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# FOURIER AND RADON TRANSFORM ON HARMONIC $NA$ GROUPS

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ABSTRACT. In this article we study the Fourier and the horocyclic Radon transform on harmonic $NA$ groups (also known as Damek-Ricci spaces). We consider the geometric Fourier transform for functions on $L^p$-spaces and prove an analogue of the $L^2$-restriction theorem. We also prove some mixed norm estimates for the Fourier transform generalizing the Hausdorff-Young and Hardy-Littlewood-Paley inequalities. Unlike Euclidean spaces the domains of the Fourier transforms are various strips in the complex plane. All the theorems are considered on these entire domains of the Fourier transforms. Finally we deal with the existence of the Radon transform on $L^p$-spaces and obtain its continuity property.

## 1. INTRODUCTION

A harmonic $NA$ group $S$ is a solvable Lie group with a canonical left invariant Riemannian structure. Their distinguished prototypes are the noncompact Riemannian symmetric spaces of rank one. It is known that the latter accounts for but a very small subclass of the class of $NA$ groups (see [1, (1.4)]). Harmonic analysis for radial functions on $S$ has had its foundations laid through the pioneering works of several authors (see e.g. [10, 11, 12, 8, 9]), whereupon further studies were taken up in [1, 2, 3, 5, 13, 14, 15, 24, 31] and the references therein. We note that unlike in the case of Riemannian symmetric spaces, the concept of radial function here is not connected with any group action and the elementary spherical functions are not necessarily matrix entries of irreducible unitary representations of the group. Nonetheless they are eigenfunctions of the Laplace-Beltrami operator and as mentioned in [1] "despite the lack of symmetry the analysis of radial functions on $S$ is quite similar to the hyperbolic space case". The aim of this paper is to go beyond the class of radial functions and address some questions of harmonic analysis of more general functions on $S$.

We adopt the definition of Fourier transform of a function $f \in C_c^\infty(S)$ as given in [4]:

$$\widetilde{f}(\lambda, n) = \int_S f(x)\mathcal{P}_\lambda(x, n)dx, \lambda \in \mathbb{R}, n \in N,$$

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which is an analogue of the Helgason Fourier transform on symmetric spaces realized in the noncompact picture. Here the role of the sphere is played by the subgroup $N$ which is not compact. The kernel defining this Fourier transform is an eigenfunction of the Laplace-Beltrami operator on $S$ distinguished by the feature that it is constant on each of a certain family of hypersurfaces parametrized by $n \in N$ (see Section 2). This makes it the object analogous to $e^{i\langle\lambda,x\rangle}$ in $\mathbb{R}^n$ which is also an eigenfunction of the Laplacian and is constant on the hypersurface $\{x \mid \langle\omega,x\rangle = \text{constant}\}$. This analogy however breaks down as we note that the kernel $\mathcal{P}_\lambda(x,n)$ is not a bounded function of $x$, an apparent obstruction for defining the Fourier transform for functions in $L^1(S)$. Nevertheless we will see that for a class of functions much larger than $L^1(S)$ the Fourier transform $\widetilde{f}(\lambda,n)$ exists as a convergent integral for almost every $n$ in $N$ (Proposition 3.1). Once this is noted, we turn our attention to the problem of extending $\widetilde{f}(\lambda,n)$ for nonreal $\lambda$ and identifying its domains of analyticity for $f$ in various function spaces. In this context the Lorentz spaces appear naturally. Indeed for radial functions (analogous to the case of symmetric spaces) the spherical Fourier transform of $L^1$-functions exists on a closed strip $S_1$, but for radial $L^p$-functions the spherical Fourier transform $\widehat{f}$ exists only on the interior of the corresponding strip, which we call $S_p$. This intriguing dissimilarity in the behaviour of $L^1$ and $L^p$-functions leads us to look at the Lorentz spaces (Theorem 3.4). As we embark upon some deeper properties of the Fourier transform of $L^p$-functions, we study the continuity of the spectral projection, that is, the operator $f \mapsto f * \phi_\lambda$ for fixed $\lambda \in S_p$ (Theorem 4.1). Next we deal with the norm inequality of the form

$$\int_N |\widetilde{f}(\lambda,n)|^q dn \leq C\|f\|_p^q.$$

As $N$ plays the role of the sphere (which is a set of zero Plancherel measure), this question is somewhat similar in spirit to the $L^2$-restriction theorem. However we may recall that the holomorphic extension of the Fourier transform is perhaps the most distinctive feature of the $NA$ groups. Our departure here is to consider the Fourier transform not only as a function on $\mathbb{R}$ but on its entire domain of definition (e.g., for $L^p$-functions it is $S_p$). Hence these results do not have analogues on $\mathbb{R}^n$. This consideration of the whole domain brings out an asymmetric behaviour of the Fourier transform vis-à-vis the lower and upper halves of the strip $S_p$ (Theorem 4.2). As a consequence of this result we get an analogue of the *Kunze-Stein phenomenon*. Theorem 4.2 naturally leads us to study the results of the genre of Hausdorff-Young and Hardy-Littlewood-Paley inequalities.

Our next object of study is the Radon transform on $S$. We are interested in the existence of the Radon transform in the sense of its existence as a lower dimensional integral. If $f$ is an integrable function, then it follows from Fubini's theorem that its restriction on almost every affine hypersubspace is integrable with respect to the induced measure and that ensures the existence of the Radon transform. Clearly the same argument cannot be applied to the restriction of an $L^p$-function. A natural question at this point is: what is the class of measurable functions on $S$ for which the Radon transform can be defined. This question is partially addressed in Theorem 5.3. We conclude the article with a mapping property of the Radon transform. For Euclidean spaces such questions were initiated in [23, 28, 26]. However, even for symmetric spaces, existence and mapping properties of horocyclic Radon transforms have not been considered so far outside integrable functions.

We may point out that several of our results in this article point to phenomena not observed in symmetric spaces.

## 2. Preliminaries

In this section we will explain the notation and gather existing results required for this paper. Most of this can be found in [4, 1]. We will also supply proofs for the facts for which we could not locate any reference.

Let $\mathfrak{n}$ be a two-step real nilpotent Lie algebra equipped with the inner product $\langle , \rangle$. Let $\mathfrak{z}$ be the centre of $\mathfrak{n}$ and $\mathfrak{v}$ its orthogonal complement. We say that $\mathfrak{n}$ is an $H$-type algebra if for every $Z \in \mathfrak{z}$ the map $J_Z : \mathfrak{v} \longrightarrow \mathfrak{v}$ defined by

$$\langle J_Z X, Y\rangle = \langle [X, Y], Z\rangle, \ X, Y \in \mathfrak{v}$$

satisfies the condition $J_Z^2 = -|Z|^2 I_{\mathfrak{v}}$, $I_{\mathfrak{v}}$ being the identity operator on $\mathfrak{v}$. A connected and simply connected Lie group $N$ is called an $H$-type group if its Lie algebra is $H$-type. Since $\mathfrak{n}$ is nilpotent, the exponential map is a diffeomorphism and hence we can parametrize the elements in $N = \exp \mathfrak{n}$ by $(X, Z)$, for $X \in \mathfrak{v}$ and $Z \in \mathfrak{z}$. It follows from the Campbell-Baker-Hausdorff formula that the group law in $N$ is given by

$$(X, Z)(X', Z') = (X + X', Z + Z' + \frac{1}{2}[X, X']).$$

The group $A = \mathbb{R}^+$ acts on an $H$-type group $N$ by nonisotropic dilation: $(X, Z) \mapsto (a^{1/2}X, aZ)$. Let $S = NA$ be the semidirect product of $N$ and $A$ under the above action. Thus the multiplication in $S$ is given by

$$(X, Z, a)(X', Z', a') = (X + a^{1/2}X', Z + aZ' + \frac{1}{2}a^{1/2}[X, X'], aa').$$

Then $S$ is a solvable, connected and simply connected Lie group having Lie algebra $\mathfrak{s} = \mathfrak{v} \oplus \mathfrak{z} \oplus \mathbb{R}$ with Lie bracket:

$$[(X, Z, \ell), (X', Z', \ell')] = (\frac{1}{2}\ell X' - \frac{1}{2}\ell' X, \ell Z' - \ell Z + [X, X'], 0).$$

We write $na = (X, Z, a)$ for the element $\exp(X + Z)a$, $X \in \mathfrak{v}, Z \in \mathfrak{z}, a \in A$. We note that for any $Z \in \mathfrak{z}$ with $|Z| = 1$, $J_Z^2 = -I_{\mathfrak{v}}$; that is, $J_Z$ defines a complex structure on $\mathfrak{v}$. Therefore $\mathfrak{v}$ is even dimensional. We suppose $\dim \mathfrak{v} = m$ and $\dim \mathfrak{z} = k$. Then $Q = \frac{m}{2} + k$ is called the *homogenous dimension* of $S$. For convenience we will also use the symbol $\rho$ for $Q/2$ and $d$ for $m + k + 1 = \dim \mathfrak{s}$.

The group $S$ is equipped with the left-invariant Riemannian metric induced by

$$\langle (X, Z, \ell), (X', Z', \ell')\rangle = \langle X, X'\rangle + \langle Z, Z'\rangle + \ell\ell'$$

on $\mathfrak{s}$. The associated left-invariant Haar measure $dx$ on $S$ is given by $dx = a^{-Q-1}dXdZda$, where $dX, dZ, da$ are the Lebesgue measures on $\mathfrak{v}, \mathfrak{z}$ and $\mathbb{R}^+$ respectively.

The group $S$ can also be realized as the unit ball

$$B(\mathfrak{s}) = \{(X, Z, \ell) \in \mathfrak{s} \mid |X|^2 + |Z|^2 + \ell^2 < 1\}$$

via a Cayley transform $C : S \longrightarrow B(\mathfrak{s})$ (see [1, pp. 646–647] for details). For an element $x \in S$, let $r(x) = d(C(x), 0)$. Then the left Haar measure in geodesic polar coordinates is given by ([1, (1.16)])

$$dx = 2^{m+k}(\sinh\frac{r}{2})^{m+k}(\cosh\frac{r}{2})^{k}\,dr\,d\omega = 2^{m}(\sinh r)^{k}(\sinh\frac{r}{2})^{m}\,dr\,d\omega, \tag{2.1}$$

where $d\omega$ denotes the surface measure on the unit sphere $\partial B(\mathfrak{s})$ in $\mathfrak{s}$.

To define the Fourier transform on $S$ we need the notion of the Poisson kernel $\mathcal{P}(x, n)$. The Poisson kernel $\mathcal{P} : S \times N \longrightarrow \mathbb{R}$ is given by $\mathcal{P}(na_t, n_1) = P_{a_t}(n_1^{-1}n)$, where

$$P_{a_t}(n) = P_{a_t}(V, Z) = Ca_t^{Q}\left(\left(a_t + \frac{|V|^2}{4}\right)^2 + |Z|^2\right)^{-Q} \tag{2.2}$$

and where $a_t = e^t, t \in \mathbb{R}$ and $n = (V, Z) \in N$. (For the precise value of $C$ so that the property (v) below holds we refer to [4, (2.6)].)

The following properties of $\mathcal{P}(x, n)$ are important for us and can be derived from (2.2):

(i) $P_a(n) = P_a(n^{-1})$.
(ii) $P_a(n) = a^{-Q}P_1(a^{-1}na)$.
(iii) For each fixed $a \in A, n \mapsto P_a^{1/2}(n)$ is bounded by $Ca^{-Q}$. In fact $P_a^{1/2}(\cdot) \in L^q(N)$, $1 < q \leq \infty$.
(iv) $\mathcal{P}(x, n) = \mathcal{P}(n_1x, n_1n)$, for $n, n_1 \in N$, $x \in S$.
(v) $\int_N P_a(n)dn = 1$.
(vi) For each fixed $n \in N$, the function $x \mapsto \mathcal{P}(x, n)$ is an unbounded function on $S$.

For $\lambda \in \mathbb{C}$, the complex power of the Poisson kernel is defined as

$$\mathcal{P}_\lambda(x, n) = \mathcal{P}(x, n)^{\frac{1}{2}-\frac{i\lambda}{Q}}.$$

It follows from (v) that for each fixed $x \in S$, $\mathcal{P}_\lambda(x, \cdot) \in L^p(N)$ for $1 \leq p \leq \infty$ if $\lambda = i\gamma_p\rho$, where $\gamma_p = (2/p - 1)$.

A distinguishing feature of the function $x \mapsto \mathcal{P}_\lambda(x, n)$ is that it is constant on certain hypersurfaces which are analogues of hyperplanes in $\mathbb{R}^n$ and horospheres in Riemannian symmetric spaces of noncompact type. We need to introduce the notion of the geodesic inversion to illustrate this feature.

