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Fourier multipliers and estimates of the Fourier transform of measures carried by smooth curves in $\boldsymbol{R}^2$

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Abstract. Assume $\alpha > 0$ and let $m(x)$ be defined for $x \in \boldsymbol{R}^2$ by $m(x) = (1-|x|^2)^\alpha$, $|x| \leqslant 1$, and $m(x) = 0$, $|x| > 1$. It is then known for what values of p m is a Fourier multiplier for $L^p(\boldsymbol{R}^2)$. In this article this result is extended to more general functions m.

It is also given an L^p estimate of the Fourier transform of measures carried by smooth curves in $\boldsymbol{R}^2$, which extends a result of C. Fefferman and E. M. Stein [4].

Introduction. Let m be a bounded measurable complex-valued function on $\boldsymbol{R}^2$. Define an operator T by setting $(Tf)^\wedge = m\hat{f}$, $f \in C_0^\infty(\boldsymbol{R}^2)$, where $\hat{f}$ is the Fourier transform of f, given by $\hat{f}(x) = \int_{\boldsymbol{R}^2} e^{-ix\cdot t} f(t)\,dt$, $x \in \boldsymbol{R}^2$, and C_0^∞ denotes the class of infinitely differentiable complex-valued functions with compact support. We say that m is a multiplier for $L^p(\boldsymbol{R}^2)$ if $\|Tf\|_{L^p(\boldsymbol{R}^2)} \leqslant C_p \|f\|_{L^p(\boldsymbol{R}^2)}$, $f \in C_0^\infty(\boldsymbol{R}^2)$, for some constant C_p depending only on m and p.

The following theorems are the main results of this paper.

THEOREM 1. *Let Γ be a C^∞ curve in $\boldsymbol{R}^2$ which is simple and closed and has a tangent at each point. Denote the region inside Γ by Ω. For $x \in \boldsymbol{R}^2$ let $\delta(x)$ denote the distance from x to Γ and let α be a positive number. Assume that m is a function on $\boldsymbol{R}^2$ with the following properties:*

(i) *The restriction of m to Ω belongs to $C^2(\Omega)$.*

(ii) *There exists a neighbourhood Ω' of Γ such that $m(x) = \delta(x)^\alpha$ if $x \in \Omega \cap \Omega'$.*

(iii) *m vanishes outside Ω.*

Then, if $0 < \alpha \leqslant 1/2$, m is a multiplier for $L^p(\boldsymbol{R}^2)$ if and only if $4/(3+2\alpha) < p < 4/(1-2\alpha)$. If $\alpha > 1/2$ m is a multiplier for $L^p(\boldsymbol{R}^2)$ for $1 \leqslant p \leqslant \infty$.

THEOREM 2. (i) *Let $I_0 = [0,1]$, assume that γ_1 and γ_2 are real and belong to $C^\infty(I_0)$ (i.e. they are infinitely differentiable in the interior of I_0 and have one-sided derivatives of all orders at the endpoints), and that $\gamma_1'(t)^2 + \gamma_2'(t)^2 \neq 0$ for $t \in I_0$. Let Γ denote the curve $\{(\gamma_1(t), \gamma_2(t)) \in \boldsymbol{R}^2;\ t \in I_0\}$,* *let dS denote the arc length measure on Γ and set*

$$Sf(x) = \int_\Gamma e^{-ix\cdot t} f(t)\,dS(t), \qquad x \in \mathbf{R}^2,\ f \in L^1(\Gamma; dS).$$

Then

$$\|Sf\|_{L^q(\mathbf{R}^2)} \leqslant C_{q,\gamma} \|f|K|^{-\gamma}\|_{L^p(\Gamma; dS)},$$

if $4 < q < \infty$, $q/(q-3) \leqslant p \leqslant \infty$ and $\gamma > 1/q$, where $K(t)$ denotes the curvature of Γ at a point $t \in \Gamma$.

(ii) *If furthermore $K(t) \geqslant 0$ for $t \in \Gamma$, then it is sufficient to assume that $\gamma_i \in C^2(I_0)$, $i = 1, 2$, and in this case the above inequality holds also for $\gamma = 1/q$.*

In the case when Γ is the unit circle Theorem 1 is well known (see Bochner [1], Herz [7], Stein [9], Fefferman [4] and Carleson and Sjölin [3]). In particular it was proved in [4] that the condition on p is sufficient for $\alpha > 1/6$ and then in [3] that it is sufficient for $\alpha > 0$. The author has also proved that this result can be extended to the case when the tangent to Γ has everywhere finite order of contact. A simplification of the proof in [3] and an easy proof of the extension just mentioned are contained in Hörmander [8]. An alternative proof in the case when Γ is the unit circle is given in Fefferman [6].

We also want to remark that if we set $\alpha = 0$ in the definition of m in Theorem 1, then it follows from Fefferman's counterexample in [5] that m is multiplier for $L^p(\mathbf{R}^2)$ if and only if $p = 2$.

The basic idea in the proof of Theorem 1 is the following. To treat the case when Γ is convex we make a partition of the curve which leads to a splitting of $\hat{m}$ with properties similar to those of the splitting carried out by Fefferman in [6] in the case of the unit circle. The main difficulty is to find a suitable partition of Γ. We then use a property of C^∞ functions (see Lemma 1) to pass to the general case.

Theorem 2 is well known in the case when the curvature of Γ never vanishes (see [4] and cf. [3], [8] and Zygmund [11]). It is also known that already in this case the conditions $q > 4$ and $q/(q-3) \leqslant p$ can not be weakened.

The proof of Theorem 2 in the case $K \geqslant 0$ is a generalization of the proof in the case of non-vanishing curvature and to pass to the C^∞ result we use Lemma 1 once more. We shall also give examples of curves Γ for which the conditions on γ in Theorem 2 can not be relaxed.

