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Four types of (super)conformal mechanics: D-module reps and invariant actions

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Abstract

(Super)conformal mechanics in one dimension is induced by parabolic or hyperbolic/trigonometric transformations, either homogeneous (for a scaling dimension λ) or inhomogeneous (at $\lambda = 0$, with ρ an inhomogeneity parameter). Four types of inequivalent (super)conformal actions are thus obtained. With the exclusion of the homogeneous parabolic case, dimensional constants are present.

Both the inhomogeneity and the insertion of λ generalize the construction of Papadopoulos [CQG 30 (2013) 075018; arXiv:1210.1719].

Inhomogeneous D-module reps are presented for the $d = 1$ superconformal algebras $osp(1|2)$, $sl(2|1)$, $B(1,1)$ and $A(1,1)$. For centerless superVirasoro algebras D-module reps are presented (in the homogeneous case for $\mathcal{N} = 1,2,3,4$; in the inhomogeneous case for $\mathcal{N} = 1,2,3$).

The four types of $d = 1$ superconformal actions are derived for $\mathcal{N} = 1,2,4$ systems. When $\mathcal{N} = 4$, the homogeneously-induced actions are $D(2,1;\alpha)$-invariant (α is critically linked to λ); the inhomogeneously-induced actions are $A(1,1)$-invariant.

In $d = 2$, for a single bosonic field, the homogeneous transformations induce a conformally invariant power-law action, while the inhomogeneous transformations induce the conformally invariant Liouville action.

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1 Introduction

In this paper we prove the existence of four types of conformally invariant actions for one-dimensional mechanical systems. In [1] Papadopoulos realized that under hyperbolic/trigonometric transformations of the fields, extra potential terms entered the conformal Lagrangians (these extra potential terms are not present when the standard, *parabolic*, realization of the conformal transformations is considered).

We generalize here the results of [1] in two distinct ways. At first we point out that a scaling dimension λ is associated with the parabolic and hyperbolic/trigonometric D-module reps of the conformal algebra $sl(2)$. In [1] λ was only taken at the given fixed value which produces constant kinetic terms.

The scaling parameter λ, however, cannot be so easily dismissed. In the supersymmetric generalizations (starting from the $\mathcal{N}=4$ supersymmetric extension) it acquires a critical property. Depending on the given supermultiplet, see e.g. formula (36), it specifies under which of the exceptional $D(2,1;\alpha)$ (parametrized by α) supersymmetry algebras, the system under consideration is superconformally invariant.

Our second extension concerns the generalization to the *inhomogeneous* parabolic and hyperbolic/trigonometric conformal transformations of the fields (in [1] only homogeneous transformations were considered).

We point out, see Appendix **A**, the existence of two inequivalent classes of one-dimensional conformal transformations (and their supersymmetric extensions), the homogeneous ones, depending on the critical scaling λ, and the inhomogeneous ones, which are parametrized by the constant ρ.

Hyperbolic versus trigonometric transformations are mutually recovered via an analytic continuation. The passage from parabolic to hyperbolic transformations, see e.g. formula (12), requires a singular change of variable. Under this change of variable the properties of their respective D-module reps (the scaling λ or, in the inhomogeneous case, the parameter ρ) are easily recovered. The singularity of the change of variable is, on the other hand, responsible for the appearance in the Lagrangians of the extra potential terms that we mentioned before.

On algebraic grounds the crucial difference between the hyperbolic and the parabolic $sl(2)$ transformations is the following. In the parabolic case the operator proportional to a time-derivative (the "Hamiltonian") is given by the (positive or negative) $sl(2)$ root, while in the hyperbolic case this Hamiltonian operator is associated with the $sl(2)$ Cartan generator.

Therefore we end up, for one-dimensional conformal systems and their supersymmetric extensions, with four types of D-module reps and their associated (super)conformally invariant actions, namely the *homogeneous parabolic*, *inhomogeneous parabolic*, *homogeneous hyperbolic/trigonometric* and *inhomogeneous hyperbolic/trigonometric* cases.

Only for the very special homogeneous parabolic case the conformally invariant actions are based on power-law and contain no dimensional parameter. In all remaining cases we have at disposal at least one dimensional constant to play with.

In the following we present all four types of (super)conformal actions in various exemplifying $d=1$ situations: the $sl(2)$-invariant actions of a single boson, the $osp(1|2)$-invariant ($sl(2|1)$-invariant) actions of an $\mathcal{N}=1$ ($\mathcal{N}=2$) supermultiplet. For a given set of $\mathcal{N} = 4$ supermultiplets, the actions are $D(2, 1; \alpha)$-invariant in the two homogeneous cases and $A(1, 1)$-invariant in the two inhomogeneous cases.

The construction is also applied, in the Lagrangian setting, to (super)conformal actions in $d = 2$ dimensions. The invariance in this case is under a single copy (for the chiral models) or the direct sum of two copies (the full conformal invariance) of the Witt algebra (the centerless Virasoro algebra) and its supersymmetric extensions.

Contrary to the $d = 1$ case, in $d = 2$, hyperbolic and parabolic active transformations of the field(s) produce the same conformally invariant output. The difference between homogeneous versus inhomogeneous transformations, on the other hand, is retained. As an example, the inhomogeneous transformations applied on a single boson induce, see (32), the conformally invariant Liouville action, while the homogeneous transformations induce a power-law conformally invariant action, see formula (31).

From the point of view of representation theory we extend here in two directions the results of [2] and [3] on D-module reps of finite $d = 1$ superconformal algebras. We enlarge the D-module reps of the $osp(1|2)$, $sl(2|1)$, $B(1, 1) = osp(3|2)$ and $A(1, 1) = sl(2|2)/\mathbb{Z}$ superalgebras to the class of *inhomogeneous* (ρ-dependent) D-module reps, see (42).

We further construct the D-module reps of the $\mathcal{N} = 1, 2, 3, 4$ centerless superVirasoro algebras, both in the homogeneous case (they are summarized in (43)) and, for $\mathcal{N} = 1, 2, 3$, inhomogeneous case (these results are summarized in (44)). The explicit construction of these D-module reps is presented in Appendix **B**.

Conformal mechanics based on the $sl(2)$ algebra has been investigated since the work of de Alfaro, Fubini and Furlan [4]. Models of superconformal mechanics have been presented in [5]–[13] (for an updated review on superconformal mechanics and a list of recent references see, e.g., [14]). There are several reasons to study one-dimensional superconformal mechanics (more on that in the Conclusions). Here it is sufficient to mention the applications to test particles moving in the proximity of the horizon of certain black holes, see [12]. In [2] and [3] it was advocated the point of view that superconformal mechanics, in the Lagrangian setting, could be derived from the D-module reps of superconformal algebras. In most of the papers in the literature and all works cited in the [14] review, the superconformal actions are based on power laws, being dependent only on dimensionless constants (apart the optional addition of oscillatorial terms, what is known as the DFF trick [4]). This is what to be expected for the homogeneous parabolic D-module reps. The possibilities offered by the three remaining types of D-module reps (presenting dimensional constants), on the other hand, greatly enlarge the class of available superconformal systems. One should confront, for instance, the power law $\mathcal{N} = 4$ superconformal systems with $A(1, 1)$ or $D(2, 1; \alpha)$ invariance investigated in [15, 16] and [17]–[22], respectively, with the $\mathcal{N} = 4$ actions presented in Section **9**.

The scheme of the paper is as follows: in Section **2** we introduce the *homogeneous* parabolic and hyperbolic D-module reps of the Witt algebra and its $sl(2)$ subalgebra. The *inhomogeneous* D-module reps of the $sl(2)$ and Witt algebras are discussed in Section **3**. In Section **4** we derive the different types of conformally invariant actions for a single boson. In Section **5** we extend the analysis to the bosonic, conformally invariant actions in $d = 2$ dimensions. In Section **6** we collect the main properties of the finite $d = 1$ superconformal algebras (together with their known D-module reps) and of the $\mathcal{N} = 1, 2, 3, 4$ centerless superVirasoro algebras. The new results on D-module reps for the finite $d = 1$ superconformal algebras and for the centerless superVirasoro algebras are summarized in Section **7**. Different types of $\mathcal{N} = 1, 2$ superconformal actions in $d = 1$ and supersymmetric chiral actions in $d = 2$ are given in Section **8**. In Section **9** we present the four types of $\mathcal{N} = 4$ superconformally invariant actions associated with the class of $(1, 4, 3)$ supermultiplets. In the Conclusions we point out the possible applications of our results and the future lines of investigations. The paper is complemented by two Appendices. A discussion about homogeneous versus inhomogeneous D-module reps is given in Appendix **A**. In Appendix **B** we present the explicit construction of the new supersymmetric D-module reps discussed in the main text.

2 The bosonic case: homogeneous D-module reps of the $sl(2)$ and $Witt$ algebras

The $sl(2)$ algebra is the conformal algebra in $d = 1$ dimensions. Its three generators D, H, K satisfy the commutation relations

$$\begin{aligned} [D, H] &= H, \\ [D, K] &= -K, \\ [H, K] &= 2D. \end{aligned} \tag{1}$$

The Cartan generator D is the dilatation operator.

A (parabolic) D-module representation of (1) is given by the differential operators (depending on a single variable t which, in application to physics, plays the role of time)

$$\begin{aligned} H &= \partial_t, \\ D &= -t\partial_t - \lambda, \\ K &= -t^2\partial_t - 2\lambda t. \end{aligned} \tag{2}$$

The constant λ is the scaling parameter. The above D-module rep is non-critical because the commutators (1) close for any value of λ.

