. □

The class of groups considered in this lemma includes free groups, PD_2-groups and the groups $Z*_m$. Note however that if π is a PD_2-group w need not be the canonical orientation character.

6.3 Stabilization and h-cobordism

It has long been known that many results of high dimensional differential topology hold for smooth 4-manifolds after stabilizing by connected sum with copies of $S^2 \times S^2$ [CS71, FQ80, La79, Qu83]. In particular, if M and N are h-cobordant closed smooth 4-manifolds then $M\sharp(\sharp^k S^2 \times S^2)$ is diffeomorphic to $N\sharp(\sharp^k S^2 \times S^2)$ for some $k \geq 0$. In the spin case $w_2(M) = 0$ this is an elementary consequence of the existence of a well-indexed handle decomposition of the h-cobordism [Wl64]. In Chapter VII of [FQ] it is shown that 5-dimensional TOP cobordisms have handle decompositions relative to a component of their boundaries, and so a similar result holds for h-cobordant closed TOP 4-manifolds. Moreover, if M is a TOP 4-manifold then $KS(M) = 0$ if and only if $M\sharp(\sharp^k S^2 \times S^2)$ is smoothable for some $k \geq 0$ [LS71].

These results suggest the following definition. Two 4-manifolds M_1 and M_2 are *stably homeomorphic* if $M_1\sharp(\sharp^k S^2 \times S^2)$ and $M_2\sharp(\sharp^l S^2 \times S^2)$ are homeomorphic, for some $k,\ l \geq 0$. (Thus h-cobordant closed 4-manifolds are stably homeomorphic.) Clearly $\pi_1(M)$, $w_1(M)$, the orbit of $c_{M*}[M]$ in $H_4(\pi_1(M); Z^{w_1(M)})$ under the action of $Out(\pi_1(M))$, and the parity of $\chi(M)$ are invariant under stabilization. If M is orientable $\sigma(M)$ is also invariant.

Kreck has shown that (in any dimension) classification up to stable homeomorphism (or diffeomorphism) can be reduced to bordism theory. There are three cases: If $w_2(\widetilde{M}) \neq 0$ and $w_2(\widetilde{N}) \neq 0$ then M and N are stably homeomorphic if and only if for some choices of orientations and identification of the fundamental groups the invariants listed above agree (in an obvious manner). If $w_2(M) = w_2(N) = 0$ then M and N are stably homeomorphic if and only if for some choices of orientations, Spin structures and identification of the fundamental group they represent the same element in $\Omega_4^{SpinTOP}(K(\pi,1))$. The most complicated case is when M and N are not Spin, but the universal covers are Spin. (See [Kr99, Te] for expositions of Kreck's ideas, and see [Po13] for an application to 4-manifolds determined by Tietze-equivalent presentations.)

We shall not pursue this notion of stabilization further (with one minor exception, in Chapter 14), for it is somewhat at odds with the tenor of this book. The manifolds studied here usually have minimal Euler characteristic, and often are aspherical. Each of these properties disappears after stabilization. We may however also stabilize by cartesian product with the real line R, and there is then the following simple but satisfying result.

Lemma 6.10 *Closed 4-manifolds M and N are h-cobordant if and only if $M \times R$ and $N \times R$ are homeomorphic.*

Proof If W is an h-cobordism from M to N (with fundamental group $\pi = \pi_1(W)$) then $W \times S^1$ is an h-cobordism from $M \times S^1$ to $N \times S^1$. The torsion is 0 in $Wh(\pi \times Z)$, by Theorem 23.2 of [Co], and so there is a homeomorphism from $M \times S^1$ to $N \times S^1$ which carries $\pi_1(M)$ to $\pi_1(N)$. Hence $M \times R \cong N \times R$. Conversely, if $M \times R \cong N \times R$ then $M \times R$ contains a copy of N disjoint from $M \times \{0\}$, and the region W between $M \times \{0\}$ and N is an h-cobordism. □

6.4 Manifolds with π_1 elementary amenable and $\chi = 0$

In this section we shall show that closed manifolds satisfying the hypotheses of Theorem 3.17 and with torsion-free fundamental group are determined up to homeomorphism by their homotopy type. As a consequence, closed 4-manifolds with torsion-free elementary amenable fundamental group and Euler characteristic 0 are homeomorphic to mapping tori. We also estimate the structure sets for RP^2-bundles over T or Kb. In the remaining cases involving torsion computation of the surgery obstructions is much more difficult. We shall comment briefly on these cases in Chapters 10 and 11.

