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FOUR PHOTON PARAMETRIC AMPLIFICATION (*)

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Résumé. — Nous présentons l'analyse de l'amplification paramétrique à quatre photons dans une jonction Josephson non polarisée. L'idée centrale de cette théorie est de représenter la jonction Josephson comme une inductance non linéaire. Une analyse linéaire en régime de petits signaux est appliquée au modèle à « deux fluides » de la jonction. Le gain, le produit gain \times bande passante, la fréquence limite et la température effective de bruit sont calculés pour un amplificateur à cavité travaillant en réflexion. L'analyse est étendue à un réseau de jonctions (connectées en série) et au pompage subharmonique.

Abstract. — An analysis is presented describing four-photon parametric amplification in an unbiased Josephson junction. Central to the theory is the model of the Josephson effect as a non-linear inductance. Linear, small signal analysis is applied to the « two fluid » model of the Josephson junction. The gain, gain-bandwidth product, high frequency limit and effective noise temperature is calculated for a cavity reflection amplifier. The analysis is extended to multiple (series-connected) junctions and sub-harmonic pumping.

1. Introduction. — The ideal Josephson-effect, governed by eq. (1.1) and (1.2) represents a lossless, non-linear inductance [1], [2]

$$I = I_J \sin \varphi \quad (1.1)$$

$$\frac{d\varphi}{dt} = \frac{2e}{\hbar} V. \quad (1.2)$$

The exact form depends on one's definition of inductance, and we choose the following form

$$V(t) = \frac{d}{dt} [L(I) \cdot I] \quad (1.3)$$

which yields

$$L(I) = \frac{\hbar}{2eI_J} \left[\frac{\sin^{-1}(I/I_J)}{(I/I_J)} \right]. \quad (1.4)$$

We note that the lossless nature of this inductance can be demonstrated by calculating the change in energy as the current is varied through the junction

$$\Delta E = \int_0^t I(t') \cdot V(t') dt' = \frac{\hbar I_J}{2e} [1 - \cos \varphi(t)]. \quad (1.5)$$

Conserving energy, this variable energy storage element, a dual to the varactor diode, can be used as the active element in a parametric amplifier [3], [4] or frequency converter [5].

The fundamental inductance is the quantity

$$L_J = \frac{\hbar}{2eI_J}. \quad (1.6)$$

This Josephson inductance for example yields a reactance of 2Ω for $I_J = 10 \mu\text{A}$ and $\omega = 2 \times 10^{10}$ rad/s. The inductive reactance cannot be made arbitrarily large, however, because the Josephson coupling energy

$$\Delta E_{\max} = \frac{\hbar I_J}{2e} \quad (1.7)$$

must be greater than the thermal fluctuation energy kT , at least.

2. Admittance matrix approach to parametric amplification. — We will assume that three sinusoidal voltages closely spaced in frequency are impressed across the junction. All other frequencies will be presented with short circuit or open circuit loads, so that no power will flow

$$V = \sum_k V_k \quad V_k = v_k \cos(\omega_k t + \varphi_k) \quad k = 0, 1, 2. \quad (2.1)$$

Integrating eq. (1.1) using eq. (1.2) and (2.1)

$$I(t) = I_J \sum_{k,l,m=-\infty}^{\infty} J_k(\alpha_0) J_l(\alpha_1) J_m(\alpha_2) \times \sin[(k\omega_0 + l\omega_1 + m\omega_2)t + k\varphi_0 + l\varphi_1 + m\varphi_2] \quad (2.2)$$