I wish to express my gratitude to Charles Fefferman for valuable conversations.

1. The multiplier theorem. We shall need the following property of C^∞ functions.

LEMMA 1. *Let I be a compact interval on $\mathbf{R}$, assume that $\varphi \in C^\infty(I)$ and is real-valued and let ε be a positive number. Set $E = \{x \in I; \varphi(x) = 0\}$ and let $\{I_n\}_{n=1}^\infty$ be the component intervals of $I \setminus E$. Then $\sum_{n=1}^\infty (\sup_{I_n} |\varphi|)^\varepsilon$ is convergent.*

Proof. Let F be set of points of accumulation of E and let $\{J_m\}_{m=1}^\infty$ be the component intervals of $I \setminus F$. To prove the lemma it is sufficient ot prove that

$$\sum_{I_n \subset J_m} (\sup_{I_n} |\varphi|)^\varepsilon \leqslant C_{\varphi,\varepsilon} |J_m| \tag{1}$$

for each m, where $|J_m|$ denotes the length of J_m.

First let k be the smallest integer which is larger than $1/\varepsilon$. At least one of the end points of each J_m is contained in F and if follows from Taylor's formula that

$$|\varphi(x)| \leqslant (\sup_{J_m} |\varphi^{(k)}|)\,|J_m|^k, \qquad x \in J_m.$$

If at most k of the intervals I_n are included in J_m the above estimate yields (1) with $C_{\varphi,\varepsilon} = k(\sup_I |\varphi^{(k)}|)^\varepsilon |I|^{k\varepsilon-1}$. If J_m includes more than k intervals I_n we make a partition of J_m into subintervals $J_{m,l}$, $l = 1, 2, \ldots$, such that each $J_{m,l}$ includes at least k and at most $2k$ intervals I_n. From Rolle's theorem it follows that each $\varphi^{(j)}$, $j = 1, 2, \ldots, k-1$, has at least one zero in each $J_{m,l}$. Repeated use of the mean value theorem yields

$$\sup_{J_{m,l}} |\varphi| \leqslant (\sup_{J_{m,l}} |\varphi'|)\,|J_{m,l}| \leqslant \ldots \leqslant (\sup_{J_{m,l}} |\varphi^{(k)}|)\,|J_{m,l}|^k$$

and hence

$$\sum_{I_n \subset J_{m,l}} (\sup_{I_n} |\varphi|)^\varepsilon \leqslant C_{\varphi,\varepsilon} |J_{m,l}|.$$

Summing this inequality over l we obtain (1) also in this case and the proof of the lemma is complete.

We introduce some notation. We let $|E|$ denote the Lebesgue measure of a set E in $\mathbf{R}$ or $\mathbf{R}^2$ and set $\lambda E = \{\lambda x; x \in E\}$, $\lambda > 0$.

If ω is an interval on $\mathbf{R}$, $f \in L^1(\omega)$ and $\alpha \in \mathbf{R}$ set

$$c_\alpha(\omega; f) = \frac{1}{|\omega|} \int_\omega e^{-i2\pi|\omega|^{-1}\alpha t} f(t)\,dt$$

and

$$C_\alpha(\omega; f) = \sum_{\nu=-\infty}^{\infty} (1+|\nu|^2)^{-1} |c_{\alpha+\nu/3}(\omega; f)|.$$

Finally set $Q = \{(x, y) \in \mathbf{R}^2;\ |x| \leqslant 10,\ |y| \leqslant 10\}$.

We shall now prove the main lemma in the proof of Theorem 1.

LEMMA 2. *Let I be a compact interval on $\boldsymbol{R}$, let φ and $\psi \in C^\infty(I)$ and assume that ψ is real-valued. Set*

$$K_N(x, y) = N \int_I e^{iN(xu+y\psi(u))}\varphi(u)\,du, \quad (x, y) \in \boldsymbol{R}^2,\ N \geqslant 2.$$

and

$$T_N f(x, y) = \int_0^1 K_N(x-t, y) f(t)\,dt, \quad f \in L^1(0,1),\ (x, y) \in \boldsymbol{R}^2.$$

Then if $4 < q \leqslant \infty$ there exists a constant C_q depending only on I, ψ, φ and q such that

$$\|T_N f\|_{L^q(Q)} \leqslant C_q N^{1/2-2/q} (\log N)^4 \|f\|_{L^q(0,1)}.$$

Proof. First set $A = 10\max(\sup_I |\psi'|, \sup_I |\psi''|, 1)$. Starting from the left endpoint of I we make a partition of I into intervals ω_k, $k = 1, 2, \ldots$ $\ldots, K$, such that $|\omega_k| \int_{\omega_k} |\psi''|\,du = AN^{-1}$ for $k < K$ and $|\omega_K| \int_{\omega_K} |\psi''|\,du \leqslant AN^{-1}$. It follows that $|\omega_k| \geqslant N^{-1/2}$ for $k < K$ and that $K \leqslant CN^{1/2}$.

We set $E = \{u \in I;\ \psi''(u) = 0\}$ and let $\{I_n\}_{n=1}^\infty$ be the component intervals of $I \setminus E$. If there exist intervals I_n for which there is at least one value of k such that $\omega_k \subset I_n$, we denote the corresponding intervals $\bigcup_{\omega_k \subset I_n} \omega_k$ by Ω_m, $m = 1, 2, \ldots, M_0$. The intervals ω_k which are not included in $\bigcup_{m=1}^{M_0} \Omega_m$ are denoted by Ω_m, $m = M_0+1, \ldots, M$. We have constructed a partition $\{\Omega_m\}_{m=1}^M$ of I with the following properties:

(2) $$|\Omega_m| \int_{\Omega_m} |\psi''|\,du \geqslant AN^{-1} \quad (\text{unless } \Omega_m = \omega_K).$$

(3) If more than one interval ω_k is included in Ω_m, then ψ'' has constant sign in Ω_m.

