FOURIER-PADÉ APPROXIMANTS FOR ANGELESCO SYSTEMS

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1. Introduction

In this paper we study linear and non-linear Fourier-Padé approximation for Angelesco systems of functions. This construction is similar to that of Hermite-Padé approximation. Instead of considering power series expansions of the functions in the system, we take their expansion in a series of orthogonal polynomials.

In [6] and [7], S. P. Suetin obtained convergence results for rows of Fourier-Padé approximation extending to this setting classical results of the theory of Padé approximation.

Diagonal sequences of Fourier-Padé approximation were studied by A. A. Gonchar, E. A. Rakhmanov, and S. P. Suetin in [2] when the function to be approximated is of Markov type; that is, the Cauchy transform of a measure supported on the real line. They give the rate of convergence of diagonal sequences of linear and non-linear Padé approximants in terms of the equilibrium measures of a related potential theoretic problem. We

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generalize those results to the case when a system of Markov functions is given defined by measures whose supports do not intersect.

Let $\mathcal{M}(\Delta)$ denote the class of all finite, Borel measures with compact support consisting of an infinite set of points contained in an interval Δ of the real line $\mathbb{R}$. Given $\sigma \in \mathcal{M}(\Delta)$, let

$$\widehat{\sigma}(z) = \int \frac{d\sigma(x)}{z-x}$$

be the associated *Markov* function. Let $\Delta_k, k = 1, \ldots, m$, be intervals of the real line such that

$$\Delta_k \cap \Delta_j = \emptyset\,, \qquad k \neq j\,,$$

and $\sigma_k \in \mathcal{M}(\Delta_k), k = 1, \ldots, m$. We say that $\sigma = (\sigma_1, \ldots, \sigma_m)$ forms an Angelesco system of measures and $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_m)$ is the associated Angelesco system of functions.

Let $\sigma_0 \in \mathcal{M}(\Delta_0)$. Likewise, we will assume that

$$\Delta_0 \cap \Delta_k = \emptyset\,, \qquad k = 1, \ldots, m\,.$$

Consider the sequence $\{\ell_n\}, n \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$, of orthonormal polynomials with respect to σ_0 with positive leading coefficient. Take a multi-index $\mathbf{n} = (n_1, \ldots, n_m) \in \mathbb{Z}_+^m$. Set

$$|\mathbf{n}| = n_1 + \cdots + n_m\,.$$

Let $Q_{\mathbf{n}}, P_{\mathbf{n},1}, \ldots, P_{\mathbf{n},m}$, be polynomials such that:

i) $\deg Q_{\mathbf{n}} \leq |\mathbf{n}|, Q_{\mathbf{n}} \not\equiv 0\,, \deg P_{\mathbf{n},j} \leq |\mathbf{n}| - 1, j = 1, \ldots, m\,.$

ii) For each $j = 1, \ldots, m$, and $k = 0, \ldots, |\mathbf{n}| + n_j - 1$

$$c_k(Q_{\mathbf{n}}\widehat{\sigma}_j - P_{\mathbf{n},j}) = \int (Q_{\mathbf{n}}\widehat{\sigma}_j - P_{\mathbf{n},j})(x)\ell_k(x)d\sigma_0(x) = 0\,.$$

Notice that

$$P_{\mathbf{n},j}(z) = \sum_{i=0}^{|\mathbf{n}|-1} c_i(Q_{\mathbf{n}}\widehat{\sigma}_j)\ell_i(z)\,. \tag{1}$$

The $|\mathbf{n}| + 1$ coefficients of $Q_{\mathbf{n}}$ satisfy a homogeneous linear system of $|\mathbf{n}|$ equations given by

$$c_k(Q_{\mathbf{n}}\widehat{\sigma}_j) = 0\,, \qquad j = 1, \ldots, m\,, \qquad k = |\mathbf{n}|, \ldots, |\mathbf{n}| + n_j - 1\,.$$

Therefore, a non-trivial solution is guaranteed.

In Section 2 we will prove that every solution to i)-ii) has $\deg Q_{\mathbf{n}} = |\mathbf{n}|$. This being the case, $(Q_{\mathbf{n}}, P_{\mathbf{n},1}, \ldots, P_{\mathbf{n},m})$ is uniquely determined up to a constant factor. In fact, let us assume that $(Q_{\mathbf{n}}, P_{\mathbf{n},1}, \ldots, P_{\mathbf{n},m})$, and $(\widetilde{Q}_{\mathbf{n}}, \widetilde{P}_{\mathbf{n},1}, \ldots, \widetilde{P}_{\mathbf{n},m})$, are solutions of i)-ii). Without loss of generality, we can assume that $Q_{\mathbf{n}}$ and $\widetilde{Q}_{\mathbf{n}}$ are monic (with leading coefficient equal to one). Obviously, if $Q_{\mathbf{n}} - \widetilde{Q}_{\mathbf{n}} \not\equiv 0$ then $(Q_{\mathbf{n}} - \widetilde{Q}_{\mathbf{n}}, P_{\mathbf{n},1} - \widetilde{P}_{\mathbf{n},1}, \ldots, P_{\mathbf{n},m} - \widetilde{P}_{\mathbf{n},m})$ is also a solution with $\deg Q_{\mathbf{n}} - \widetilde{Q}_{\mathbf{n}} < |\mathbf{n}|$ which contradicts our assumption. Hence $Q_{\mathbf{n}} \equiv \widetilde{Q}_{\mathbf{n}}$ and by (1) it follows that $P_{\mathbf{n},j} \equiv \widetilde{P}_{\mathbf{n},j}, j = 1 \ldots, m$.

The rational vector function $\left(\frac{P_{\mathbf{n},1}}{Q_{\mathbf{n}}}, \ldots, \frac{P_{\mathbf{n},m}}{Q_{\mathbf{n}}}\right)$ constructed from any solution of i)-ii) is called the **n**-th linear Fourier-Padé approximant for the Angelesco system $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_m)$ with respect to σ_0. We shall see that for all $\mathbf{n} \in \mathbb{Z}_+^m$ the linear Fourier-Padé approximant of an Angelesco system is unique.

Non-linear Fourier-Padé approximants are determined as follows. Given $\mathbf{n} \in \mathbb{Z}_+^m$, we must find polynomials $T_{\mathbf{n}}$, $S_{\mathbf{n},1}, \ldots, S_{\mathbf{n},m}$ such that

- i') $\deg T_{\mathbf{n}} \leq |\mathbf{n}|, T_{\mathbf{n}} \not\equiv 0\,, \deg(S_{\mathbf{n},j}) \leq |\mathbf{n}| - 1$, $j = 1, \ldots, m$.
- ii') For each $j = 1, \ldots, m$, and $k = 0, \ldots, |\mathbf{n}| + n_j - 1$

$$c_k\left(\widehat{\sigma}_j - \frac{S_{\mathbf{n},j}}{T_{\mathbf{n}}}\right) = \int \left(\widehat{\sigma}_j - \frac{S_{\mathbf{n},j}}{T_{\mathbf{n}}}\right)(x)\ell_k(x)d\sigma_0(x) = 0\,.$$

This system of equations is non-linear in the coefficients of the polynomials. We shall prove that for each $\mathbf{n} \in \mathbb{Z}_+^m$, the system has a solution but we have not been able to show that it is unique. For any solution of i')-ii'), the vector rational function $\left(\frac{S_{\mathbf{n},1}}{T_{\mathbf{n}}}, \ldots, \frac{S_{\mathbf{n},m}}{T_{\mathbf{n}}}\right)$ is called an **n**-th non-linear Fourier-Padé approximant for the Angelesco system $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_m)$ with respect to σ_0.

In this paper, we obtain the rate of convergence (divergence) of linear and non-linear Fourier-Padé approximants for Angelesco systems such that the measures $\sigma_0, \ldots, \sigma_m$ are in the class **Reg** of regular measures. For different equivalent forms of defining regular measures see sections 3.1 to 3.3 in [5]. In particular, $\sigma_0 \in \mathbf{Reg}$ if and only if

$$\lim_n |\ell_n(z)|^{1/n} = \exp\{g_{\Omega_0}(z;\infty)\} ,$$

uniformly on compact subsets of the complement of the smallest interval containing the support, $\operatorname{supp}(\sigma_0)$, of σ_0 and $g_{\Omega_0}(\cdot;\infty)$ denotes Green's function for the region $\Omega_0 = \mathbb{C} \setminus \operatorname{supp}(\sigma_0)$ with singularity at ∞. Analogously, one defines regularity for the other measures $\sigma_1, \ldots, \sigma_m$. In the sequel, we write $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \mathbf{Reg}$ to mean that $\sigma_k \in \mathbf{Reg}, k = 0, \ldots, m$. The system $(\sigma_1, \ldots, \sigma_m)$ will be used to construct the Angelesco system of functions whereas σ_0 will determine the system of orthogonal polynomials with respect to which the Fourier expansions will be taken. Therefore, for all $0 \le j, k \le m$, we assume that

$$\Delta_j \cap \Delta_k = \emptyset , \qquad j \neq k .$$

In Theorems 1 and 2 below, we find the rate of convergence of the $|\mathbf{n}|$th root of the error of approximation of the functions $\widehat{\sigma}_k$ by linear and non-linear Fourier-Padé approximants, respectively. The answers are given in terms of extremal solutions of certain vector valued equilibrium problems for the logarithmic potential. Before stating Theorems 1 and 2, we need to introduce some notation and results from potential theory.

