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Four-Person Envy-Free Chore Division

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In this article we explore the problem of *chore division*, which is closely related to a classical question, due to Steinhaus [10], of how to cut a cake fairly. We focus on *constructive* solutions, that is, those obtained via a well-defined procedure or algorithm. Among the many notions of fairness is *envy-freeness*: an envy-free cake division is a set of cuts and an allocation of the pieces that gives each person what she feels is the *largest* piece. It is non-trivial to find such a division, since the cake may not be homogeneous and player valuations on subsets of cake will differ, in general. Much progress has been made on finding constructive algorithms for achieving envy-free cake divisions; most recently, Brams and Taylor [3] produced the first general n-person procedure. The recent books by Brams and Taylor [4] and Robertson and Webb [8] give surveys on the cake-cutting literature.

In contrast to cakes, which are desirable, the dual problem of chore division is concerned with dividing an object deemed *undesirable*. Here, each player would like to receive what he considers to be the *smallest* piece of, say, a set of chores. This problem appears to have been first introduced by Martin Gardner [6].

Much less work has been done to develop algorithms for chore division than for cake-cutting. Of course, for 2 people, the familiar *I-cut-you-choose* cake-cutting procedure also works for dividing chores: one cuts the chores and the other chooses what she feels is the smallest piece. Oskui [8, p. 73] gave the first envy-free solutions for chore division among 3 people. Su [12] developed an envy-free chore-division algorithm for an arbitrary number of players; however, it does not yield an exact solution, but only an ϵ-approximate one. There appear to be no exact envy-free chore-division algorithms for more than three players in the literature; in unpublished manuscripts, Brams and Taylor [2] and Peterson and Su [7] offer n-person algorithms but these are not bounded in the number of steps they require.

In this article, we develop a simple and bounded procedure for envy-free chore division among 4 players. The reader will find that many of the ideas involved—moving knives, trimming and lumping, and a notion of "irrevocable advantage"—provide a nice introduction to similar techniques that arise in the literature on *fair division* problems. As a warm-up to some of these ideas, we also present a 3-person solution that is simpler and more symmetrical than the procedure of Oskui.

We assume throughout this paper that chores are infinitely divisible. This is not unreasonable, as a finite set of chores can be partitioned by dividing up each chore (for instance, a lawn to be mowed could be divided just as if it were a cake), or dividing the time spent on them. We also assume that player valuations over subsets of the chores

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are additive, that is, no value is destroyed or created by cutting or lumping pieces together. (The proper context for modeling player valuations is measure theory, but we can avoid that for the purposes of this article.)

A 3-person chore-division procedure   We first describe a simpler 3-person chore-division procedure, which introduces some ideas that are important later.

Our 3-person chore-division procedure relies on Austin's procedure [1] for dividing a cake into two pieces so that each of two players believes it is a 50-50 split. For completeness, we review it here. Let one player hold two knives over the cake, with one at the left edge, such that the portion of cake between them is what she believes to be *exactly* half. If the second player agrees that it is exactly half, we are done. Otherwise, let the first player move the knives across the cake from left to right, keeping the portion between them exactly half (in her estimation), until the second player agrees it is exactly half. (There must be such a point because when the rightmost knife reaches the right edge, the leftmost knife must be where the rightmost knife began, hence the second player must by that point have changed preferences.) At this point cuts are made and the pieces of cake outside the knives are lumped together, yielding two pieces that both players agree are exactly equal.

Austin's procedure is an example of what is sometimes called a "moving-knife" procedure in the cake-cutting literature [5]. Our 3-person chore-division algorithm is also a moving-knife procedure. The key idea is to divide the chores into *six* pieces and assign each player two of the pieces that he feels are at least as small as each pair of pieces the other players receive.

A THREE-PERSON ENVY-FREE CHORE-DIVISION PROCEDURE

Step 1. Divide the chores into three *portions* using any 3-person envy-free cake-division procedure (that guarantees players a piece they think is largest), such as the Stromquist moving-knife procedure [11]. Now label each portion by the name of the player to whom the cake-division procedure would assign that portion (this player believes that portion is *largest*).