The Virasoro algebra *Vir* is the central extension of the algebra of one-dimensional diffeomorphisms (known as "Witt algebra"). Its infinite generators L_n ($n \in \mathbb{Z}$) satisfy the commutation relations

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m+n,0}. \tag{3}$$

The Virasoro algebra contains $sl(2)$ as a subalgebra. It is obtained by restricting $n = \pm 1, 0$.

In the centerless case the Witt algebra admits a parabolic D-module rep. Indeed

$$L_n^{par.} = -t^{n+1}\partial_t - \lambda_n t^n \tag{4}$$

give the commutators (3) with $c = 0$ provided that the λ_m's satisfy the set of equations

$$m\lambda_m - n\lambda_n = (m-n)\lambda_{m+n}. \tag{5}$$

A solution is recovered for

$$\lambda_n \quad = \quad n\tilde{\lambda}+\tilde{\gamma}, \tag{6}$$

with $\tilde{\lambda},\tilde{\gamma}$ arbitrary constants.

For the $sl(2)$ generators we obtain, in particular,

$$\begin{aligned} L_{-1}^{par.} &= -\partial_t+(\tilde{\lambda}-\tilde{\gamma})\frac{1}{t},\\ L_0^{par.} &= -t\partial_t-\tilde{\gamma},\\ L_1^{par.} &= -t^2\partial_t-(\tilde{\lambda}+\tilde{\gamma})t. \end{aligned} \tag{7}$$

The special value $\tilde{\gamma}=\tilde{\lambda}$ allows us to identify, for $\lambda=\tilde{\lambda}$,

$$\begin{aligned} L_{-1}^{par.} &\equiv -H,\\ L_0^{par.} &\equiv D,\\ L_1^{par.} &\equiv K. \end{aligned} \tag{8}$$

At this special value of $\tilde{\gamma}$ one of the root generators of $sl(2)$ is proportional to a time-derivative and, in physics, can be identified with the Hamiltonian.

The constant $\tilde{\gamma}$ is arbitrary and can be changed via a similarity transformation. Indeed, for $f(t)=sgn(t)\hat{\gamma}ln|t|$, we have

$$L_n^{par.} \quad \mapsto \quad L_n^f=e^fL_n^{par.}e^{-f}=L_n^{par.}+\hat{\gamma}t^n.$$

Therefore, $\tilde{\gamma}\mapsto\tilde{\gamma}-\hat{\gamma}$.

For the special choice $\tilde{\gamma}=\tilde{\lambda}$, the parabolic D-module rep of the Witt algebra is

$$L_n^{par.} \quad = \quad -t^{n+1}\partial_t-(n+1)\tilde{\lambda}t^n. \tag{9}$$

By using hyperbolic/trigonometric functions, a hyperbolic/trigonometric D-module rep of the Witt algebra can be given. In the hyperbolic case the $c=0$ commutators (3) are satisfied for

$$L_n^{hyp.} \quad = \quad -\frac{1}{\mu}e^{n\mu\tau}(\partial_\tau+\overline{\lambda}_n), \tag{10}$$

if $\overline{\lambda}_n=n\overline{\lambda}+\overline{\gamma}$. The dimensional constant μ has been introduced here for dimensional reasons. Without loss of generality we can fix it at the $\mu=1$ value. In most of the cases, nevertheless, it is convenient to explicitly keep it in the equations in order to facilitate a dimensional analysis.

The $sl(2)$ generators read now

$$\begin{aligned} L_1^{hyp.} &= -\frac{1}{\mu}e^{\mu\tau}(\partial_\tau+\overline{\lambda}+\overline{\gamma}),\\ L_0^{hyp.} &= -\frac{1}{\mu}(\partial_\tau+\overline{\gamma}),\\ L_{-1}^{hyp.} &= -\frac{1}{\mu}e^{-\mu\tau}(\partial_\tau-\overline{\lambda}+\overline{\gamma}). \end{aligned} \tag{11}$$

In the hyperbolic case the generator proportional to the time-derivative (the "Hamiltonian") coincides, for $\overline{\gamma}=0$, with the $sl(2)$ Cartan generator $L_0^{hyp.}$.
Just like the parabolic case, the constant $\overline{\gamma}$ can be shifted by a similarity transformation.

At this point it is important to stress that the parabolic and the hyperbolic D-module reps of the Witt algebras are singled out, among the most general class of D-module reps, by the aforementioned very special property. Namely, that for a specific value of the constant parameter (either $\tilde{\gamma}$ or $\overline{\gamma}$), one of the $sl(2)$ generators is proportional to the Hamiltonian. The mathematical difference between the parabolic and the hyperbolic D-module reps can be stated as follows. In the parabolic case, the Hamiltonian is identified with the (positive or negative) $sl(2)$ root generator while, in the hyperbolic case, the Hamiltonian is identified with the $sl(2)$ Cartan generator. This difference proves to be crucial in the construction of conformally invariant actions.

From an algebraic point of view the hyperbolic D-module rep can be recovered from the parabolic D-module rep via a singular transformation. Let us call, for simplicity, $\overline{L}_n = L_n^{hyp.}$ when we fix the values $\mu=1$ and $\overline{\gamma}=0$. Therefore $\overline{L}_n = -e^{n\tau}(\partial_\tau + n\overline{\lambda})$. For $t>0$ the change of variable

$$t \mapsto \tau(t) = ln(t) \tag{12}$$

allows to recover the parabolic rep $\overline{L}_n = -t^{n+1}\partial_t - n\overline{\lambda}t^n$ at the specific values, for its constants, $\tilde{\lambda}=\overline{\lambda}$ and $\tilde{\gamma}=0$.

The

$$\tilde{\lambda} \;=\; \overline{\lambda} \tag{13}$$

relation is of particular importance. Extended to superconformal algebras with $\mathcal{N}\geq 4$ (the ones, as discussed in the Introduction, where the criticality of the scale parameter plays a role), it implies that the same critical scaling is recovered in both parabolic and hyperbolic cases (we will see this property at work in the following of the paper).

The singularity of the transformation connecting parabolic and hyperbolic D-module reps has the consequence, for the respective conformal invariant actions, that they are not (at least trivially) related. With respect to the parabolic case, in the hyperbolic case extra potential terms appear due to the presence of the dimensional constant μ (and due to the different identification of the Hamiltonian operator with the given $sl(2)$ generator).

The connection of the trigonometric case (that we do not need here to write down explicitly) with the hyperbolic case is simply given by an analytic continuation. One can perform a Wick rotation of the time coordinate τ by identifying a new periodic variable θ ($\tau\equiv i\theta$). Alternatively, the analytic continuation can also be obtained by performing a Wick rotation of the dimensional constant μ, mapping $\mu\mapsto i\mu$. It will be shown in the following that the extra potential terms entering the conformally invariant actions in the hyperbolic case are not bounded below, due to a "wrong" sign. Since they are proportional to μ^2, the correct sign can be recovered through the latter Wick rotation. The conformally invariant actions based on the trigonometric D-module transformations have therefore well-defined, bounded from below, potentials.

As recalled in [1], the group of diffeomorphisms $Diff(\mathbb{R})$ of the real line induced by the hyperbolic D-module rep is promoted, in the trigonometric case, to the group of diffeomorphisms $Diff(\mathbf{S}^1)$ of the $\mathbf{S}^1$ circle.

3 Inhomogeneous D-module reps of the $sl(2)$ and Witt algebras

Besides distinguishing Witt algebra's D-module reps into the two classes of *parabolic* versus *hyperbolic/trigonometric* representations, another discrimination can be introduced. It concerns the *homogeneous* versus the *inhomogeneous* representations.

Let $\varphi(t)$ be a time-dependent field. In the homogeneous case, the action of the Witt generators is written down as

$$L_n(\varphi) \quad = \quad a_n\dot{\varphi} + b_n\varphi. \tag{14}$$

In the inhomogeneous case the generators act as

$$L_n(\varphi) \quad = \quad a_n\dot{\varphi} + d_n. \tag{15}$$

In both cases the closure of the $c = 0$ (3) commutators is guaranteed, provided that the coefficients a_n, b_n and a_n, d_n are fixed to proper values (the coefficients b_n, d_n coincide; for clarity reasons in application to conformal actions, it will be however convenient to denote with different letters their respective normalization constants).

The parabolic subcase requires

$$a_n = -t^{n+1}, \qquad b_n = \tilde{\lambda}\dot{a}_n, \qquad d_n = \tilde{\rho}\dot{a}_n. \tag{16}$$

The hyperbolic subcase requires

$$a_n = -\frac{1}{\mu}e^{n\mu\tau}, \qquad b_n = \overline{\lambda}\dot{a}_n, \qquad d_n = \overline{\rho}\dot{a}_n. \tag{17}$$

Taking into account the discussion in Appendix **A**, the overall result is the existence of four types of D-module representations of the Witt algebra, labelled as follows:

I (*Hom. par.*) - the *homogeneous parabolic* rep,
II (*Inh. par.*) - the *inhomogeneous parabolic* rep,
III (*Hom. hyp.*) - the *homogeneous hyperbolic* rep,
IV (*Inh. hyp.*) - the *inhomogeneous hyperbolic* rep.