The geodesic inversion $\sigma : S \longrightarrow S$ is an involutive, measure-preserving, diffeomorphism which is explicitly given by ([9, 25]):

$$\sigma(V, Z, a_t) = \left(\left(e^t + \frac{|V|^2}{4}\right)^2 + |Z|^2\right)^{-1}\left(\left(-\left(e^t + \frac{|V|^2}{4}\right) + J_Z\right)V, -Z, a_t\right). \tag{2.3}$$

For $x = na_t \in S$, let $A(x) = \log a_t = t$. As $A$ normalizes $N$, it follows that $A(x^{-1}) = -t$. The geodesic inversion $\sigma$ has the following important property:

$$\mathcal{P}(\sigma(x), e) = Ce^{QA(x)},\ x \in S. \tag{2.4}$$

Here is a brief sketch of the proof. Let $\alpha = \left(e^t + \frac{|V|^2}{4}\right)^2 + |Z|^2$ and $\beta = -(e^t + \frac{|V|^2}{4})$. Then $\beta^2 + |Z|^2 = \alpha$. Then it follows from (2.3) that

$$\sigma(V, Z, a_t) = \frac{1}{\alpha}\left((\beta + J_Z)V, -Z, a_t\right).$$

Using the fact that $\langle J_Z X, Y\rangle = \langle X, -J_Z Y\rangle$, we get $|(\beta + J_Z)V|^2 = (\beta^2 + |Z|^2)|V|^2$. A straightforward computation using (2.2) now yields

$$\mathcal{P}(\sigma(V, Z, a_t), e) = C\left(\frac{e^t}{\alpha}\right)^Q \left(\left(\frac{e^t}{\alpha} + \frac{1}{4}\left|(\beta + J_Z)\frac{V}{\alpha}\right|^2\right)^2 + \frac{|Z|^2}{\alpha^2}\right)^{-Q} = Ce^{tQ}.$$

This completes the proof. As $A$ normalizes $N$ it also follows from above that the function $x \mapsto \mathcal{P}(\sigma(x), e)$ is an $N$-bi-invariant function on $S$.

Let $\mathcal{L}$ denote the Laplace-Beltrami operator of $S$. Then for every fixed $n_1 \in N$, the function $\mathcal{P}_\lambda(x, n_1)$ is an eigenfunction of $\mathcal{L}$ with eigenvalue $-(\lambda^2 + Q^2/4)$ ([4, p. 411]). It is clear from (2.4) that $\mathcal{P}_\lambda(x, n_1)$ is constant on the hypersurfaces $H_{n_1, a_t} = \{n_1\sigma(a_t n) \mid n \in N\}$. In view of this it is natural to define the Fourier transform of a function $f \in C_c^\infty(S)$ as ([4, p. 406]):

$$\widetilde{f}(\lambda, n) = \int_S f(x)\mathcal{P}_\lambda(x, n)dx.$$

From (2.4), (iv) and the fact that $\sigma$ is an involution it follows that

$$\mathcal{P}_\lambda(x, n) = \mathcal{P}_\lambda(\sigma\sigma(n^{-1}x), e) = e^{(\rho - i\lambda)A(\sigma(n^{-1}x))}. \tag{2.5}$$

Hence the definition of $\widetilde{f}(\lambda, n)$ can be rewritten as:

$$\widetilde{f}(\lambda, n) = \int_S f(x)e^{(\rho - i\lambda)A(\sigma(n^{-1}x))}dx.$$

This shows the structural similarity of $\widetilde{f}(\lambda, n)$ with the Helgason Fourier transform $\widetilde{f}(\lambda, k) = \int_X f(x)e^{(i\lambda - \rho)H(x^{-1}k)}$ on a Riemannian symmetric space $X$ (see [20]).

A function $f$ on $S$ is called *radial* if, for all $x \in S$, $f(x) = f(r(x))$. For a suitable function $f$ on $S$ its radialization $Rf$ is defined as

$$Rf(x) = \int_{S\nu} f(y)d\sigma_\nu(y), \tag{2.6}$$

where $\nu = r(x)$ and $d\sigma_\nu$ is the surface measure induced by the left-invariant Riemannian metric on the geodesic sphere $S_\nu = \{y \in S \mid d(y, e) = \nu\}$ normalized by $\int_{S_\nu} d\sigma_\nu(y) = 1$. It is clear that if $f$ is radial, then $Rf = f$. It is known that ([4, p. 410])

$$R(\mathcal{P}_\lambda(\cdot, n))(x) = \phi_\lambda(x)\mathcal{P}_\lambda(e, n), \tag{2.7}$$

where $\phi_\lambda(x) = \int_N P_\lambda(x, n)\mathcal{P}_{-\lambda}(e, n)dn$ is the elementary spherical function ([4, Proposition 4.2]). It follows that $\phi_\lambda$ is a radial eigenfunction of $\mathcal{L}$ with eigenvalue $-(\lambda^2 + Q^2/4)$ satisfying $\phi_\lambda(x) = \phi_{-\lambda}(x)$, $\phi_\lambda(x) = \phi_\lambda(x^{-1})$ and $\phi_\lambda(e) = 1$. We have the following basic estimate of $\phi_\lambda$ ([1, (3.5)]):

$$\phi_{i(\frac{1}{p} - \frac{1}{2})Q}(r) \asymp \begin{cases} e^{-\frac{Q}{p'}r} & \text{if } 1 \le p < 2, \\ (1 + r)e^{-\rho r} & \text{if } p = 2. \end{cases} \tag{2.8}$$

Here and everywhere in this article $A \asymp B$ for two positive expressions $A$ and $B$ means $C_1 B \le A \le C_2 B$ for two positive constants $C_1$ and $C_2$.

We define the spherical Fourier transform of a suitable radial function $f$ as

$$\widehat{f}(\lambda) = \int_S f(x)\phi_\lambda(x)dx.$$

It follows from (2.6) and (2.7) that if $f$ is a radial function, then, unlike the case of Riemannian symmetric spaces, its Helgason Fourier transform does not boil down to its spherical Fourier transform; indeed, they are related as

$$\widetilde{f}(\lambda, n) = \widehat{f}(\lambda)\mathcal{P}_\lambda(e, n). \tag{2.9}$$

For $f \in C_c^\infty(S)$ we have:

(1) Inversion formula ([4, Theorem 4.4]):

$$f(x) = C\int_{-\infty}^{\infty}\int_N \widetilde{f}(\lambda, n)\mathcal{P}_{-\lambda}(x, n)|c(\lambda)|^{-2}d\lambda\, dn.$$

(2) Plancherel Theorem ([4, p. 419]): The Fourier transform extends to an isometry from $L^2(S)$ onto the space $L^2(\mathbb{R}^+ \times N, |c(\lambda)|^{-2}d\lambda\, dn$. For the precise value of the constants, we refer to [4, p. 414].

A constant multiple of the function $|c(\lambda)|^{-2}$ is given by the following ([1, (2.31)]):

(2.10)

$$\prod_{0<j<\frac{m}{4}}(\lambda^2 + j^2)^2\lambda^3\coth\pi\lambda \ \text{ if } k = 1, \frac{m}{2} \text{ odd},$$

$$\prod_{0<j \text{ odd } <\frac{m}{2}}(\lambda^2 + \frac{j^2}{4})^2 \prod_{\frac{m}{2}<j \text{ odd } <\frac{m}{2}+k}(\lambda^2 + \frac{j^2}{4})\lambda\tanh\pi\lambda \ \text{ if } k \text{ odd}, \frac{m}{2} \text{ even},$$

$$\prod_{0\le j\le\frac{m}{2}}(\lambda^2 + \frac{j^2}{4}) \prod_{\frac{m}{2}<j \text{ even } <\frac{m}{2}+k}(\lambda^2 + \frac{j^2}{4}) \ \text{ if } k \text{ even}, \frac{m}{2} \text{ even},$$

where $j \in \mathbb{Z}$. Anker et al. [1] showed that these three cases exhaust all $NA$ groups of rank one.

We also need the following definitions and results for the Lorentz spaces (see [19, 30] for details). Let $(M, m)$ be a $\sigma$-finite measure space, $f : M \longrightarrow \mathbb{C}$ be a measurable function and $p \in [1, \infty)$, $q \in [1, \infty]$. We define

$$\|f\|^*_{p,q} = \begin{cases} \left(\frac{q}{p}\int_0^\infty [f^*(t)t^{1/p}]^q \frac{dt}{t}\right)^{1/q} & \text{when } q < \infty, \\ \sup_{t>0} t d_f(t)^{1/p} & \text{when } q = \infty. \end{cases}$$

Here $d_f$ is the distribution function of $f$ and $f^*(t) = \inf\{s \mid d_f(s) \le t\}$ is the *nonincreasing rearrangement* of $f$ ([19, p. 45]). We take $L^{p,q}(M)$ to be the set of all measurable $f : M \longrightarrow \mathbb{C}$ such that $\|f\|^*_{p,q} < \infty$. By $L^{\infty,\infty}(M)$ and $\|\cdot\|_{\infty,\infty}$ we mean respectively the space $L^\infty(M)$ and the norm $\|\cdot\|_\infty$. For $p, q \in [1, \infty)$ the following identity gives an alternative expression of $\|\cdot\|^*_{p,q}$, which we will use (see e.g. [32, p. 104], [7]):

$$\frac{q}{p}\int_0^\infty (t^{1/p}f^*(t))^q \frac{dt}{t} = q\int_0^\infty (t d_f(t)^{1/p})^q \frac{dt}{t}.$$

It is easy to check the identity for nonnegative simple functions of the type $s(x) = \sum_{j=1}^n a_j\chi_{E_j}(x)$, where $a_1 > a_2 > \cdots > a_n > 0$, the $E_j$'s are pairwise disjoint measurable sets of finite measures and $\chi_{E_j}$ is the characteristic function of $E_j$. As any nonnegative function can be approximated by a sequence of simple functions like $s$ above and as $s_n \uparrow |f|$ almost everywhere implies $d_{s_n} \uparrow d_f$ and $s_n^* \uparrow f^*$ ([19, p. 10, 47]), using the monotone convergence theorem we get the identity for all measurable functions.

For $p, q$ in the range above, $L^{p,p}(M) = L^p(M)$ and if $q_1 \leq q_2$, then $\|f\|^*_{p,q_2} \leq \|f\|^*_{p,q_1}$ and consequently $L^{p,q_1}(M) \subset L^{p,q_2}(M)$. The following version of Hölder's inequality for Lorentz spaces ([19], p. 74) will be useful for us. Let $0 < p, q, r \leq \infty$, $0 < s_1, s_2 \leq \infty$. Then

$$\|f.g\|^*_{r,s} \leq C_{p,q,s_1,s_2} \|f\|^*_{p,s_1} \|g\|^*_{q,s_2}, \tag{2.11}$$

where $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$, $\frac{1}{s_1} + \frac{1}{s_2} = \frac{1}{s}$. We recall that for $1 < p < \infty$ and $1 \leq q < \infty$, the dual of $L^{p,q}(M)$ is $L^{p',q'}(M)$, where $\frac{1}{p} + \frac{1}{p'} = 1 = \frac{1}{q} + \frac{1}{q'}$, and the dual space of $L^{1,q}(M)$ is $\{0\}$ for $1 < q < \infty$ (see [19, p. 52]). Everywhere in this article any $p \in [1, \infty)$ is related to $p'$ by the relation above.

For a complex number $z$, we will use $\Re z$ and $\Im z$ to denote respectively the real and imaginary parts of $z$. We will follow the standard practice of using the letter $C$ for a constant, whose value may change from one line to another. Occasionally the constant $C$ will be suffixed to show its dependency on important parameters.

## 3. Existence of the Fourier transform

We recall that for $f \in C_c^\infty(S)$, its Helgason Fourier transform is defined as

$$\widetilde{f}(\lambda, n) = \int_S f(x) \mathcal{P}_\lambda(x, n) dx.$$

As $\mathcal{P}_\lambda(x, n)$ is not a bounded function of $x \in S$ (except when $\Im\lambda = -i\rho$ (see Section 2)), the definition of the Fourier transform does not extend naturally to functions in $L^1(S)$. On the other hand we will see below that for a large class of functions containing $L^1(S)$ the Fourier transform $\widetilde{f}(\lambda, n)$ can be defined for almost every $n \in N$.

Let us consider the function space

$$L^1_0(S) = \{f : S \longrightarrow \mathbb{C} \ \text{ measurable } \ | \ f.\phi_0 \in L^1(S)\}.$$

**Proposition 3.1.** *If $f \in L^1_0(S)$, then there exists a subset $N_0$ of $N$ of full Haar measure, depending only on $f$, such that $\widetilde{f}(\lambda, n)$ exists for all $n \in N_0$ and $\lambda \in \mathbb{R}$.*

*Proof.* Using $|\mathcal{P}_\lambda(x, n)| = \mathcal{P}_0(x, n)$ it follows that for $\lambda \in \mathbb{R}$,

$$\begin{aligned}
\left| \int_N \widetilde{f}(\lambda, n) \mathcal{P}_0(e, n) dn \right| &\leq \int_N \left| \int_S f(y) \mathcal{P}_\lambda(y, n) dy \right| |\mathcal{P}_0(e, n)| dn \\
&\leq \int_N \int_S |f(y)| |\mathcal{P}_\lambda(y, n)| dy \mathcal{P}_0(e, n) dn \\
&= \int_N \int_S |f(y)| \mathcal{P}_0(y, n) dy \mathcal{P}_0(e, n) dn \\
&= \int_S |f(y)| \int_N \mathcal{P}_0(y, n) \mathcal{P}_0(e, n) dn \, dy \\
&= \int_S |f(y)| \phi_0(y) dy < \infty.
\end{aligned}$$

Therefore there exists a set $N_0 \subset N$ such that $N \setminus N_0$ is of Haar measure 0 and $\widetilde{f}(\lambda, n)$ exists for all $n \in N_0$. $\square$

We are interested in certain subspaces of $L^1_0(S)$. We will see below that $L^1(S)$ and the Lorentz spaces $L^{p,q}(S)$, $1 < p < 2$, $1 \leq q \leq \infty$ are subspaces of $L^1_0(S)$. The crux of the matter is the following behaviour of the elementary spherical function. For $p \in [1, 2)$ we define:

$$S_p = \{\lambda \in \mathbb{C} \mid |\Im\lambda| \leq \gamma_p \rho\}.$$

By $S_p^\circ$ and $\partial S_p$ we denote respectively the interior and the boundary of $S_p$. Since for $\lambda \in \mathbb{C}$,

$$|\phi_\lambda(x)| \leq \phi_{i\mu}(x), \quad \text{when } |\Im\lambda| \leq \mu \tag{3.1}$$

([1, (4.12)]) and $\phi_{i\rho} \equiv 1$, it follows that if $\lambda \in S_1$, then $\|\phi_\lambda\|_\infty \leq 1$. Thus $L^1(S) \subset L^1_0(S)$.