Step 2. Let player i divide portion i into 2 pieces (which she feels is exactly half) and assign those pieces to the *other* two players such that they each feel they have received no more than half of portion i. (This can be achieved via Austin's procedure: letting player i and one other player, say j, agree on a 50-50 split, let the remaining player choose the half she thinks is smallest, and give the other half to j.)

Step 3. Repeat Step 2 for each player, then end the procedure.

We now verify that each player has been assigned *two* out of six total pieces such that each feels her share is *smallest*.

Call the players i, j, and k. Player i will not envy player j because one piece of each of their pairs came from the portion labelled k, and i feels her half of that portion was no larger than j's. As for her other piece, player i feels it was no more than half of the portion it came from, and therefore cannot be as large as player j's other piece, which i felt was exactly half of the *largest* portion. The same argument holds for any permutation of i, j, and k. See FIGURE 1.

This procedure requires at most 8 cuts (Step 1 uses 2 cuts, and Austin's procedure uses at most 2 cuts each time it is applied; in FIGURE 1, some pieces may have been reassembled for simplicity). It is also less complicated than the discrete procedure of Oskui [8]. There are some 3-person moving-knife schemes that require fewer cuts [8, 9], but our approach is distinguished by being symmetric with respect to the players and being based on a cake-cutting procedure. The former property simplifies

i	j	k
to j	*to i*	*to i*
to k	*to k*	*to j*

Figure 1   An envy-free assignment of six pieces (of chores) to three people

the verification of envy-freeness, while the latter property may help in generalizing the scheme to more players via known cake-cutting algorithms.

A 4-Person Chore-division Procedure   We now describe our 4-person chore-division procedure, which is also a moving-knife procedure and requires at most 16 cuts. It draws ideas from both the Brams-Taylor-Zwicker 4-person envy-free moving-knife scheme for cakes [5] and the Oskui 3-person envy-free discrete chore-division scheme [8]. We also show how the notion of *irrevocable advantage*, important in cake-cutting [4], can be applied in chore division.

Suppose the players are named Alice, Betty, Carl, and Debbie. For convenience, we assume the chores are a rectangular block that may be divided by vertical cuts. Let Alice and Betty divide the chores into four pieces they both agree are all equal, by performing three applications of Austin's procedure (using at most 6 cuts).

Call the pieces X_1, X_2, X_3, and X_4. Note that if Debbie and Carl disagree on which piece is the smallest, we can immediately allocate the pieces. Thus we may assume they agree that one piece is *strictly* smaller than the others, say X_4. Then each person thinks the following:

$$\text{Alice:} \quad X_1 = X_2 = X_3 = X_4$$
$$\text{Betty:} \quad X_1 = X_2 = X_3 = X_4$$
$$\text{Carl:} \quad X_4 < X_1, X_2, X_3$$
$$\text{Debbie:} \quad X_4 < X_1, X_2, X_3.$$

Now, for each of X_1, X_2, and X_3, let Debbie and Carl mark how they would trim them to make them the same size as X_4. As each piece is rectangular, assume the trimmings are marked from the top edge, so that a person receives the piece below her mark. See FIGURE 2. Hence, we can speak of one mark as being *higher than* another. The following procedure will yield an envy-free chore division (we've already described Step 1):

A FOUR-PERSON ENVY-FREE CHORE-DIVISION PROCEDURE

Step 1. Let Alice and Betty use three applications of Austin's procedure (6 cuts) to obtain 4 pieces (X_1, X_2, X_3, X_4) that Alice and Betty believe are exactly equal in size. If Carl and Debbie disagree on which piece is smallest, then allocate the pieces accordingly and end the procedure. Otherwise, call X_4 the piece that Carl and Debbie agree is smallest.