Let $F_k, k = 1, \ldots, N$, be (not necessarily distinct) closed bounded intervals of the real line and $\mathcal{C} = (c_{j,k})$ be a real, positive definite, symmetric matrix of order N. $\mathcal{C}$ will be called the interaction matrix. By $\mathcal{M}_1(F_k), k = 1, \ldots, N$, we denote the subclass of probability measures of $\mathcal{M}(F_k)$ and

$$\mathcal{M}_1 = \mathcal{M}_1(F_1) \times \cdots \times \mathcal{M}_1(F_N) .$$

Given a vector measure $\mu \in \mathcal{M}_1$ and $j = 1, \ldots, N$, we define the combined potential

$$W_j^{\mu}(x) = \sum_{k=1}^{N} c_{j,k} V^{\mu_k}(x)\,, \qquad x \in \Delta_j\,, \tag{2}$$

where

$$V^{\mu_k}(x) = \int \log \frac{1}{|x-t|}\, d\mu_k(t)$$

denotes the standard logarithmic potential of μ_k. We denote

$$w_j^{\mu} = \text{ínf}\{W_j^{\mu}(x) : x \in \Delta_j\}\,.$$

In Chapter 5 of [3] (see Propositions 4.5, 4.6, and Theorem 4.1) the authors prove (we state the result in a form convenient for our purpose)

Lemma 1. *Let $\mathcal{C}$ be a real, positive definite, symmetric matrix of order N. If there exists $\overline{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_N) \in \mathcal{M}_1$ such that for each $j = 1, \ldots, N$*

$$W_j^{\overline{\mu}}(x) = w_j^{\overline{\mu}}\,, \qquad x \in supp(\overline{\mu}_j)\,,$$

then $\overline{\mu}$ is unique. Moreover, if $c_{j,k} \geq 0$ when $F_j \cap F_k \neq \emptyset$ then $\overline{\mu}$ exists.

The vector measure $\overline{\mu} \in \mathcal{M}_1$ is called the equilibrium solution for the vector potential problem determined by $\mathcal{C}$ on the system of intervals $F_j\,, j = 1, \ldots, N\,.$

In the sequel $\Lambda = \Lambda(p_1, \ldots, p_m) \subset \mathbb{Z}_+^m$ is an infinite system of distinct multi-indices such that

$$\lim_{\mathbf{n}\in\Lambda} \frac{n_j}{|\mathbf{n}|} = p_j \in (0,1)\,, \qquad j = 1, \ldots, m\,.$$

Let us define the block matrix

$$\mathcal{C}_1 = \begin{pmatrix} \mathcal{C}_{1,1} & \mathcal{C}_{1,2} \\ \mathcal{C}_{2,1} & \mathcal{C}_{2,2} \end{pmatrix},$$

where

$$\mathcal{C}_{1,1} = \begin{pmatrix} 2p_1^2 & p_1p_2 & \cdots & p_1p_m \\ p_2p_1 & 2p_2^2 & \cdots & p_2p_m \\ \vdots & \vdots & \ddots & \vdots \\ p_mp_1 & p_mp_2 & \cdots & 2p_m^2 \end{pmatrix},$$

and $\mathcal{C}_{1,2}, \mathcal{C}_{2,1}, \mathcal{C}_{2,2}$ are diagonal matrices given by

$$\mathcal{C}_{1,2} = \mathcal{C}_{2,1} = \mathrm{diag}\{-p_1(1+p_1), -p_2(1+p_2), \cdots, -p_m(1+p_m)\},$$

and

$$\mathcal{C}_{2,2} = \mathrm{diag}\{2(1+p_1)^2, 2(1+p_2)^2, \cdots, 2(1+p_m)^2\}.$$

$\mathcal{C}_1$ satisfies all the assumptions of Lemma 1 on the system of intervals $F_j = \Delta_j, j = 1,\ldots,m, F_j = \Delta_0, j = m+1,\ldots,2m$, including $c_{j,k} \geq 0$ when $F_j \cap F_k \neq \emptyset$. The only non-trivial property is its positive definiteness and we shall prove this in Section 2. Let $\overline{\mu} = \overline{\mu}(\mathcal{C}_1)$ be the equilibrium solution for the corresponding vector potential problem. We have

Theorem 1. *Let* $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \mathbf{Reg}$ *and consider a sequence of multi-indices* $\Lambda = \Lambda(p_1,\ldots,p_m)$. *Let* $\left(\frac{P_{\mathbf{n},1}}{Q_{\mathbf{n}}}, \ldots, \frac{P_{\mathbf{n},m}}{Q_{\mathbf{n}}}\right), \mathbf{n} \in \Lambda$, *be the associated sequence of linear Fourier-Padé approximants for the Angelesco system of functions* $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_m)$ *with respect to* σ_0. *Then,*

$$\lim_{\mathbf{n}\in\Lambda} \left| \widehat{\sigma}_j(z) - \frac{P_{\mathbf{n},j}(z)}{Q_{\mathbf{n}}(z)} \right|^{1/|\mathbf{n}|} = G_j(z), \qquad j = 1,\ldots,m, \tag{3}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus (\cup_{k=0}^m \Delta_k)$, *where*

$$G_j(z) = \exp\left((W_j^{\overline{\mu}}(z) - \omega_j^{\overline{\mu}})/p_j \right),$$

$\overline{\mu} = \overline{\mu}(\mathcal{C}_1)$, *and the combined potentials* $W_j^{\overline{\mu}}$ *are defined by* (2) *using* $\mathcal{C}_1$.

Set

$$G_j^{\pm} = \{x \in \overline{\mathbb{C}} \setminus (\cup_{k=0}^m \Delta_k) : \pm \left(\omega_j^{\overline{\mu}} - W_j^{\overline{\mu}}(x) \right) > 0\}.$$

An immediate consequence of Theorem 1 is

Corollary 1. *Under the assumptions of Theorem 1,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}} = \widehat{\sigma}_j\,, \qquad j = 1,\ldots,m\,,$$

uniformly on compact subsets of G_j^+ *and diverges to infinity at each point of* G_j^-.

Non-linear Fourier Padé approximants require the solution of a different vector potential equilibrium problem. Let

$$\mathcal{C}_2 = \begin{pmatrix} \mathcal{C}_{1,1} & \mathcal{C}_{1,2} \\ \mathcal{C}_{2,1} & \mathcal{C}_{2,2}^2 \end{pmatrix},$$

where $\mathcal{C}_{1,1}, \mathcal{C}_{1,2}, \mathcal{C}_{2,1}$ are as before and

$$\mathcal{C}_{2,2}^2 = \begin{pmatrix} \frac{2m(1+p_1)^2}{m+1} & \frac{-2(1+p_1)(1+p_2)}{m+1} & \cdots & \frac{-2(1+p_1)(1+p_m)}{m+1} \\ \frac{-2(1+p_2)(1+p_1)}{m+1} & \frac{2m(1+p_2)^2}{m+1} & \cdots & \frac{-2(1+p_2)(1+p_m)}{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{-2(1+p_m)(1+p_1)}{m+1} & \frac{-2(1+p_m)(1+p_2)}{m+1} & \cdots & \frac{2m(1+p_m)^2}{m+1} \end{pmatrix}.$$

$\mathcal{C}_2$ is a real, positive definite, symmetric matrix of order $2m$. We take the system of intervals $F_j = \Delta_j, j = 1,\ldots,m, F_j = \Delta_0, j = m+1,\ldots,2m$. $\mathcal{C}_2$ does not satisfy that $c_{j,k} \geq 0$ when $F_j \cap F_k \neq \emptyset$. In Theorem 4 of Section 3, we prove that the corresponding equilibrium problem has at least one solution and that $\mathcal{C}_2$ is positive definite. Therefore, according to Lemma 1 the solution is unique. Let $\overline{\mu} = \overline{\mu}(\mathcal{C}_2)$ be the equilibrium solution for the corresponding vector potential problem. In Lemma 5 we show that for each $\mathbf{n} \in \mathbb{Z}_+^m$ there exists at least one non-linear Fourier- Padé approximant but we have not been able to prove that it is unique. We have

Theorem 2. *Let* $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \mathbf{Reg}$ *and consider a sequence of multi-indices* $\Lambda = \Lambda(p_1, \ldots, p_m)$. *Let* $\left(\frac{S_{\mathbf{n},1}}{T_{\mathbf{n}}}, \ldots, \frac{S_{\mathbf{n},m}}{T_{\mathbf{n}}}\right), \mathbf{n} \in \Lambda$, *be an associated sequence of non-linear Fourier-Padé approximants for the Angelesco system of functions* $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_m)$ *with respect to* σ_0. *Then,*

$$\lim_{\mathbf{n}\in\Lambda} \left| \widehat{\sigma}_j(z) - \frac{S_{\mathbf{n},j}(z)}{T_{\mathbf{n}}(z)} \right|^{1/|\mathbf{n}|} = H_j(z)\,, \qquad j = 1,\ldots,m\,, \tag{4}$$

uniformly on each compact subset of $\overline{\mathbb{C}} \setminus (\cup_{k=0}^{m} \Delta_j)$, *where*

$$H_j(z) = \exp\left((W_j^{\overline{\mu}}(z) - \omega_j^{\overline{\mu}})/p_j\right) ,$$

$\overline{\mu} = \overline{\mu}(\mathcal{C}_2)$ *and the combined potentials* $W_j^{\overline{\mu}}$ *are defined by* (2) *using* $\mathcal{C}_2$.