**Lemma 3.2.** (i) *Let $1 < p < 2$. If $\lambda \in S_p^\circ$, then $\phi_\lambda \in L^{p',q}(S)$ for any $q \in [1, \infty]$.*

(ii) *If $1 \leq p < 2$ and $\lambda \in \partial S_p$, then $\phi_\lambda \in L^{p',\infty}(S)$.*

(iii) *If $p = 2$ and $\lambda \in S_2 = \mathbb{R}$, then $\phi_\lambda/(1 + r(\cdot)) \in L^{2,\infty}(S)$.*

*Proof.* (i) Let $\lambda \in S_p^\circ \setminus \mathbb{R}$. Then $|\Im\lambda| = \gamma_{p_1}\rho$ for some $p_1 \in (p, 2)$. We have from (3.1) and (2.8) that $|\phi_\lambda(r)| \leq e^{-\frac{Q}{p_1'}r}$. We will show that the function $f(r) = e^{-\frac{Q}{p_1'}r}$, defined for $r \geq 0$ is in $L^{p',1}(S)$.

Since for $\alpha > 0$, $d_f(\alpha) = m\{r \mid e^{-\frac{Q}{p_1'}r} > \alpha\}$ we have $d_f(\alpha) = 0$ if $\alpha \geq 1$. Therefore we assume that $\alpha < 1$. Now

$$\begin{aligned} d_f(\alpha) &= m\{r \mid r < \frac{p_1'}{Q}\log\frac{1}{\alpha}\} \\ &\leq \int_0^{\frac{p_1'}{Q}\log\frac{1}{\alpha}} e^{Qr}dr \quad (\text{as } \sinh r \leq e^r) \\ &= \frac{1}{Q}\left(e^{p_1'\log\frac{1}{\alpha}} - 1\right) \leq \frac{1}{Q\alpha^{p_1'}}. \end{aligned}$$

As $p_1 > p$ the integral

$$\int_0^\infty \alpha d_f(\alpha)^{1/p'}\frac{d\alpha}{\alpha} \leq \frac{1}{Q^{1/p'}}\int_0^\infty \frac{1}{\alpha^{p_1'/p'}} < \infty.$$

Therefore $\phi_\lambda \in L^{p',1}(S) \subset L^{p',q}(S)$ for $1 < q \leq \infty$.

When $\lambda \in \mathbb{R}$ then also the argument above works as, by (2.8) for $p_1 \in (p, 2)$,

$$|\phi_\lambda(r)| \leq C(1 + r)e^{-\frac{Q}{2}r} \leq Ce^{-\frac{Q}{p_1'}r}.$$

This completes the proof of (i).

(ii) The case $p = 1$ has already been discussed. So we assume $1 < p < 2$. For this case we consider the function $f(r) = e^{-\frac{Q}{p'}r}$, where $p \in (1, 2]$. The calculations in (i) show that $d_f(\alpha) \leq \frac{1}{Q\alpha^{p'}}$ and hence $\sup_{0<\alpha<1} \alpha d_f(\alpha)^{\frac{1}{p'}} \leq Q^{-1/p'} < \infty$. Using (2.8) it now follows that $\phi_\lambda \in L^{p'\infty}(S)$ when $\lambda \in \partial S_p$ and $p \in (1, 2)$.

(iii) When $\lambda \in \mathbb{R}$ then by (2.8) we have $\phi_0/(1 + r(\cdot) \asymp e^{-\rho r} = e^{-\frac{Q}{2}r}$. Therefore (ii) above shows that $f(r) = \phi_0(r)/(1 + r)$ is in $L^{2,\infty}(S)$. $\square$

*Remark* 3.3. It is clear from Lemma 3.2 and (2.11) that $L^{p,q}(S)\subset L^1_0(S), 1<p<2$, $1\leq q<\infty$. In particular $L^p(S)\subset L^1_0(S)$ for $1\leq p<2$. Also it follows from the lemma above that if for some measurable function $f$, $f(x)(1+r(x))\in L^{2,1}(S)$, then $f\in L^1_0(S)$.

**Theorem 3.4.** *Let $f$ be a measurable function in the Lorentz space $L^{p,q}(S)$.*

(i) *If $1\leq p<2$ and $q=1$, then there exists a subset $N^p$ of $N$ of full Haar measure, depending only on $f$, such that $\widetilde{f}(\lambda,n)$ exists for all $n\in N^p$ and $\lambda\in S_p$.*

(ii) *If $1<p<2$ and $1<q\leq\infty$, then there exists a subset $N^p$ of $N$ of full Haar measure, depending only on $f$, such that $\widetilde{f}(\lambda,n)$ exists for all $n\in N^p$ and $\lambda\in S_p^\circ$.*

(iii) *If $p,q$ are as in* (ii), *then there exists a subset $N_p'$ of $N$ of full Haar measure, depending only on $f$, such that $\widetilde{f}(\lambda,n)$ exists for all $n\in N_p'$ and almost every $\lambda\in\partial S_p$.*

*Proof.* Let $f\in L^{p,1}(S)$ and $\lambda\in S_p$. Then by the lemma above, as $\phi_{i\Im\lambda}\in L^{p',\infty}(S)$,

$$\left|\int_N \widetilde{f}(\lambda,n)\mathcal{P}_{-\lambda}(e,n)dn\right|\leq\int_S|f(x)|\phi_{i\Im\lambda}(x)dx<\infty.$$

Thus $\widetilde{f}(\lambda,n)$ exists for all $n\in N_\lambda\subset N$, where the Haar measure of $N\setminus N_\lambda$ is zero. Let $N^p=N_{i\gamma_p\rho}\cap N_{-i\gamma_p\rho}$. Thus the Haar measure of $N\setminus N^p$ is zero and for all $n\in N^p$ we have

$$\int_S f(x)\mathcal{P}_{\pm i\gamma_p\rho}(x,n)dx<\infty. \tag{3.2}$$

Let us assume that $f\geq 0$. We fix an $n\in N^p$. Now

$$|\widetilde{f}(\lambda,n)|\leq\int_S f(x)\mathcal{P}_{i\Im\lambda}(x,n)dx.$$

Let

$$S_+=\{x\in S\mid\mathcal{P}_0(x,n)\geq 1\}\text{ and }S_-=\{x\in S\mid\mathcal{P}_0(x,n)<1\}.$$

Since $\lambda\in S_p$ we have

$$0\leq\frac{1}{2}-\gamma_p\rho.\frac{1}{Q}\leq\frac{1}{2}+\frac{\Im\lambda}{Q}\leq\frac{1}{2}+\gamma_p\rho.\frac{1}{Q}.$$

Thus for $x\in S_+$, $\mathcal{P}_{i\Im\lambda}(x,n)\leq\mathcal{P}_{i\gamma_p\rho}(x,n)$ and for $x\in S_-$, $\mathcal{P}_{i\Im\lambda}(x,n)\leq\mathcal{P}_{-i\gamma_p\rho}(x,n)$. Then as $n\in N^p$ by (3.2),

$$\begin{aligned}|\widetilde{f}(\lambda,n)|&\leq\int_{S_+}f(x)\mathcal{P}_{i\gamma_p\rho}(x,n)dx+\int_{S_-}f(x)\mathcal{P}_{-i\gamma_p\rho}(x,n)dx\\&\leq\int_S f(x)(\mathcal{P}_{i\gamma_p\rho}(x,n)+\mathcal{P}_{-i\gamma_p\rho}(x,n))dx<\infty.\end{aligned}$$

For a general function $f$, writing it as a linear combination of its positive and negative parts and using linearity it follows that for all $n\in N^p$, $\widetilde{f}(\lambda,n)$ exists for all $\lambda\in S_p$. This completes the proof of (i).

To prove (ii) we note that by Lemma 3.2, $\phi_{i\Im\lambda}\in L^{p',q'}(S)$ for $\lambda\in S_p^\circ$. Therefore as in (i), for almost every $n\in N_\lambda\subset N$, a set of full Haar measure, the Fourier transform $\widetilde{f}(\lambda,n)$ exists. Let $\{\lambda_m\}$ be an increasing sequence of positive numbers converging to $\gamma_p\rho$. Then $i\lambda_m \to i\gamma_p\rho$. Since $\lambda \in S_p^o$, there exists $m \in \mathbb{N}$ such that $|\Im\lambda| \leq \lambda_m$.

Now for $n \in \widetilde{N_m} = N_{i\lambda_m} \cap N_{-i\lambda_m}$,

$$|\widetilde{f}(\lambda, n)| \leq \int_{S_+} |f(x)|\mathcal{P}_{i\lambda_m}(x,n)dx + \int_{S_-} |f(x)|\mathcal{P}_{-i\lambda_m}(x,n)dx < \infty,$$

where $S_+, S_-$ are as defined in (i). So, the set $N^p = \bigcap_{m\in\mathbb{N}} \widetilde{N_m}$ is a set of full measure and for all $n \in N^p$, $\widetilde{f}(\lambda, n)$ exists for all $\lambda \in S_p^o$.

The proof of part (iii) is postponed until Section 4 (see Observation 4.12 D). Part (iii) will not be used until then. □

*Remark* 3.5. The following remarks are in order.

(i) In Section 5 (See Theorem 5.4) we will show that for each fixed $n \in N^p$, $\lambda \mapsto \widetilde{f}(\lambda, n)$ is continuous in its domain of definition and analytic in the interior. We will also prove an analogue of the Riemann-Lebesgue lemma in the appropriate domain.

(ii) We recall that if $f \in L^{p,q}(S), 1 < p < 2, 1 \leq q \leq \infty$, then there exists $f_1 \in L^1(S)$ and $f_2 \in L^r(S)$, where $r \in (p, 2]$ is such that $f = f_1 + f_2$. (Indeed we can take $f_1 = f\chi_{|f|\geq 1}$ and $f_2 = f\chi_{|f|<1}$.) Therefore one can first prove the existence of the Fourier transform for $L^p$-functions ($p \in [1, 2)$) and then use the decomposition above to prove it for $L^{p,q}$-functions. Even then it needs an argument (similar to what we used) to show that the Fourier transform of an $L^{p,q}$-function can be extended to $S_p^\circ$. It is also not clear whether the existence of the Fourier transform of an $L^{p,1}$-function on the boundary of $S_p$ can be obtained in this manner.

(iii) In Section 2 we have mentioned that $(L^{1,q}(S))^* = \{0\}$ for $1 < q < \infty$. Therefore $\phi_\lambda \notin (L^{1,q}(S))^*$ for $1 < q < \infty$. Hence we cannot apply the argument used in Theorem 3.4 for the case $p = 1, q > 1$. We may also point out that through a straightforward calculation it can be verified that the radial function $f(r) = r^{-(m+k+1)}\chi_{[0,1]}$ is in weak $L^1(S)$, but the integral $\int_S f(r)\phi_0(r)J(r)dr$ does not converge, where $J(r)$ is the Jacobian of the polar decomposition. This shows that while for $p > 1$ the pointwise existence of the Fourier transform is guaranteed for weak $L^p$-functions, the situation is different for weak $L^1$-functions.

The standard method of approximation using the heat kernel gives the following inversion formula for the Lorentz space functions. We recall that the heat kernel $h_t$ is a radial function defined through its spherical Fourier transform as $\widehat{h_t}(\lambda) = e^{-t(\lambda^2+Q^2/4)}$ ([1, (5.4)]).

**Proposition 3.6.** *Let $f \in L^{p,q}(S)$, $p \in (1, 2)$, $q \geq 1$ or $f \in L^1(S) \cup L^2(S)$. If $\widetilde{f} \in L^1(N \times \mathbb{R}, |c(\lambda)|^{-2}\, d\lambda\, dn)$, then for almost every $x \in S$,*

$$f(x) = C \int_{N\times\mathbb{R}} \widetilde{f}(\lambda, n)\mathcal{P}_{-\lambda}(x, n)|c(\lambda)|^{-2}d\lambda\, dn.$$

*Proof.* Let $m$ be a fixed left-invariant Haar measure on $S$. We suppose that $f = f_1 + f_2$, where $f \in L^1(S)$ and $f_2 \in L^2(S)$. Then $f * h_t = f_1 * h_t + f_2 * h_t$ for $t > 0$.

Since $g * h_t(x) \longrightarrow g(x)$ for $g \in C_c^\infty(S)$ and the heat maximal operator is weak $(1,1)$ (see [1, Theorem 5.50]) it follows that there exist measurable sets $E_1, E_2 \subset S$ with $m(E_1^c) = 0 = m(E_2^c)$ such that as $t \to 0$, $f_1 * h_t(x) \to f_1(x)$ for all $x \in E_1$ and $f_2 * h_t(x) \to f_2(x)$ for all $x \in E_2$. This implies that for all $x \in E = E_1 \cap E_2$, $f * h_t(x) \to f(x)$ as $t \to 0$ where $m(E^c) = 0$.