(4) For every n $I_n \cap \Omega_m$ is non-empty for at most three values of m.

We have

$$\int_{\Omega_m} |\psi''|\,du \leqslant \sum_{I_n \cap \Omega_m \neq \emptyset} \int_{I_n} |\psi''|\,du$$

and using (4) we obtain

(5) $$\sum_m \Big(\int_{\Omega_m} |\psi''|\,du\Big)^\varepsilon \leqslant 3 \sum_n \Big(\int_{I_n} |\psi''|\,du\Big)^\varepsilon, \quad 0 < \varepsilon \leqslant 1.$$

If ω is a subinterval of I we set $K_N^\omega(x, y) = N \int_\omega e^{iN(xu+y\psi(u))}\varphi(u)\,du$ and

$$T_N^\omega f(x, y) = \int_0^1 K_N^\omega(x-t, y) f(t)\,dt, \quad f \in L^1(0,1).$$

Extending f to $\boldsymbol{R}$ by setting $f(t) = 0$ for $t \in \boldsymbol{R} \setminus (0,1)$ we obtain

(6) $$T_N^\omega f(x, y) = N \int_\omega e^{iN(xu+y\psi(u))}\varphi(u)\hat{f}(Nu)\,du.$$

We are going to prove that if $\Omega_m \neq \omega_K$, then

(7) $$\|T_N^{\Omega_m} f\|_{L^q(Q)} \leqslant C\Big(\int_{\Omega_m} |\psi''|\,du\Big)^{1/4-1/q} N^{1/2-2/q} (\log N)^4 \|f\|_{L^q(0,1)},$$
$$4 < q \leqslant \infty.$$

We fix m and for each integer l let $\omega_1^l, \omega_2^l, \ldots$, denote the intervals ω_k in Ω_m for which $2^{-l-1} < |\omega_k| \leqslant 2^{-l}$ (if there is any), where ω_i^l is to the left of ω_j^l if $i < j$. Then set $T_{N,l,k}^{\Omega_m} = \sum_{j=k \pmod 4} T_N^{\omega_j^l}$, $l \in \boldsymbol{Z}$, $k = 0, 1, 2, 3$, and $F_{l,k} = (T_{N,l,k}^{\Omega_m} f)^2$. $F_{l,k}$ is the inverse Fourier transform of a measure on $E = \{(N(u_1+u_2), N(\psi(u_1)+\psi(u_2)));\ u_i \in \Omega_m, i = 1, 2\}$ and a computation shows that for every s_1

(8) $$|\{s_2;\ (s_1, s_2) \in E\}| \leqslant N \int_{\Omega_m} |\psi''|\,du\,|\Omega_m|.$$

We choose $\chi \in C^\infty(\boldsymbol{R}^2)$ such that $|\chi| \geqslant 1$ in Q and $\hat{\chi} \in C_0^\infty(\boldsymbol{R}^2)$ and has support in a unit square with center at the origin. Choosing $\hat{\chi}(x_1, x_2) = \beta(x_1)\beta(x_2)$, where β belongs to a suitable non-quasi-analytic class, we may also assume that

(9) $$\chi(x) = O(e^{-|x|^{1-\delta}}), \quad |x| \to \infty,$$

where δ is a small positive number. Using (8) and (2) we easily prove that

(10) $$|\operatorname{supp}(\chi F_{l,k})^\wedge| \leqslant CN^2 |\Omega_m|^2 \int_{\Omega_m} |\psi''|\,du.$$

From Schwarz's inequality and Plancherel's theorem it follows that

$$\|g\|_{L^\infty(\boldsymbol{R}^2)} \leqslant 2\pi |\operatorname{supp}\hat{g}|^{1/2} \|g\|_{L^2(\boldsymbol{R}^2)},$$

if $\hat{g} \in C_0^\infty(\boldsymbol{R}^2)$, and hence

(11) $$\|g\|_{L^{q/2}(\boldsymbol{R}^2)} \leqslant C |\operatorname{supp}\hat{g}|^{1/2-2/q} \|g\|_{L^2(\boldsymbol{R}^2)}.$$

We have

$$\|T_{N,l,k}^{\Omega_m} f\|_{L^q(Q)} = \|F_{l,k}\|_{L^{q/2}(Q)}^{1/2} \leqslant \|\chi F_{l,k}\|_{L^{q/2}(\boldsymbol{R}^2)}^{1/2}$$

for each l and k and using (11) with $g = \chi F_{l,k}$ and (10) we obtain

(12) $$\|T_N^{\Omega_m} f\|_{L^q(Q)} \leqslant C\Big(\int_{\Omega_m} |\psi''|\,du\Big)^{1/4-1/q} N^{1/2-2/q} \sum_{l,k} \|\chi F_{l,k}\|_{L^2(\boldsymbol{R}^2)}^{1/2}.$$

We now fix l and write ω_j instead of ω_j^l. We have

$$\chi F_{l,k} = \sum_{j,j'=k(\mathrm{mod}4)} \chi(T_N^{\omega_j}f)(T_N^{\omega_{j'}}f)$$

and shall prove that two terms $\chi(T_N^{\omega_j}f)(T_N^{\omega_{j'}}f)$ and $\chi(T_N^{\omega_i}f)(T_N^{\omega_{i'}}f)$ in this sum are orthogonal in $L^2(\boldsymbol{R}^2)$ if $j \leqslant j'$, $i \leqslant i'$ and $(j, j') \neq (i, i')$.