Notice that the limit only depends on Λ and not on the non-linear Fourier-Padé approximants selected (in case that they were not uniquely determined). Set

$$H_j^{\pm} = \{x \in \overline{\mathbb{C}} \setminus (\cup_{j=0}^{m} \Delta_j : \pm \left(\omega_j^{\overline{\mu}} - W_j^{\overline{\mu}}(x)\right) > 0\}.$$

As a consequence of Theorem 2, we obtain

Corollary 2. *Under the assumptions of Theorem 2,*

$$\lim_{\mathbf{n}\in\Lambda} \frac{S_{\mathbf{n},j}}{T_{\mathbf{n}}} = \widehat{\sigma}_j\,, \qquad j = 1, \ldots, m\,,$$

uniformly on compact subsets of H_j^{+} *and diverges to infinity at each point of* H_j^{-}.

Section 2 is dedicated to the proof of Theorem 1 and Section 3 to that of Theorem 2. Section 4 is dedicated to the justification of Lemma 1 as stated here since in [3] the assumption $c_{j,k} \geq 0$ if $F_j \cap F_k \neq \emptyset$ is assumed in general. In the sequel, we maintain the notation introduced above.

2. Proof of Theorem 1

From the definition of the linear Fourier-Padé approximant immediately follows that for each $j = 1, \ldots, m$

$$(5) \qquad \int x^k (Q_{\mathbf{n}}(x)\widehat{\sigma}_j(z) - P_{\mathbf{n},j}(x)) d\sigma_0(x) = 0, \quad k = 0, \ldots, |\mathbf{n}| + n_j - 1\,.$$

Since the function $Q_{\mathbf{n}}(z)\widehat{\sigma}_j(z) - P_{\mathbf{n},j}(z)$ is continuous on Δ_0, from (5) we have that $Q_{\mathbf{n}}(z)\widehat{\sigma}_j(z) - P_{\mathbf{n},j}(z)$ has at least $|\mathbf{n}| + n_j$ sign changes on Δ_0.

Let $W_{\mathbf{n},j}$ be the monic polynomial whose zeros are the points where $Q_{\mathbf{n}}(z)\widehat{\sigma}_j(z) - P_{\mathbf{n},j}(z)$ changes sign on the interval Δ_0. Obviously, $\deg W_{\mathbf{n},j} \geq$

$|\mathbf{n}|+n_j$ and

$$\frac{Q_{\mathbf{n}}(z)\widehat{\sigma}_j(z)-P_{\mathbf{n},j}(z)}{W_{\mathbf{n},j}(z)}\in\mathcal{H}(\overline{\mathbb{C}}\setminus\operatorname{supp}(\sigma_j)),\quad j=1,\ldots,m\,,$$

is analytic on the indicated region. Thus, linear Fourier-Padé approximants satisfy interpolation conditions on Δ_0. A similar statement holds for the non-linear Fourier-Padé approximants. In our proofs, we will use certain orthogonality relations satisfied by vector rational interpolants.

Lemma 2. *Let* $(\widehat{\sigma}_1,\ldots,\widehat{\sigma}_m)$ *be an Angelesco system,* $\mathbf{n}=(n_1,\ldots,n_m)\in\mathbb{Z}_+^m$*, and* $(w_{\mathbf{n},1},\ldots,w_{\mathbf{n},m})$ *a system of polynomials such that* $\deg w_{\mathbf{n},j}\geq|\mathbf{n}|+n_j, j=1,\ldots,m,$ *whose zeros lie on an interval* $\Delta_0,\Delta_0\cap\Delta_j=\emptyset, j=1,\ldots,m$*. Let* $(\frac{p_{\mathbf{n},1}}{q_{\mathbf{n}}},\ldots,\frac{p_{\mathbf{n},m}}{q_{\mathbf{n}}})$ *be a vector rational function such that* $\deg p_{\mathbf{n},j}\leq|\mathbf{n}|-1, j=1,\ldots,m,\deg q_{\mathbf{n}}\leq|\mathbf{n}|, q_{\mathbf{n}}\not\equiv 0,$ *and*

$$\frac{q_{\mathbf{n}}(z)\widehat{\sigma}_j(z)-p_{\mathbf{n},j}(z)}{w_{\mathbf{n},j}(z)}\in\mathcal{H}(\overline{\mathbb{C}}\setminus supp(\sigma_j)),\quad j=1,\ldots,m\,. \tag{6}$$

Then

$$\int x^k\frac{q_{\mathbf{n}}(x)}{w_{\mathbf{n},j}(x)}d\sigma_j(x)=0\,,\quad k=0,1,\ldots,n_j-1\,,\quad j=1,\ldots,m\,. \tag{7}$$

Consequently, $\deg q_{\mathbf{n}}=|\mathbf{n}|$ *with exactly* n_j *simple zeros in the interior of* Δ_j *(in connection with intervals of the real line, the interior refers to the Euclidean topology of the real line) and* $\deg w_{\mathbf{n},j}=|\mathbf{n}|+n_j, j=1,\ldots,m$*. Let* $q_{\mathbf{n}}=q_{\mathbf{n},j}\widetilde{q}_{\mathbf{n},j}$*, where* $q_{\mathbf{n},j}$ *is the monic polynomial whose zeros are those of* $q_{\mathbf{n}}$ *lying in the interior of* Δ_j*. Then*

$$\widehat{\sigma}_j(z)-\frac{p_{\mathbf{n},j}(z)}{q_{\mathbf{n}}(z)}=\frac{w_{\mathbf{n},j}(z)}{q^2_{\mathbf{n},j}(z)\widetilde{q}_{\mathbf{n},j}(z)}\int\frac{q^2_{\mathbf{n},j}(x)}{z-x}\frac{\widetilde{q}_{\mathbf{n},j}(x)}{w_{\mathbf{n},j}(x)}d\sigma_j(x)\,. \tag{8}$$

Proof. Notice that (6) and the assumption on the degrees of the polynomials $q_{\mathbf{n}},p_{\mathbf{n},j}$, and $w_{\mathbf{n},j}$ imply that for $j=1,\ldots,m$, and $k=0,\ldots,n_j-1$,

$$z^k\frac{q_{\mathbf{n}}(z)\widehat{\sigma}_j(z)-p_{\mathbf{n},j}(z)}{w_{\mathbf{n},j}(z)}=\mathcal{O}\left(\frac{1}{z^2}\right),\quad z\to\infty\,.$$

Let Γ_j be a closed, smooth, Jordan curve that surrounds Δ_j such that all the intervals $\Delta_i, i\neq j, i=0,\ldots,m$, lie in the unbounded connected component of the complement of Γ_j. By Cauchy's Theorem, Cauchy's Integral Formula and Fubini's Theorem, it follows that

$$0 = \frac{1}{2\pi i}\int_{\Gamma_j} z^k \frac{q_{\mathbf{n}}(z)\widehat{\sigma}_j(z) - p_{\mathbf{n},j}(z)}{w_{\mathbf{n},j}(z)} dz =$$

$$\frac{1}{2\pi i}\int_{\Gamma_j} z^k \frac{q_{\mathbf{n}}(z)\widehat{\sigma}_j(z)}{w_{\mathbf{n},j}(z)} dz - \frac{1}{2\pi i}\int_{\Gamma_j} z^k \frac{p_{\mathbf{n},j}(z)}{w_{\mathbf{n},j}(z)} dz =$$

$$\frac{1}{2\pi i}\int_{\Gamma_j} z^k \frac{q_{\mathbf{n}}(z)}{w_{\mathbf{n},j}(z)} \int \frac{d\sigma_j(x)}{z-x}\, dz = \int x^k \frac{q_{\mathbf{n}}(x)}{w_{\mathbf{n},j}(x)} d\sigma_j(x),$$

for $k = 0, 1, \ldots, n_j - 1$ and $j = 1, \ldots, m$. Therefore, (7) follows.

Using standard arguments of orthogonality, from (7) we obtain that $q_{\mathbf{n}}$ must have at least n_j sign changes in the interior of Δ_j and, consequently, at least n_j zeros of odd multiplicity. Since $\deg q_{\mathbf{n}} \leq |\mathbf{n}|$, we have that $\deg q_{\mathbf{n}} = |\mathbf{n}|$, that all its zeros are simple and they are distributed in such a way that exactly n_j lie in the interior of Δ_j.

Assume that $\deg w_{\mathbf{n},j} > |\mathbf{n}| + n_j$ for some j. Then

$$z^k \frac{q_{\mathbf{n}}(z)\widehat{\sigma}_j(z) - p_{\mathbf{n},j}(z)}{w_{\mathbf{n},j}(z)} = \mathcal{O}\left(\frac{1}{z^2}\right), \quad z \to \infty, \qquad k = 0, \ldots, n_j.$$

This implies that (7) holds for all $k = 0, \ldots, n_j$. In turn, this means that $q_{\mathbf{n}}$ has at least $n_j + 1$ zeros in the interior of Δ_j against what was just proved. Therefore, $\deg w_{\mathbf{n},j} = |\mathbf{n}| + n_j$.