As $\widetilde{f} \in L^1(N \times \mathbb{R})$ we can show that for all $\phi \in C_c^\infty(S)$, $\langle f * h_t, \phi \rangle = \langle F, \phi \rangle$, where $F(x) = \int_{N\times\mathbb{R}} \widetilde{f}(\lambda, n) e^{-t(\lambda^2+Q^2/4)} \mathcal{P}_{-\lambda}(x,n)|c(\lambda)|^{-2} d\lambda\, dn$. That is,

$$f * h_t(x) = C \int_{\mathbb{R}\times N} \widetilde{f}(\lambda, n) e^{-t(\lambda^2+Q^2/4)} \mathcal{P}_{-\lambda}(x,n)|c(\lambda)|^{-2} d\lambda\, dn.$$

Using the dominated convergence theorem we now get the result. □

## 4. Properties of the Fourier transform

Existence of $\widetilde{f}(\lambda, n)$ (with $\lambda$ in the appropriate domain of definition) as a measurable function on $N$ naturally leads to the following question:

If $f \in L^p(S), 1 \le p < 2$ and $\lambda \in S_p^\circ$, then what can be said about the continuity of the operator $f \mapsto \widetilde{f}(\lambda, \cdot)$? As a step towards this we will first investigate the continuity of the spectral projection operator $f \mapsto f * \phi_\lambda$. For the Riemannian symmetric spaces and $\lambda \in \mathbb{R}$ this was essentially considered in [29].

**Theorem 4.1.** *Let $f \in C_c^\infty(S)$ and $q \in [1,2)$. Then $\|f * \phi_\lambda\|_{q'} \le C_{q,s}\|f\|_q$ for $\lambda$ in the strip $\{\lambda \in \mathbb{C} \mid |\Im\lambda| \le (\frac{2}{q} - 1)\rho s\}$ for some $s \in (0,1)$. When $q = 1$, then $C_{q,s} = 1$ and $s$ can also be 1.*

*Proof.* We shall divide the proof into three steps and in each step we will use an analytic interpolation.

**Step 1.** In this step we shall show that for $\lambda \in \mathbb{R}$ and $p \in [1,2)$,

$$\|f * \phi_\lambda\|_{p'} \le C_p \|f\|_p, \tag{4.1}$$

where $C_p > 0$ depends on $p$ and in particular $C_1 = 1$.

The case $p = 1$ of (4.1), i.e. $\|f * \phi_\lambda\|_\infty \le \|f\|_1$, is immediate since $\|\phi_\lambda\|_\infty \le 1$ for $\lambda \in \mathbb{R}$.

Now we take $p \in (1,2)$. We define an analytic family of linear operators $T_z$ from $(S, dx)$ to itself, where

$$T_z f = f * \phi_0^{1+z} \text{ for } -\frac{1}{2} \le \Re z.$$

For $z = -\frac{1}{2} + iy$,

$$\|T_z f\|_\infty \le \|f\|_1 \text{ as } \|\phi_0\|_\infty \le 1. \tag{4.2}$$

We will see below that for $z = \varepsilon + iy$ with any $\varepsilon > 0$,

$$\|T_z f\|_2 \le C_\varepsilon \|f\|_2. \tag{4.3}$$

We note that $(f * \phi_0^{1+\varepsilon})\widetilde{\ }(\lambda, n) = \widetilde{f}(\lambda, n)(\phi_0^{1+\varepsilon})\widehat{\ }(\lambda)$ for $\lambda \in \mathbb{R}, n \in N$. For $\lambda \in \mathbb{R}$ and $\varepsilon > 0$ we have by (2.8),

$$\left|(\phi_0^{1+\varepsilon})\widehat{\ }(\lambda)\right| = \left|\int_S \phi_0^{1+\varepsilon}(x)\phi_\lambda(x) dx\right| \le \int_S \phi_0(x)^{2+\varepsilon} dx < \infty.$$

Thus $(\phi_0^{1+\varepsilon})\widehat{}$ is bounded by a constant $C_\varepsilon$, say, which depends only on $\varepsilon$. Therefore using the Plancherel formula we have for $z=\varepsilon+iy$,

$$\|T_z f\|_2 \le \| \, |f| * \phi_0^{1+\varepsilon}\|_2 = \|(|f| * \phi_0^{1+\varepsilon})\widetilde{}\|_{L^2(N\times\mathfrak{a}^*)} \le C_\varepsilon \|\widetilde{f}\|_{L^2(N\times\mathfrak{a}^*)} = C_\varepsilon \|f\|_2.$$

Thus (4.3) is established.

We choose an $a$ such that $1-a+a/2=1/p$. Then for $\varepsilon=(2-p)/(4(p-1))>0$ we get $(a-1)/2+\varepsilon a=0$. Therefore by analytic interpolation of (4.2) and (4.3) we have (4.1).

**Step 2.** In this step we shall show that for $p\in[1,2)$ and $\lambda=i\gamma_p\rho$,

$$\|f*\phi_\lambda(x)^{1+\varepsilon}\|_{p'} \le C_{p,\varepsilon}\|f\|_p \quad \text{for any } \varepsilon>0. \tag{4.4}$$

For $p=1$ this is true as $\phi_{i\rho}$ is bounded. So we take $p\in(1,2)$.

We consider the analytic family of linear operators

$$S_z f = f*\psi_z^\varepsilon \quad \text{for } |\Im z|\le\rho,$$

where $\psi_z^\varepsilon(x)=(e^{-izr(x)}e^{-\rho r(x)})^{1+\varepsilon}$. Note that $\phi_{it\rho}\asymp\psi_{it\rho}^0$ (see (2.8)).

Now $S_{l+i\rho}f=f*\psi_{l+i\rho}^\varepsilon$ is clearly $(1,\infty)$ and from (4.3) it follows that at $z=l+i0$, $S_z$ is $(2,2)$. Therefore by analytic interpolation, $S_z$ is $(p,p')$ at $z=i\gamma_p\rho$. That is,

$$\|f*\psi_{i\gamma_p\rho}^\varepsilon\|_{p'} \le C_p\|f\|_p.$$

Since $\psi_{i\gamma_p\rho}^\varepsilon\asymp\phi_{i\gamma_p\rho}^{1+\varepsilon}$, we have proved (4.4).

**Step 3.** For $q,s$ as in the hypothesis we shall now show that

$$\|f*\phi_{\pm i(\frac{2}{q}-1)s\rho}\|_{q'} \le C_{q,s}\|f\|_q. \tag{4.5}$$

We choose a $p$ such that $q<p<2$ and $(\frac{2}{q}-1)s=\gamma_p$. Note that $p$ depends on the choice of $s$. Let $t=\gamma_p\rho$. We consider the analytic family of operators

$$R_z f = f*(\phi_{it})^{1+z} \text{ for } -\frac{1}{2}\le\Re z.$$

It can be verified that $R_{-\frac{1}{2}+iy}$ is $(1,\infty)$ with constant 1. Now by (4.4), $R_{\varepsilon+iy}$ is of type $(p,p')$ with constant $C_{p,\varepsilon}$ for any $\varepsilon>0$.

We find the convex combination of 1 and $p$ which gives $q$; precisely, $\frac{1-a}{1}+\frac{a}{p}=\frac{1}{q}$, i.e. $a=p'/q'<1$. We take $\varepsilon=\frac{1}{2}(\frac{q'}{p'}-1)>0$ so that $-\frac{1}{2}(1-a)+\varepsilon a=0$.

By analytic interpolation we have for $t=(\frac{2}{q}-1)s$, $s\in(0,1)$,

$$\|f*\phi_{it}\|_{q'} \le C_{q,s}\|f\|_q.$$

As $\phi_\lambda=\phi_{-\lambda}$ we have (4.5).

Finally the theorem follows from (4.5) and (3.1). □

With this preparation we now offer the following analogue of the restriction theorem. We recall that $\gamma_p=\frac{2}{p}-1$.

**Theorem 4.2.** *Let $f\in L^p(S), 1\le p<2$. Then for $p<q<p'$ and for $\lambda\in\mathbb{C}$ such that $\Im\lambda=\gamma_q\rho$,*

$$\left(\int_N |\widetilde{f}(\lambda,n)|^q dn\right)^{1/q} \le C_{p,q}\|f\|_p.$$

*Moreover, when $p=1$, then $q\in[1,\infty]$ and $C_{p,q}=C_1=1$.*

*Proof.* First we shall prove the theorem for $q = 2$. In this case $\Im\lambda = \gamma_q\rho = 0$. For $\lambda \in \mathbb{R}$ we have

$$\begin{aligned}
\|\widetilde{f}(\lambda,\cdot)\|^2_{L^2(N)} &= \left|\int_N \widetilde{f}(\lambda,n)\overline{\widetilde{f}(\lambda,n)}dn\right| \\
&= \left|\int_N \left(\int_S f(x)P_\lambda(x,n)dx\right)\overline{\widetilde{f}(\lambda,n)}dn\right| \\
&= \left|\int_S f(x)\left(\int_N \overline{\widetilde{f}(\lambda,n)}P_\lambda(x,n)dn\right)dx\right| \\
&= \left|\int_S f(x)\left(\int_N \overline{\widetilde{f}(\lambda,n)\mathcal{P}_{-\lambda}(x,n)}dn\right)dx\right| \\
&= \left|\int_S f(x)\overline{f*\phi_\lambda(x)}dx\right| \quad \text{(see [4, Lemma 4.3])} \\
&\leq \|f\|_{p_0}\|f*\phi_\lambda\|_{p_0'},\ 1 \leq p_0 < 2 \\
&\leq C_{p_0}\|f\|^2_{p_0}, \qquad (4.6)
\end{aligned}$$

where $C_1 = 1$ by (4.1).

Next we shall consider $q$ in the range $p < q < 2$. For $\lambda = r + i\rho, r \in \mathbb{R}$, we get using Fubini's theorem and the fact that $\int_N |\mathcal{P}_\lambda(x,n)|dn = 1$,

$$\int_N |\widetilde{f}(\lambda,n)|dn = \int_N \left|\int_S f(x)\mathcal{P}_\lambda(x,n)dx\right| dn \leq \|f\|_1. \qquad (4.7)$$

We consider the measure spaces $(S, dx)$ and $(N, dn)$. For $\lambda \in \mathcal{S}_1$ and $f \in C_c^\infty(S)$ we define an analytic family of linear operators:

$$T_\lambda f(n) = \widetilde{f}(\lambda,n) \text{ for } f \in C_c^\infty(S).$$

From (4.6) and (4.7) we have

$$\|T_\lambda f\|_{L^2(N)} \leq C_{p_0}\|f\|_{p_0} \text{ for } \lambda \in \mathbb{R} \quad \text{and} \qquad (4.8)$$

$$\|T_\lambda f\|_{L^1(N)} \leq \|f\|_1 \text{ for } \lambda = r + i\rho, r \in \mathbb{R}. \qquad (4.9)$$

By analytic interpolation of (4.8) and (4.9) we get

$$\|\widetilde{f}(\lambda,\cdot)\|_{L^q(N)} \leq C_{p,q}\|f\|_p, \text{ for } p > 1 \text{ and } \|\widetilde{f}(\lambda,\cdot)\|_{L^q(N)} \leq \|f\|_1,$$

where $\Im\lambda = i\gamma_q\rho$ and $\frac{1}{q} = (\frac{1}{p} - \frac{1}{p_0})\frac{p_0'}{2} + \frac{1}{2}$.

Using the expression above of $\frac{1}{q}$ and the fact that $p_0 < 2$, it follows that $p < q$. By varying $p_0$ we see that the range of $q$ is $(p, 2]$.

To prove the theorem for $2 < q < p'$ we note that for $\Im\lambda = -\rho$, we have

$$|\widetilde{f}(\lambda,n)| = \left|\int_S f(x)\mathcal{P}(x,n)^{\frac{1}{2}-\frac{i}{Q}(\Re\lambda - \frac{iQ}{2})}dx\right| \leq \int_S |f(x)|dx = \|f\|_1.$$

Therefore in terms of the analytic family of operators $T_\lambda$ described above we have

$$\|T_\lambda f\|_{L^\infty(N)} \leq \|f\|_1 \text{ for } \lambda = r - i\rho, r \in R. \qquad (4.10)$$

Interpolating between the estimates (4.8) and (4.10) we have

$$\|\widetilde{f}(\lambda,\cdot)\|_{L^q(N)} \leq C_{p,q}\|f\|_p \text{ for } p > 1 \text{ and } \|\widetilde{f}(\lambda,\cdot)\|_{L^q(N)} \leq \|f\|_1,$$

where $\Im\lambda = \gamma_q\rho$ and $p$, $q$ are given by $\frac{2}{q} = \frac{p_0'}{p'}$ (and hence $q < p'$ and $q > 2$).

Again varying $p_0$ we see that the range of $q$ is $[2, p')$. □

The following mapping property of the Poisson transform follows from the theorem above and a standard duality argument. The Poisson transform of a function $F$ on $N$ is defined as (see [4])

$$\mathfrak{P}_\lambda F(x) = \int_N F(n)\mathcal{P}_\lambda(x,n)dn.$$

**Corollary 4.3.** *For $1 \leq p < 2$, $p < q < p'$ and $\lambda \in \mathbb{C}$ with $\Im\lambda = \gamma_q \rho$ we have for $F$ as above,*

$$\|\mathfrak{P}_\lambda F\|_{L^{p'}(S)} \leq C_{p,q}\|F\|_{L^{q'}(N)}.$$

*Proof.* Let $\lambda$ be as in the hypothesis. For suitable functions $f$ on $S$ and $F$ on $N$, we have by Fubini's theorem,

$$\begin{aligned}\int_N \widetilde{f}(\lambda,n)F(n)dn &= \int_N\int_S f(x)\mathcal{P}_\lambda(x,n)dxF(n)dn \\ &= \int_S f(x)\int_N \mathcal{P}_\lambda(x,n)F(n)dn\,dx \\ &= \int_S f(x)\mathfrak{P}_\lambda F(x)dx.\end{aligned}$$

On the other hand, Hölder's inequality and Theorem 4.2 give

$$\left|\int_N \widetilde{f}(\lambda,n)F(n)dn\right| \leq \|\widetilde{f}(\lambda,\cdot)\|_{L^q(N)}\|F\|_{L^{q'}(N)} \leq C_{p,q}\|f\|_p\|F\|_{L^{q'}(N)}.$$

Thus

$$\left|\int_S f(x)\,\mathfrak{P}_\lambda F(x)dx\right| \leq C_{p,q}\|f\|_p\|F\|_{L^{q'}(N)}.$$

The corollary now follows by duality. □

See [21, p. 207] for some related results on the Poisson transform on symmetric spaces.