Theorem 6.11 *Let M be a closed 4-manifold with $\chi(M) = 0$ and whose fundamental group π is torsion-free, coherent, locally virtually indicable and restrained. Then M is determined up to homeomorphism by its homotopy type. If moreover $h(\pi) = 4$ then every automorphism of π is realized by a self homeomorphism of M.*

Proof By Theorem 3.17 either $\pi \cong Z$ or $Z *_m$ for some $m \neq 0$, or M is aspherical, π is virtually poly-Z and $h(\pi) = 4$. Hence $Wh(\pi) = 0$, in all cases. If $\pi \cong Z$ or $Z *_m$ then the surgery obstruction homomorphisms are epimorphisms, by Lemma 6.9. We may calculate $L_4(\pi, w)$ by means of Theorem 12.6 of [Wl], or more generally §3 of [St87], and we find that if $\pi \cong Z$ or $Z *_{2n}$ then $\sigma_4(M)$ is in fact an isomorphism. If $\pi \cong Z *_{2n+1}$ then there are two normal cobordism classes of homotopy equivalences $h : X \to M$. Let ξ generate the image of $H^2(\pi; \mathbb{F}_2) \cong Z/2Z$ in $H^2(M; \mathbb{F}_2) \cong (Z/2Z)^2$, and let $j : S^2 \to M$ represent the unique nontrivial spherical class in $H_2(M; \mathbb{F}_2)$. Then $\xi^2 = 0$, since $c.d.\pi = 2$, and $\xi \cap j_*[S^2] = 0$, since $c_M j$ is nullhomotopic. It follows that $j_*[S^2]$ is Poincaré dual to ξ, and so $v_2(M) \cap j_*[S^2] = \xi^2 \cap [M] = 0$. Hence $j^* w_2(M) = j^* v_2(M) + (j^* w_1(M))^2 = 0$ and so f_j has nontrivial normal invariant, by Lemma 6.5. Therefore each of these two normal cobordism classes contains a self homotopy equivalence of M.

If M is aspherical, π is virtually poly-Z and $h(\pi) = 4$ then $S_{TOP}(M)$ has just one element, by Theorem 2.16 of [FJ]. The theorem now follows. □

Corollary 6.11.1 *Let M be a closed 4-manifold with $\chi(M)=0$ and fundamental group $\pi\cong Z$, Z^2 or $Z\rtimes_{-1}Z$. Then M is determined up to homeomorphism by π and $w(M)$.*

Proof If $\pi\cong Z$ then M is homotopy equivalent to $S^1\times S^3$ or $S^1\tilde{\times}S^3$, by Corollary 4.5.3, while if $\pi\cong Z^2$ or $Z\rtimes_{-1}Z$ it is homotopy equivalent to the total space of an S^2-bundle over T or Kb, by Theorem 5.10. □

The homotopy type of a closed 4-manifold M is also determined by π and $w(M)$ if $\chi(M)=0$ and $\pi\cong Z*_m$ for m even [Hi13c].

We may now give an analogue of the Farrell and Stallings fibration theorems for 4-manifolds with torsion-free elementary amenable fundamental group.