To show this we shall prove that their Fourier transforms have disjoint supports. It is sufficient to prove that the distance between the set

$$E_{j,j'} = \{(N(u_1+u_2), N(\psi(u_1)+\psi(u_2))); u_1 \in \omega_j, u_2 \in \omega_{j'}\}$$

and the corresponding set $E_{i,i'}$ is larger than $\sqrt{2}$. Without loss of generality we may assume that $j < i$.

Now assume that $u_1 \in \omega_j$, $u_2 \in \omega_{j'}$, $v_1 \in \omega_i$, $v_2 \in \omega_{i'}$ and that $|N(u_1+u_2)-N(v_1+v_2)| \leqslant \sqrt{2}$. It follows that $i' < j'$. Setting $\varrho = \min(v_1-u_1, u_2-v_2)$ and using the definition of A and the intervals ω_j we obtain

$$\begin{aligned} |N(\psi(u_1)+\psi(u_2))-N(\psi(v_1)+\psi(v_2))| &= N\Big|\int_{v_2}^{u_2}\psi'\,d\xi - \int_{u_1}^{v_1}\psi'\,d\xi\Big| \\ &\geqslant N\Big|\int_{v_2}^{v_2+\varrho}\psi'\,d\xi - \int_{u_1}^{u_1+\varrho}\psi'\,d\xi\Big| - N(\sqrt{2}/N)(A/10) \\ &= N\int_{u_1}^{u_1+\varrho}|\psi'(\xi+v_2-u_1)-\psi'(\xi)|\,d\xi - A\sqrt{2}/10 \\ &\geqslant N\int_{\omega_{j+1}}\Big(\int_{\omega_{j+2}}|\psi''|\,du\Big)d\xi - A/5 \geqslant A/2 - A/5 > \sqrt{2}, \end{aligned}$$

which is the desired estimate.

From the orthogonality it follows that

$$\|\chi F_{l,k}\|^2_{L^2(\boldsymbol{R}^2)} \leqslant 2\sum_{j,j'}\|\chi(T_N^{\omega_j}f)(T_N^{\omega_{j'}}f)\|^2_{L^2(\boldsymbol{R}^2)}$$

for each k and using the rapid decrease of χ and trivial estimates of $T_N^{\omega_j}f$ we obtain

(13) $$\|\chi F_{l,k}\|^2_{L^2(\boldsymbol{R}^2)} \leqslant C\sum_{j,j'}\|(T_N^{\omega_j}f)(T_N^{\omega_{j'}}f)\|^2_{L^2(Q_N)} + CN^{-10}\|f\|^4_{L^4(0,1)},$$

where $Q_N = (\log N)^{1+2\delta}Q$.

We are now going to estimate $T_N^{\omega_j}f$ and shall first study $K_N^{\omega_j}$. Letting u_j denote the left endpoint of ω_j and setting

$$\varrho(u) = \varrho_j(u; y) = e^{iNy(\psi(u+u_j)-\psi'(u_j)u)}\varphi(u+u_j)$$

we obtain

$$K_N^{\omega_j}(x, y) = e^{iNxu_j}N\int_0^{|\omega_j|} e^{iN(x+y\psi'(u_j))u}\varrho(u)\,du.$$

We also set

$$g(a) = N\int_0^{|\omega_j|} e^{iNau}\varrho(u)\,du$$

and then have

$$K_N^{\omega_j}(x, y) = e^{iNxu_j}g(x+y\psi'(u_j)).$$

From the definition of ω_j it follows that

$$|\varrho(u)| \leqslant C \quad\text{and}\quad |\varrho'(u)| \leqslant C(\log N)^{1+2\delta}2^l \quad \text{for } 0 \leqslant u \leqslant |\omega_j|$$

if $|y| \leqslant 10(\log N)^{1+2\delta}$ and integrating by parts in the integral defining g we can prove that

(14) $$|g^{(s)}(a)| \leqslant C(\log N)^{1+2\delta}(N2^{-l})^s\min(N2^{-l}, |a|^{-1}), \quad s = 0, 1, 2.$$

Setting $\varkappa_i = ((i-1)2\pi N^{-1}2^l, i2\pi N^{-1}2^l)$, $i \in \boldsymbol{Z}$, we obtain

$$|T_N^{\omega_j}f(x, y)| \leqslant \sum_{i=-\infty}^{\infty}\Big|\int_{\varkappa_i} g(x+y\psi'(u_j)-t)e^{-iNu_jt}f(t)\,dt\Big|$$

We also set $n = n(x, y) = [(2\pi)^{-1}N2^{-l}(x+y\psi'(u_j))]$, where [ ] denotes the integral part. It then follows from (14) that

$$\Big|\frac{\partial^s g}{\partial t^s}(x+y\psi'(u_j)-t)\Big| \leqslant C(\log N)^{1+2\delta}(N2^{-l})^{s+1}(1+|i-n|)^{-1},$$

$t \in \varkappa_i$, $s = 0, 1, 2$, and hence

$$g(x+y\psi'(u_j)-t) = \sum_{\nu=-\infty}^{\infty}\gamma_\nu e^{-iN2^{-l}3^{-1}\nu t}, \quad t \in \varkappa_i, (x, y) \in Q_N$$

where

$$|\gamma_\nu| \leqslant C(1+|\nu|^2)^{-1}(\log N)^{1+2\delta}N2^{-l}(1+|i-n|)^{-1}, \quad \nu \in \boldsymbol{Z}$$

(see [2], Lemma 3).

Using this representation of g we obtain

$$\begin{aligned} |T_N^{\omega_j}f(x, y)| &\leqslant C(\log N)^{1+2\delta}\sum_{i=-\infty}^{\infty}(1+|i-n|)^{-1}C_{2^lu_j}(\varkappa_i; f) \\ &= C(\log N)^{1+2\delta}\sum_{|\mu|\leqslant N^2}(1+|\mu|)^{-1}C_{2^lu_j}(\varkappa_{n+\mu}; f), \quad (x, y) \in Q_N, \end{aligned}$$

since f vanishes outside the interval $(0, 1)$.