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where we have used the equality

$$\exp(j\alpha \sin \varphi) = \sum_{k=-\infty}^{\infty} J_k(\alpha) \exp(jk\varphi) \quad (2.3)$$

and

$$\alpha_k = \frac{2 e v_k}{\hbar \omega_k}. \quad (2.4)$$

We now introduce two constraints. First, we choose

$$\omega_2 = 2 \omega_0 - \omega_1. \quad (2.5)$$

Second, we require

$$\alpha_1, \alpha_2 \ll 1 \quad (\alpha_0 \text{ arbitrary}). \quad (2.6)$$

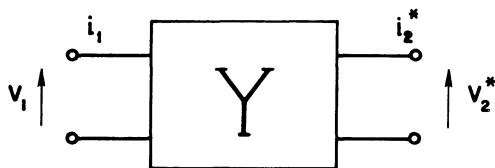
We now look for current oscillating at one of these three frequencies. The second constraint eliminates second and higher order products in ω_1 , and ω_2 . Following Russer [6], we write down the three currents in complex representation

$$i_0 = -2 j I_J J_1(\alpha_0) e^{j\varphi_0} \quad (2.7a)$$

$$i_1 = -j \frac{2 e I_J}{\hbar} \left\{ J_0(\alpha_0) \frac{v_1}{\omega_1} - J_2(\alpha_0) \frac{v_2^*}{\omega_2} e^{2j\varphi_0} \right\} \quad (2.7b)$$

$$i_2 = -j \frac{2 e I_J}{\hbar} \left\{ J_0(\alpha_0) \frac{v_2}{\omega_2} - J_2(\alpha_0) \frac{v_1^*}{\omega_1} e^{2j\varphi_0} \right\}. \quad (2.7c)$$

We have approximated $J_1(\alpha_k) \approx (\alpha_k/2)$, $k = 1, 2$ due to eq. (2.6) and the i_k and v_k now contain the original phase information from eq. (2.1). We see that the equations for the weak signal and idler have been linearized : i_1 couples to v_2^* and i_2^* couples to



$$\begin{pmatrix} i_1 \\ i_2^* \end{pmatrix} = Y \begin{pmatrix} v_1 \\ v_2^* \end{pmatrix}$$

FIG. 1. — Four terminal equivalent circuit admittance matrix Y for the signal frequency (ω_1) and idler frequency (ω_2).

v_1 , with the pump voltage as a parameter (see Fig. 1). We can formally express this relation in matrix form

$$\begin{bmatrix} i_1 \\ i_2^* \end{bmatrix} = \begin{bmatrix} Y_{11} & Y_{12}^* \\ Y_{21} & Y_{22}^* \end{bmatrix} \begin{bmatrix} v_1 \\ v_2^* \end{bmatrix} \quad (2.8)$$

where

$$Y_{11} = -j \frac{J_0(\alpha_0)}{L_J \omega_1}$$

$$Y_{12} = -j \frac{J_2(\alpha_0)}{L_J \omega_2} e^{-2j\varphi_0}$$

$$Y_{21} = -j \frac{J_2(\alpha_0)}{L_J \omega_1} e^{-2j\varphi_0}$$

$$Y_{22} = -j \frac{J_0(\alpha_0)}{L_J \omega_2}.$$

3. The doubly degenerate parametric amplifier. — Figure 2 illustrates a circuit that parallels the well-known varactor diode cavity reflection parametric amplifier. One cavity will resonate pump, signal and idler simultaneously. A source G_0 is coupled to the cavity and the cavity is coupled to the load G_0 through a four port circulator. The Josephson junction is terminated with a capacitor C , to resonate the linear inductance, and a shunt conductance G_J which represents internal losses. This termination is presented identically to both frequencies.

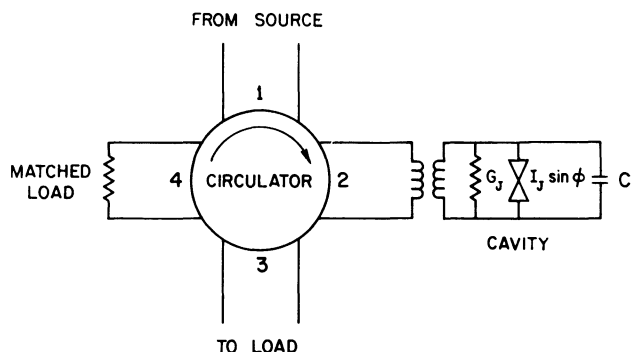


FIG. 2. — Circuit model for the cavity reflection amplifier.