Theorem 6.12 *Let M be a closed 4-manifold whose fundamental group π is torsion-free and elementary amenable. A map $f:M\to S^1$ is homotopic to a fibre bundle projection if and only if $\chi(M)=0$ and f induces an epimorphism from π to Z with finitely generated kernel.*

Proof The conditions are clearly necessary. Suppose that they hold. Let $\nu=\mathrm{Ker}(\pi_1(f))$, let M_ν be the infinite cyclic covering space of M with fundamental group ν and let $t:M_\nu\to M_\nu$ be a generator of the group of covering transformations. By Corollary 4.5.2 either $\nu=1$ (so $M_\nu\simeq S^3$) or $\nu\cong Z$ (so $M_\nu\simeq S^2\times S^1$ or $S^2\tilde{\times}S^1$) or M is aspherical. In the latter case π is a torsion-free virtually poly-Z group, by Theorem 1.11 and Theorem 9.23 of [Bi]. Thus in all cases there is a homotopy equivalence f_ν from M_ν to a closed 3-manifold N. Moreover the self homotopy equivalence $f_\nu t f_\nu^{-1}$ of N is homotopic to a homeomorphism, g say, and so f is fibre homotopy equivalent to the canonical projection of the mapping torus $M(g)$ onto S^1. It now follows from Theorem 6.11 that any homotopy equivalence from M to $M(g)$ is homotopic to a homeomorphism. □

The structure sets of the RP^2-bundles over T or Kb are also finite.

Theorem 6.13 *Let M be the total space of an RP^2-bundle over T or Kb. Then $S_{TOP}(M)$ has order at most 32.*

Proof As M is nonorientable $H^4(M;\mathbb{Z})=Z/2Z$ and as $\beta_1(M;\mathbb{F}_2)=3$ and $\chi(M)=0$ we have $H^2(M;\mathbb{F}_2)\cong(Z/2Z)^4$. Hence $[M;G/TOP]$ has order 32. Let $w=w_1(M)$. It follows from the Shaneson-Wall splitting theorem (Theorem

12.6 of [Wl]) that $L_4(\pi, w) \cong L_4(Z/2Z, -) \oplus L_2(Z/2Z, -) \cong (Z/2Z)^2$, detected by the Kervaire-Arf invariant and the codimension-2 Kervaire invariant. Similarly $L_5(\pi, w) \cong L_4(Z/2Z, -)^2$ and the projections to the factors are Kervaire-Arf invariants of normal maps induced over codimension-1 submanifolds. (In applying the splitting theorem, note that $Wh(Z \oplus (Z/2Z)) = Wh(\pi) = 0$, by Theorem 6.1 above.) Hence $S_{TOP}(M)$ has order at most 128.

The Kervaire-Arf homomorphism c is onto, since $c(\hat{g}) = (w^2 \cup \hat{g}^*(k_2)) \cap [M]$, $w^2 \neq 0$ and every element of $H^2(M; \mathbb{F}_2)$ is equal to $\hat{g}^*(k_2)$ for some normal map $\hat{g} : M \to G/TOP$. Similarly there is a normal map $f_2 : X_2 \to RP^2$ with $\sigma_2(f_2) \neq 0$ in $L_2(Z/2Z, -)$. If $M = RP^2 \times B$, where $B = T$ or Kb is the base of the bundle, then $f_2 \times id_B : X_2 \times B \to RP^2 \times B$ is a normal map with surgery obstruction $(0, \sigma_2(f_2)) \in L_4(Z/2Z, -) \oplus L_2(Z/2Z, -)$. We may assume that f_2 is a homeomorphism over a disc $\Delta \subset RP^2$. As the nontrivial bundles may be obtained from the product bundles by cutting M along $RP^2 \times \partial\Delta$ and regluing via the twist map of $RP^2 \times S^1$, the normal maps for the product bundles may be compatibly modified to give normal maps with nonzero obstructions in the other cases. Hence σ_4 is onto and so $S_{TOP}(M)$ has order at most 32. $\square$

In each case $H_2(M; \mathbb{F}_2) \cong H_2(\pi; \mathbb{F}_2)$, so the argument of Lemma 6.5 does not apply. However we can improve our estimate in the abelian case.

Theorem 6.14 *Let M be the total space of an RP^2-bundle over T. Then $S_{TOP}(M)$ has order 8.*

Proof Since π is abelian the surgery sequence may be identified with the algebraic surgery sequence of [Rn], which is an exact sequence of abelian groups. Thus it shall suffice to show that $L_5(\pi, w)$ acts trivially on the class of id_M in $S_{TOP}(M)$.