Set $q_{\mathbf{n}} = q_{\mathbf{n},j}\widetilde{q}_{\mathbf{n},j}$, where $q_{\mathbf{n},j}$ is the monic polynomial whose zeros are those of $q_{\mathbf{n}}$ lying in the interior of Δ_j. Notice that $\widetilde{q}_{\mathbf{n},j} d\sigma_j / w_{\mathbf{n},j}$ is a real measure with constant sign on Δ_j. For future reference, notice that with this notation the orthogonality relations (7) may be expressed as

$$\text{(9)} \qquad \int x^k q_{\mathbf{n},j}(x)|\widetilde{q}_{\mathbf{n},j}(x)| \frac{d\sigma_j(x)}{|w_{\mathbf{n},j}(x)|} = 0, \quad k = 0, 1, \ldots, n_j - 1.$$

Hence, for each $j = 1, \ldots, m$, $q_{\mathbf{n},j}$ is the monic orthogonal polynomial of degree n_j with respect to the varying measure $\frac{|\widetilde{q}_{\mathbf{n},j}|}{|w_{\mathbf{n},j}|} d\sigma_j$.

Notice that

$$\frac{[q_{\mathbf{n},j}(q_{\mathbf{n}}\widehat{\sigma}_j - p_{\mathbf{n},j})](z)}{w_{\mathbf{n},j}(z)} = \mathcal{O}\left(\frac{1}{z}\right), \quad z \to \infty.$$

Choose Γ_j as before. Using Cauchy's integral formula, Cauchy's Theorem, and Fubini's Theorem, we obtain that for each $j = 1, \ldots, m$

$$\frac{[q_{\mathbf{n},j}(q_{\mathbf{n}}\widehat{\sigma}_j - p_{\mathbf{n},j})](z)}{w_{\mathbf{n},j}(z)} = \frac{1}{2\pi i}\int_{\Gamma_j} \frac{[q_{\mathbf{n},j}(q_{\mathbf{n}}\widehat{\sigma}_j - p_{\mathbf{n},j})](\zeta)}{w_{\mathbf{n},j}(\zeta)(z-\zeta)} d\zeta =$$

$$\int \frac{1}{2\pi i}\int_{\Gamma_j} \frac{(q_{\mathbf{n},j}q_{\mathbf{n}})(\zeta)}{w_{\mathbf{n},j}(\zeta)(z-\zeta)(\zeta - x)} d\zeta d\sigma_j(x) = \int \frac{q^2_{\mathbf{n},j}(x)}{z-x}\frac{\widetilde{q}_{\mathbf{n},j}(x)}{w_{\mathbf{n},j}(x)} d\sigma_j(x)\,,$$

which is equivalent to (8). □

The vector rational function $(\frac{p_{\mathbf{n},1}}{q_{\mathbf{n}}}, \ldots, \frac{p_{\mathbf{n},m}}{q_{\mathbf{n}}})$ is called a multipoint vector Padé approximant of the Angelesco system $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_m)$. According to Lemma 2 a necessary condition for their existence is that $\deg w_{\mathbf{n},j} \leq |\mathbf{n}| + n_j, j = 1, \ldots, m$. Solving a homogeneous linear system of equations one sees that this condition is also sufficient. When $\deg w_{\mathbf{n},j} = |\mathbf{n}| + n_j, j = 1, \ldots, m$, uniqueness follows because then $\deg q_{\mathbf{n}} = |\mathbf{n}|$ as we have seen.

Remark 1. Applying this Lemma to linear Fourier-Padé approximants, we have that $\deg(Q_{\mathbf{n}}) = |\mathbf{n}|$. Thus, for each $\mathbf{n} \in \mathbb{Z}^m_+$, they are uniquely determined as claimed.

Let us return to linear Fourier-Padé approximants. In this case, $Q_{\mathbf{n}} = q_{\mathbf{n}}, Q_{\mathbf{n},j} = q_{\mathbf{n},j}, \widetilde{Q}_{\mathbf{n},j} = \widetilde{q}_{\mathbf{n},j}$ and $W_{\mathbf{n},j} = w_{\mathbf{n},j}$.

Lemma 3. *For each* $j = 1, \ldots, m$, *and* $k = 0, \ldots, |\mathbf{n}| + n_j - 1$

$$\int t^k \frac{W_{\mathbf{n},j}(t)}{|Q_{\mathbf{n},j}(t)|}\left(\int \frac{Q^2_{\mathbf{n},j}(x)}{|t-x|}\frac{|\widetilde{Q}_{\mathbf{n},j}(x)|}{|W_{\mathbf{n},j}(x)|} d\sigma_j(x)\right) d\sigma_0(t) = 0\,. \tag{10}$$

Moreover, $\deg W_{\mathbf{n},j} = |\mathbf{n}| + n_j, j = 1, \ldots, m$; *that is,* $Q_{\mathbf{n}}(z)\widehat{\sigma}_j(z) - P_{\mathbf{n},j}(z)$ *has exactly* $|\mathbf{n}| + n_j$ *sign changes in the interior of* Δ_0.

Proof. From (8) and the definition of the linear Fourier-Padé approximant, (10) follows directly. The assertion concerning the degree of $W_{\mathbf{n},j}$ is also contained in Lemma 2. □

Let $\{\mu_l\} \subset \mathcal{M}(\mathcal{K})$ be a sequence of measures, where $\mathcal{K}$ is a compact subset of the complex plane and $\mu \in \mathcal{M}(\mathcal{K})$. We write

$$* \lim_{l} \mu_l = \mu \,, \qquad \mu \in \mathcal{M}(\mathcal{K}) \,,$$

if for every continuous function $f \in \mathcal{C}(\mathcal{K})$

$$\lim_{l} \int f d\mu_l = \int f d\mu \,;$$

that is, when the sequence of measures converges to μ in the weak star topology. Given a polynomial q_l of degree $l \geq 1$, we denote the associated normalized zero counting measure by

$$\nu_{q_l} = \frac{1}{l} \sum_{q_l(x)=0} \delta_x \,,$$

where δ_x is the Dirac measure with mass 1 at x (in the sum the zeros are repeated according to their multiplicity).

In order to prove our main results we need Theorem 3.3.3 of [5]. We present it in the form stated in [1] which is more adequate for our purpose. In [1], it was proved under stronger assumptions on the measure.

Lemma 4. *Let $\{\phi_l\}, l \in \Gamma \subset \mathbb{Z}_+$, be a sequence of positive continuous functions on a bounded closed interval $\Delta \subset \mathbb{R}$, $\sigma \in \mathbf{Reg} \cap \mathcal{M}(\Delta)$, and let $\{q_l\}, l \in \Gamma$, be a sequence of monic polynomials such that* $\deg q_l = l$ *and*

$$\int q_l(t) t^k \phi_l(t) d\sigma(t) = 0, \quad k = 0, \dots, l-1.$$

Assume that

$$\lim_{l \in \Gamma} \frac{1}{2l} \log \frac{1}{|\phi_l(x)|} = v(x),$$

uniformly on Δ. Then

$$* \lim_{l \in \Gamma} \nu_{q_l} = \overline{\nu},$$

and

$$\lim_{l \in \Gamma} \left(\int |q_l|^2 \phi_l d\mu \right)^{1/2l} = e^{-\omega},$$

where $\overline{\nu} \in \mathcal{M}_1(\Delta_1)$ *is the unique solution of the equilibrium problem*

$$V^{\overline{\nu}}(x) + v(x) \begin{cases} = \omega, & x \in supp(\overline{\nu})\,, \\ \geq \omega, & x \in \Delta_1\,, \end{cases}$$

in the presence of the external field v.

Using this result, we can obtain the asymptotic limit distribution of the zeros of the polynomials $Q_{\mathbf{n},j}$ and $W_{\mathbf{n},j}$.

Theorem 3. *Let* $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \mathbf{Reg}$ *and consider a sequence of multi-indices* $\Lambda = \Lambda(p_1, \ldots, p_m)$. *Then, for each* $j = 1, \ldots, m$

$$* \lim_{\mathbf{n}\in\Lambda} \nu_{Q_{\mathbf{n},j}} = \overline{\mu}_j\,, \qquad * \lim_{\mathbf{n}\in\Lambda} \nu_{W_{\mathbf{n},j}} = \overline{\mu}_{m+j}\,,$$

where $\overline{\mu} = \overline{\mu}(\mathcal{C}_1) \in \mathcal{M}_1$ *is the vector equilibrium measure determined by the matrix* $\mathcal{C}_1$ *on the system of intervals* $F_j = \Delta_j, j = 1, \ldots, m\,,\; F_j = \Delta_0, j = m+1, \ldots, 2m\,.$

Proof. The unit ball in the cone of positive Borel measures is weakly compact; therefore, it is sufficient to show that the sequences of measures $\{\nu_{Q_{\mathbf{n},j}}\}$ and $\{\nu_{W_{\mathbf{n},j}}\}, \mathbf{n} \in \Lambda$, have only one accumulation point which coincide, respectively, with the components of the vector measure $\overline{\mu}(\mathcal{C}_1)$. Let $\Lambda' \subset \Lambda$ be a subsequence of multi-indices such that for each $j = 1, \ldots, m$

$$* \lim_{\mathbf{n}\in\Lambda'} \nu_{Q_{\mathbf{n},j}} = \nu_j\,, \qquad * \lim_{\mathbf{n}\in\Lambda'} \nu_{W_{\mathbf{n},j}} = \nu_{m+j}\,.$$

(Notice that $\nu_j \in \mathcal{M}_1(\Delta_j), j = 1, \ldots, m$, and $\nu_j \in \mathcal{M}_1(\Delta_0), j = m+1, \ldots, 2m$.) Therefore,

$$\lim_{\mathbf{n}\in\Lambda'} |Q_{\mathbf{n},j}(z)|^{\frac{1}{n_j}} = \exp(-V^{\nu_j}(z)), \tag{11}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$, and

$$\lim_{\mathbf{n}\in\Lambda'} |W_{\mathbf{n},j}(z)|^{\frac{1}{|\mathbf{n}|+n_j}} = \exp(-V^{\nu_{m+j}}(z)), \tag{12}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_0$.