Step 2. Let Carl and Debbie mark X_1, X_2, X_3 where they would cut them to create ties for smallest with X_4. Without loss of generality, suppose Debbie has more marks higher than Carl's. Trim the pieces at the higher marks (3 cuts), and set aside the trimmings.
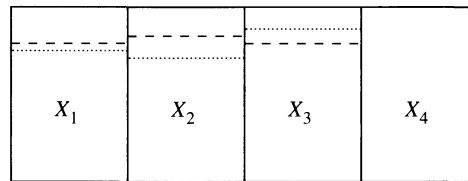
- - - - - - - **Debbie's marks**
.... **Carl's marks**

Figure 2 This figure shows a possible set of markings made in Step 1 of our 4-person procedure

Step 3. Let Betty add back to one piece some of the corresponding trimming (1 cut) to create a two-way tie for smallest piece.

Step 4. Let players choose from these pieces in the order Alice-Betty-Carl-Debbie, with Betty required to take the added-back piece if Alice didn't, and Carl required to choose a piece trimmed at his marking if it is available. This will allocate everything except the trimmings in an envy-free fashion.

Step 5. Divide the trimmings (6 cuts), exploiting an irrevocable advantage of Betty over whomever receives X_4. (The concept of an irrevocable advantage is defined and explained later.) This will allocate the trimmings in an envy-free fashion.

We now verify that this yields an envy-free solution. We assumed in Step 2 that Debbie has more higher marks than Carl (if not, just reverse the roles of Carl and Debbie in what follows). This produces two cases, the first in which Debbie has three higher marks and the second in which Debbie has two.

Case I: Debbie has three higher marks. Assume that Debbie's marks are all at or above Carl's marks. Following Step 2, let Debbie trim X_1, X_2, and X_3 at her marks to obtain a four-way tie for the smallest piece. Call the trimmed pieces X_1', X_2', and X_3', and the trimmings T_1, T_2, and T_3, which are set aside for later. At this point, each person thinks:

$$\text{Alice:} \quad X_1', X_2', X_3' < X_4$$

$$\text{Betty:} \quad X_1', X_2', X_3' < X_4$$

$$\text{Carl:} \quad X_4 \leq X_1', X_2', X_3'$$

$$\text{Debbie:} \quad X_1' = X_2' = X_3' = X_4.$$

Carl must believe X_4 is the smallest, or tied for the smallest, because his marks were all at or below Debbie's (meaning he believes more should be trimmed to make them equal to X_4).

Of the remaining pieces X_1', X_2', X_3', suppose without loss of generality that Betty believes $X_3' \leq X_2' \leq X_1'$. Following Step 3, let Betty return some of the trimmings T_3 to X_3' to create a two-way tie for the smallest piece. (We still call the modified piece X_3', and its trimmings T_3.) Thus player valuations change:

$$\text{Alice:} \quad X_1', X_2', X_3' < X_4$$

$$\text{Betty:} \quad X_2' = X_3' \leq X_1' < X_4$$

$$\text{Carl:} \quad X_4 \leq X_1', X_2', X_3'$$

$$\text{Debbie:} \quad X_1' = X_2' = X_4 < X_3'.$$

Following Step 4, Alice chooses first (hence is envy-free), then Betty, who is required to take X_3' if it was not chosen by Alice. Since Betty has at least one of X_2', X_3' to choose from, she is envy-free. Then Carl chooses, and will clearly still have his smallest piece X_4 available. Debbie will have one of her three smallest pieces available because Betty took X_3' if Alice did not. Thus all are non-envious of the portions they have recieved so far.

Dividing the Trimmings. The trimmings still need to be divided and assigned. Without loss of generality, suppose that Betty chose X_3' in the procedure above. Then Betty thinks

$$\text{Betty:} \quad T_1, T_2 \le T_3.$$

Note that because Betty believed $X_3 = X_4$, she could receive all of T_3 and still not envy Carl. In fact, by the above inequality she could receive $\frac{1}{3}(T_1 + T_2 + T_3) \le T_3$ and still not envy him. We will say that Betty has an *irrevocable advantage* over Carl with respect to the trimmings.