The gain of the reflection amplifier is calculated using the voltage reflection coefficient

$$\text{Power Gain} = \mathfrak{J} = |\rho_v|^2 = \left| \frac{Y_T^{(1)} - G_0}{Y_T^{(1)} + G_0} \right|^2. \quad (3.1)$$

The admittance $Y_T^{(1)}$ is the admittance seen at port 2, at frequency ω_1 , when frequency ω_2 is terminated with the conductance G_J plus load conductance G_0

$$\begin{aligned} Y_L^{(1)} &= G_J + j\omega_1 C \\ Y_L^{(2)} &= G_J + G_0 + j\omega_2 C \\ Y_T^{(1)} &= G_J + Y_{11} + j\omega_1 C - \end{aligned} \quad (3.2)$$

$$- \frac{(Y_{12} Y_{21}^*) (Y_{22} + j\omega_2 C + G_J + G_0)}{|Y_{22} + j\omega_2 C + G_J + G_0|^2}.$$

Following Russer, we define the Q value, detuning parameter δ , and cavity frequency $\omega_c = \omega_0$,

$$\begin{aligned} Q &= \frac{J_0(\alpha_0)}{\omega_c L_J (G_0 + G_J)} \\ \delta_{1,2} &= \left(\frac{\omega_{1,2}}{\omega_c} - \frac{\omega_c}{\omega_{1,2}} \right) \quad \omega_c^2 = \frac{J_0(\alpha_0)}{L_J C}. \end{aligned} \quad (3.3)$$

In the limit of small detuning, $\delta = \delta_2 = -\delta_1$. We further define

$$\begin{aligned}\xi &= Q\delta \ll 1 \\ \eta &= \left[\frac{J_2(\alpha_0)}{\omega_c L_J (G_0 + G_J)} \right]^2 \\ \zeta &= \frac{G_0 - G_J}{G_0 + G_J}\end{aligned} \quad (3.4)$$

and finally have for the frequency-dependent gain

$$\mathfrak{J} = \frac{\left[\zeta + \frac{\eta}{1 + \xi^2} \right]^2 + \xi^2 \left[1 + \frac{\eta}{1 + \xi^2} \right]^2}{\left[1 - \frac{\eta}{1 + \xi^2} \right]^2 + \xi^2 \left[1 + \frac{\eta}{1 + \xi^2} \right]^2}. \quad (3.5)$$

For zero detuning ($\xi = 0$) and high gain ($\eta \lesssim 1$), the gain reduces to

$$\mathfrak{J}_{\max} = \frac{(\zeta + \eta)^2}{(1 - \eta)^2}. \quad (3.6)$$

Since both ζ and η in eq. (3.6) are functions of G_0 , it is not clear how to optimize this design. Isolating the dependence on the load conductance G_0 by defining two new variables representing normalized load ρ and a modified power parameter η_J , viz.

$$\rho = \frac{G_0}{G_J} \quad \eta_J = \left(\frac{J_2(\alpha_0)}{\omega_c L_J G_J} \right)^2 \quad (3.7)$$

eq. (3.6) is rewritten in terms of these new variables

$$\mathfrak{J}_{\max} = \frac{[(\rho^2 - 1) + \eta_J]^2}{[(\rho + 1)^2 - \eta_J]^2}. \quad (3.8)$$