Let $\lambda_1, \lambda_2 : \pi \to Z$ be epimorphisms generating $Hom(\pi, Z)$ and let $t_1, t_2 \in \pi$ represent a dual basis for $\pi/(torsion)$ (i.e., $\lambda_i(t_j) = \delta_{ij}$ for $i = 1, 2$). Let u be the element of order 2 in π and let $k_i : Z \oplus (Z/2Z) \to \pi$ be the monomorphism defined by $k_i(a, b) = at_i + bu$, for $i = 1, 2$. Define splitting homomorphisms p_1, p_2 by $p_i(g) = k_i^{-1}(g - \lambda_i(g)t_i)$ for all $g \in \pi$. Then $p_i k_i = id_{Z \oplus (Z/2Z)}$ and $p_i k_{3-i}$ factors through $Z/2Z$, for $i = 1, 2$. The orientation character $w = w_1(M)$ maps the torsion subgroup of π onto $Z/2Z$, by Theorem 5.13, and t_1 and t_2 are in $\mathrm{Ker}(w)$. Therefore p_i and k_i are compatible with w, for $i = 1, 2$. As $L_5(Z/2Z, -) = 0$ it follows that $L_5(k_1)$ and $L_5(k_2)$ are inclusions of complementary summands of $L_5(\pi, w) \cong (Z/2Z)^2$, split by the projections $L_5(p_1)$ and $L_5(p_2)$.

Let γ_i be a simple closed curve in T which represents $t_i \in \pi$. Then γ_i has a product neighbourhood $N_i \cong S^1 \times [-1,1]$ whose preimage $U_i \subset M$ is homeomorphic to $RP^2 \times S^1 \times [-1,1]$. As in Theorem 6.13 there is a normal map $f_4 : X_4 \to RP^2 \times [-1,1]^2$ (*rel* boundary) with $\sigma_4(f_4) \neq 0$ in $L_4(Z/2Z,-)$. Let $Y_i = (M \setminus intU_i) \times [-1,1] \cup X_4 \times S^1$, where we identify $(\partial U_i) \times [-1,1] = RP^2 \times S^1 \times S^0 \times [-1,1]$ with $RP^2 \times [-1,1] \times S^0 \times S^1$ in $\partial X_4 \times S^1$. If we match together $id_{(M\setminus intU_i)\times[-1,1]}$ and $f_4 \times id_{S^1}$ we obtain a normal cobordism Q_i from id_M to itself. The image of $\sigma_5(Q_i)$ in $L_4(\mathrm{Ker}(\lambda_i), w) \cong L_4(Z/2Z,-)$ under the splitting homomorphism is $\sigma_4(f_4)$. On the other hand its image in $L_4(\mathrm{Ker}(\lambda_{3-i}), w)$ is 0, and so it generates the image of $L_5(k_{3-i})$. Thus $L_5(\pi,w)$ is generated by $\sigma_5(Q_1)$ and $\sigma_5(Q_2)$, and so acts trivially on id_M. $\square$

Does $L_5(\pi,w)$ act trivially on each class in $S_{TOP}(M)$ when M is an RP^2-bundle over Kb? If so, then $S_{TOP}(M)$ has order 8 in each case. Are these manifolds determined up to homeomorphism by their homotopy type?

6.5 Bundles over aspherical surfaces

The fundamental groups of total spaces of bundles over hyperbolic surfaces all contain nonabelian free subgroups. Nevertheless, such bundle spaces are determined up to s-cobordism by their homotopy type, except when the fibre is RP^2, in which case we can only show that the structure sets are finite.