Let $[t] = [\tau] = -1$ be the scaling dimension of the time coordinate(s) (therefore $[\mu] = 1$). Let us furthemore set the scaling dimension of the field φ being given by $[\varphi] = s$.

For consistency, in the respective cases, the scaling dimensions of the $\tilde{\lambda}$, $\tilde{\rho}$, $\overline{\lambda}$, $\overline{\rho}$ parameters are

$$I : [\tilde{\lambda}] = 0, \qquad II : [\tilde{\rho}] = s, \qquad III : [\overline{\lambda}] = 0, \qquad IV : [\overline{\rho}] = s. \tag{18}$$

For $s \neq 0$ the *Hom. par.* rep contains no dimensional parameter, while one dimensional parameter ($\tilde{\rho}$) is found in the *Inh. par.* rep, one dimensional parameter (μ) in the *Hom. hyp.* rep and two dimensional parameters ($\mu, \overline{\rho}$) in the *Inh. hyp.* rep.

Similarly to the homogeneous case, the change of variable (12) allows to connect the inhomogeneous parabolic and hyperbolic D-module reps. Under this transformation the relation

$$\tilde{\rho} \;=\; \overline{\rho} \tag{19}$$

is verified.

Since no confusion will arise, in both parabolic and hyperbolic cases, we denote in the following, for simplicity, the scaling parameter of the homogeneous D-module rep as "λ" and the parameter of the inhomogeneous D-module rep as "ρ".

4 Conformal actions in $d = 1$

We are looking at first for conformally invariant actions depending on a single field $\varphi(t)$. The Lagrangian has the form

$$\mathcal{L} \;=\; g(\varphi)\dot{\varphi}^2 + h(\varphi), \tag{20}$$

where $g(\varphi)$ is a (one-dimensional) metric and $h(\varphi)$ is a potential term. The conformal invariance puts restrictions on both g and h.

We present here the general results for the four types of conformal transformations (homogeneous/inhomogeneous and parabolic/hyperbolic) introduced in Sections **2** and **3**.

In the homogeneous parabolic case, the invariance under the L_n transformations (14) requires solving the system of equations

$$\begin{aligned} \dot{a}_n[(1+2\lambda)g + \lambda g_\varphi \varphi] &= 0, \\ 2\lambda g \ddot{a}_n \varphi + h_\varphi a_n + N_\varphi^{(n)} &= 0, \\ \lambda h_\varphi \varphi \dot{a}_n + N_t^{(n)} &= 0, \end{aligned} \tag{21}$$

with a_n given in (16). The set of functions $N^{(n)}(\varphi, t)$ has to be determined; it reflects the arbitrariness of the invariance of the Lagrangian up to a total time-derivative.

The same system is derived in the homogeneous hyperbolic case with a_n given in (17). In the hyperbolic case we have the relation

$$\ddot{a}_n \;=\; n^2\mu^2 a_n, \tag{22}$$

which is not present in the parabolic case.

Under the inhomogeneous transformations (15) the system of equations

$$\begin{aligned} \dot{a}_n[g + \rho g_\varphi] &= 0, \\ 2\rho g \ddot{a}_n + h_\varphi a_n + N_\varphi^{(n)} &= 0, \\ \rho h_\varphi \dot{a}_n + N_t^{(n)} &= 0 \end{aligned} \tag{23}$$

is derived for both parabolic and hyperbolic cases; a_n is given, respectively, by (16) and (17).

Solving the above systems for all four cases is straightforward.

In the *Hom. par.* case, for instance, the first set of equations in (21) gives for the metric the solution $g = C_1\varphi^{-\frac{(1+2\lambda)}{\lambda}}$ (C_1 is a normalization constant). The third set of (21) equations allows to write $N^{(n)} = -\lambda h_\varphi \varphi a_n + M^{(n)}$, where $M^{(n)}(\varphi)$ are arbitrary functions of φ which do not explicitly depend on the time coordinate t. By plugging this result into the second set of equations, together with the (16) position for a_n, we end up with the following system: $-2\lambda(n+1)nt^{n-1}g\varphi - t^{n+1}[(1-\lambda)h_\varphi - \lambda h_{\varphi\varphi}\varphi] + M_\varphi^{(n)} = 0$.

The vanishing of the term inside square brackets gives the solution $h = C_2\varphi^{\frac{1}{\lambda}}$ (C_2 is the normalization constant). The first term in the left hand side is vanishing for $n = 0, -1$, while it can be reabsorbed by a suitable choice of $M^{(1)}(\varphi)$ for $n = 1$.

Therefore, the (21) system of equations cannot be nontrivially solved, simultaneously, for all $n \in \mathbb{Z}$, but at most for the $sl(2)$ subalgebra.

Deriving the solution for the three remaining cases proceeds along similar lines. In the two hyperbolic cases, the (22) relation for the a_n's induces an extra term in the potential, proportional to the metric normalization constant C_1, which is not present in the parabolic cases.

The overall results can be summarized as follows. We obtain four $d = 1$ conformal actions, invariant under different realizations of the $sl(2)$ active transformations of the single bosonic field $\varphi(t)$. Their respective Lagrangians are given by

I - Homogeneous parabolic case:

$$\mathcal{L} = C_1\varphi^{-\frac{(1+2\lambda)}{\lambda}}\dot{\varphi}^2 + C_2\varphi^{\frac{1}{\lambda}}. \tag{24}$$

II - Inhomogeneous parabolic case:

$$\mathcal{L} = C_1 e^{-\frac{1}{\rho}\varphi}\dot{\varphi}^2 + C_2 e^{\frac{1}{\rho}\varphi}. \tag{25}$$

III - Homogeneous hyperbolic case:

$$\mathcal{L} = C_1[\varphi^{-\frac{(1+2\lambda)}{\lambda}}\dot{\varphi}^2 + \mu^2\lambda^2\varphi^{-\frac{1}{\lambda}}] + C_2\varphi^{\frac{1}{\lambda}}. \tag{26}$$

IV - Inhomogeneous hyperbolic case:

$$\mathcal{L} = C_1 e^{-\frac{1}{\rho}\varphi}[\dot{\varphi}^2 + \mu^2\rho^2] + C_2 e^{\frac{1}{\rho}\varphi}. \tag{27}$$

In order to have a dimensionless action $\mathcal{S}$ ($[\mathcal{S}] = 0$), the scaling dimension of the Lagrangian is $[\mathcal{L}] = 1$, if we assign the time coordinate to have scaling dimension -1. Taking into account $[\mu] = 1$ and the relations (18), we end up with the following dimensional analysis:
- in both homogeneous cases (*I* and *III*), $[C_1] = [C_2] = 0$, provided that $[\varphi] = \lambda$;
- in both inhomogeneous cases (*II* and *IV*), $[C_1] = -1 - 2s$, $[C_2] = 1$, $[\varphi] = [\rho] = s$, with s arbitrary.

The homogeneous parabolic case is the only one not containing dimensional constants in the conformal action (in the *Hom. hyp.* case the constant μ is present).

The C_1, C_2 constants are arbitrary. On the other hand, in the two hyperbolic cases, extra terms for the potential appear with respect to the parabolic cases. Their normalization constant ($C_1\mu^2$) is linked with the metric normalization constant. Since μ^2 is positive, these potential terms have a "wrong" sign and are not bounded below.

A consistent action with a correct, bounded below, potential is obtained by allowing μ to be a complex variable and performing the $\mu \mapsto i\mu$ Wick rotation. As recalled in Section **2**, this is tantamount to pass from the hyperbolic to the trigonometric version of the D-module representation. In the latter case, for consistency, the field φ as well needs to be complexified. A simple inspection shows that conformal, $sl(2)$-invariant actions based on the trigonometric D-module reps are given by

i - Homogeneous trigonometric case:

$$\mathcal{L} = C_1(|\varphi|^{-\frac{(1+2\lambda)}{\lambda}}\dot{\varphi}^*\dot{\varphi} - \mu^2\lambda^2|\varphi|^{-\frac{1}{\lambda}}) + C_2|\varphi|^{\frac{1}{\lambda}}; \tag{28}$$

ii - Inhomogeneous trigonometric case:

$$\mathcal{L} = C_1 e^{-\frac{\varphi+\varphi^*}{2\rho}}(\dot{\varphi}^*\dot{\varphi} - \mu^2\rho^2) + C_2 e^{\frac{\varphi+\varphi^*}{2\rho}}. \tag{29}$$

In both cases the correct, bounded below potentials are obtained by choosing $C_1 > 0$ and $C_2 \leq 0$.

Our results should be compared with the ones derived by Papadopoulos in [1]. In that paper only homogeneous transformations were considered. Furthermore, only constant metrics were discussed. This amounts to set $\lambda = -\frac{1}{2}$ in the homogeneous parabolic Lagrangian (24) and in the homogeneous hyperbolic Lagrangian (26) (the results of [1] are recovered, as it should be, in these special cases). These restrictions, however, can no longer be justified for the $\mathcal{N}$-extended superconformal actions with $\mathcal{N} \geq 4$. As already pointed out in Section **2**, in the $\mathcal{N} \geq 4$ cases, the parameter λ becomes critical. It specifies under which of the inequivalent superconformal algebras the mechanical system is invariant.

5 Conformal actions in $d = 2$

It is instructive to extend the previous analysis to $d = 2$ conformally invariant actions. In the Lagrangian framework and classical case, the infinite-dimensional conformal algebra is $\mathfrak{witt} \oplus \mathfrak{witt}$, the direct sum of two copies of the Witt algebra $\mathfrak{witt}$.