From Schwarz's inequality it follows that

$$(15)\qquad |T_N^{\omega_j} f(x,y)|^2 \leqslant C(\log N)^{3+4\delta} \sum_{|\mu|\leqslant N^2} (1+|\mu|)^{-1} C_{2^l u_j}(\varkappa_{n+\mu}; f)^2, \qquad (x,y)\in Q_N.$$

We shall now estimate the sum in (13) using the above inequality. The technique is similar to the proof in [6].

It follows from the definition of the intervals ω_j that $|\psi'(u_j)-\psi'(u_{j'})| \geqslant N^{-1}2^l$ if $j\neq j'$ and we may also assume that the above difference is less than a small constant for all j, j'. We let s_0 be the smallest integer such that $N^{-1}2^l > 2^{-s_0}$ and conclude that $s_0 \leqslant C\log N$. We also set

$$\mathscr{A}_s = \{(j,j');\ 2^{-s-1} < |\psi'(u_j)-\psi'(u_{j'})| \leqslant 2^{-s}\},$$

$s\in \boldsymbol{Z}$, $s<s_0$, and $\mathscr{A}_{s_0} = \{(j,j');\ \omega_j = \omega_{j'}\}$.

Setting $n_j(x,y) = \big[(2\pi)^{-1}N2^{-l}\big(x+y\psi'(u_j)\big)\big]$ and defining $n_{j'}(x,y)$ analogously we see from a geometrical argument that

$$|\{(x,y)\in Q_N;\ n_j(x,y) = n,\ n_{j'}(x,y) = n'\}| \leqslant C(\log N)^{1+2\delta} N^{-2} 2^{2l+s}$$

for all integers n, n' if $(j,j')\in\mathscr{A}_s$. Also $(x,y)\in Q_N$, $(j,j')\in\mathscr{A}_s$ implies that

$$|n_j(x,y)-n_{j'}(x,y)| \leqslant C(\log N)^{1+2\delta} N 2^{-l-s}.$$

Hence

$$(16)\qquad \sum_{j,j'} \|(T_N^{\omega_j} f)(T_N^{\omega_{j'}} f)\|^2_{L^2(Q_N)} \leqslant \sum_{|\mu|\leqslant N^2}\ \sum_{|\nu|\leqslant N^2} (1+|\mu|)^{-1}(1+|\nu|)^{-1} S(\mu,\nu),$$

where

$$S(\mu,\nu) = C(\log N)^{6+8\delta} \sum_{s\leqslant s_0}\ \sum_{(j,j')\in\mathscr{A}_s} I_{j,j'}$$

and

$$I_{j,j'} = \iint_{Q_N} C_{2^l u_j}(\varkappa_{n_j+\mu}; f)^2 C_{2^l u_{j'}}(\varkappa_{n_{j'}+\nu}; f)^2\, dx\, dy$$
$$\leqslant D_s \sum_{|n-n'|\leqslant C_s} C_{2^l u_j}(\varkappa_{n+\mu}; f)^2 C_{2^l u_{j'}}(\varkappa_{n'+\nu}; f)^2,$$

where

$$D_s = C(\log N)^{1+2\delta} N^{-2} 2^{2l+s} \quad\text{and}\quad C_s = C(\log N)^{1+2\delta} N 2^{-l-s}.$$

From Parseval's formula it follows that

$$(17)\qquad S(\mu,\nu) \leqslant C(\log N)^{7+10\delta} \sum_{s\leqslant s_0}\ \sum_{|n-n'|\leqslant C_s} 2^s \Big(\int_{\varkappa_{n+\mu}} |f|^2\,dt\Big)\Big(\int_{\varkappa_{n'+\nu}} |f|^2\,dt\Big).$$

We set

$$B_k = \{n\in\boldsymbol{Z};\ |n-kC_s|\leqslant C_s\}, \qquad k\in\boldsymbol{Z},$$

and

$$A_{\mu,k} = \bigcup_{n\in B_k} \varkappa_{n+\mu}, \qquad k\in\boldsymbol{Z}.$$

Hence $|A_{\mu,k}| \leqslant C(\log N)^{1+2\delta} 2^{-s}$ and the last sum in (17) is majorized by

$$2^{s+1}\sum_k \sum_{n,n'\in B_k} \Big(\int_{\varkappa_{n+\mu}} |f|^2\,dt\Big)\Big(\int_{\varkappa_{n'+\nu}} |f|^2\,dt\Big) = 2^{s+1}\sum_k \Big(\int_{A_{\mu,k}} |f|^2\,dt\Big)\Big(\int_{A_{\nu,k}} |f|^2\,dt\Big)$$
$$\leqslant C2^s \sum_k \Big(\int_{A_{\mu,k}} |f|^4\,dt\,|A_{\mu,k}| + \int_{A_{\nu,k}} |f|^4\,dt\,|A_{\nu,k}|\Big)$$
$$\leqslant C(\log N)^{1+2\delta}\int_0^1 |f|^4\,dt \leqslant C(\log N)^{1+2\delta}\|f\|^4_{L^4(0,1)}.$$

Hence the left-hand side of (16) is less than $C(\log N)^{11+12\delta}\|f\|^4_{L^4(0,1)}$ and (7) follows if we use (12) and (13).

In the case $\Omega_m = \omega_K$ the above argument yields (7) with the first factor after the constant removed and an application of (5) and Lemma 1 completes the proof of Lemma 2.

We shall now use the above lemma to prove the multiplier theorem.