So, lump all the trimmings together (say, $T = T_1 + T_2 + T_3$), and let Alice and Debbie use Austin's procedure to divide T into four pieces that they both agree are all equal. Then let the players choose in the order Carl, Betty, and then (in any order) Debbie and Alice.

With respect to the trimmings, Carl will envy no one because he chooses first. Betty, choosing the smallest of the remaining three pieces, will have a piece that she believes is at most $\frac{1}{3}T$ and therefore will not envy Carl. Alice and Debbie will not envy Betty or Carl because they think all four pieces are equal. Thus the trimmings can be divided in an envy-free fashion.

Case II: Debbie has two higher marks. Assume now that Debbie has two marks at or above Carl's marks. Without loss of generality suppose that Carl has a higher mark on X_3 than Debbie, as in FIGURE 2. Following Step 2, let cuts be made at all three highest marks. Then

$$\text{Alice:} \quad X_1', X_2', X_3' < X_4$$
$$\text{Betty:} \quad X_1', X_2', X_3' < X_4$$
$$\text{Carl:} \quad X_3' = X_4 \le X_1', X_2'$$
$$\text{Debbie:} \quad X_1' = X_2' = X_4 < X_3'.$$

Following Step 3, let Betty create a two-way tie for the smallest piece (as before) by returning to the smallest piece some of the corresponding trimmings. She may add either to X_1', X_2' or X_3'. The X_1' and X_2' cases are equivalent, so we have two subcases.

Suppose Betty adds to X_3' until it is as large as, say, X_2'. Then

$$\text{Alice:} \quad X_1', X_2', X_3' < X_4$$
$$\text{Betty:} \quad X_2' = X_3' \le X_1' < X_4$$
$$\text{Carl:} \quad X_4 \le X_1', X_2', X_3'$$
$$\text{Debbie:} \quad X_1' = X_2' = X_4 \le X_3'.$$

These inequalities are identical to those in Case I, and thus our procedure works in the same way. Moreover, the trimmings can be handled just as before, since Betty has an irrevocable advantage over Carl (who receives X_4).

Otherwise, suppose Betty adds to X_1' until it is as large as, say, X_2' (the X_3' case is similar). Then

$$\text{Alice:} \quad X_1', X_2', X_3' < X_4$$
$$\text{Betty:} \quad X_1' = X_2' \leq X_3' < X_4$$
$$\text{Carl:} \quad X_3' = X_4 \leq X_1', X_2'$$
$$\text{Debbie:} \quad X_2' = X_4 \leq X_1', X_3'.$$

Following Step 4, let Alice choose first. When Betty chooses, she will have one of her two smallest pieces available (and will take X_1' if available as the procedure requires). Next, the procedure requires Carl to take X_3' if available (since it was the piece trimmed at his marking) and otherwise he recieves X_4; either way he believes he has the smallest piece. As X_1' and X_3' are allocated by this point, we know that Debbie will receive either X_2' or X_4, and hence is also envy-free. For the trimmings, note that Betty has an irrevocable advantage over whomever receives the X_4 piece, so the trimmings can be divided using the method discussed earlier.

This concludes the verification of envy-freeness for all cases. Note that we could alternately have presented tables for each case that list envy-free assignments for Betty, Carl, and Debbie given what Alice chose first. However, remembering those tables would not be as easy as remembering the steps of our procedure.

A bounded n-person procedure?   Our procedure gives the first known bounded procedure for 4-person envy-free chore division, requiring at most 16 cuts. Actually, this can be reduced to 15 cuts with a modification much like Brams-Taylor-Zwicker's 5-cut modification [5] of the triple application of Austin's procedure.

Although the reader may be tempted to try to further reduce the number of cuts needed for 4-person envy-free chore division, progress in this direction is not as important as the more compelling problem of finding any bounded procedure for more than 4 players. While there do exist finite n-person envy-free chore-division procedures ([2], [7]), these are *not bounded* in the number of steps or cuts, that is, depending on player preferences, they could take arbitrarily long to resolve. For cake-cutting as well as chore division, the existence of bounded n-person envy-free division procedures remains a major unsolved problem that will probably require new techniques.

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