Let $x_{1,2}$ be the coordinates of the plane. The $L_n^{\pm}$ generators of a D-module rep of $\mathfrak{witt} \oplus \mathfrak{witt}$ can be written in terms of the chiral/antichiral coordinates $z_{\pm} = x_1 \pm x_2$. The $L_n^{\pm}$ generators can be recovered from the $d = 1$ L_n generators introduced in Section **2** after replacing either t or τ (in the respective cases) with $z_{\pm}$. The chiral/antichiral decomposition implies the vanishing of the commutators $[L_n^+, L_m^-] = 0$ for any $n, m \in \mathbb{Z}$.

The two-dimensional conformal actions have a Lagrangian of the form

$$\mathcal{L} = g\varphi_+\varphi_- + h, \tag{30}$$

where the $\pm$ suffix denotes the partial derivative with respect to $z_{\pm}$.
Looking for conformal invariance under the assumption that homogeneous/homogeneous or inhomogeneous/inhomogeneous active D-module transformations of $\varphi(z_{\pm})$ apply on both chiral/antichiral sectors, we are led to the following results. Contrary to the $d =$

1 case, the parabolic and hyperbolic D-module reps produce the same output for the Lagrangians, while the actions are invariant under the whole infinite set of $L_n^{\pm}$ generators.

The two surviving inequivalent cases correspond to the Homogeneous or the Inhomogeneous transformations, respectively.

In the Homogeneous case g, h are restricted so that the conformally invariant Lagrangian is

$$\mathcal{L} = \frac{C_1}{\varphi^2}\varphi_+\varphi_- + C_2\varphi^{\frac{1}{\lambda}}, \tag{31}$$

with C_1, C_2 arbitrary constants.

The resulting action is invariant under the $\delta_n^{\pm}(\varphi) = -z_{\pm}^{n+1}\varphi_{\pm} - \lambda(n+1)z_{\pm}^n\varphi$ transformations.

In the Inhomogeneous case we recover the Liouville action. The metric needs to be a constant, while the potential is the exponential Liouville potential. We have

$$\mathcal{L} = C_1\varphi_+\varphi_- + C_2 e^{\frac{\varphi}{\rho}}. \tag{32}$$

The corresponding action is invariant under the $\delta_n^{\pm}(\varphi) = -z_{\pm}^{n+1}\varphi_{\pm} - \rho(n+1)z_{\pm}^n$ transformations.

By requiring the action being dimensionless, and assuming $[z_{\pm}] = -1$, we obtain the Lagrangian scaling dimension $[\mathcal{L}] = 2$.

In the Homogeneous case the scaling dimensions are fixed to be

$$[\varphi] = 2\lambda, \quad [C_1] = [C_2] = [\lambda] = 0 \tag{33}$$

(therefore, no dimensional parameter is present in the theory).
In the Inhomogenous (Liouville) case, for an arbitrary value s, the scaling dimensions are

$$[\varphi] = [\rho] = s, \quad [C_1] = -2s, \quad [C_2] = 2. \tag{34}$$

One should note that the classical Liouville action is invariant under two separate copies of the *centerless* Virasoro algebra. Even in this case, on the other hand, the associated Noether charges, endowed with a Poisson brackets structure, necessarily close the centrally extended version of the algebra, the full $Vir \oplus Vir$ algebra. It is a consequence of a non-equivariant moment map applied to the Liouville theory (see [23] for details).

6 On superconformal algebras

We discuss here two types of superconformal algebras, the supersymmetric extensions of the $d = 1$ conformal algebra $sl(2)$ and the supersymmetric extensions of the Virasoro algebra.

The finite one-dimensional superconformal algebras belong to the simple Lie superalgebras classified in [24, 25, 26] and satisfy special properties. A $d = 1$ superconformal algebra $\mathcal{G}$ admits a grading [27] $\mathcal{G} = \mathcal{G}_{-1} \oplus \mathcal{G}_{-\frac{1}{2}} \oplus \mathcal{G}_0 \oplus \mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_1$. Its even sector $\mathcal{G}_{even} = \mathcal{G}_0 \oplus \mathcal{G}_{-1} \oplus \mathcal{G}_1$ is isomorphic to $sl(2) \oplus R$, where the subalgebra R is known as

R-symmetry. The odd sector ($\mathcal{G}_{\frac{1}{2}} \oplus \mathcal{G}_{-\frac{1}{2}}$) is spanned by $2\mathcal{N}$ generators ($\mathcal{N}$ is the number of extended supersymmetries).
At fixed $\mathcal{N}$ the positive sector $\mathcal{G}_{>0}$ is isomorphic to the $d=1$ superPoincaré algebra (the algebra of the $\mathcal{N}$-extended supersymmetric quantum mechanics [28]).

If we denote, see (1), the $sl(2)$ generators as D, H, K, we have that $\mathcal{G}_1$ ($\mathcal{G}_{-1}$) is spanned by the positive (negative) root H (K), while $\mathcal{G}_0 = D\mathbb{C} \oplus R$.

We are especially interested in the $\mathcal{N} = 1, 2, 4, 8$ extensions. The corresponding list of $d=1$ superconformal algebras is given by $osp(1|2)$ for $\mathcal{N}=1$ and $sl(2|1)$ for $\mathcal{N}=2$. For $\mathcal{N}=4$ we have the exceptional superalgebras $D(2,1;\alpha)$, depending on the complex parameter $\alpha \neq 0, -1$ and $A(1,1) = sl(2|2)/\mathbb{Z}$ (it can be recovered for $\alpha = 0, -1$). Four distinct simple Lie superalgebras exist for $\mathcal{N}=8$: $A(3,1)$, $D(4,1)$, $D(2,2)$ and the exceptional superalgebra $F(4)$.

The exceptional superalgebras $D(2,1;\alpha)$, $D(2,1;\alpha')$ are isomorphic iff α' belongs to an S_3-group orbit generated by the moves $\alpha \mapsto \frac{1}{\alpha}$ and $\alpha \mapsto -(1+\alpha)$, i.e. if α' takes one of the six values

$$\alpha, \quad \tfrac{1}{\alpha}, \quad -(1+\alpha), \quad -\tfrac{1}{(1+\alpha)}, \quad -\tfrac{(1+\alpha)}{\alpha}, \quad -\tfrac{\alpha}{(1+\alpha)}. \tag{35}$$

The (homogeneous and parabolic) D-module reps of the above $d=1$ superconformal algebras have been constructed in [2] (the $\mathcal{N}=1,2,4$ cases and one $\mathcal{N}=8$ example) and [3] (the remaining $\mathcal{N}=8$ cases). The construction relies upon the classification, presented in [29] and [30], of the $d=1$ superPoincaré (the $\mathcal{G}_{>0}$ subalgebra) D-module reps.

Concerning the $d=1$ superPoincaré D-module reps for $\mathcal{N}=1,2,4,8$, the results can be summarized as follows. The differential operators act on $\mathcal{N}$ bosonic and $\mathcal{N}$ fermionic fields (the supermultiplet). For any $k=0,1,\ldots,\mathcal{N}$, we have k fields with scaling dimension λ (they are known as the "propagating bosons"), $\mathcal{N}$ fields (the fermions) with scaling dimension $\lambda+\frac{1}{2}$ and the remaining $\mathcal{N}-k$ fields (the so-called "auxiliary bosons") with scaling dimension $\lambda+1$. Both a supermultiplet and the associated $d=1$ superPoincaré D-module rep will be denoted with the symbol "$(k,\mathcal{N},\mathcal{N}-k)_\lambda$".

The extension to a $d=1$ superconformal algebra D-module rep requires introducing (in compatible way, so that to close the (anti)commutation relations) the extra differential operators associated to the $\mathcal{G}_{\leq 0}$ generators.

The [2] and [3] results can be summarized as follows:
i) for $\mathcal{N}=1,2$ and any value of the scaling dimension λ (no criticality), the $(k,\mathcal{N},\mathcal{N}-k)_\lambda$ supermultiplet induces a D-module rep for $osp(1|2)$ and $sl(2|1)$, respectively;
ii) for $\mathcal{N}=4$ the $(k,\mathcal{N},\mathcal{N}-k)_\lambda$ supermultiplet induces a D-module rep for the $D(2,1;\alpha)$ superalgebra with the identification

$$\alpha \;=\; (2-k)\lambda \tag{36}$$

(since α, up to the (35) relations, parametrizes inequivalent superalgebras, we already encounter here the criticality of the scaling dimension);
iii) for $\mathcal{N}=8$ the $(k,\mathcal{N},\mathcal{N}-k)_\lambda$ supermultiplet induces a D-module rep for a superconformal algebra only for $k\neq 4$ and at the critical scaling dimensions

$$\lambda \equiv \lambda_k = \tfrac{1}{k-4}; \tag{37}$$

the given superalgebras are $D(4,1)$for $k=0,8$, $F(4)$ for $k=1,7$, $A(3,1)$ for $k=2,6$ and $D(2,2)$ for $k=3,5$.