The coupling of the resonant circuit to the source and load is represented by ρ . The modified power parameter η_J is a function of pump power and

$$\gamma = (\omega_c L_J G_J)^{-1}.$$

γ is a figure of merit for this amplifier, determining the high frequency cutoff of the gain of this parametric amplifier. The reason for this is that for many Josephson junctions [7], [8] there exists a simple relationship between the quasiparticle conductance G_J and the critical current:

$$\frac{\hbar}{L_J G_J} = \frac{2eI_J}{G_J} \simeq 4(1 - t) \hbar \omega_g \quad (3.9)$$

where $t = T/T_c$, the reduced temperature, and $\omega_g = 2\Delta(0)/\hbar$ the gap frequency, we have

$$\gamma \approx 4(1 - t) \frac{\omega_g}{\omega_c}. \quad (3.10)$$

Figure 3 illustrates the dependence of the gain on ρ and η_J . Gain greater than unity is possible for any value of ρ as long as η_J can be adequately adjusted. However,

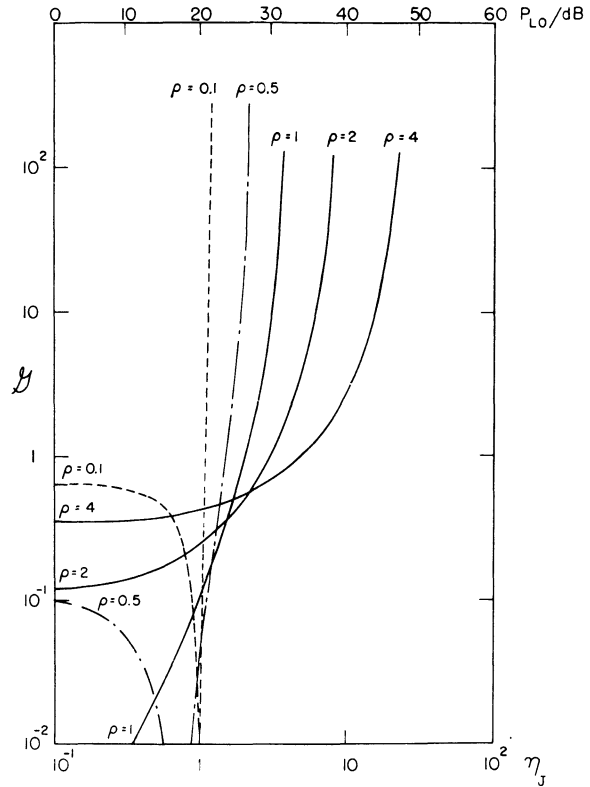


FIG. 3. — Plot of power gain \mathfrak{J} , of a reflection cavity amplifier, as a function of η_J the reduced power parameter. The upper abscissa is scaled in relative pump power assuming $\alpha_0 \ll 1$.

note that since $J_2(\alpha_0)$ has an absolute maximum, η_J cannot exceed a certain limit, though the pump power be arbitrarily increased. The sensitivity of the gain (at large gain) to fluctuations in pump power is seen to be more severe for $\rho < 1$ than for $\rho > 1$. In addition, for $\rho < 1$ the gain can actually go to zero for $\eta_J = |\rho^2 - 1|$. The conductance G_J , intrinsic to all Josephson junctions, appears as a dual to the parasitic series resistance found in varactor diodes. It is desirable therefore that the amplifier be overcoupled ($\rho > 1$) as much as is permitted by gain requirements so that the gain sensitivity is minimized.

4. Gain-bandwidth product and noise. — Assuming that high gain is obtained at zero detuning, the gain-bandwidth in the overcoupled case ($\rho > 1$) is calculated

$$\mathfrak{J}_{\max}^{1/2} \left(\frac{\Delta\omega}{\omega_c} \right) = \frac{\rho}{2 Q_J} \quad (4.1)$$

where

$$Q_J = \frac{J_0(\alpha_0)}{\omega_c L_J G_J} = \gamma J_0(\alpha_0). \quad (4.2)$$

It is seen that a high Q , larger than that obtained by resonating out the linear inductance, is undesirable since it lowers the gain-bandwidth product. Note also that G_J cancels out in the gain-bandwidth product.