For each fixed $j = 1\ldots,m$, the polynomials $Q_{\mathbf{n},j}$ satisfy the orthogonality relations (9). Using (11) and (12) it follows that

$$\lim_{\mathbf{n}\in\Lambda'} \frac{1}{2n_j}\log\frac{|W_{\mathbf{n},j}(x)|}{|\widetilde{Q}_{\mathbf{n},j}(x)|} = -\frac{1+p_j}{2p_j}V^{\nu_{m+j}}(x) + \sum_{k\neq j}\frac{p_k}{2p_j}V^{\nu_k}(x),$$

uniformly on Δ_j. By Lemma 4, ν_j is the unique equilibrium measure for the extremal problem

$$V^{\nu_j}(x) + \sum_{k\neq j}\frac{p_k}{2p_j}V^{\nu_k}(x) - \frac{1+p_j}{2p_j}V^{\nu_{m+j}}(x) \geq \theta_j, \quad x\in\Delta_j\,, \tag{13}$$

with equality for all $x\in\operatorname{supp}(\nu_j)$. Additionally,

$$\lim_{\mathbf{n}\in\Lambda'}\left(\int |Q_{\mathbf{n},j}(x)|^2\frac{|\widetilde{Q}_{\mathbf{n},j}(x)|d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|}\right)^{\frac{1}{2n_j}} = e^{-\theta_j}. \tag{14}$$

On the other hand, for each fixed $j = 1,\ldots,m$, the polynomials $W_{\mathbf{n},j}$ satisfy the orthogonality relations (10) and we can apply once more Lemma 4. Notice that for all $t\in\Delta_0$

$$\int\frac{|Q^2_{\mathbf{n},j}(x)|}{|t-x|}\frac{|\widetilde{Q}_{\mathbf{n},j}(x)|d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|} \leq \frac{1}{\delta_j}\int|Q_{\mathbf{n},j}(x)|^2\frac{|\widetilde{Q}_{\mathbf{n},j}(x)|d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|}, \tag{15}$$

where $\delta_j = \inf\{|t-x| : t\in\Delta_0,\, x\in\Delta_j\}$ and

$$\int\frac{|Q^2_{\mathbf{n},j}(x)|}{|t-x|}\frac{|\widetilde{Q}_{\mathbf{n},j}(x)|d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|} \geq \frac{1}{\delta^*_j}\int|Q_{\mathbf{n},j}(x)|^2\frac{|\widetilde{Q}_{\mathbf{n},j}(x)|d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|}, \tag{16}$$

with $\delta^*_j = \max\{|t-x| : t\in\Delta_0,\, x\in\Delta_j\}$. From (11), (12), (14), (15), and (16), we obtain

$$\lim_{\mathbf{n}\in\Lambda'}\frac{1}{2(|\mathbf{n}|+n_j)}\log\frac{|Q_{\mathbf{n},j}(x)|}{\int\frac{|Q_{\mathbf{n},j}(t)|^2}{|x-t|}\frac{|\widetilde{Q}_{\mathbf{n},j}(t)|d\sigma_j(t)}{|W_{\mathbf{n},j}(t)|}}$$
$$= -\frac{p_j}{2(1+p_j)}V^{\nu_j}(x) + \frac{p_j}{1+p_j}\theta_j,$$

uniformly on Δ_0. Using Lemma 4, ν_{m+j} is the unique extremal solution for the equilibrium problem

$$V^{\nu_{m+j}}(x) - \frac{p_j}{2(1+p_j)}V^{\nu_j}(x) + \frac{p_j}{1+p_j}\theta_j \geq \theta_{m+j}, \quad x\in\Delta_0\,, \tag{17}$$

with equality for all $x\in\operatorname{supp}(\nu_{m+j})$.

Rewriting (13) and (17) conveniently, we see that the vector measure $(\nu_1,\ldots,\nu_{2m}) \in \mathcal{M}_1$ is the unique solution for the vector equilibrium problem determined by the system of extremal problems

$$(18)\quad 2p_j^2 V^{\nu_j}(x) + \sum_{k\neq j} p_j p_k V^{\nu_k}(x) - p_j(1+p_j)V^{\nu_{m+j}}(x) \geq \omega_j, \quad x \in \Delta_j\,,$$

$(2p_j^2\theta_j = \omega_j)$ with equality for all $x \in \operatorname{supp}(\nu_j)$, and

$$(19)\qquad 2(1+p_j)^2 V^{\nu_{m+j}}(x) - p_j(1+p_j)V^{\nu_j}(x) \geq \omega_{m+j}\,, \quad x \in \Delta_0\,,$$

with equality for all $x \in \operatorname{supp}(\nu_{m+j})$. That is, it is the equilibrium measure $\overline{\mu} \in \mathcal{M}_1$ for the vector potential problem determined by $\mathcal{C}_1$ on the system of intervals $F_j = \Delta_j, j = 1,\ldots,m$, $F_j = \Delta_0, j = m+1,\ldots,2m$. The condition $c_{j,k} \geq 0$ if $F_j \cap F_k \neq \emptyset$ is fulfilled. According to Lemma 1, this equilibrium vector measure is uniquely determined if $\mathcal{C}_1$ is positive definite. Let us prove this.

For $j \in \{1,\ldots,m\}$ the principle minor $\mathcal{C}_1^{(j)}$ of order j of $\mathcal{C}_1$ is

$$\det(\mathcal{C}_1^{(j)}) = (p_1\cdots p_j)^2 \begin{vmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{vmatrix}_{j\times j} = (p_1\cdots p_j)^2(j+1) > 0\,.$$

For $j \in \{m+1,\ldots,2m\}$ the principle minor $\mathcal{C}_1^{(j)}$ of order j of $\mathcal{C}_1$ can be calculated as follows. For each $k = 1,\ldots,m$, factor out p_k from the kth row and kth column of $\mathcal{C}_1^{(j)}$. From the row and column $m+k, k = 1,\ldots,j-m$, factor out $1+p_k$. In the resulting determinant, for each $k = 1,\ldots,j-m$, add the kth row to the $(m+k)$th row and then to the resulting determinant add the kth column to the $(m+k)$th column. We obtain

$$\det(\mathcal{C}_1^{(j)}) = [p_1\cdots p_m(1+p_1)\cdots(1+p_{j-m})]^2 \begin{vmatrix} 2 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 \end{vmatrix}_{j\times j} =$$

$$[p_1 \cdots p_m (1+p_1) \cdots (1+p_{j-m})]^2 (j+1) > 0\,.$$

With this we conclude the proof. □

Proof of Theorem 1. From (8), (15), and (16), the asymptotic behavior of the function $\widehat{\sigma}_j - \frac{P_{\mathbf{n},j}}{Q_{\mathbf{n}}}$ depends on the behavior of $W_{\mathbf{n},j}$, $Q_{\mathbf{n},j}$, $\widetilde{Q}_{\mathbf{n},j}$, and $\gamma_{\mathbf{n},j}$, where

$$\frac{1}{\gamma_{\mathbf{n},j}^2} = \min_{Q} \left\{ \int |Q(x)|^2 \frac{|\widetilde{Q}_{\mathbf{n},j}(x)| d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|} \,:\, Q(x) = x^{n_j} + \cdots \right\}$$
$$= \int |Q_{\mathbf{n},j}(x)|^2 \frac{|\widetilde{Q}_{\mathbf{n},j}(x)| d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|}.$$

From Theorem 3, for each $j = 1, \ldots, m$, we have

$$\lim_{\mathbf{n}\in\Lambda} |W_{\mathbf{n},j}(x)|^{1/|\mathbf{n}|} = \exp\{-(1+p_j) V^{\overline{\mu}_{m+j}}(x)\}, \tag{20}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_0$, and

$$\lim_{\mathbf{n}\in\Lambda} |Q_{\mathbf{n},j}(x)|^{1/|\mathbf{n}|} = \exp\{-p_j V^{\overline{\mu}_j}(x)\}, \tag{21}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$, where $\overline{\mu} = \overline{\mu}(\mathcal{C}_1)$ Using (14) (see also parenthesis after (18)), it follows that

$$\lim_{|\mathbf{n}|\to\infty} \left(\frac{1}{\gamma_{\mathbf{n},j}^2} \right)^{1/|\mathbf{n}|} = \exp\{-2p_j\theta_j\} = \exp\{-\omega_j/p_j\}\,. \tag{22}$$

Combining (8), (20), (21), and (22), we conclude that (3) holds true uniformly on compact subsets of the indicated region. □

3. Proof of Theorem 2

We begin by proving the existence of non-linear Fourier-Padé approximants.