The D-module reps for the $\mathcal{N}=4$ $d=1$ superconformal algebra $A(1,1)$ (recovered from the $\alpha=0,-1$ values) are, in particular, obtained at the critical values

$$\lambda=0 \qquad \text{and} \qquad \lambda=\frac{1}{k-2} \quad (k\neq 2) \tag{38}$$

for, respectively, the supermultiplets

$$(k,4,4-k)_{\lambda=0} \quad \forall k=0,1,2,3,4 \quad \text{and} \quad (k,4,4-k)_{\lambda=\frac{1}{k-2}}, \quad k\neq 2. \tag{39}$$

The singular transformation, discussed in Section **2**, which relates the parabolic and hyperbolic D-module reps of $sl(2)$ (producing, in particular, the (13) equality between the respective scaling dimensions) is applicable in all supersymmetric cases. As a consequence, homogeneous hyperbolic D-module reps, with the same criticality properties of the corresponding parabolic cases, are immediately obtained for all the above listed superalgebras.

For what concerns the supersymmetric extensions of the Virasoro algebras, it is known [31] that non-trivial central charges can only exist up to $\mathcal{N}=4$. Since in the following we are dealing with D-module reps, here we only need to consider the *centerless* ($c=0$) $\mathcal{N}=1,2,4$ superVirasoro algebras which generalize the Witt algebra.

The centerless $\mathcal{N}=4$ superVirasoro algebra is spanned by the even generators L_n, J^i_n and by the odd generators Q^I_r, where $I=0,1,2,3$ and $i=1,2,3$. The centerless $\mathcal{N}=1,2$ superVirasoro algebras are its subalgebras, obtained by restricting $I=0,1$ and $i=1$ for $\mathcal{N}=2$ and $I=0$ for $\mathcal{N}=1$ (the latter case includes only the L_n, Q^0_r generators). Two variants of the superalgebras exist [32], the Ramond (R) and the Neveu-Schwarz (NS) versions. In both cases n is an integer ($n\in\mathbb{Z}$); in the Ramond version r is also an integer ($r\in\mathbb{Z}$), while in the Neveu-Schwarz version r is a half-integer number ($r\in\frac{1}{2}+\mathbb{Z}$).

The (anti)commutators of the centerless $\mathcal{N}=4$ superVirasoro algebra are explicitly given by

$$\begin{array}{rcl}
[L_m,L_n] &=& (m-n)L_{m+n},\\
{[L_n,Q^I_r]} &=& (\frac{n}{2}-r)Q^I_{n+r},\\
{[L_n,J^i_m]} &=& -mJ^i_{n+m},\\
\{Q^0_r,Q^0_s\} &=& 2L_{r+s},\\
\{Q^0_r,Q^i_s\} &=& 2(r-s)J^i_{r+s},\\
\{Q^i_r,Q^j_s\} &=& 2\delta^{ij}L_{r+s}+2\epsilon^{ijk}(r-s)J^k_{r+s},\\
{[Q^0_r,J^i_n]} &=& \frac{1}{2}Q^i_{n+r},\\
{[Q^i_r,J^j_n]} &=& -\frac{1}{2}\delta^{ij}Q^0_{n+r}-\frac{1}{2}\epsilon^{ijk}Q^k_{n+r},\\
{[J^i_n,J^j_m]} &=& -\epsilon^{ijk}J^k_{n+m}.
\end{array} \tag{40}$$

The finite $d=1$ $\mathcal{N}=4$ superconformal algebra $A(1,1)$ is recovered as a subalgebra.

In the Ramond version the $A(1,1)$ generators are $L_0, L_{\pm 2}, Q^I_{\pm 1}, J^i_0$;
in the Neveu-Schwarz version they are $L_{\pm 1}, L_0, Q^I_{\pm \frac{1}{2}}, J^i_0$.
The $osp(1|2)$ subalgebra is given by the generators
$L_0, L_{\pm 1}, Q^0_{\pm \frac{1}{2}}$ (NS) or $L_0, L_{\pm 2}, Q^0_{\pm 1}$ (R).
The $sl(2|1)$ subalgebra is given by the generators
$L_0, L_{\pm 1}, Q^0_{\pm \frac{1}{2}}, Q^1_{\pm \frac{1}{2}}, J^1_0$ (NS) or $L_0, L_{\pm 2}, Q^0_{\pm 1}, Q^1_{\pm 1}, J^1_0$ (R).

An extra centerless superVirasoro case which does not fit into the above scheme and contains an extra set of odd generators (W_r) is given by the $\mathcal{N}=3$ extension. The even generators of the $\mathcal{N}=3$ centerless superVirasoro are L_n and J^i_n, while the odd generators are Q^i_r and W_r ($i=1,2,3$). The (anti)commutators are explicitly given by

$$\begin{array}{rcl}
[L_m, L_n] &=& (m-n)L_{m+n}, \\
[L_n, Q^i_r] &=& (\frac{n}{2}-r)Q^i_{n+r}, \\
[L_n, J^i_m] &=& -mJ^i_{n+m}, \\
[L_n, W_r] &=& -(\frac{n}{2}+r)W_{n+r}, \\
\{Q^i_r, Q^j_s\} &=& 2\delta^{ij}L_{r+s} + 2\epsilon^{ijk}(r-s)J^k_{r+s}, \\
[Q^i_r, J^j_n] &=& -\frac{1}{2}\epsilon^{ijk}Q^k_{n+r} - \frac{n}{2}\delta^{ij}W_{n+r}, \\
\{Q^i_r, W_s\} &=& 2J^i_{r+s}, \\
[J^i_n, J^j_m] &=& -\frac{1}{2}\epsilon^{ijk}J^k_{n+m}, \\
[J^i_n, W_r] &=& 0, \\
\{W_r, W_s\} &=& 0.
\end{array} \tag{41}$$

The finite subalgebra consisting of the twelve generators $L_0, L_{\pm 1}, J^i_0, Q^i_{\pm \frac{1}{2}}$ (please note the absence of the W_r's generators) is the $d=1$ $\mathcal{N}=3$ superconformal algebra $B(1,1) = osp(3|2)$.

In [3] a D-module rep for $B(1,1)$ was constructed. It acts on the $(1,3,3,1)$ supermultiplet which contains one bosonic field of scaling dimension λ, three fermionic fields of scaling dimension $\lambda+\frac{1}{2}$, three bosonic fields of scaling dimension $\lambda+1$ and one fermionic field of scaling dimension $\lambda+\frac{3}{2}$. This D-module rep (existing for an arbitrary λ) is non-critical.

7 New results for superconformal D-module reps

In this Section we concentrate all new results concerning D-module reps of superconformal algebras. Our analysis heavily used algebraic computations with *Mathematica*.

Two classes of results are presented. At first we extend the construction presented in [2] and [3] of the *homogeneous* D-module reps to the case of the *inhomogeneous* D-module reps of the $d=1$ finite superconformal algebras (the ones we introduced in Section **6**). Next, we extend the [2] and [3] results to the case of D-module reps (both *homogeneous* and *inhomogeneous*) of the centerless $\mathcal{N}=1,2,3,4$ superVirasoro algebras.

The explicit presentation of these D-module reps is given in Appendix **B**.

As discussed in Appendix **A**, the new class of inhomogeneous D-module reps are obtained for $\lambda = 0$ and $\rho \neq 0$.

Since the presence of at least a propagating boson is required to construct the inhomogeneous term, the inhomogeneous supermultiplets $(k, \mathcal{N}, \mathcal{N}-k)_{\lambda=0,\rho\neq 0}$ can only exist for $k \geq 1$.

The list of inhomogeneous D-module reps for the finite $d = 1$ superconformal algebras of Section **6** is given by

$$\begin{array}{lcllllll}
\mathcal{N}=1 & : & osp(1|2) & \text{with} & (1,1)_{0,\rho}, & & & \\
\mathcal{N}=2 & : & sl(2|1) & \text{with} & (1,2,1)_{0,\rho}, & (2,2)_{0,\rho}, & & \\
\mathcal{N}=3 & : & B(1,1) & \text{with} & (1,3,3,1)_{0,\rho}, & & & \\
\mathcal{N}=4 & : & A(1,1) & \text{with} & (1,4,3)_{0,\rho}, & (2,4,2)_{0,\rho}, & (3,4,1)_{0,\rho}, & (4,4,0)_{0,\rho}, \\
\mathcal{N}=8 & : & \text{none} & & & & &
\end{array} \tag{42}$$

(the last result is a consequence of the fact that, for $\mathcal{N} = 8$, the $d = 1$ finite superconformal algebras have critical scalings $\lambda \neq 0$).