Thermal noise will always be present and the effective noise temperature of the system will be at least

$$T_{\text{eff}} = \frac{T_0 G_0 + T_J G_J}{G_0 + G_J} = \frac{T_0 \rho + T_J}{1 + \rho} \quad (4.3)$$

where T_0 and T_J refer to the physical temperatures of G_0 and G_J respectively. To obtain low noise amplification of a cold source it is again preferable to have ρ as large as possible, consistent with adequate gain. When $\hbar\omega \gg kT_{\text{eff}}$ the limiting noise temperature becomes [9]

$$T_n \approx \frac{\hbar\omega}{k}. \quad (4.4)$$

5. Multiply connected junctions. — The basic lumped-circuit equivalent for the pumped Josephson element is a complex admittance

$$Y_{\text{in}}^{(1)} = G_{\text{in}}^{(1)} + jB_{\text{in}}^{(1)}. \quad (5.1)$$

For N identical junctions in series

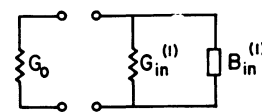
$$Y_{\text{in}}^{(1)} = N^{-1} G_{\text{in}}^{(1)} + jN^{-1} B_{\text{in}}^{(1)}. \quad (5.2)$$

Alternatively the preceeding gain analysis can be carried over in its entirety by replacing G_0 by NG_0 . This approach is shown in figure 4.

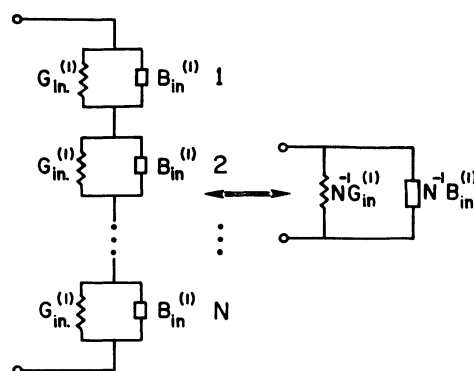
Several advantages accrue with multiple junctions, since the effective value of ρ is increased N -fold. First, the gain stability (at equal gain) increases as $N\rho$ increases. Second, the gain-bandwidth product increases. This assumes that $\eta_J(V_0)$ can be readjusted for equal gain as $N\rho$ increases. A third point is concerned with the maximum available power. From eq. (1.7) the maximum power from a single Josephson junction, in any mode of operation, is less than

$$P_{\text{max}} = \Delta E_{\text{max}} \cdot \frac{\omega}{2\pi}. \quad (5.3)$$

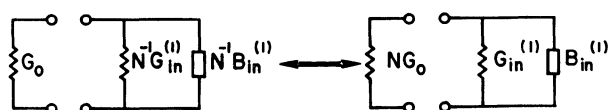
For multiple junctions the gain saturation power level increases at least as fast as N^2 . This third point will be more important in other circuits employing Josephson junctions, such as frequency converters and oscillators. It should be stressed that it is not at all necessary for the junctions to be identical in this parametric scheme.



(a) ONE JUNCTION



(b) N JUNCTIONS IN SERIES



(c) EQUIVALENCE

FIG. 4. — Representation of the generalization from a single junction to multiple, series-connected junctions.

6. Sub-harmonic pumping. — Instead of the frequency scheme chosen in eq. (2.5) we could have chosen

$$\omega_2 = 2n\omega_0 - \omega_1 \quad n = 1, 2, 3. \quad (6.1)$$

The preceeding analysis can be modified by the simple substitution

$$J_2(\alpha_0) \rightarrow J_{2n}(\alpha_0). \quad (6.2)$$

This sub-harmonic pumping scheme could be useful in the millimeter, sub-millimeter range where monochromatic sources have not been developed to the degree that they have been at other frequencies.

There are potential drawbacks, however. Since the gain term η_J is proportional to $J_{2n}^2(\alpha_0)$, more pump power is needed to achieve the same gain as pumping at the fundamental frequency ω_0 . Further, instability or frequency-modulation on the pump becomes magnified by the factor n .

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