We notice that in Theorem 4.2 the norm estimate of $\widetilde{f}(\lambda,\cdot)$ depends on $\Im\lambda$. The following corollary gives instead a norm estimate which is uniform over a strip $S_q, p < q \leq 2$.

Let $L^q(N,P_1) = \{f \text{ measurable on } N \mid \int_N |f(n)|^q P_1(n)dn < \infty\}$.

**Corollary 4.4.** *Let $1 \leq p < q \leq 2$ and $1 \leq r \leq q$. If $f \in L^p(S)$, then*

$$\|\widetilde{f}(\lambda,\cdot)\|_{L^r(N,P_1)} \leq C_{p,q}\|f\|_p$$

*for any $\lambda$ in the strip $\mathcal{S}_q = \{z \in \mathbb{C} \mid |\Im z| \leq \gamma_q\rho\}$.*

*Proof.* Since $p < q \leq 2$, $q' \in [2,p')$ and $\gamma_{q'} = -\gamma_q$, we use Theorem 4.2 for $q'$ to get

$$\|\widetilde{f}(-i\gamma_q\rho,\cdot)\|_{L^{q'}(N,P_1)} \leq C_{p,q}\|f\|_p.$$

Now as $P_1(n) \leq 1$ for all $n \in N$, $\int_N P_1(n)dn = 1$ and as $q \leq q'$ we have from above,

$$\|\widetilde{f}(-i\gamma_q\rho,\cdot)\|_{L^q(N,P_1)} \leq C_{p,q}\|f\|_p.$$

On the other hand, a direct application of Theorem 4.2 for $q$ (such that $p < q \leq 2$) gives

$$\|\widetilde{f}(i\gamma_q\rho,\cdot)\|_{L^q(N,P_1)} \leq C_{p,q}\|f\|_p.$$

Bringing together the two inequalities above we get for $1 \leq p < q \leq 2$,

$$\|\widetilde{f}(\pm i\gamma_q\rho, \cdot)\|_{L^q(N,P_1)} \leq C_{p,q}\|f\|_p.$$

As the weight $P_1(n)$ makes $N$ a finite measure space, it follows that for $1 \leq r \leq q$,

$$\|\widetilde{f}(\pm i\gamma_q\rho, \cdot)\|_{L^r(N,P_1)} \leq C_{p,q}\|f\|_p. \tag{4.11}$$

Now the result follows by an analytic interpolation of the linear operator $T_z$ between the measure spaces $(S, dx)$ and $(N, P_1)$, where $T_z$ is given by

$$T_z f(n) = \widetilde{f}(z, n), \text{ for } f \in C_c^\infty(X).$$

By (4.11), $T_z$ is $(p, r)$ on the line $\mathbb{R} \pm i\gamma_q\rho$. Hence by interpolation, $T_z$ is $(p, r)$ on the part of the imaginary axis from $-i\gamma_q\rho$ to $i\gamma_q\rho$. This proves the result. □

We notice that the corollary above remains true if we replace $P_1$ by $P_1^\alpha$, where $\alpha > 1/2$.

As another application of Theorem 4.2 we shall prove an instance of the *Kunze-Stein phenomenon* on $S$ for right convolution with radial functions.

**Corollary 4.5.** *Let $1 \leq q < p \leq 2$. Then there exists a positive constant $C$ such that for all $F \in C_c^\infty(S)$ and radial $f \in C_c^\infty(S)$,*

$$\text{(i)} \quad \|F * f\|_p \leq C_{p,q}\|F\|_p\|f\|_q,$$
$$\text{(ii)} \quad \|F * f\|_p \leq C_{p,q}\|F\|_q\|f\|_p.$$

*Proof.* (i) is proved in [1, p. 657]. We shall take up (ii). Since $\|f * g\|_1 \leq \|f\|_1\|g\|_1$ for $f, g \in L^1(S)$ and we can use the bilinear interpolation (see [19, p. 273]), it is enough to prove the result only for $p = 2$. That is, we need to show that for $F \in C_c^\infty(S)$ and a radial function $f \in C_c^\infty(S)$,

$$\|F * f\|_2 \leq C_q\|F\|_q\|f\|_2.$$

Since $(F * f)\tilde{}(\lambda, n) = \widetilde{F}(\lambda, n)\widehat{f}(\lambda)$ (see [4, p. 412]) we have

$$\begin{aligned}
\|F * f\|_2^2 &= \int_{\mathfrak{a}^*}\int_N |\widetilde{F}(\lambda, n)|^2|\widehat{f}(\lambda)|^2|c(\lambda)|^{-2}d\lambda dn \\
&= \int_{\mathfrak{a}^*} |\widehat{f}(\lambda)|^2\|\widetilde{F}(\lambda, n)\|_{L^2(N)}^2|c(\lambda)|^{-2}d\lambda \\
&\leq C_q\|F\|_q^2\|\widehat{f}\|_2^2 \text{ (using (4.6))} \\
&= C_q\|F\|_q^2\|f\|_2^2.
\end{aligned}$$

□

We will now present some results which compare the size of the function with that of its Fourier transform. More precisely we will prove some theorems of the genre of Hausdorff-Young and Hardy-Littlewood-Paley inequalities. On symmetric spaces some versions of these theorems are proved in [16, 17, 18, 22]. However the results in this section are very different in nature. The precise comparison is as follows: the version of the Hausdorff-Young inequality in [22] does not include the Plancherel theorem. The Hausdorff-Young inequality in [17] deals only with radial functions while the version in [16] is for $K$-finite functions and excludes both the $p = 1$ and $p = 2$ cases. We offer the following analogue:

**Theorem 4.6.** *For $1 \leq p \leq 2$ and $p \leq q \leq p'$,*

$$\left(\int_{\mathbb{R}} \left(\int_N |\widetilde{f}(\lambda + i\gamma_q \rho, n)|^q dn\right)^{\frac{p'}{q}} |c(\lambda)|^{-2} d\lambda\right)^{\frac{1}{p'}} \leq C_{p,q}\|f\|_p.$$

The case $p = q = 2$ is a weakening of the Plancherel theorem. We also note that when $q = p'$ the result best resembles the classical Hausdorff-Young inequality at the lower boundary of the strip $S_p$ (see [27, p. 147] for the statement in $\mathbb{R}^n$).

The following result is immediate from Theorem 4.2 and Theorem 4.6:

**Corollary 4.7.** *For $1 \leq p < 2$ and $p < q < p'$,*

$$\left(\int_{\mathbb{R}} \left(\int_N |\widetilde{f}(\lambda + i\gamma_q \rho, \cdot)|^q dn\right)^{\frac{r}{q}} |c(\lambda)|^{-2} d\lambda\right)^{\frac{1}{r}} \leq C_{p,q}\|f\|_p,$$

*for all $r \in [p', \infty]$.*

We need the following preparation for proving Theorem 4.6. Let $(X_i, \mu_i), i = 1, 2$, be two $\sigma$-finite measure spaces and $(X, \mu)$ their product space. We take an ordered pair $P = (p_1, p_2) \in [1, \infty] \times [1, \infty]$ and a measurable function $f(x_1, x_2)$ on $(X, \mu)$. We define the mixed norm $(p_1, p_2)$ of $f$ as

$$\|f\|_P = \|f\|_{(p_1,p_2)} = \left(\int_{X_1} \left(\int |f(x_1, x_2)|^{p_1} dx_2\right)^{p_2/p_1} dx_1\right)^{1/p_2}.$$

For details about the mixed normed spaces we refer to [6]. For two such ordered pairs $P = (p_1, p_2)$ and $Q = (q_1, q_2)$, we write $1/R = (1-t)/P + t/Q, 0 < t < 1$ to mean $1/r_1 = (1-t)/p_1 + t/q_1$ and $1/r_2 = (1-t)/p_2 + t/q_2$, where $R = (r_1, r_2)$. We have the following analytic interpolation for the mixed normed spaces (see [6]).

We consider two product measure spaces $M = X_1 \times X_2$ and $N = Y_1 \times Y_2$. Let $dx$ and $dy$ denote respectively the (product) measures on $M$ and $N$. Let $\{T_z \mid z \in \mathbb{C}\}$ be a family of linear operators between $M$ and $N$ of admissible growth. We suppose that for all finite linear combinations of characteristic functions of rectangles of finite measures $f$ on $M$:

$$\|T_{iy}(f)\|_{Q_1} \leq A_1(y)\|f\|_{P_1} \text{ and } \|T_{1+iy}(f)\|_{Q_2} \leq A_2(y)\|f\|_{P_2}$$

for $P_1, P_2, Q_1, Q_2 \in [1, \infty] \times [1, \infty]$ such that $\log |A_i(y)| \leq Ae^{a|y|}, a < \pi, i = 1, 2$. Let $1/R = (1-t)/P_1 + t/P_2$ and $1/S = (1-t)/Q_1 + t/Q_2, 0 < t < 1$. Then

$$\|T_t(f)\|_S \leq A_t\|f\|_R.$$

We also need the following estimate of the Plancherel density $|c(\lambda)|^{-2}$, which we obtain from its explicit expression given in (2.10).

**Lemma 4.8.** $|c(\lambda)|^{-2} \asymp \lambda^2(1 + |\lambda|)^{m+k-2}$ *for $\lambda \in \mathbb{R}$.*

We include here a brief sketch of the proof.

*Proof.* For a set $A$ let $|A|$ be its cardinality. It is easy to see that:
If $n \in 2\mathbb{Z}$, $n > 0$, then

$$|\{j \mid 0 < j < n, j \in 2\mathbb{Z}+1\}| = \frac{n}{2} \text{ and } |\{j \mid 0 < j < n, j \in 2\mathbb{Z}\}| = \frac{n}{2} - 1.$$

If $n \in 2\mathbb{Z}+1, n > 0$, then

$$|\{j \mid 0 < j < n, j \in 2\mathbb{Z}+1\}| = \frac{n-1}{2} \text{ and } |\{j \mid 0 < j < n, j \in 2\mathbb{Z}\}| = \frac{n-1}{2}.$$

We also have

$$\lambda \tanh \pi\lambda \asymp \frac{\lambda^2}{1+\lambda}, \lambda \geq 0.$$

Using these, the following estimates can be obtained:
When $k = 1, \frac{m}{2} = 2l+1 \in 2\mathbb{Z}+1, l \geq 0:$

$$\prod_{0<j \text{ integer } <\frac{m}{4}} \{\lambda^2 + j^2\}^2 \lambda^3 \coth \pi\lambda \asymp (1+|\lambda|)^{4l} \lambda^2 (1+|\lambda|) = \lambda^2 (1+|\lambda|)^{m+k-2}.$$

When $k \in 2\mathbb{Z}+1$, $\frac{m}{2} \in 2\mathbb{Z}$:

$$\prod_{0<j \text{ odd } <\frac{m}{2}} \{\lambda^2 + (\frac{j}{2})^2\}^2 \prod_{\frac{m}{2}<j \text{ odd } <\frac{m}{2}+k} \{\lambda^2 + (\frac{j}{2})^2\} |\lambda| \tanh \pi|\lambda|$$

$$\asymp \quad (1+|\lambda|)^{4.\frac{m}{4}} (1+|\lambda|)^{2.\frac{k-1}{2}} \frac{\lambda^2}{1+|\lambda|} = \lambda^2 (1+|\lambda|)^{m+k-2}.$$

When $k \in 2\mathbb{Z}, \frac{m}{2} \in 2\mathbb{Z}$:

$$\prod_{0\leq j \text{ integer } \leq\frac{m}{2}} \{\lambda^2 + (\frac{j}{2})^2\} \prod_{\frac{m}{2}<j \text{ even } <\frac{m}{2}+k} \{\lambda^2 + (\frac{j}{2})^2\}$$

$$\asymp \quad \lambda^2 (1+|\lambda|)^{2.\frac{m}{2}} (1+|\lambda|)^{2.(\frac{k}{2}-1)} = \lambda^2 (1+|\lambda|)^{m+k-2}. \qquad \square$$

*Proof of Theorem* 4.6. Let us fix a $p \in [1,2]$. We shall first handle the cases $q = p$ and $q = p'$ of the theorem. We consider the measure spaces $(S, dx)$ and $Y = (\mathbb{R} \times N, (1+|\lambda|)^{\beta} d\lambda dn)$, where $\beta = m+k$. We define an analytic family of linear operators for $f \in C_c^\infty(S)$ by

$$T_z f(\lambda, n) = \widetilde{f}(\lambda+z, n)(1+|\lambda|)^{-1}(\lambda+z), \text{ where } |\Im z| \leq \rho.$$

We will prove the following mixed norm estimates:

(a) $\|T_z f\|_{(2,2)} \leq C_z \|f\|_2$ for $z \in \mathbb{R}$.
(b) $\|T_z f\|_{(1,\infty)} \leq C_z \|f\|_1$ for $z \in \mathbb{C}$ such that $\Im z = \rho$.
(c) $\|T_z f\|_{(\infty,\infty)} \leq C_z \|f\|_1$ for $z \in \mathbb{C}$ such that $\Im z = -\rho$.