Concerning the centerless superVirasoro algebras, the homogeneous supermultiplets are encountered for

$$\begin{array}{lllll}
\mathcal{N}=1 \;\; \text{SVir}: & (k,1,1-k)_\lambda, & k=0,1 & \text{with} & \lambda \;\; \text{arbitrary}, \\
\mathcal{N}=2 \;\; \text{SVir}: & (k,2,2-k)_\lambda, & k=0,1,2 & \text{with} & \lambda \;\; \text{arbitrary}, \\
\mathcal{N}=3 \;\; \text{SVir}: & (1,3,3,1)_\lambda, & & \text{with} & \lambda \;\; \text{arbitrary}, \\
\mathcal{N}=4 \;\; \text{SVir}: & (k,4,4-k)_\lambda, & k=0,1,2,3,4 & \text{with} & \lambda=0 \;\; \text{or} \;\; \lambda=\frac{1}{k-2}(k\neq 2).
\end{array} \tag{43}$$

The inhomogeneous D-module reps of the centerless superVirasoro algebras are only encountered for $\mathcal{N} = 1, 2, 3$ but not for $\mathcal{N} = 4$:

$$\begin{array}{ll}
\mathcal{N}=1 \;\; \text{SVir}: & (1,1)_{0,\rho}, \\
\mathcal{N}=2 \;\; \text{SVir}: & (2,2,0)_{0,\rho} \;\; \text{and} \;\; (1,2,1)_{0,\rho}, \\
\mathcal{N}=3 \;\; \text{SVir}: & (1,3,3,1)_{0,\rho}, \\
\mathcal{N}=4 \;\; \text{SVir}: & \text{none}.
\end{array} \tag{44}$$

It is instructive to show the reason for the absence of the inhomogeneous D-module reps for the centerless $\mathcal{N} = 4$ superVirasoro. It is due to the fact that, in particular, the closure of the algebra requires the commutators $[J^3_n, Q^3_r]$ to be proportional to the Q^4_{n+r} generators. Let's take, as an example, the $(4, 4, 0)_{\lambda,\rho}$ supermultiplet. We are led to a system of equations to be solved:

$$\begin{aligned}
2\lambda - \frac{1}{2} &= A, \\
(\frac{1}{2} - 2\lambda)\partial_t + \lambda(-n + r(1-4\lambda)) &= A(-\partial_t - 2(n+r)\lambda), \\
-(n + r(4\lambda - 1))\rho &= -2A(n+r)\rho,
\end{aligned} \tag{45}$$

where A is a proportionality constant.

To solve the system for all n, r, either one has to set $\lambda = \frac{1}{2}$ and ρ arbitrary (which is equivalent to the homogeneous representation $(4,4,0)_{\frac{1}{2}}$) or $\lambda = \rho = 0$.

By restricting the conditions to $n = 0$ and $r = \pm\overline{r}$ (the case of the $A(1,1)$ subalgebra), the system is solved for arbitrary values of λ and ρ.

The inspection of the consistency conditions induced by all (anti)commutators leads to the results that we have presented in this Section.

It is worth pointing out, as a last comment, that the inhomogeneous D-module reps discussed here consist of a different and inequivalent class of transformations with respect to the inhomogenous D-module reps discussed in [2] and [3]. There, the $sl(2)$ generators act homogeneously and the representations are only obtained at the critical value $\lambda = -1$.

8 Some examples of $\mathcal{N} = 1, 2$ superconformal actions in $d = 1, 2$ dimensions

We illustrate here an application of supersymmetry with the construction of some $\mathcal{N} = 1, 2$ superconformal actions in $d = 1, 2$ dimensions.

In $d = 1$ we obtain the following $osp(1|2)$-invariant actions for the $\mathcal{N} = 1$ supermultiplet $(1,1)$ (a single boson φ and a single fermion ψ):

I - for the *homogeneous parabolic* case the Lagrangian is

$$\mathcal{L} = C\varphi^{-\frac{1+2\lambda}{\lambda}}(\dot{\varphi}^2 + \psi\dot{\psi}), \tag{46}$$

with dimensions $[\varphi] = \lambda$, $[\psi] = \lambda + \frac{1}{2}$, $[C] = [\lambda] = 0$;
II - for the *inhomogeneous parabolic* case the Lagrangian is

$$\mathcal{L} = Ce^{-\frac{\varphi}{\rho}}(\dot{\varphi}^2 + \psi\dot{\psi}), \tag{47}$$

with dimensions $[\varphi] = [\rho] = s$, $[\psi] = s + \frac{1}{2}$, $[C] = -1 - 2s$;
III - for the *homogeneous hyperbolic* case the Lagrangian is

$$\mathcal{L} = C[\varphi^{-\frac{1+2\lambda}{\lambda}}(\dot{\varphi}^2 + \mu\psi\dot{\psi}) + \mu^2\lambda^2\varphi^{-\frac{1}{\lambda}}], \tag{48}$$

with dimensions $[\varphi] = [\psi] = \lambda$, $[\mu] = 1$, $[C] = [\lambda] = 0$;
IV - for the *inhomogeneous hyperbolic* case the Lagrangian is

$$\mathcal{L} = Ce^{-\frac{\varphi}{\rho}}(\dot{\varphi}^2 + \mu\psi\dot{\psi} + \mu^2\rho^2), \tag{49}$$

with dimensions $[\varphi] = [\psi] = [\rho] = s$, $[\mu] = 1$, $[C] = -1 - 2s$.

We note a similarity and a difference with respect to the purely bosonic results. Just like the bosonic actions, the hyperbolic cases present a potential term, proportional to $C_1\mu^2$, which is absent in the parabolic cases. On the other hand, the potential terms proportional to C_2 and appearing in (24-27) are now excluded due to the supersymmetry constraint.

In $d = 2$, for a single boson φ and a single fermion ψ, we obtain $\mathcal{N} = 1$ chiral (antichiral) actions, invariant under a *single* (*chiral/antichiral*) *copy* of the centerless superVirasoro algebra. The Lagrangians are given by

$$\mathcal{L}^{\pm} = \frac{C}{\varphi^2}(\varphi_+\varphi_- + \psi\psi_{\mp}), \tag{50}$$

for the homogeneous case and

$$\mathcal{L}^{\pm} = C(\varphi_+\varphi_- + \psi\psi_{\mp}) \tag{51}$$

(the constant kinetic term) for the inhomogeneous case.

It should be pointed out that, in order to get the superLiouville extension, a second fermion needs to be added, see [33] and [34].

As an $\mathcal{N} = 2$ example in $d = 1$ we present the $sl(2|1)$-invariant actions for the supermultiplet $(1, 2, 1)$ (a propagating boson φ, two fermions ψ_1, ψ_2 and an auxiliary bosonic field g). The Lagrangians are:

I - for the *homogeneous parabolic* case

$$\mathcal{L} = A(\dot{\varphi}^2 + \psi_i\dot{\psi}_i + g^2) - \frac{1}{2}A_\varphi\epsilon^{ij}\psi_i\psi_j g, \qquad A = C\varphi^{-\frac{1+2\lambda}{\lambda}}; \tag{52}$$

II - for the *inhomogeneous parabolic* case

$$\mathcal{L} = A(\dot{\varphi}^2 + \psi_i\dot{\psi}_i + g^2) - \frac{1}{2}A_\varphi\epsilon^{ij}\psi_i\psi_j g, \qquad A = Ce^{-\frac{\varphi}{\rho}}; \tag{53}$$

III - for the *homogeneous hyperbolic* case

$$\mathcal{L} = A(\dot{\varphi}^2 + \mu\psi_i\dot{\psi}_i + \mu^2 g^2) - \frac{1}{2}\mu^2 A_\varphi\epsilon^{ij}\psi_i\psi_j g + \mu^2\lambda^2 A\varphi^2, \qquad A = C\varphi^{-\frac{1+2\lambda}{\lambda}}; \tag{54}$$

IV - for the *inhomogeneous hyperbolic* case

$$\mathcal{L} = A(\dot{\varphi}^2 + \mu\psi_i\dot{\psi}_i + \mu^2 g^2) - \frac{1}{2}\mu^2 A_\varphi\epsilon^{ij}\psi_i\psi_j g + \mu^2\rho^2 A, \qquad A = Ce^{-\frac{\varphi}{\rho}}. \tag{55}$$

9 On $\mathcal{N} = 4$ $d = 1$ superconformal actions with exceptional $D(2, 1; \alpha)$ invariance

The supermultiplet $(1, 4, 3)_\lambda$ consists of a single propagating boson φ, four fermions ψ_0, ψ_i and three auxiliary bosons g_i (here $i = 1, 2, 3$; in the formulas below we also use the index $I = 0, 1, 2, 3$). For this supermultiplet, see formula (36), we have $\alpha = \lambda$. We list here its superconformally invariant actions.
For homogeneous transformations the Lagrangians of the $D(2, 1; \alpha)$-invariant actions are, in the *homogeneous parabolic* case,

$$\mathcal{L} = A(\dot{\varphi}^2 + \psi_I\dot{\psi}_I + g_i^2) + A_\varphi(\psi_0\psi_i g_i + \frac{1}{2}\epsilon^{ijk}\psi_i\psi_j g_k) + \frac{1}{6}A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k,$$
$$\text{with} \quad A = C\varphi^{-\frac{1+2\alpha}{\alpha}} \tag{56}$$

and, in the *homogeneous hyperbolic* case,

$$\begin{aligned}\mathcal{L} &= A(\dot{\varphi}^2+\mu\psi_I\dot{\psi}_I+\mu^2g_i^2)+\mu^2A_\varphi(\psi_0\psi_ig_i+\frac{1}{2}\epsilon^{ijk}\psi_i\psi_jg_k)+\\ &\quad \frac{1}{6}\mu^2A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k+\mu^2\alpha^2A\varphi^2,\\ \text{with} &\quad A=C\varphi^{-\frac{1+2\alpha}{\alpha}}. \end{aligned} \tag{57}$$

For inhomogeneous transformations, the requirement that $\rho\neq 0$ with $\lambda=0$ implies that the superconformal actions are only invariant under the $A(1,1)$ superalgebra.
For inhomogeneous transformations the Lagrangians of the $A(1,1)$-invariant actions are, in the *inhomogeneous parabolic* case,

$$\begin{aligned}\mathcal{L} &= A(\dot{\varphi}^2+\psi_I\dot{\psi}_I+g_i^2)+A_\varphi(\psi_0\psi_ig_i+\frac{1}{2}\epsilon^{ijk}\psi_i\psi_jg_k)+\frac{1}{6}A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k,\\ \text{with} &\quad A=Ce^{-\frac{\varphi}{\rho}} \end{aligned} \tag{58}$$

and, in the *inhomogeneous hyperbolic* case,

$$\begin{aligned}\mathcal{L} &= A(\dot{\varphi}^2+\mu\psi_I\dot{\psi}_I+\mu^2g_i^2)+\mu^2A_\varphi(\psi_0\psi_ig_i+\frac{1}{2}\epsilon^{ijk}\psi_i\psi_jg_k)+\\ &\quad \frac{1}{6}\mu^2A_{\varphi\varphi}\epsilon^{ijk}\psi_0\psi_i\psi_j\psi_k+\mu^2\rho^2A,\\ \text{with} &\quad A=Ce^{-\frac{\varphi}{\rho}}. \end{aligned} \tag{59}$$

10 Conclusions

We summarize the results of the paper. For what concerns representations, besides the results on the bosonic $sl(2)$ and Witt algebras, we introduced the new class of inhomogeneous D-module reps for the finite simple Lie superalgebras $osp(1|2)$, $sl(2|1)$, $B(1,1)$ and $A(1,1)$. These new reps, at the scaling dimension $\lambda=0$, depend on the parameter ρ which measures the inhomogeneity.