Proof of Theorem 1. Cover Γ with finitely many small open discs D_j, $j = 1, 2, \ldots, n$, so that δ is C^∞ and $m = \delta^a$ in $\Omega\cap D_j$ for each j. Choose $\varphi_j\in C_0^\infty(\boldsymbol{R}^2)$ such that $\operatorname{supp}\varphi_j\subset D_j$, $j = 1, 2, \ldots, n$, and $\sum_1^n \varphi_j = 1$ in a neighbourhood of Γ.

Writing $m = m(1-\sum_1^n\varphi_j)+\sum_1^n m\varphi_j$ we observe that the first term is C^2 and has compact support and thus is a multiplier for $L^p(\boldsymbol{R}^2)$ for $1\leqslant p\leqslant\infty$. We then fix j and shall study $m\varphi_j$.

Performing a rotation we may assume that $\operatorname{supp}\varphi_j\subset I\times\boldsymbol{R}$, where I is a compact interval on $\boldsymbol{R}$ and that δ equals the distance to a curve $\{(u,v)\in\boldsymbol{R}^2;\ u\in I,\ v = \psi(u)\}$, where $\psi\in C^\infty(I)$, in $\operatorname{supp}\varphi_j$. Since $\big(\delta(u,v)/|v-\psi(u)|\big)^a$ is C^∞ in a neighbourhood of $\operatorname{supp}\varphi_j$ it is sufficient to prove that $\big(v-\psi(u)\big)_+^a\varphi_j(u,v)$ is a multiplier (here $x_+ = \max(x,0)$, $x\in\boldsymbol{R}$). We may also assume (following Hörmander [8]) that $\varphi_j(u,v) = \varphi(u)\varrho\big(v-\psi(u)\big)$, where $\varphi\in C^\infty(I)$ and $\varrho\in C_0^\infty(\boldsymbol{R})$.

Letting K denote the inverse Fourier transform of $\big(v-\psi(u)\big)_+^a\varphi_j(u,v)$ we get

$$(18)\qquad K(x,y) = (2\pi)^{-2}\iint_{I\times\boldsymbol{R}} e^{i(xu+yv)}\varphi(u)\varrho\big(v-\psi(u)\big)\big(v-\psi(u)\big)_+^a\,du\,dv$$
$$= (2\pi)^{-2}\int_I e^{i(xu+y\psi(u))}\varphi(u)\,du\int_0^\infty e^{iyv}\varrho(v)v^a\,dv.$$

We let Q' and Q'' be two squares in the plane with sides parallel to the coordinate axes and side length $1/8$ and assume that the distance between them is $\geqslant 1/8$ and $\leqslant 2$. Then let f have support in Q' and set

$$S_N f(x, y) = \iint_{Q'} N^2 K\big(N(x-t), N(y-s)\big) f(t, s)\, dt\, ds, \quad (x, y) \in Q'', N \geqslant 2.$$

We shall prove that

$$(19) \qquad \|S_N f\|_{L^q(Q'')} \leqslant C_q N^{1/2-2/q-a} (\log N)^4 \|f\|_{L^q(Q')}, \quad 4 < q \leqslant \infty.$$

The last integral in (18) equals $Cy^{-1-a} + O(y^{-2-a})$, $y \to +\infty$, and it follows from Lemma 2 that

$$(20) \quad \Big(\iint_{\{(x,y)\in Q''; |y-s| \geqslant c_0\}} \Big| \int_{\boldsymbol{R}} N^2 K\big(N(x-t), N(y-s)\big) f(t, s)\, dt \Big|^q dx\, dy \Big)^{1/q} \leqslant C_q N^{1/2-2/q-a} (\log N)^4 \Big(\int_{\boldsymbol{R}} |f(t, s)|^q\, dt \Big)^{1/q}, \quad 4 < q < \infty,$$

for all values of s if c_0 is a positive constant and an analogous estimate holds for $q = \infty$.

If $|y-s| < c_0$ and c_0 is chosen small enough, then it follows from repeated partial integrations in the first integral on the right-hand side of (18) that

$$\big| N^2 K\big(N(x-t), N(y-s)\big) \big| \leqslant CN^{-a}, \quad (x, y) \in Q'', \quad (t, s) \in Q'$$

and hence (20) holds with $|y-s| \geqslant c_0$ replaced by $|y-s| < c_0$. Minkowski's inequality for integrals yields (19) and Theorem 1 can be obtained from the following standard argument.

Choose $\Phi \in C_0^\infty(\boldsymbol{R})$, non-vanishing only in the interval $(1/2, 2)$, such that $\sum_0^\infty \Phi(2^{-k} t) = 1$ for $t \geqslant 1$. Set $K_k(x) = K(x)\Phi(2^{-k}|x|)$, $x \in \boldsymbol{R}^2$, $k = 0, 1, 2, \ldots$

If f has support in a square with side length 2^{k-3} it follows from (19) with $N = 2^k$ and a change of scale that

$$\|K_k * f\|_{L^q(\boldsymbol{R}^2)} \leqslant C_q 2^{k(1/2-2/q-a)} k^4 \|f\|_{L^q(\boldsymbol{R}^2)}, \quad 4 < q \leqslant \infty,$$

and the same estimate can be obtained for a general f by writing $f = \sum_i f\chi_i$, where χ_i are characteristic functions of squares with side length 2^{k-3}.

If $0 < a \leqslant 1/2$ and $4 < q < 4/(1-2a)$ or $a > 1/2$ and $4 < q \leqslant \infty$, $\sum_0^\infty 2^{k(1/2-2/q-a)} k^4$ converges and hence m is a multiplier for $L^q(\boldsymbol{R}^2)$. The sufficiency of the condition on p in Theorem 1 then follows from interpolation and duality.

That the condition is also necessary follows from essentially the same simple argument as in the case when Γ is the unit circle (see e.g. [4], pp. 10–11).

The following result on summability of Fourier integrals is a consequence of Theorem 1.