Then we will use analytic interpolation of mixed norm spaces.

For $z=\xi\in\mathbb{R}$, by Lemma 4.8 and the Plancherel theorem we have

$$\begin{aligned}\|T_\xi f\|^2_{L^2(Y)} &= \int_{\mathbb{R}}\int_N |\widetilde{f}(\lambda+\xi,n)|^2|\lambda+\xi|^2(1+|\lambda|)^{\beta-2}d\lambda\,dn\\ &= \int_{\mathbb{R}}\int_N |\widetilde{f}(\lambda,n)|^2|\lambda|^2(1+|\lambda-\xi|)^{\beta-2}d\lambda\,dn\\ &\le \int_{\mathbb{R}}\int_N |\widetilde{f}(\lambda,n)|^2|\lambda|^2(1+|\lambda|)^{\beta-2}(2+|\xi|)^{\beta-2}d\lambda\,dn\\ &\le C(2+|\xi|)^{\beta-2}\int_{\mathbb{R}}\int_N |\widetilde{f}(\lambda,n)|^2|c(\lambda)|^{-2}d\lambda\,dn\\ &= C(2+|\xi|)^{\beta-2}\|f\|^2_{L^2(S)}. \qquad (4.12)\end{aligned}$$

In one of the steps above we have used the fact that $\beta>2$. This proves (a).

If $z=\xi+i\rho,\xi\in\mathbb{R}$, then using the fact that $\mathcal{P}_{-i\rho}(x,n)\equiv 1$ we have

$$\begin{aligned}\int_N |T_{\xi+i\rho}f(\lambda,n)|dn &\le \int_N |\widetilde{f}(\xi+i\rho+\lambda,n)||\lambda+\xi+i\rho|(1+|\lambda|)^{-1}dn\\ &\le |\lambda+\xi+i\rho|(1+|\lambda|)^{-1}\left(\int_S |f(x)|\left(\int_N \mathcal{P}_{i\rho}(x,n)dn\right)dx\right)\\ &\le (1+|\xi|+\rho)\int_S |f(x)|\phi_{i\rho}(x)dx\\ &\le (1+|\xi|+\rho)\|f\|_1.\end{aligned}$$

This implies that

$$\sup_{\lambda\in\mathbb{R}}\|T_{\xi+i\rho}f(\lambda,\cdot)\|_{L^1(N)}\le(1+|\xi|+\rho)\|f\|_1. \qquad (4.13)$$

This proves (b).

For $z=\xi-i\rho,\xi\in\mathbb{R}$,

$$\begin{aligned}|T_{\xi-i\rho}f(\lambda,n)| &= |\widetilde{f}(\lambda+\xi-i\rho,n)||\lambda+\xi-i\rho|(1+|\lambda|)^{-1}\\ &\le |\lambda+\xi-i\rho|(1+|\lambda|)^{-1}\int_S |f(x)|\mathcal{P}_{-i\rho}(x,n)dx\\ &\le (1+|\xi|+\rho)\|f\|_1. \qquad (4.14)\end{aligned}$$

This implies that

$$\sup_{\lambda\in\mathbb{R},n\in N}|T_{\xi-i\rho}f(\lambda,n)|\le(1+|\xi|+\rho)\|f\|_1. \qquad (4.15)$$

This proves (c). From (4.12) and (4.13) it follows by analytic interpolation for mixed normed spaces that

$$\|T_{i\gamma_p\rho}f\|_{(p,p')}\le C\|f\|_p,\quad 1\le p\le 2.$$

That is,

$$\left(\int_{\mathbb{R}}\left(\int_N|\widetilde{f}(\lambda+i\gamma_p\rho,n)|^p dn\right)^{\frac{p'}{p}}|\lambda+i\gamma_p\rho|^{p'}(1+|\lambda|)^{\beta-p'}d\lambda\right)^{\frac{1}{p'}}\le C\|f\|_p.$$

From this it follows easily by Lemma 4.8 that for $1 \le p \le 2$,

$$\left(\int_{\mathbb{R}} \left(\int_N |\widetilde{f}(\lambda + i\gamma_p\rho, n)|^p dn\right)^{\frac{p'}{p}} |c(\lambda)|^{-2} d\lambda\right)^{\frac{1}{p'}} \le C_p \|f\|_p. \tag{4.16}$$

Similarly from (4.12), (4.15) and Lemma 4.8, using again the interpolation of mixed normed spaces, we get for $1 \le p \le 2$,

$$\left(\int_{\mathbb{R}} \int_N |\widetilde{f}(\lambda - i\gamma_p\rho, n)|^{p'} |c(\lambda)|^{-2} d\lambda dn\right)^{\frac{1}{p'}} \le C_p \|f\|_p. \tag{4.17}$$

This completes the proof for the cases $q = p$ and $q = p'$.

For the cases $p < q < p'$ we need to interpolate again. We take the measure spaces $(S, dx)$ and $X = (\mathbb{R} \times N, |c(\lambda)|^{-2} d\lambda dn)$. We fix a $q \in (p, 2]$ and define two linear operators $R_1$ and $R_2$ from $S$ to $X$ by

$$R_1 f(\lambda, n) = \widetilde{f}(\lambda + i\gamma_q\rho, n) \text{ and } R_2 f(\lambda, n) = \widetilde{f}(\lambda - i\gamma_q\rho, n).$$

As (4.16) and (4.17) are true for all $p \in [1, 2]$ we have

$$\|R_1 f\|_{(q,q')} \le C_q \|f\|_q, \tag{4.18}$$

$$\|R_2 f\|_{(q',q')} \le C_q \|f\|_q. \tag{4.19}$$

As for $\lambda \in \mathbb{R}$ and $x = n_1 a, n_1 \in N, a \in A$, $|\mathcal{P}_{\lambda + i\gamma_q\rho}(x, n)|^q = P_a(n_1^{-1} n)$, we also have for $\lambda \in \mathbb{R}$,

$$\begin{aligned} \left(\int_N |\widetilde{f}(\lambda + i\gamma_q\rho, n)|^q dn\right)^{\frac{1}{q}} &\le \left(\int_N \left(\int_S |f(x)||\mathcal{P}_{\lambda + i\gamma_q\rho}(x, n)| dx\right)^q dn\right)^{\frac{1}{q}} \\ &\le \int_S |f(x)| \left(\int_N |\mathcal{P}_{\lambda + i\gamma_q\rho}(x, n)|^q dn\right)^{\frac{1}{q}} dx \\ &\le \|f\|_1. \end{aligned}$$

That is,

$$\|R_1 f\|_{(q,\infty)} \le \|f\|_1. \tag{4.20}$$

Also as for $\lambda \in \mathbb{R}$ and $x = n_1 a$, $n \in N, a \in A$, $|\mathcal{P}_{\lambda - i\gamma_q\rho}(x, n)|^{q'} = P_a(n_1^{-1} n)$, through similar steps we get

$$\|R_2 f\|_{(q',\infty)} \le \|f\|_1. \tag{4.21}$$

From (4.18) and (4.20) we have by interpolation of mixed normed spaces

$$\|R_1 f\|_{(q,p')} \le C_{p,q} \|f\|_p$$

and similarly from (4.19) and (4.21) we have

$$\|R_2 f\|_{(q',p')} \le C_{p,q} \|f\|_p.$$

This proves the theorem. □

*Observation* 4.9. From the theorem above we can infer the following behaviour of the Fourier transform of a function $f \in L^p(S)$ on $\partial S_p$ for $p \in [1, 2]$: For almost every $\lambda \in \mathbb{R}$, $\widetilde{f}(\lambda + i\gamma_p\rho, \cdot)$ exists and defines an $L^p$-function on $N$ and $\widetilde{f}(\lambda - i\gamma_p\rho, \cdot)$ exists and defines an $L^{p'}$-function on $N$.

We will now generalize the cases $q = p$ and $q = p'$ in another direction. Particular cases of this result can be viewed as analogues of Hardy-Littlewood-Paley inequalities. For $1 < p \leq 2$ the Hardy-Littlewood-Paley inequality on $\mathbb{R}^n$ states that

$$\int_{\mathbb{R}^n} |\widehat{f}(x)|^p |x|^{n(p-2)} dx \leq c_p \|f\|_p^p. \tag{4.22}$$

Using the multiplication formula for the Fourier transform, it then follows from the duality argument that for $2 \leq p < \infty$,

$$\|\widehat{f}\|_p^p \leq C_p \int_{\mathbb{R}^n} |f(x)|^p |x|^{n(p-2)} dx. \tag{4.23}$$

(See [27, p. 175].) For the lack of multiplication formula in $S$ it appears that one has to prove possible analogues of these two theorems independently. It is necessary to have a natural interpretation of the weight $|x|^n$. If we write $w(x)$ for $|x|^{n-1}$, then we see that the weight in (4.22) and (4.23) is $|x|w(x)$, which is also a constant multiple of the first primitive of $w(x)$. This leads us to consider the weight of the form $|\lambda||c(\lambda)|^{-2}$. In fact the primitive of the function $|c(\lambda)|^{-2}$ satisfies the same estimate as $|\lambda||c(\lambda)|^{-2}$. For symmetric spaces, an analogue of (4.23) was proved in [18] for radial functions and while an analogue of (4.22) was obtained in [22] the theorem does not accommodate the Plancherel theorem as a special case. Moreover, in both papers, the domain of the $L^p$ Fourier transform is considered to be $\mathbb{R}$. We need the following definition for the theorem:

**Definition 4.10.** For $p > 2$ the space $L^{(p)}(S)$ is the set of all measurable functions $f$ on $S$ such that $\|f\|_{(p)} < \infty$, where

$$\|f\|_{(p)} = \left( \int_S |f(x)|^p J(x)^{(p-2)} dx \right)^{1/p}$$

and $J(x) = (\sinh \frac{r(x)}{2})^m (\sinh r(x))^k$ is essentially the Jacobian of the polar decomposition.

Since $L^{(p)}(S) \subset L^{p',p}(S)$ (see (4.28) and (4.29) below), the Fourier transform $\widetilde{f}(\lambda, n)$ of a function $f \in L^{(p)}(S)$ exists for $\lambda \in S_{p'}^{\circ}$ and for almost every $n \in N$.

**Theorem 4.11.** (i) *Let $1 < q \leq 2$ be fixed. Then for $f \in L^p(S)$, $1 < p \leq q$,*

$$\begin{aligned}
\left( \int_{\mathbb{R}} \|\widetilde{f}(\lambda + i\gamma_q \rho, \cdot)\|_{L^q(N)}^r (|\lambda||c(\lambda)|^{-2})^{(r/p'-1)} |c(\lambda)|^{-2} d\lambda \right)^{\frac{1}{r}} &\leq C_{p,q} \|f\|_p \quad \textit{and} \\
\left( \int_{\mathbb{R}} \|\widetilde{f}(\lambda - i\gamma_q \rho, \cdot)\|_{L^{q'}(N)}^r (|\lambda||c(\lambda)|^{-2})^{(r/p'-1)} |c(\lambda)|^{-2} d\lambda \right)^{\frac{1}{r}} &\leq C_{p,q} \|f\|_p,
\end{aligned}$$

*where $\frac{1}{r} = 1 - \frac{q'-1}{p'}$.*

(ii) *Let $2 \leq q < \infty$ be fixed. Then for $f \in L^{(p)}(S)$ with $q \leq p < \infty$,*

$$\begin{aligned}
\left( \int_{\mathbb{R}} \|\widetilde{f}(\lambda + i\gamma_{q'} \rho, \cdot)\|_{L^{q'}(N)}^p |c(\lambda)|^{-2} d\lambda \right)^{\frac{1}{p}} &\leq C_{p,q} \|f\|_{(p)} \quad \textit{and} \\
\left( \int_{\mathbb{R}} \|\widetilde{f}(\lambda - i\gamma_{q'} \rho, \cdot)\|_{L^{q}(N)}^p |c(\lambda)|^{-2} d\lambda \right)^{\frac{1}{p}} &\leq C_{p,q} \|f\|_{(p)}.
\end{aligned}$$

*Proof.* (i) We fix a $q$ such that $1 < q \leq 2$ and consider the measure spaces $(S, dx)$ and $(\mathbb{R} \setminus \{0\}, d\mu(\lambda))$, where $d\mu(\lambda) = |\lambda|^{-q}|c(\lambda)|^{-2(1-q)}d\lambda$. We define a sublinear operator $T$ for $f \in L^1(S) + L^q(S)$ by

$$Tf(\lambda) = \|\widetilde{f}(\lambda + i\gamma_q\rho, .)\|_{L^q(N)}(|\lambda||c(\lambda)|^{-2})^{q/q'}.$$

We note that by Observation 4.9, $T$ can be defined on $L^1(S) + L^q(S)$ and hence for any $L^p(S)$ with $1 \leq p \leq q$.