Lemma 5. *Given $(\sigma_0; \sigma_1, \ldots, \sigma_m)$, for each $\mathbf{n} \in \mathbb{Z}_+^m$ there exists an $\mathbf{n}$-th non-linear Fourier-Padé approximant of $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_m)$ with respect to σ_0.*

Proof. In the proof we make use of multipoint Hermite-Padé approximation. Fix $\mathbf{n} \in \mathbb{Z}_+^m$. For each $j \in \{1, \dots, m\}$, choose an arbitrary set of $|\mathbf{n}|+n_j$ points contained in Δ_0

$$X_{\mathbf{n},j} = (x_{\mathbf{n},j,1}, \dots, x_{\mathbf{n},j,|\mathbf{n}|+n_j}) \in \Delta_{\mathbf{n},j}\,,$$

where

$$\Delta_{\mathbf{n},j} = \{(x_1, \dots, x_{|\mathbf{n}|+n_j}) \in \Delta_0^{|\mathbf{n}|+n_j} : x_1 \le \cdots \le x_{|\mathbf{n}|+n_j}\}\,.$$

Let

$$w_{\mathbf{n},j}(x) = (x - x_{\mathbf{n},j,1}) \cdots (x - x_{\mathbf{n},j,|\mathbf{n}|+n_j})\,,$$

and consider the simultaneous multipoint Padé approximant which interpolates the functions $\widehat{\sigma}_j, j = 1, \dots, m,$ at the zeros of $w_{\mathbf{n},j}$ respectively. That is, $(p_{\mathbf{n},1}/q_{\mathbf{n}}, \dots, p_{\mathbf{n},m}/q_{\mathbf{n}})$ is a vector rational function such that $\deg(p_{\mathbf{n},j}) \le |\mathbf{n}| - 1$, $j = 1, \dots, m$, $\deg(q_{\mathbf{n}}) \le |\mathbf{n}|$, $q_{\mathbf{n}} \not\equiv 0$, and

$$\frac{q_{\mathbf{n}}\widehat{\sigma}_j - p_{\mathbf{n},j}}{w_{\mathbf{n},j}} \in \mathcal{H}(\mathbb{C} \setminus \operatorname{supp}(\sigma_j)). \tag{23}$$

From Lemma 2 we have (7) and (8). Once we have determined $q_{\mathbf{n}}$, for each $j = 1, \dots, m$, we define the monic polynomial $\Omega_{\mathbf{n},j}$, $\deg(\Omega_{\mathbf{n},j}) = |\mathbf{n}| + n_j$, by the orthogonality relations

$$\int y^k \Omega_{\mathbf{n},j}(y) \left(\frac{1}{q^2_{\mathbf{n},j}(y)\widetilde{q}_{\mathbf{n},j}(y)} \int \frac{q^2_{\mathbf{n},j}(x)}{y - x} \frac{\widetilde{q}_{\mathbf{n},j}(x) d\sigma_j(x)}{w_{\mathbf{n},j}(x)} \right) d\sigma_0(y) = 0, \tag{24}$$

$k = 0, \dots, |\mathbf{n}| + n_j - 1$. For each $j = 1, \dots, m$, these relations determine a unique $\Omega_{\mathbf{n},j}$ since the (varying) measures involved have constant sign on Δ_0.

The polynomial $\Omega_{\mathbf{n},j}$ has exactly $|\mathbf{n}| + n_j$ simple zeros in the interior of Δ_0. Set

$$Y_{\mathbf{n},j} = (y_{\mathbf{n},j,1}, \dots, y_{\mathbf{n},j,|\mathbf{n}|+n_j}) \in \Delta_{\mathbf{n},j}\,,$$

where $y_{\mathbf{n},j,1} < \cdots < y_{\mathbf{n},j,|\mathbf{n}|+n_j}$ are the zeros of $\Omega_{\mathbf{n},j}$.

Since for each $j = 1, \ldots, m$, the distance between Δ_j and Δ_0 is greater than zero, the correspondence

$$(X_{\mathbf{n},1}, \ldots, X_{\mathbf{n},m}) \longrightarrow (Y_{\mathbf{n},1}, \ldots, Y_{\mathbf{n},m}),$$

defines a continuous function from $\Delta_{\mathbf{n},1} \times \cdots \times \Delta_{\mathbf{n},m}$ into itself with the Euclidean norm. The continuity of this function is an easy consequence of the fact that $\Delta_0 \cap \Delta_j = \emptyset, j = 1, \ldots, m$. By Brouwer's fixed point Theorem (see page 364 of [4]) this function has at least one fixed point. Choose a fixed point. Then, $w_{\mathbf{n},j} = \Omega_{\mathbf{n},j}$, $j = 1, \ldots, m$. Consequently (24) can be rewritten as

$$\int y^k w_{\mathbf{n},j}(y) \left(\frac{1}{q^2_{\mathbf{n},j}(y) \widetilde{q}_{\mathbf{n},j}(y)} \int \frac{q^2_{\mathbf{n},j}(x)}{y - x} \frac{\widetilde{q}_{\mathbf{n},j}(x) d\sigma_j(x)}{w_{\mathbf{n},j}(x)} \right) d\sigma_0(y) = 0 \,, \tag{25}$$

$k = 0, \ldots, |\mathbf{n}| + n_j - 1$, and taking into consideration (8) we obtain that for each $j = 1, \ldots, m$,

$$\int \left(\widehat{\sigma}_j(x) - \frac{p_{\mathbf{n},j}(x)}{q_{\mathbf{n}}(x)} \right) x^k d\sigma_j(x) = 0 \,, \qquad k = 0, \ldots, |\mathbf{n}| + n_j - 1 \,.$$

From the definition, it follows that $(p_{\mathbf{n},1}/q_{\mathbf{n}}, \ldots, p_{n_m}/q_{\mathbf{n}})$ is an $\mathbf{n}$th non linear Fourier-Padé approximant for the Angelesco system, taking $S_{\mathbf{n},j} = p_{\mathbf{n},j}$, $j = 1, \ldots, m$, and $T_{\mathbf{n}} = q_{\mathbf{n}}$. □

Let $\left(\frac{S_{\mathbf{n},1}}{T_{\mathbf{n}}}, \ldots, \frac{S_{\mathbf{n},m}}{T_{\mathbf{n}}} \right)$ be any non-linear Fourier-Padé approximant with respect to the Angelesco system $(\widehat{\sigma}_1, \ldots, \widehat{\sigma}_m)$. From ii') it follows that $\widehat{\sigma}_j(z) - \frac{S_{\mathbf{n},j}(z)}{T_{\mathbf{n}}(z)}$ has at least $|\mathbf{n}| + n_j$ sign changes on Δ_0. Let $W_{\mathbf{n},j}$ be the monic polynomial whose zeros are the points where this function changes sign on Δ_0. Obviously, $\deg W_{\mathbf{n},j} \geq |\mathbf{n}| + n_j$ and

$$\frac{T_{\mathbf{n}}(z) \widehat{\sigma}_j(z) - S_{\mathbf{n},j}(z)}{W_{\mathbf{n},j}(z)} \in \mathcal{H}(\overline{\mathbb{C}} \setminus \operatorname{supp}(\sigma_j)), \quad j = 1, \ldots, m \,, \tag{26}$$

is analytic on the indicated region. (These polynomials $W_{\mathbf{n},j}$ do not coincide with those of the linear case.) Using Lemma 2 it follows that

$$\int x^k \frac{|T_{\mathbf{n}}(x)|}{|W_{\mathbf{n},j}(x)|} d\sigma_j(x) = 0 \,, \quad k = 0, 1, \ldots, n_j - 1 \,, \quad j = 1, \ldots, m \,. \tag{27}$$

and

$$\widehat{\sigma}_j(z) - \frac{S_{\mathbf{n},j}(z)}{T_{\mathbf{n}}(z)} = \frac{W_{\mathbf{n},j}(z)}{T^2_{\mathbf{n},j}(z)\widetilde{T}_{\mathbf{n},j}(z)} \int \frac{T^2_{\mathbf{n},j}(x)}{z-x} \frac{\widetilde{T}_{\mathbf{n},j}(x)}{W_{\mathbf{n},j}(x)} d\sigma_j(x)\,, \tag{28}$$

where $T_{\mathbf{n},j}$ is the monic polynomial whose zeros are the n_j zeros of $T_{\mathbf{n}}$ lying in the interior of Δ_j. Combining (28) with ii') we obtain

(29)

$$\int y^k W_{\mathbf{n},j}(y) \left(\frac{1}{T^2_{\mathbf{n},j}(y)|\widetilde{T}_{\mathbf{n},j}(y)|} \int \frac{T^2_{\mathbf{n},j}(x)}{|y-x|} \frac{|\widetilde{T}_{\mathbf{n},j}(x)| d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|} \right) d\sigma_0(y) = 0\,.$$

The proof of Theorem 2 is similar to that of Theorem 1. First, we study the asymptotic zero distribution of the polynomials $T_{\mathbf{n},j}$ and $W_{\mathbf{n},j}$. Then, we use this result to obtain the asymptotic behavior of the remainder in the approximation.