COROLLARY 1. *Let Γ, Ω and m satisfy the conditions of Theorem 1 and suppose that $0 \in \Omega$ and $m(0) = 1$. Assume that either $0 < a \leqslant 1/2$ and $4/(3+2a) < p \leqslant 2$ or $a > 1/2$ and $1 \leqslant p \leqslant 2$. For $R > 0$ define the operator S_R on $L^p(\boldsymbol{R}^2)$ by $(S_R f)^\wedge = m_R \hat{f}$, where $m_R(x) = m(R^{-1}x)$, $x \in \boldsymbol{R}^2$. Then $S_R f$ converges to f in $L^p(\boldsymbol{R}^2)$ when R tends to infinity if $f \in L^p(\boldsymbol{R}^2)$.*

Proof. There exist positive numbers d_1 and d_2 such that $\Gamma \subset \{x \in \boldsymbol{R}^2;\ d_1 < |x| < d_2\}$. We choose φ and ψ in $C_0^\infty(\boldsymbol{R}^2)$ such that $\varphi(x) = 1$ in a neighbourhood of the origin, $\operatorname{supp}\varphi \subset \{x \in \boldsymbol{R}^2;\ |x| < d_1\}$ and $\varphi(x) + \psi(x) = 1$ for $|x| \leqslant d_2$. Let $f \in L^p(\boldsymbol{R}^2)$ and write $S_R f = S_R' f + S_R'' f$, where $(S_R' f)^\wedge = \varphi_R m_R \hat{f}$, $(S_R'' f)^\wedge = \psi_R m_R \hat{f}$ and φ_R and ψ_R are defined in the same way as m_R.

Since $\varphi m \in C^2$, we have $\lim_{R\to\infty} \|S_R' f - f\|_{L^p(\boldsymbol{R}^2)} = 0$.

A dilation shows that the functions m_R are multipliers for $L^p(\boldsymbol{R}^2)$ of uniformly bounded norm and using the fact that ψ is smooth and vanishes in a neighbourhood of the origin we conclude that $\lim_{R\to\infty} \|S_R'' f\|_{L^p(\boldsymbol{R}^2)} = 0$, which completes the proof of the corollary.

A similar results on summability of Fourier series can also be obtained from Theorem 1, since a continuous multiplier for $L^p(\boldsymbol{R}^2)$ corresponds to a multiplier for $L^p(\boldsymbol{T}^2)$ (see [10], p. 260).

2. Proof of Theorem 2. We shall use the following lemma.

LEMMA 3. *Let I be a compact interval on $\boldsymbol{R}$, let $\psi \in C^2(I)$ and assume that ψ is real-valued and $\psi''(t) \geqslant 0$ for $t \in I$. Set*

$$Sf(x, y) = \int_I e^{-i(xt + y\psi(t))} f(t)\, dt, \quad (x, y) \in \boldsymbol{R}^2, f \in L^1(I).$$

Then

$$(21) \qquad \|Sf\|_{L^q(\boldsymbol{R}^2)} \leqslant C_q |I|^{1-1/p-3/q} \|f \psi''^{-1/q}\|_{L^p(I)}, \qquad 4 < q < \infty, \quad q/(q-3) \leqslant p \leqslant \infty,$$

where C_q does not depend on I or ψ.

Proof. We first assume that $4 < q < \infty$, $p = q/(q-3)$ and that the right-hand side of (21) is finite. We use the method in [3], pp. 289–290. We have

$$\big(Sf(x, y)\big)^2 = 2 \iint_{\{(t,s)\in I\times I; t<s\}} e^{-i(x(t+s) + y(\psi(t)+\psi(s)))} f(t) f(s)\, dt\, ds,$$

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and setting $u = t+s$, $v = \psi(t)+\psi(s)$ we get

$$(Sf(x,y))^2 = 2\iint_D e^{-i(xu+yv)} f(t)f(s)|\psi'(t)-\psi'(s)|^{-1}du\,dv,$$

where t and s are functions of u and v and D is the image in the (u,v)-plane of $I\times I$ under the above mapping.

Defining r by $2/q+1/r = 1$, using Hausdorff–Young's inequality and changing variables once more we obtain

$$\|Sf\|^2_{L^q(\mathbf{R}^2)} \leqslant C\Big(\iint_{I\times I} |f(t)|^r|f(s)|^r|\psi'(t)-\psi'(s)|^{1-r}dt\,ds\Big)^{1/r}.$$

We set $\xi = \psi'(t)$, $\eta = \psi'(s)$ and it follows that

$$\|Sf\|_{L^q(\mathbf{R}^2)} \leqslant C\Big(\iint_{\psi'(I)\times\psi'(I)} |f(t)|^r|f(s)|^r|\xi-\eta|^{1-r}(\psi''(t))^{-1}(\psi''(s))^{-1}d\xi d\eta\Big)^{1/2r}.$$

We now use Hölder's inequality and the theorem on fractionary integrals as in the case of non-vanishing curvature (cf. [8]) and conclude that

$$\|Sf\|_{L^q(\mathbf{R}^2)} \leqslant C_q\Big(\int_{\psi'(I)} |f(t)|^{p_0r}(\psi''(t))^{-p_0}d\xi\Big)^{1/p_0r} = C_q\|f\psi''^{-1/q}\|_{L^p(I)}, \tag{22}$$

where $p_0 = p/r$. Hence (21) is proved in the case $p = q/(q-3)$ and the remaining case follows from Hölder's inequality.

Proof of Theorem 2. The result in Theorem 2, case (ii) follows immediately from Lemma 3 and it remains to treat the C^∞ case. We may assume that $\Gamma = \{(u,v)\in\mathbf{R}^2;\ u\in I,\ v = \psi(u)\}$, where I is a compact interval and $\psi\in C^\infty(I)$. We set

$$S'f(x,y) = \int_I e^{-i(xt+y\psi(t))}f(t)\,dt$$

and

$$S_nf(x,y) = \int_{I_n} e^{-i(xt+y\psi(t))}f(t)\,dt, \quad n = 1,2,3,\ldots,$$

where I_n are the component intervals of $\{t\in I;\ \psi''(t)\neq 0\}$.