Theorem 6.15 *Let M be a closed 4-manifold which is homotopy equivalent to the total space E of an F-bundle over B where B and F are aspherical closed surfaces. Then M is s-cobordant to E and $\widetilde{M}$ is homeomorphic to R^4.*

Proof If $\chi(B) = 0$ then $\pi \times Z$ is an extension of a poly-Z group (of Hirsch length 3) by $\pi_1(F)$. Otherwise, $\pi_1(B) \cong F *_Z F'$, where the amalgamated subgroup Z is square-root closed in each of the free groups F and F'. (See the final paragraph on page 120.) In all cases $\pi \times Z$ is a square root closed generalised free product with amalgamation of groups in *Cl*. Comparison of the Mayer-Vietoris sequences for $\mathbb{L}_0$-homology and L-theory (as in Proposition 2.6 of [St84]) shows that $S_{TOP}(E\times S^1)$ has just one element. (Note that even when $\chi(B) = 0$ the groups arising in intermediate stages of the argument all have trivial Whitehead groups.) Hence $M \times S^1 \cong E \times S^1$, and so M is s-cobordant to E by Lemma 6.10 and Theorem 6.2.

The final assertion follows from Corollary 7.3B of [FQ] since M is aspherical and π is 1-connected at ∞ [Ho77]. $\square$

Davis has constructed aspherical 4-manifolds whose universal covering space is not 1-connected at ∞ [Da83].

Theorem 6.16 *Let M be a closed 4-manifold which is homotopy equivalent to the total space E of an S^2-bundle over an aspherical closed surface B. Then M is s-cobordant to E, and $\widetilde{M}$ is homeomorphic to $S^2 \times R^2$.*

Proof Let $\pi = \pi_1(E) \cong \pi_1(B)$. Then $Wh(\pi) = 0$, and $H_*(\pi; \mathbb{L}_0^w) \cong L_*(\pi, w)$, as in Lemma 6.9. Hence $L_4(\pi, w) \cong Z \oplus (Z/2Z)$ if $w = 0$ and $(Z/2Z)^2$ otherwise. The surgery obstruction map $\sigma_4(E)$ is onto, by Lemma 6.9. Hence there are two normal cobordism classes of maps $h : X \to E$ with $\sigma_4(h) = 0$. The kernel of the natural homomorphism from $H_2(E; \mathbb{F}_2) \cong (Z/2Z)^2$ to $H_2(\pi; \mathbb{F}_2) \cong Z/2Z$ is generated by $j_*[S^2]$, where $j : S^2 \to E$ is the inclusion of a fibre. As $j_*[S^2] \neq -0$, while $w_2(E)(j_*[S^2]) = j^*w_2(E) = 0$ the normal invariant of f_j is nontrivial, by Lemma 6.5. Hence each of these two normal cobordism classes contains a self homotopy equivalence of E.

Let $f : M \to E$ be a homotopy equivalence (necessarily simple). Then there is a normal cobordism $F : V \to E \times [0, 1]$ from f to some self homotopy equivalence of E. As I_π^+ is an isomorphism, by Lemma 6.9, there is an s-cobordism W from M to E, as in Corollary 6.7.2.

The universal covering space $\widetilde{W}$ is a proper s-cobordism from $\widetilde{M}$ to $\widetilde{E} \cong S^2 \times R^2$. Since the end of $\widetilde{E}$ is tame and has fundamental group Z we may apply Corollary 7.3B of [FQ] to conclude that $\widetilde{W}$ is homeomorphic to a product. Hence $\widetilde{M}$ is homeomorphic to $S^2 \times R^2$. $\square$

Let ρ be a PD_2-group. As $\pi = \rho \times (Z/2Z)$ is square-root closed accessible from $Z/2Z$, the Mayer-Vietoris sequences of [Ca73] imply that $L_4(\pi, w) \cong L_4(Z/2Z, -) \oplus L_2(Z/2Z, -)$ and that $L_5(\pi, w) \cong L_4(Z/2Z, -)^\beta$, where $w = pr_2 : \pi \to Z/2Z$ and $\beta = \beta_1(\rho; \mathbb{F}_2)$. Since these L-groups are finite the structure sets of total spaces of RP^2-bundles over aspherical surfaces are also finite. (Moreover the arguments of Theorems 6.13 and 6.14 can be extended to show that σ_4 is an epimorphism and that most of $L_5(\pi, w)$ acts trivially on id_E, where E is such a bundle space.)

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