Now

$$\begin{aligned}\|Tf\|_{q'}^{q'} &= \int_{\mathbb{R}} |Tf(\lambda)|^{q'} d\mu(\lambda) \\ &= \int_{\mathbb{R}} \|\widetilde{f}(\lambda + i\gamma_q\rho, .)\|_{L^q(N)}^{q'} |c(\lambda)|^{-2} d\lambda \\ &\leq C_q \|f\|_q^{q'}.\end{aligned}$$

In the last step we have used Theorem 4.6, the case $p = q$ when $q < 2$ and the Plancherel theorem when $q = 2$. Therefore $T$ is of type $(q, q')$. Now we will show that $T$ is of weak type $(1, 1)$. For $t \geq 0$ we define $E_t = \{\lambda \in \mathbb{R} : Tf(\lambda) > t\}$. Then

$$\begin{aligned}E_t &= \{\lambda \in \mathbb{R} : (|\lambda||c(\lambda)|^{-2})^{q/q'} \|\widetilde{f}(\lambda + i\gamma_q\rho, .)\|_{L^q(N)} > t\} \\ &\subset \{\lambda \in \mathbb{R} : (|\lambda||c(\lambda)|^{-2})^{q/q'} \|f\|_1 > t\} \\ &= \{\lambda \in \mathbb{R} : (|\lambda||c(\lambda)|^{-2}) > (t/\|f\|_1)^{q'/q}\} = A_t, \text{ say.}\end{aligned}$$

We denote $(t/\|f\|_1)^{q'/q}$ by $\alpha_t$. Let

$$A_t^+ = \{\lambda \geq 0 \mid (|\lambda||c(\lambda)|^{-2}) > \alpha_t\} \text{ and } A_t^- = \{\lambda \leq 0 \mid (|\lambda||c(\lambda)|^{-2}) > \alpha_t\}.$$

Since $(|\lambda||c(\lambda)|^{-2})^{-q}$ is an even function we note that $A_t^- = -A_t^+$ and $\mu(A_t^+) = \mu(-A_t^+)$. So we have to estimate only the quantity $\mu(A_t^+)$.

Let $G(\lambda) = \lambda^3(1+\lambda)^{\beta-2}$, where $\beta = m + k$. Then for $\lambda \geq 0$, $G'(\lambda) = \lambda^2(1+\lambda)^{\beta-3}(3 + (\beta+1)\lambda) \geq \lambda^2(1+\lambda)^{\beta-2}$ and $\lambda|c(\lambda)|^{-2} \asymp G(\lambda)$ by Lemma 4.8.

We have

$$\begin{aligned}\mu(A_t^+) &= \int_{\mathbb{R}} \chi_{A_t^+}(\lambda)(|\lambda||c(\lambda)|^{-2})^{-q}|c(\lambda)|^{-2} d\lambda \\ &\leq C \int_{A_t^+} \lambda^{-q}\lambda^{2(1-q)}(1+\lambda)^{(\beta-2)(1-q)} d\lambda \\ &= C \int_{A_t^+} (\lambda^3(1+\lambda)^{\beta-2})^{-q}\lambda^2(1+\lambda)^{\beta-2} d\lambda \\ &\leq C \int_{A_t^+} G(\lambda)^{-q} G'(\lambda) d\lambda \\ &= C \int_{c\alpha_t}^{\infty} z^{-q} dz,\end{aligned}$$

where $C$ and $c$ are positive constants.

Therefore

$$\mu(A_t) \leq C_q \lim_{b \to \infty} ((c\alpha_t)^{1-q} - b^{1-q}) = C_q \alpha_t^{1-q} = C_q \|f\|_1/t.$$

This proves that $T$ is weak $(1, 1)$.

Now we use an interpolation theorem from ([19, p. 62, Cor. 1.4.21]). If $\frac{1}{p} = 1 - t + \frac{t}{q}$, then $t = \frac{q'}{p'}$. Hence $\frac{1}{r} = 1 - t + \frac{t}{q'} = \frac{p'q - q'}{p'q}$. So $T$ is of the type $(p, r)$, that is,

$$\left( \int_{\mathbb{R}} \|\widetilde{f}(\lambda + i\gamma_q \rho, \cdot)\|^r_{L^q(N)} (|\lambda||c(\lambda)|^{-2})^s |c(\lambda)|^{-2} d\lambda \right)^{\frac{1}{r}} \leq C_{p,q} \|f\|_p,$$

where $s = \frac{qr}{q'} - q$. Substituting the value of $r$, we get $s = \frac{r}{p'} - 1$. This completes the proof of the first inequality in (i).

To prove the second inequality in (i) we need to define the operator

$$T_1 f(\lambda) = \|\widetilde{f}(\lambda - i\gamma_q \rho, .)\|_{L^{q'}(N)} (|\lambda||c(\lambda)|^{-2})^{q/q'},$$

and proceed in an analogous way.

(ii) We consider the measure spaces $(S, dx)$ and $(\mathbb{R}, |c(\lambda)|^{-2} d\lambda)$. For $f \in L^1(S) + L^{p',1}(S)$ we define the sublinear operator $T$ between the measure spaces by $Tf(\lambda) = \|\widetilde{f}(\lambda + i\gamma_{q'}\rho, \cdot)\|_{L^{q'}(N)}$. The domain of definition of $T$ is justified by Observation 4.9 as $L^{p',1}(S) \subset L^{p'}(S)$ and $p' \leq 2$.

First we will show that

$$\|T\|^*_{\infty,\infty} \leq \|f\|^*_{1,1}. \tag{4.24}$$

We note that for $\lambda \in \mathbb{R}$,

$$|\mathcal{P}_{\lambda + i\gamma_{q'}}(x, n)| = |\mathcal{P}(x, n)^{\frac{1}{2} - \frac{i}{Q}(\lambda + i\gamma_{q'}\rho)}| = \mathcal{P}(x, n)^{\frac{1}{2} + \frac{\gamma_{q'}}{2}} = \mathcal{P}(x, n)^{\frac{1}{q'}}.$$

Therefore writing $x = n_1 a$, where $n_1 \in N$ and $a \in A$, we see that

$$|\mathcal{P}_{\lambda + i\gamma_{q'}}(x, n)|^{q'} = \mathcal{P}(x, n) = P_a(n_1^{-1} n).$$

As $\int_N P_a(n) dn = 1$ (see Section 2) it follows that

$$\int_N |\mathcal{P}_{\lambda + i\gamma_{q'}}(x, n)|^{q'} dn = 1.$$

Minkowski's integral inequality now yields,

$$\begin{aligned}
\|\widetilde{f}(\lambda + i\gamma_{q'}\rho, \cdot)\|_{L^{q'}(N)} &= \left( \int_N |\widetilde{f}(\lambda + i\gamma_{q'}\rho, n)|^{q'} dn \right)^{1/q'} \\
&\leq \left( \int_N \left( \int_S |f(x)||\mathcal{P}_{\lambda + i\gamma_{q'}}(x, n)| dx \right)^{q'} dn \right)^{1/q'} \\
&\leq \int_S |f(x)| \left( \int_N |\mathcal{P}_{\lambda + i\gamma_{q'}}(x, n)|^{q'} dn \right)^{1/q'} dx \\
&= \|f\|_1.
\end{aligned}$$

That is, $|Tf(\lambda)| \leq \|f\|_1$ for all $\lambda \in \mathbb{R}$ and hence $\|Tf\|_\infty \leq \|f\|_1$. We recall (see Section 2) that $\|Tf\|_\infty = \|Tf\|^*_{\infty,\infty}$ and $\|f\|_1 = \|f\|^*_{1,1}$. This proves (4.24).

Next we claim that

$$\|Tf\|^*_{q,\infty} \leq C_q \|f\|^*_{q',1}. \tag{4.25}$$

As a step we first note that

$$\|Tf\|_q \leq C_q \|f\|_{q'} \text{ for } q \geq 2. \tag{4.26}$$

Indeed when $q = 2$, then $\gamma_{q'} = 0$ and (4.26) is a consequence of the Plancherel theorem. When $q > 2$, then $q' < q$ and we use Theorem 4.6 to obtain $\|Tf\|_q \leq C_q \|f\|_{q'}$.

By properties of the Lorentz norm (see section 2) we have as $q < \infty$ and $q' > 1$,

$$\|Tf\|^*_{q,\infty} \leq \|Tf\|^*_{q,q}, \|f\|_{q'} = \|f\|^*_{q',q'} \leq \|f\|^*_{q',1} \text{ and } \|Tf\|^*_{q,q} = \|Tf\|_q.$$

Combining these with (4.26) we get

$$\|Tf\|^*_{q,\infty} \leq \|Tf\|^*_{q,q} = \|Tf\|_q \leq C_q \|f\|_{q'} \leq C_q \|f\|^*_{q',1},$$

which proves (4.25).

Hence by interpolation ([30, Theorem 3.15, p. 197]) we have for $q \leq p < \infty$, $1 \leq s \leq \infty$,

$$\|Tf\|^*_{p,s} \leq C_{p,q} \|f\|^*_{p',s}, q \leq p < \infty, \quad 1 \leq s \leq \infty. \tag{4.27}$$

Let $u = p/(p-2)$ and $g(x) = f(x)J(x)^{\frac{1}{u}}$. Then $g \in L^p(S)$ and $\|g\|_p = \|f\|_{(p)}$. Since $J(x) = J(r(x)) \leq Ce^{Qr(x)}$ it follows from a straightforward calculation that $m(\{x \in S \mid J(x) \leq t\}) \leq Ct$ for all $t > 0$ for some constant $C$, where $m$ is the Haar measure of $S$. As a result, $m(\{x \mid J(x)^{-\frac{1}{u}} > t\}) \leq Ct^{-u}$ and hence $J(x)^{-\frac{1}{u}} \in L^{u,\infty}(S)$. Using Hölder's inequality (see (2.11)) we get

$$\|f\|^*_{p',p} \leq C_p \|g\|_p \|J^{-\frac{1}{u}}\|_{u,\infty}. \tag{4.28}$$

Taking $s = p$ in (4.27) and using (4.28), we get

$$\|Tf\|^*_{p,p} \leq C_{p,q} \|g\|_p = C_{p,q} \left( \int_S |f(x)|^p J(x)^{p-2} dx \right)^{1/p}, \tag{4.29}$$

which proves the first inequality in (ii). The second inequality can be proved in an analogous way, substituting $L^{q'}(N)$ by $L^q(N)$ and $\gamma_{q'}$ by $\gamma_q$ in the definition of the operator. □

*Observation* 4.12. We have the following observations.

A. In Theorem 4.11(i), if $q = 2$, then $r = p$ and the power of $|\lambda||c(\lambda)|^{-2}$ is $r/p' - 1 = (p-2)$. Then the theorem somewhat resembles the classical Hardy-Littlewood-Paley inequality for $p \leq 2$. (See (4.22).)

B. In Theorem 4.11(i), the condition $r = p'$ is equivalent to the condition $p = q$. In this case the first inequality in (i) gives back the case $q = p$ of Theorem 4.6. Similarly the second inequality in (i) gives back the case $q = p'$ of Theorem 4.6.

C. If $p = q = 2$, then $\gamma_q = \gamma_{q'} = 0$ and all four inequalities in Theorem 4.11 are a weakening of the Plancherel theorem.

D. If we put $p = q = \alpha'$ in Theorem 4.11 (ii), then by (4.27) and its analogue for the second inequality in (ii) it follows that for $1 \leq \alpha < 2$, the Fourier transform $\widetilde{f}(\lambda, n)$ of any function $f \in L^{\alpha,s}(S)$, $0 < s \leq \infty$, exists on the boundaries of the strip $S_\alpha$ for almost every $n \in N$. This settles part (iii) of Theorem 3.4. Note that this does not follow from the argument of breaking a function in $L^{\alpha,s}(S)$ into $L^1$ and $L^p$ parts with $p > \alpha$ (see Remark 3.5 (ii)).

## 5. Existence and mapping properties of the Radon transform

We recall that for a suitable function $f$ on $S$ we have the following integral formula:

$$\int_S f(x)dx = \int_N \int_{\mathbb{R}} f(na_s)e^{-2\rho s}dnds.$$

For a measurable function $f$ on $S$ its Radon transform $\mathcal{R}f$ on $A \times N$ is defined by

$$\mathcal{R}f(a_s, n) = e^{-s\rho} \int_N f(n\sigma(n_1 a_s))dn_1, \tag{5.1}$$

wherever the integral makes sense.