If q, p and γ satisfy the conditions in Theorem 2 it follows from Lemma 3 and Lemma 1 that

$$\|S'f\|_{L^q(\mathbf{R}^2)} \leqslant \sum_{n=1}^\infty \|S_nf\|_{L^q(\mathbf{R}^2)} \leqslant C_q\sum_{n=1}^\infty \|f|\psi''|^{-1/q}\|_{L^p(I_n)}$$
$$\leqslant C_q\sum_{n=1}^\infty (\sup_{I_n}|\psi''|)^{\gamma-1/q}\|f|\psi''|^{-\gamma}\|_{L^p(I_n)} \leqslant C_{q,\gamma}\|f|\psi''|^{-\gamma}\|_{L^p(I)}$$

and Theorem 2 is proved.

The following result on restrictions of Fourier transforms follows from Theorem 2 by duality.

COROLLARY 2. (i) *If Γ satisfies the conditions of case* (i) *in Theorem* 2, *then*

$$\|\hat f|K|^\gamma\|_{L^p(\Gamma;ds)} \leqslant C_{q,\gamma}\|f\|_{L^q(\mathbf{R}^2)},$$

if $1 < q < 4/3$, $1 \leqslant p \leqslant q/3(q-1)$ and $\gamma > (q-1)/q$.

(ii) *If Γ satisfies the conditions of case* (ii) *in Theorem* 2, *then the above inequality holds also for $\gamma = (q-1)/q$.*

The following estimate follows from Corollary 2 if we apply Hölder's inequality.

COROLLARY 3. *Let Γ be a C^{n+1} curve in $\mathbf{R}^2$, for some integer $n\geqslant 3$, which has non-vanishing curvature except at finitely many points. Assume that the highest order of contact of the tangent at these points is $n-1$. Then*

$$\|\hat f\|_{L^p(\Gamma;ds)} \leqslant C_{p,q}\|f\|_{L^q(\mathbf{R}^2)},$$

if $1\leqslant p\leqslant\infty$, $1\leqslant q\leqslant\infty$ and $1/(n+1)p+1/q > 1$.

We shall finally give examples of curves Γ for which the conditions on γ in Theorem 2 cannot be weakened. We begin with case (ii) and let $\Gamma = \{(u,v)\in\mathbf{R}^2;\ 0\leqslant u\leqslant 1/2,\ v = \psi(u)\}$, where $\psi(t) = e^{-1/t}$, $0 < t\leqslant 1/2$, and $\psi(0) = 0$. Assume that $4 < q < \infty$, $q/(q-3)\leqslant p < \infty$ and that

$$\|Sf\|_{L^q(\mathbf{R}^2)} \leqslant C_{p,q,\gamma}\|f\psi''^{-\gamma}\|_{L^p(0,1/2)}.$$

We shall prove that then necessarily $\gamma\geqslant 1/q$. We set $f(t) = (\psi''(t))^\beta$, $0\leqslant t\leqslant\varepsilon$, and $f(t) = 0$, $\varepsilon < t\leqslant 1/2$, where $\beta = \gamma p/(p-1)$ and ε is a small positive number. It follows that

$$|Sf(x,y)| \geqslant \frac{1}{10}\int_0^\varepsilon (\psi'')^\beta dt, \quad |x|\leqslant\frac{1}{10\varepsilon}, \quad |y|\leqslant\frac{1}{10\psi(\varepsilon)},$$

and hence

$$\int_0^\varepsilon (\psi'')^\beta dt\,(\varepsilon\psi(\varepsilon))^{-1/q} \leqslant C_{p,q,\gamma}\Big(\int_0^\varepsilon (\psi'')^{\beta p-\gamma p}dt\Big)^{1/p}.$$

Using the choice of β we obtain

$$\Big(\int_0^\varepsilon (\psi'')^{\gamma p/(p-1)}dt\Big)^{(p-1)/p} \leqslant C_{p,q,\gamma}(\varepsilon\psi(\varepsilon))^{1/q},$$

and a calculation shows that this can hold for small values of ε only if $\gamma\geqslant 1/q$.

The same argument works also in the case $p = \infty$. We then let Γ be given by the function ψ, defined by $\psi(t) = e^{-1/t}\sin(1/t^k)$, $0 < t\leqslant c$, and $\psi(0) = 0$, where k is a large positive integer and c a small constant. We assume that $4 < q < \infty$, $q/(q-3) \leqslant p < \infty$ (the same argument works for $p = \infty$) and shall prove that there is no constant $C_{p,q}$ such that.

$$\|Sf\|_{L^q(\mathbf{R}^2)} \leqslant C_{p,q} \|f|\psi''|^{-1/q}\|_{L^p(0,c)}.$$

We set $f(t) = |\psi''(t)|^{\beta}$, $0 \leqslant t \leqslant 1/n$, and $f(t) = 0$ otherwise, where $\beta = p/q(p-1)$ and n is a large positive integer, and the above inequality yields

$$\Big(\int_0^{1/n} |\psi''|^{\beta} dt\Big)^{(p-1)/p} \leqslant C_{p,q} n^{-1/q} e^{-n/q}.$$

A computation shows that the last integral is larger than $c_0 n^{2(k+1)\beta-2} e^{-\beta n}$, where $c_0 > 0$, and we obtain a contradiction if k is chosen large enough, e.g. $k > q$.

We finally remark that a counterexample constructed in a similar way shows that if $1/(n+1)p + 1/q < 1$, then the inequality in Corollary 3 does not hold.

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