Theorem 4. *Let $(\sigma_0; \sigma_1, \ldots, \sigma_m) \in \mathbf{Reg}$ and consider a sequence of multi-indices $\Lambda = \Lambda(p_1, \ldots, p_m)$. Then, there exists $\overline{\mu} = (\overline{\mu}_1, \ldots, \overline{\mu}_{2m}) \in \mathcal{M}_1$ such that for each $j = 1, \ldots, m$*

$$*\lim_{\mathbf{n}\in\Lambda} \nu_{T_{\mathbf{n},j}} = \overline{\mu}_j\,, \qquad *\lim_{\mathbf{n}\in\Lambda} \nu_{W_{\mathbf{n},j}} = \overline{\mu}_{m+j}\,.$$

Moreover, $\overline{\mu} = \overline{\mu}(\mathcal{C}_2)$ is the vector equilibrium measure determined by the matrix $\mathcal{C}_2$ on the system of intervals $F_j = \Delta_j, j = 1, \ldots, m\,,\ F_j = \Delta_0, j = m+1, \ldots, 2m\,.$

Proof. Let us show that the sequences of measures $\{\nu_{T_{\mathbf{n},j}}\}$ and $\{\nu_{w_{\mathbf{n},j}}\}, \mathbf{n} \in \Lambda$, have only one accumulation point. Let $\Lambda' \subset \Lambda$ be a subsequence of indices such that for each $j = 1, \ldots, m$

$$*\lim_{\mathbf{n}\in\Lambda'} \nu_{T_{\mathbf{n},j}} = \nu_j\,, \qquad *\lim_{\mathbf{n}\in\Lambda'} \nu_{W_{\mathbf{n},j}} = \nu_{m+j}\,.$$

(Notice that $\nu_j \in \mathcal{M}_1(\Delta_j), j = 1, \ldots, m$, and $\nu_j \in \mathcal{M}_1(\Delta_0), j = m+1, \ldots, 2m$.) Therefore,

$$\lim_{\mathbf{n}\in\Lambda'} |T_{\mathbf{n},j}(z)|^{\frac{1}{n_j}} = \exp(-V^{\nu_j}(z))\,, \tag{30}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$, and

$$\lim_{\mathbf{n}\in\Lambda'} |W_{\mathbf{n},j}(z)|^{\frac{1}{|\mathbf{n}|+n_j}} = \exp(-V^{\nu_{m+j}}(z)), \tag{31}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_0$.

As we have seen, $T_{\mathbf{n},j}$ is orthogonal with respect to the varying measure $\frac{|\widetilde{T}_{\mathbf{n},j}|}{|W_{\mathbf{n},j}|} d\sigma_j$. Using (30) and (31), we obtain

$$\lim_{\mathbf{n}\in\Lambda'} \frac{1}{2n_j} \log \frac{|W_{\mathbf{n},j}(x)|}{|\widetilde{T}_{\mathbf{n},j}(x)|} = -\frac{1+p_j}{2p_j} V^{\nu_{m+j}}(x) + \sum_{k\neq j} \frac{p_k}{2p_j} V^{\nu_k}(x)\,,$$

uniformly in Δ_j. By (27) and Lemma 4, ν_j is the unique equilibrium measure for the extremal problem

$$V^{\nu_j}(x) + \sum_{k\neq j} \frac{p_k}{2p_j} V^{\nu_k}(x) - \frac{1+p_j}{2p_j} V^{\nu_{m+j}}(x) \geq \eta_j\,, \qquad x \in \Delta_j\,, \tag{32}$$

with equality for all $x \in \operatorname{supp}(\nu_j)$, and

$$\lim_{\mathbf{n}\in\Lambda'} \left(\int |T^2_{\mathbf{n},j}(x)| \frac{|\widetilde{T}_{\mathbf{n},j}(x)| d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|} \right)^{\frac{1}{2n_j}} = e^{-\eta_j}. \tag{33}$$

These relations are completely similar to those obtained for the linear case (see (13) and (14)). On the other hand, $W_{\mathbf{n},j}$ satisfies the orthogonality relations (29). We can apply once more Lemma 4 obtaining that, for each $j = 1,\ldots,m$, ν_{m+j} is the unique equilibrium measure for the extremal problem

$$V^{\nu_{m+j}}(x) - \frac{p_j}{1+p_j} V^{\nu_j}(x) - \sum_{k\neq j} \frac{p_k}{2(1+p_j)} V^{\nu_k}(x) \geq \eta_{m+j}\,, \qquad x \in \Delta_0\,, \tag{34}$$

with equality for all $x \in \operatorname{supp}(\nu_{m+j})$. These relations differ from those obtained for the linear case (see (17))

If we look at the matrix corresponding to this system of equations we see that it is not symmetric. Let us rewrite the system as follows. Multiply equations (32) times $2p_j^2$ and we obtain for each $j = 1,\ldots,m$,

$$2p_j^2 V^{\nu_j}(x) + \sum_{k\neq j} p_j p_k V^{\nu_k}(x) - p_j(1+p_j) V^{\nu_{m+j}}(x) \geq 2\eta_j p_j^2 = w_j\,, \qquad x \in \Delta_j\,, \tag{35}$$

with equality for all $x \in \mathrm{supp}(\nu_j)$. With equations (34) we have to work harder. First, let us multiply them times $2(1+p_j)$ thus obtaining for each $j = 1, \ldots, m$,

$$(36) \qquad -2p_j V^{\nu_j}(x) - \sum_{k \neq j} p_k V^{\nu_k}(x) + 2(1+p_j) V^{\nu_{m+j}}(x) \geq$$

$$2\eta_{m+j}(1+p_j) = \eta'_{m+j}\,, \quad x \in \Delta_0\,,$$

with equality for all $x \in \mathrm{supp}(\nu_{m+j})$.

Let us show that in this second group of equations we have equality for all $x \in \Delta_0$ ($\mathrm{supp}(\nu_{m+j}) = \Delta_0$). In fact, notice that $2(1+p_j)\nu_{m+j}$ is a measure on Δ_0 of total mass equal to $2(1+p_j)$. On the other hand

$$2p_j \nu_j + \sum_{k \neq j} p_k \nu_k$$

is a measure of total mass $p_j + \sum_{k=1}^m p_k = 1 + p_j < 2(1+p_j)$ supported on the set $\cup_{k=1}^m \Delta_k$ which is disjoint from Δ_0. Therefore,

$$2(1+p_j)\nu_{m+j} = (2p_j \nu_j + \sum_{k \neq j} p_k \nu_k)' + (1+p_j)\omega_{\Delta_0}\,,$$

where $(\cdot)'$ denotes the balayage onto Δ_0 of the indicated measure and ω_{Δ_0} is the equilibrium measure on Δ_0 (without external field). Since these two measures are supported on all Δ_0 so is their sum. Thus, $\mathrm{supp}(\nu_{m+j}) = \Delta_0\,.$

The idea now is to take row transformations on the system of equations (36) to transform it conveniently. The matrix of this system of equations is

$$\begin{pmatrix} -2p_1 & -p_2 & \cdots & -p_m & 2(1+p_1) & 0 & \cdots & 0 \\ -p_1 & -2p_2 & \cdots & -p_m & 0 & 2(1+p_2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -p_1 & -p_2 & \cdots & -2p_m & 0 & 0 & \cdots & 2(1+p_m) \end{pmatrix}.$$

Since each column has a common factor we will carry out the operations without the common factor and afterwards place them back. Thus in columns

$k = 1, \ldots, m$ we factor out $-p_k$ and in columns $k = m+1, \ldots, 2m$ we factor out $2(1 + p_{k-m})$, respectively. The resulting matrix is

$$\begin{pmatrix} 2 & 1 & \cdots & 1 & 1 & 0 & \cdots & 0 \\ 1 & 2 & \cdots & 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 & 0 & 0 & \cdots & 1 \end{pmatrix} = \begin{pmatrix} \mathcal{B} & \mathcal{I} \end{pmatrix},$$

where $\mathcal{I}$ denotes the identity matrix of order m. We know that the submatrix $\mathcal{B}$ is positive definite and through row operations can be reduced to the identity. This is the same as multiplying $\begin{pmatrix} \mathcal{B} & \mathcal{I} \end{pmatrix}$ on the left by $\mathcal{B}^{-1}$. Doing this we obtain the block matrix

$$\begin{pmatrix} \mathcal{I} & \mathcal{B}^{-1} \end{pmatrix}.$$

It is easy to check that

$$\mathcal{B}^{-1} = \frac{1}{m+1} \begin{pmatrix} m & -1 & \ldots & -1 \\ -1 & m & \ldots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & m \end{pmatrix}.$$

Multiplying back the factors we extracted we obtain the matrix

$$\begin{pmatrix} -p_1 & 0 & \cdots & 0 & \frac{2m(1+p_1)}{m+1} & \frac{-2(1+p_2)}{m+1} & \cdots & \frac{-2(1+p_m)}{m+1} \\ 0 & -p_2 & \cdots & 0 & \frac{-2(1+p_1)}{m+1} & \frac{2m(1+p_2)}{m+1} & \cdots & \frac{-2(1+p_m)}{m+1} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & -p_m & \frac{-2(1+p_1)}{m+1} & \frac{-2(1+p_2)}{m+1} & \cdots & \frac{2m(1+p_m)}{m+1} \end{pmatrix}.$$

Therefore, the system of equations (36) is equivalent to

$$-p_j V^{\nu_j}(x) + \frac{2m(1+p_j)}{m+1} V^{\nu_{m+j}}(x) - \sum_{k \neq j} \frac{2(1+p_k)}{m+1} V^{\nu_{m+j}}(x) = \eta''_{m+j}, \quad x \in \Delta_0, \tag{37}$$

where

$$(\eta''_{m+1}, \ldots, \eta''_{2m})^t = \mathcal{B}^{-1}(\eta'_{m+1}, \ldots, \eta'_{2m})^t.$$

Finally, multiply the jth equation in (37) times $(1+p_j)$ to obtain

$$-p_j(1+p_j)V^{\nu_j}(x)+\frac{2m(1+p_j)^2}{m+1}V^{\nu_{m+j}}(x)- \tag{38}$$

$$\sum_{k\neq j}\frac{2(1+p_k)(1+p_j)}{m+1}V^{\nu_{m+j}}(x)=\eta''_{m+j}(1+p_j)=w_{m+j}\,,\quad x\in\Delta_0\,.$$

The system of equilibrium problems defined by (35) and (38) has the interaction matrix

$$\mathcal{C}_2=\begin{pmatrix}\mathcal{C}_{1,1} & \mathcal{C}_{1,2}\\ \mathcal{C}_{2,1} & \mathcal{C}^2_{2,2}\end{pmatrix}$$

defined in Section 1. Thus, the corresponding equilibrium problem has at least one solution given by $(\nu_1,\ldots,\nu_m)$. According to Lemma 1, $(\nu_1,\ldots,\nu_m)$ is uniquely determined if we prove that $\mathcal{C}_2$ is positive definite.