**Definition 5.1.** For measurable functions $g$ on $N \times \mathbb{R}$ and $h$ on $\mathbb{R}$ we define the *adjoint of the Radon transform of the first and second kind*, $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ respectively, as:

$$\begin{aligned} \mathcal{R}_1^* g(x) = & \int_N g(n, A(\sigma(n^{-1}x)))\mathcal{P}_0(x,n)dn \\ = & \int_N g(n, A(\sigma(n^{-1}x)))e^{\rho A(\sigma(n^{-1}x))}dn, \end{aligned} \tag{5.2}$$

whenever the integral makes sense and

$$\mathcal{R}_2^* h(x) = h(A(\sigma(x)))\mathcal{P}_0(x,e) = h(A(\sigma(x)))e^{\rho A(\sigma(x))}. \tag{5.3}$$

It is clear that if for almost every $t \in \mathbb{R}$, $g(\cdot, t)$ is a bounded function on $N$, then $\mathcal{R}_1^* g$ exists. The motivation for defining $\mathcal{R}_1^*$ and $\mathcal{R}_2^*$ comes from the following:

**Proposition 5.2.** *Let $f$ be a measurable function on $S$.*
(a) *For a measurable function $g$ on $N \times \mathbb{R}$,*

$$\int_{N\times\mathbb{R}} \mathcal{R}f(a_s, n)g(n,s)dnds = \int_S f(x)\mathcal{R}_1^* g(x)dx.$$

(b) *For a measurable function $h$ on $\mathbb{R}$,*

$$\int_{\mathbb{R}} \mathcal{R}f(a_s,n)h(s)ds = \int_S f(x)L_n(\mathcal{R}_2^*)h(x)dx, \text{ where } L_n(\mathcal{R}_2^* h)(x) = \mathcal{R}_2^* h(n^{-1}x).$$

*Proof.* The proof follows by writing down the definitions and interchanging integrals. Indeed for (a),

$$\begin{aligned}
&\int_{N\times\mathbb{R}} \mathcal{R}f(a_s,n)g(n,s)dsdn \\
= &\int_{N\times\mathbb{R}} e^{-s\rho}\int_N f(n\sigma(n_1a_s))dn_1 g(n,s)ds\,dn \\
= &\int_N \left(\int_S f(n\sigma(y))g(n,A(y))e^{\rho A(y)}dy\right)dn \quad \text{(substituting } n_1a_s=y\text{)} \\
= &\int_N \left(\int_S f(nz)g(n,A(\sigma(z)))e^{\rho A(\sigma(z))}dz\right)dn \quad \text{(substituting } z=\sigma(y)\text{)} \\
= &\int_N \left(\int_S f(x)g(n,A(\sigma(n^{-1}x)))e^{\rho A(\sigma(n^{-1}x))}dx\right)dn \quad \text{(substituting } z=n^{-1}x\text{)} \\
= &\int_S f(x)\left(\int_N g(n,A(\sigma(n^{-1}x)))e^{\rho A(\sigma(n^{-1}x))}dn\right)dx \\
= &\int_S f(x)\mathcal{R}_1^*g(x)dx.
\end{aligned}$$

Similarly for (b) we have

$$\begin{aligned}
\langle \mathcal{R}f(\cdot,n),g\rangle_{L^2(A)} &= \int_A \mathcal{R}f(a_s,n)g(s)ds \\
&= \int_{\mathbb{R}}\left(\int_N e^{-s\rho}f(n\sigma(n_1a_s))dn_1\right)g(s)ds \\
&= \int_{\mathbb{R}\times N} e^{-sQ}e^{s\rho}f(n\sigma(n_1a_s))g(s)dn_1ds \\
&= \int_S e^{\rho A(y)}f(n\sigma(y))g(A(y))dy \quad \text{(substituting } n_1a_s=y\text{)} \\
&= \int_S e^{\rho A(\sigma(x))}f(nx)g(A(\sigma(x)))dx \quad \text{(substituting } \sigma(y)=x\text{)} \\
&= \int_S e^{\rho A(\sigma(n^{-1}z))}f(z)g(A(\sigma(n^{-1}z)))dz \quad \text{(substituting } nx=z\text{)} \\
&= \int_S f(z)\mathcal{R}_2^*g(n^{-1}z)dz.
\end{aligned}$$

□

A natural question at this point is: Is it possible to characterize the class of functions for which the integral defining the Radon transform exists? We offer a partial answer to this question. For a suitable function $g$ on $\mathbb{R}$ let $\mathcal{F}g$ denote its Euclidean Fourier transform, i.e. $\mathcal{F}(g)(\lambda)=\int_{\mathbb{R}} g(x)e^{-i\lambda x}dx$.

**Theorem 5.3.** (a) *If $f\in L^1_0(S)$, then $\mathcal{R}f\in L^1(A\times N, P_1(n)^{1/2}dtdn)$. If $f$ is nonnegative, then the converse is also true.*

(b) *If $f\in L^1_0(S)$, then $\mathcal{F}(\mathcal{R}f(\cdot,n))(\lambda)=\widetilde{f}(\lambda,n)$ for $\lambda\in\mathbb{R}$. The relation above remains valid for $\lambda\in S_p^\circ$ and for $\lambda\in S_p$ when $f\in L^{p,q}(S), 1<p<2, 1<q\leq\infty$ and $f\in L^{p,1}(S), 1\leq p<2$ respectively.*

(c) *If* $f \in L^{p,q}(S), 1 < p < 2, 1 < q \leq \infty$ *and* $\alpha \in [0, \gamma_p \rho)$, *then for almost every* $n \in N$,

$$\int_{\mathbb{R}} \mathcal{R}|f|(a_s, n) e^{\alpha|s|} ds < \infty.$$

*If* $q = 1$ *and* $p$ *is as above, then* $\alpha$ *can be equal to* $\gamma_p \rho$. *If* $f \in L^1(S)$, *then the inequality above holds with* $\alpha = \rho$.

(d) *If* $f \in L^1(S)$ *or* $f \in L^{p,q}(S)$ *with* $1 < p < 2, 1 \leq q \leq \infty$ *and* $\mathcal{R}f \equiv 0$, *then* $f \equiv 0$.

*Proof.* (a) For $\lambda \in \mathbb{R}$ we define $g_\lambda(a_t, n) = e^{-i\lambda t} \mathcal{P}_{-\lambda}(e, n)$. Using the fact that for each fixed $t \in \mathbb{R}$, $g_\lambda(a_t, n)$ is a bounded function in $n$, we have

$$\begin{aligned} \mathcal{R}_1^* g_\lambda(x) &= \int_N e^{\rho A(\sigma(n^{-1}x))} e^{-i\lambda A(\sigma(n^{-1}x))} \mathcal{P}_{-\lambda}(e, n) dn \\ &= \int_N \mathcal{P}_\lambda(x, n) \mathcal{P}_{-\lambda}(e, n) dn = \phi_\lambda(x). \end{aligned}$$

In particular $\mathcal{R}_1^* g_0(x) = \phi_0(x)$ as $g_0(a_t, n) = \mathcal{P}_0(e, n)$.

Using Proposition 5.2 (a) we have

$$\int_S |f(x)| \phi_0(x) dx = \int_S |f(x)| \mathcal{R}_1^* g_0(x) dx = \int_{R \times N} \mathcal{R}|f|(a_t, n) \mathcal{P}_0(e, n) dt dn$$

whenever the integrals involved make sense. This shows that if $f$ is nonnegative, then $f \in L_0^1(S)$ if and only if $\mathcal{R}f(\cdot, \cdot) \in L^1(A \times N, P_1(n)^{1/2}\, dt\, dn)$.

(b) We first note that since $f \in L_0^1(S)$, by (a) we have $\mathcal{R}f(\cdot, n) \in L^1(\mathbb{R})$ for almost every $n \in N$ and hence it makes sense to talk about its Euclidean Fourier transform. Now let $g(t) = e^{-i\lambda t}$. Then by (5.3) and (2.5),

$$\mathcal{R}_2^* g(n^{-1}x) = (e^{A(\sigma(n^{-1}x))})^{\rho - i\lambda} = \mathcal{P}_\lambda(x, n).$$

Therefore by Proposition 5.2 (b) we have:

$$\mathcal{F}(\mathcal{R}f(\cdot, n))(\lambda) = \int_{\mathbb{R}} \mathcal{R}f(a_s, n) e^{-i\lambda s} ds = \int_S f(x) \mathcal{P}_\lambda(x, n) dx = \widetilde{f}(\lambda, n) \tag{5.4}$$

for almost every $n \in N$. The rest of the assertions follow from a similar argument.

(c) We take $g_+(s) = e^{\alpha s}$ and $g_-(s) = e^{-\alpha s}$. Then $L_n(\mathcal{R}_2^* g_+)(x) = \mathcal{P}_{i\alpha}(x, n)$ and $L_n(\mathcal{R}_2^* g_-)(x) = \mathcal{P}_{-i\alpha}(x, n)$. Hence by Proposition 5.2 (b) and Theorem 3.4, we have for almost every $n \in N$,

$$\begin{aligned} \int_{\mathbb{R}} \mathcal{R}|f|(a_s, n) e^{\alpha s} ds = \int_S |f|(x) \mathcal{P}_{i\alpha}(x, n) dx < \infty \text{ and} \\ \int_{\mathbb{R}} \mathcal{R}|f|(a_s, n) e^{-\alpha s} ds = \int_S |f|(x) \mathcal{P}_{-i\alpha}(x, n) dx < \infty, \end{aligned}$$

for $\alpha$ as in the hypotheses. For $f \in L^1(S)$, the argument is similar.

(d) As $\mathcal{R}f \equiv 0$, by (5.4) we have $\widetilde{f} \equiv 0$ on $\mathbb{R} \times N$. Therefore we can apply Proposition 3.6 to get the result. □

The following properties of the Fourier transform follow from the theorem above.

**Theorem 5.4.** *Let $1 \leq p < 2$. If $f \in L^{p,1}(S)$, then for almost every fixed $n \in N$, the map $\lambda \mapsto \widetilde{f}(\lambda, n)$ is continuous on $S_p$ and analytic on $S_p^\circ$. Furthermore*

$$\lim_{|\xi|\to\infty} \widetilde{f}(\xi + i\eta, n) = 0$$

*uniformly in $\eta \in [-\gamma_p\rho, \gamma_p\rho]$.*

*For functions in $L^{p,q}(S)$, $q > 1$, the assertions above remain valid for $\lambda \in S_p^\circ$ and for $\eta \in [-(\gamma_p\rho - \delta), (\gamma_{p_1}\rho - \delta)]$ for any $0 < \delta < \gamma_p$.*

*Proof.* We recall that for a suitable function $g$ on $\mathbb{R}$, $\mathcal{F}(g)$ is the Euclidean Fourier transform of $g$. Since $f \in L^{p,1}(S)$ and $|\mathcal{R}f| \leq \mathcal{R}|f|$, Theorem 5.3 (c) implies that $\int_{\mathbb{R}} |\mathcal{R}f|(a_s, n)e^{\gamma_p\rho|s|}ds$ is finite for almost every $n \in N$. We fix an $n \in N$ for which $\mathcal{R}f$ is defined and call $\mathcal{R}f(a_s, n) = g(s)$. Thus $g$ is in the weighted space $L^1(\mathbb{R}, w)$ with the weight $w = e^{\gamma_p\rho|s|}$. By parts (b) and (c) of Theorem 5.3, $\mathcal{F}(g)(\lambda) = \widetilde{f}(\lambda, n)$ for $\lambda \in S_p$. This reduces the theorem to the Riemann-Lebesgue Lemma for functions on $\mathbb{R}$ which are integrable with an exponential weight.

A standard use of Fubini's theorem, Morera's theorem and the dominated convergence theorem yields that $\lambda \mapsto \mathcal{F}(g)(\lambda) = \widetilde{f}(\lambda, n)$ is continuous on $S_p$ and analytic on $S_p^\circ$. It is also clear that for $\eta$ as in the hypotheses, $|\mathcal{F}(g)(\xi + i\eta)| \leq \int_{\mathbb{R}} |g(s)|e^{|\eta||s|}ds \leq \|g\|_{w,1}$, where $\|g\|_{w,1} = \int_{\mathbb{R}} g(x)e^{\gamma_p\rho|x|}dx$, the weighted $L^1$-norm of $g$.

To complete the proof of the assertion we now approximate $g$ in $L^1(\mathbb{R}, w)$ by a finite sum $h$ of step functions, use $\mathcal{F}h(\xi + i\eta) \to 0$ as $|\xi| \to \infty$ uniformly and note that

$$\begin{aligned} |\mathcal{F}(g)(\xi + i\eta)| &\leq |\mathcal{F}(g)(\xi + i\eta) - \mathcal{F}(h)(\xi + i\eta)| + |\mathcal{F}(h)(\xi + i\eta)| \\ &\leq \|g - h\|_{w,1} + |\mathcal{F}(h)(\xi + i\eta)|. \end{aligned}$$

An easy modification of the argument above proves the assertion for functions in $L^{p,q}(S)$. □

*Remark* 5.5. We note that if for some function $f$, $f(x)(1 + r(x)) \in L^{2,1}(S)$, then the conclusions of Theorem 5.3 and Theorem 5.4 remain valid in the appropriate domain.

We conclude this section with the following mapping property of the Radon transform.

**Theorem 5.6.** *If $1 \leq r < q < 2$, $1 \leq p \leq q$, then*

$$\left(\int_N \left(\int_{\mathbb{R}} |\mathcal{R}f(n,t)|e^{\gamma_q\rho|t|}dt\right)^p P_1(n)dn\right)^{1/p} \leq C\|f\|_r.$$

*Proof.* Let us write $\alpha$ for $\gamma_q\rho$. It is clear that $\pm i\alpha \in S_r^\circ$. Using Minkowski's inequality, (5.4) and Corollary 4.4 we have

$$\begin{aligned}
&\left(\int_N \left(\int_{\mathbb{R}} |\mathcal{R}f(n,t)|e^{\alpha|t|}dt\right)^p P_1(n)dn\right)^{1/p}\\
&\leq \left(\int_N \left(\int_{\mathbb{R}_+} \mathcal{R}|f|(n,t)e^{\alpha t}dt + \int_{\mathbb{R}_-} \mathcal{R}|f|(n,t)e^{-\alpha t}dt\right)^p P_1(n)dn\right)^{1/p}\\
&\leq \left(\int_N \left(\widetilde{|f|}(i\alpha,n) + \widetilde{|f|}(-i\alpha,n)\right)^p P_1(n)dn\right)^{1/p}\\
&\leq \left(\int_N \widetilde{|f|}(i\alpha,n)^p P_1(n)dn\right)^{1/p} + \left(\int_N \widetilde{|f|}(-i\alpha,n)^p P_1(n)dn\right)^{1/p}\\
&\leq C_{p_1,q_1,q}\|f\|_r + C'_{p_1,q_1,q}\|f\|_r = C''_{p_1,q_1,q}\|f\|_r. \qquad \square
\end{aligned}$$

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