D-module reps have also been constructed for the centerless superVirasoro algebras: homogeneous reps for the $\mathcal{N}=1,2,3,4$ extensions and inhomogeneous reps for the $\mathcal{N}=1,2,3$ extensions. They are based on the $(k,\mathcal{N},\mathcal{N}-k)$ supermultiplets (for $\mathcal{N}=1,2,4$) and on the $(1,3,3,1)$ supermultiplet (for $\mathcal{N}=3$).

We pointed out that, for both homogeneous and inhomogeneous reps, two variants of the D-module reps can be presented: parabolic and hyperbolic/trigonometric. They are mutually related by a singular transformation.

We ended up with four different types of active (super)conformal transformations (*hom. par.*, *inhom. par.*, *hom. hyp.* and *inhom. hyp.*) that can be used, in the Lagrangian setting, to construct inequivalent (super)conformal actions in $d=1$ and $d=2$ dimensions.

For systems with $\mathcal{N}=0,1,2,4$ we presented the four types of $d=1$ actions invariant under their respective finite (super)conformal algebra.

In $d=2$ the two classes of homogeneous and inhomogeneous conformal actions are exemplified by (31) and by the Liouville action (32), respectively.

The $d = 1$ (super)conformal actions contain no dimensional constants only in the homogeneous parabolic case. This is the class of theories discussed in the [14] review. New classes of superconformally invariant theories can therefore be constructed from the three remaining types of transformations.

It is rather straightforward to extend the results here presented to more complicated cases. For $\mathcal{N} = 4$, for instance, one can investigate multi-particle systems by applying our construction to a certain number of supermultiplets in interactions. The provision is that the supermultiplets should carry a representation of the same superconformal algebra (either $A(1,1)$ or $D(2,1;\alpha)$ for a fixed α).

One of the possible interesting applications of our work concerns the investigations on the $CFT(1)/AdS(2)$ correspondence (see [35] and [36]). Our results shed a new light on the left side (the conformal side) of the correspondence.

A very promising field of investigation concerns the extension to non-relativistic conformal Galilei or conformal Newton-Hooke systems (see [37, 38]). Recently, a lot of activity in constructing models for this kind of theories has been motivated by the CFT/AdS correspondence applied to non-relativistic systems like the ones appearing in condensed matter (see, e.g., [39, 40]). Unlike the $(1+0)$-dimensional theories considered here, these conformal systems live in $(1+d)$-dimension, d being the number of space coordinates. A recent paper [41] proved how the (homogeneous parabolic) $\mathcal{N} = 2$ superconformal D-module reps in $(1+0)$ can be enlarged to induce $\mathcal{N} = 2$ ℓ-conformal Galilei superalgebras in $(1+d)$ dimensions. It is tempting to extend the new class of one-dimensional superconformal D-module reps discussed here to the $(1+d)$-dimensional case.

Appendix A: On D-module reps and the interpolation of *Hom* and *Inhom* conformal actions

In principle one can "mix" the homogeneous and inhomogeneous D-module reps of the Witt algebra by allowing the couple of parameters (λ, ρ) being simultaneously non-vanishing. In the parabolic case, for instance, the general Witt algebra transformations applied on the field φ are

$$L_m^{par}(\varphi) \;=\; -t^{m+1}\dot{\varphi} - \lambda(m+\gamma)t^m\varphi - \rho(m+\beta)t^m. \qquad \text{(A.1)}$$

L_{-1}^{par} is proportional to the Hamiltonian if we set $\gamma = 1$ and $\beta = 1$.

For $\lambda \neq 0$ we can write

$$L_m^{par}(\varphi) \;=\; -t^{m+1}\dot{\varphi} - \lambda(m+1)t^m(\varphi + \frac{\rho}{\lambda}), \qquad \text{(A.2)}$$

so that the action of the homogeneous transformation with scaling dimension $\lambda \neq 0$ is recovered for the shifted field $\overline{\varphi} = \varphi + \frac{\rho}{\lambda}$. Therefore the (λ, ρ) transformations with $\lambda \neq 0$ are equivalent to the pure homogeneous transformations with scaling parameter λ and $\rho = 0$. The same is true in the hyperbolic case.

(A.1) fails to interpolate the two cases, leaving us with the two inequivalent classes of

i) $(\lambda, 0)$ homogeneous and

ii) $(0, \rho)$ inhomogeneous transformations.

The (A.1) transformations, on the other hand, are useful to interpolate the conformally invariant actions. An $sl(2)$-invariant action (for $m=\pm1,0$) based on (A.1) is given by the Lagrangian

$$\mathcal{L} \;=\; K_1(\lambda\varphi+\rho)^{-\frac{1+2\lambda}{\lambda}}\dot{\varphi}^2+K_2(\lambda\varphi+\rho)^{\frac{1}{\lambda}}, \tag{A.3}$$

with K_1, K_2 arbitrary constants.

The homogeneous Lagrangian (24) is recovered for $\rho=0$.

The inhomogeneous Lagrangian (25) is recovered in the $\lambda\rightarrow 0$ limit by suitably rescaling the constants K_1, K_2. This is accomplished by expressing the (A.3) Lagrangian as

$$\mathcal{L}=K_1\rho^{-\frac{1+2\lambda}{\lambda}}\left(1+\frac{\lambda\varphi}{\rho}\right)^{-\frac{1}{\lambda}-2}\dot{\varphi}^2+K_2\rho^{\frac{1}{\lambda}}\left(1+\frac{\lambda\varphi}{\rho}\right)^{\frac{1}{\lambda}}$$

and by taking $K_1=C_1\rho^{\frac{1+2\lambda}{\lambda}}$ and $K_2=C_2\rho^{-\frac{1}{\lambda}}$.

The possibility offered by the interpolation allows simplifying the constructions of the homogeneous and inhomogeneous conformally invariant actions, since both actions can be derived at one stroke.

The extension of the properties here discussed to the supersymmetric cases is immediate.

Appendix B: Explicit presentation of the new supersymmetric D-module reps

We present here, for completeness, the explicit constructions of the new D-module reps introduced in Section **7**. They are

i) the *inhomogeneous* D-module reps of the finite $d=1$ superconformal algebras $osp(1|2)$, $sl(2|1)$, $B(1,1)$, $A(1,1)$ and

ii) the (both *homogeneous* and *inhomogeneous*) D-module reps of the centerless $\mathcal{N}=1,2,3,4$ superVirasoro algebras.

The D-module reps with $\mathcal{N}=1,2,4$ act on the $(\mathcal{N}+1|\mathcal{N})$ supermultiplets m, $m^T=(\varphi_1,\ldots,\varphi_k,g_1,\ldots,g_{\mathcal{N}-k},1|\psi_1,\ldots,\psi_{\mathcal{N}})$, with component fields $\varphi_a, g_i, \psi_\alpha$ and constant entry 1 in the $(\mathcal{N}+1)$-th position. The $\mathcal{N}=3$ D-module rep acts on a $(5|4)$ supermultiplet with 1 in the 5-th position. The homogeneous D-module reps are recovered by deleting the row and the column associated with the constant entry 1 in the supermultiplet.

For $\mathcal{N}=1$, in matrix form and in the hyperbolic presentation, we can write for the centerless superVirasoro generators

$$\begin{aligned} Q_r^0 &= e^{rt}\left(\begin{array}{ccc} 0 & 0 & 1\\ 0 & 0 & 0\\ -\partial_t-2r\lambda & -2r\rho & 0\end{array}\right),\\ L_n &= e^{nt}\left(\begin{array}{ccc} -\partial_t-n\lambda & -n\rho & 0\\ 0 & 0 & 0\\ 0 & 0 & -\partial_t-\frac{1}{2}n(1+2\lambda)\end{array}\right).\end{aligned} \tag{B.1}$$

The inhomogeneous D-module rep of $osp(1|2)$ is recovered for $n=0,\pm1$, $r=\pm\frac{1}{2}$ and by setting $\lambda=0$.

To save space, in the remaining cases we limit to present here the odd generators (the even generators are recovered, see (40), from their anticommutators) and write them in terms of the E_{ij} matrices, whose entries are 1 in the i-th row, j-th column and vanishing otherwise. We have,

for $\mathcal{N}=2$:
the $(2,2,0)$ rep is constructed from

$$\begin{aligned} Q_r^0 &= e^{rt}[E_{14}+E_{25}-2(E_{43}+E_{53})r\rho-(E_{41}+E_{52})(\partial_t+2r\lambda)],\\ Q_r^1 &= e^{rt}[E_{15}-E_{24}+2(E_{43}-E_{53})r\rho+(E_{42}-E_{51})(\partial_t+2r\lambda)]; \end{aligned}\tag{B.2}$$

the $(1,2,1)$ rep is constructed from

$$\begin{aligned} Q_r^0 &= e^{rt}[E_{14}+E_{52}-2E_{43}r\rho-E_{25}r-(E_{25}+E_{41})(\partial_t+2r\lambda)],\\ Q_r^1 &= e^{rt}[E_{15}-E_{42}-2E_{53}r\rho+E_{24}r+(E_{24}-E_{51})(\partial_t+2r\lambda)]; \end{aligned}\tag{B.3}$$

the $(0,2,2)$ rep is constructed from

$$\begin{aligned} Q_r^0 &= e^{rt}[E_{41}+E_{52}-(E_{14}+E_{25})(\partial_t+r+2r\lambda)],\\ Q_r^1 &= e^{rt}[E_{51}-E_{42}+(E_{24}-E_{15})(\partial_t+r+2r\lambda)]; \end{aligned}\tag{B.4}$$

for $\mathcal{N}=4$:
the $(4,4,0)$ rep is constructed from

$$\begin{aligned} Q_r^0 &= e^{rt}[E_{16}+E_{27}+E_{38}+E_{49}-2(E_{65}+E_{75}+E_{85}+E_{95})r\rho\\ &\quad -(E_{61}+E_{72}+E_{83}+E_{94})(\partial_t+2r\lambda)],\\ Q_r^1 &= e^{rt}[E_{17}-E_{26}-E_{39}+E_{48}+2(E_{65}-E_{75}-E_{85}+E_{95})r\rho\\ &\quad +(E_{62}-E_{71}-E_{84}+E_{93})(\partial_t+2r\lambda)],\\ Q_r^2 &= e^{rt}[E_{18}+E_{29}-E_{36}-E_{47}+2(E_{65}+E_{75}-E_{85}-E_{95})r\rho\\ &\quad +(E_{63}+E_{74}-E_{81}-E_{92})(\partial_t+2r\lambda)],\\ Q_r^3 &= e^{rt}[E_{19}-E_{28}+E_{37}-E_{46}+2(E_{65}-E_{75}+E_{85}-E_{95})r\rho\\ &\quad +(E_{64}-E_{73}+E_{82}-E_{91})(\partial_t+2r\lambda)]; \end{aligned}\tag{B.5}$$

the $(3,4,1)$ rep is constructed from

$$\begin{aligned} Q_r^0 &= e^{rt}[E_{16}+E_{27}+E_{38}+E_{94}-2(E_{65}+E_{75}+E_{85})r\rho-E_{49}r\\ &\quad -(E_{49}+E_{61}+E_{72}+E_{83})(\partial_t+2r\lambda)],\\ Q_r^1 &= e^{rt}[E_{17}-E_{26}-E_{39}+E_{84}+2(E_{65}-E_{75}+E_{95})r\rho-E_{48}r\\ &\quad +(E_{62}-E_{48}-E_{71}+E_{93})(\partial_t+2r\lambda)],\\ Q_r^2 &= e^{rt}[E_{18}+E_{29}-E_{36}-E_{74}+2(E_{65}-E_{85}-E_{95})r\rho+E_{47}r\\ &\quad +(E_{47}+E_{63}-E_{81}-E_{92})(\partial_t+2r\lambda)],\\ Q_r^3 &= e^{rt}[E_{19}-E_{28}+E_{37}-E_{64}+2(E_{85}-E_{75}-E_{95})r\rho+E_{46}r\\ &\quad +(E_{46}-E_{73}+E_{82}-E_{91})(\partial_t+2r\lambda)]; \end{aligned}\tag{B.6}$$

the $(2,4,2)$ rep is constructed from

$$\begin{aligned}
Q_r^0 &= e^{rt}[E_{16}+E_{27}+E_{83}+E_{94}-2(E_{65}+E_{75})r\rho-(E_{38}+E_{49})r \\
&\quad -(E_{38}+E_{49}+E_{61}+E_{72})(\partial_t+2r\lambda)], \\
Q_r^1 &= e^{rt}[E_{17}-E_{26}+E_{84}-E_{93}+2(E_{65}-E_{75})r\rho+(E_{39}-E_{48})r \\
&\quad +(E_{39}-E_{48}+E_{62}-E_{71})(\partial_t+2r\lambda)], \\
Q_r^2 &= e^{rt}[E_{18}+E_{29}-E_{63}-E_{74}-2(E_{85}+E_{95})r\rho+(E_{36}+E_{47})r \\
&\quad +(E_{36}+E_{47}-E_{81}-E_{92})(\partial_t+2r\lambda)], \\
Q_r^3 &= e^{rt}[E_{19}-E_{28}-E_{64}+E_{73}+2(E_{85}-E_{95})r\rho+(E_{46}-E_{37})r \\
&\quad +(E_{46}-E_{37}+E_{82}-E_{91})(\partial_t+2r\lambda)];
\end{aligned} \tag{B.7}$$

the $(1,4,3)$ rep is constructed from

$$\begin{aligned}
Q_r^0 &= e^{rt}[E_{16}+E_{72}+E_{83}+E_{94}-2E_{65}r\rho \\
&\quad -(E_{27}+E_{38}+E_{49})r-(E_{27}+E_{38}+E_{49}+E_{61})(\partial_t+2r\lambda)], \\
Q_r^1 &= e^{rt}[E_{17}-E_{62}+E_{84}-E_{93}-2E_{75}r\rho+(E_{26}+E_{39}-E_{48})r \\
&\quad +(E_{26}+E_{39}-E_{48}-E_{71})(\partial_t+2r\lambda)], \\
Q_r^2 &= e^{rt}[E_{18}-E_{63}-E_{74}+E_{92}-2E_{85}r\rho+(E_{36}-E_{29}+E_{47})r \\
&\quad +(E_{36}-E_{29}+E_{47}-E_{81})(\partial_t+2r\lambda)], \\
Q_r^3 &= e^{rt}[E_{19}-E_{64}+E_{73}-E_{82}-2E_{95}r\rho+(E_{28}-E_{37}+E_{46})r \\
&\quad +(E_{28}-E_{37}+E_{46}-E_{91})(\partial_t+2r\lambda)];
\end{aligned} \tag{B.8}$$

the $(0,4,4)$ rep is constructed from

$$\begin{aligned}
Q_r^0 &= e^{rt}[E_{61}+E_{72}+E_{83}+E_{94}-(E_{16}+E_{27}+E_{38} \\
&\quad +E_{49})(\partial_t+r+2r\lambda)], \\
Q_r^1 &= e^{rt}[E_{71}-E_{62}+E_{84}-E_{93} \\
&\quad +(E_{26}-E_{17}+E_{39}-E_{48})(\partial_t+r+2r\lambda)], \\
Q_r^2 &= e^{rt}[E_{81}-E_{63}-E_{74}+E_{92} \\
&\quad +(E_{36}-E_{18}-E_{29}+E_{47})(\partial_t+r+2r\lambda)], \\
Q_r^3 &= e^{rt}[E_{73}-E_{64}-E_{82}+E_{91} \\
&\quad +(E_{28}-E_{19}-E_{37}+E_{46})(\partial_t+r+2r\lambda)].
\end{aligned} \tag{B.9}$$

The above operators produce D-module reps for the respective superalgebras only at the critical values, which have been presented in Section **7**, for λ and ρ. The inhomogeneous D-module reps of the finite $d=1$ superconformal algebras are obtained for $r=\pm\frac{1}{2}$ and $\lambda=0$.

The $\mathcal{N}=3$ D-module rep of the centerless superVirasoro algebra (41) exists for arbitrary values of λ and ρ. At $\lambda=0$ the restriction to the $B(1,1)$ subalgebra generators produces the $(1,3,3,1)_{0,\rho}$ inhomogeneous D-module rep of $B(1,1)$.

To reconstruct the full D-module rep is sufficient to present the three Q^i's operators.

The $\mathcal{N}=3$ SuperVirasoro $(1,3,3,1)$ rep is obtained from

$$
\begin{aligned}
Q_r^1 &= e^{rt}[E_{17}-E_{39}-E_{62}+E_{84}-2E_{75}r\rho \\
&\quad +(E_{26}-E_{48}+2E_{93})r+(E_{26}-E_{48}-E_{71}+E_{93})(\partial_t+2r\lambda)],\\
Q_r^2 &= e^{rt}[E_{18}+E_{29}-E_{63}-E_{74}-2E_{85}r\rho \\
&\quad +(E_{36}+E_{47}-2E_{92})r+(E_{36}+E_{47}-E_{81}-E_{92})(\partial_t+2r\lambda)],\\
Q_r^3 &= e^{rt}[E_{16}+E_{49}+E_{72}+E_{83}-2E_{65}r\rho-(E_{27}+E_{38}+2E_{94})r \\
&\quad -(E_{27}+E_{38}+E_{61}+E_{94})(\partial_t+2r\lambda)].
\end{aligned}
\tag{B.10}
$$

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