Let us show that $\mathcal{C}_2$ is positive definite. The first m principal minors of $\mathcal{C}_1$ and $\mathcal{C}_2$ coincide and we already know that they are positive. Let $\mathcal{C}_2^{(j)}$ denote the principal minor of $\mathcal{C}_2$ of order j where $j\in\{m+1,\ldots,2m\}$. For each $k=1,\ldots,m$, factor out p_k from the kth row and kth column of $\mathcal{C}_2^{(j)}$. From the row and column $m+k, k=1,\ldots,j-m$, factor out $1+p_k$. In the resulting determinant, for each $k=1,\ldots,j-m$, add the kth row to the $(m+k)$th row and then to the resulting determinant add the kth column to the $(m+k)$th column. We obtain

$$\det(\mathcal{C}_2^{(j)})=[p_1\cdots p_m(1+p_1)\cdots(1+p_{j-m})]^2\times$$

$$\begin{vmatrix}
2 & 1 & \cdots & 1 & 1 & 1 & \cdots & 1\\
1 & 2 & \cdots & 1 & 1 & 1 & \cdots & 1\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
1 & 1 & \cdots & 2 & 1 & 1 & \cdots & 1\\
1 & 1 & \cdots & 1 & \frac{2m}{m+1} & \frac{m-1}{m+1} & \cdots & \frac{m-1}{m+1}\\
1 & 1 & \cdots & 1 & \frac{m-1}{m+1} & \frac{2m}{m+1} & \cdots & \frac{m-1}{m+1}\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
1 & 1 & \cdots & 1 & \frac{m-1}{m+1} & \frac{m-1}{m+1} & \vdots & \frac{2m}{m+1}
\end{vmatrix}$$

In the determinant above, delete the row $m+1$ from the following ones and in the resulting determinant add to the column $m+1$ those after it and we get

$$\begin{vmatrix} 2 & 1 & \cdots & 1 & j-m & 1 & \cdots & 1 \\ 1 & 2 & \cdots & 1 & j-m & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 2 & j-m & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 & \frac{(m+1)+(j-m)(m-1)}{m+1} & \frac{m-1}{m+1} & \cdots & \frac{m-1}{m+1} \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{vmatrix} =$$

$$\begin{vmatrix} 2 & 1 & \cdots & 1 & j-m \\ 1 & 2 & \cdots & 1 & j-m \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 2 & j-m \\ 1 & 1 & \cdots & 1 & j-m \end{vmatrix} + \begin{vmatrix} 2 & 1 & \cdots & 1 & 0 \\ 1 & 2 & \cdots & 1 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 1 & 1 & \cdots & 2 & 0 \\ 1 & 1 & \cdots & 1 & \frac{(m+1)-2(j-m)}{m+1} \end{vmatrix} =$$

$$(j-m) + (m+1) - 2(j-m) = 2m+1-j > 0\,.$$

With this we conclude the proof. □

We are ready to prove Theorem 2.

Proof of Theorem 2. From (28) the asymptotic behavior of $\widehat{\sigma}_j(z) - \frac{S_{\mathbf{n},j}(z)}{T_{\mathbf{n}}(z)}$ can be expressed in terms of that of the sequences of polynomials $W_{\mathbf{n},j}$, $T_{\mathbf{n},j}$, and $\zeta_{\mathbf{n},j}$, where

$$\frac{1}{\zeta_{\mathbf{n},j}^2} = \operatorname{mín}\left\{ \int |Q(x)|^2 \frac{|\widetilde{T}_{\mathbf{n},j}(x)| d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|} : Q(x) = x^{n_j} + \cdots \right\}$$

$$= \int |T_{\mathbf{n},j}(x)|^2 \frac{|\widetilde{T}_{\mathbf{n},j}(x)| d\sigma_j(x)}{|W_{\mathbf{n},j}(x)|}\,.$$

On account of Theorem 4, we have

$$\lim_{\mathbf{n}\in\Lambda} |W_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp\{-(1+p_j)V^{\overline{\mu}_{m+j}}(z)\}\,, \tag{39}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_0$, and

$$\lim_{\mathbf{n}\in\Lambda} |T^2_{\mathbf{n},j}(z)|^{1/|\mathbf{n}|} = \exp\{-2p_j V^{\overline{\mu}_j}(z)\}\,, \tag{40}$$

uniformly on compact subsets of $\mathbb{C} \setminus \Delta_j$, where $\overline{\mu} = \overline{\mu}(\mathcal{C}_2)$. Using (33), we have

$$\lim_{\mathbf{n}\in\Lambda} \left(\frac{1}{\zeta^2_{\mathbf{n},j}}\right)^{1/|\mathbf{n}|} = \exp\{-2p_j\eta_j\} = \exp\{-w_j/p_j\}\,. \tag{41}$$

Combining (28), (39), (40), and (41), we obtain that (4) holds true uniformly on compact subsets of the indicated region. □

4. Comments on Lemma 1

Let $\mathcal{M}(F_k), k = 1,\dots,N$, be the class of all finite Borel measures on $\mathcal{M}(F_k)$ and

$$\mathcal{M} = \mathcal{M}_(F_1) \times \cdots \times \mathcal{M}(F_N)\,.$$

Given a real, symmetric, positive definite matrix $\mathcal{C} = (c_{j,k})$ of order N, define the mutual energy of two vector measures $\mu^1, \mu^2 \in \mathcal{M}$ by

$$J(\mu^1,\mu^2) = \sum_{j,k=1}^{N} \int\int c_{j,k} \ln \frac{1}{|z-x|} d\mu^1_j(z) d\mu^2_k(x). \tag{42}$$

The energy of the vector measure $\mu \in \mathcal{M}$ is

$$J(\mu) = \sum_{j,k=1}^{N} c_{j,k} I(\mu_j,\mu_k)\,, \tag{43}$$

where

$$I(\mu_j,\mu_k) = \int\int \ln \frac{1}{|z-x|} d\mu_j(z) d\mu_k(x)\,. \tag{44}$$

For $\mu \in \mathcal{M}$ define the combined potentials $W^{\mu}_j, j = 1,\dots,N$, as in the introduction, and the vector potential $W^{\mu} = (W^{\mu}_1,\dots,W^{\mu}_N)$. These formulas may be rewritten as

$$J(\mu^1,\mu^2) = \int W^{\mu^2}(z) d\mu^1(z)\,, \tag{45}$$

where

$$\int W^{\mu^2}(z)d\mu^1(z) = \sum_{i=1}^{m} \int W_i^{\mu^2}(z)d\mu_i^1(z)\,,$$

and

$$J(\mu) = \int W^{\mu}(z)d\mu(z)\,. \tag{46}$$

If $\mu, \mu^1, \mu^2 \in \mathcal{E}$ are vector charges whose components have finite energy, the energy $J(\mu)$ of μ and the mutual energy $J(\mu^1,\mu^2)$ of μ^1,μ^2 can be defined analogously by formulas (43) and (42), respectively.

In Proposition 5.4.2 of [3], using a unitary decomposition of $\mathcal{C}$, the authors prove that $J(\cdot)$ defines a nonsingular positive definite quadratic form on the linear space $\mathcal{E}$. (Here, the condition $c_{j,k} \geq 0$ when $F_j \cap F_k \neq \emptyset, j,k \in \{1,\ldots,N\}$ on the coefficients of $\mathcal{C}$ is not needed.) Therefore, if there is a vector measure $\mu^1 \in \mathcal{M}_1, J(\mu^1) < \infty$, which minimizes the functional $J(\cdot)$ on $\mathcal{M}_1$, it is unique (see, for example, Theorem 5.3.1 in [3]). If $c_{j,k} \geq 0$ when $F_j \cap F_k \neq \emptyset, j,k \in \{1,\ldots,N\}$ the functional $J(\cdot)$ is lower semicontinuous in the weak star topology of $\mathcal{M}$ (see Proposition 5.4.1 in [3]). Consequently, the functional $J(\cdot)$ attains its minimum in $\mathcal{M}_1$.

Let $0 \leq \epsilon \leq 1$ and $\mu^1,\mu^2 \in \mathcal{M}_1$. Assume that the components of μ^1,μ^2 have finite energy. Set $\widetilde{\mu} = \epsilon\mu^2 + (1-\epsilon)\mu^1 \in \mathcal{M}_1$. It is algebraically straightforward to verify that

$$J(\widetilde{\mu}) - J(\mu^1) = \epsilon^2 J(\mu^2-\mu^1) + 2\epsilon \int W^{\mu^1}(x)d(\mu^2-\mu^1)(x)\,. \tag{47}$$

Assume that μ^1 minimizes $J(\cdot)$ on $\mathcal{M}_1$. Dividing by ϵ and letting ϵ tend to zero, it follows that

$$\int W^{\mu^1}(x)d(\mu^2-\mu^1)(x) \geq 0. \tag{48}$$

for all $\mu^2 \in \mathcal{M}_1$. Reciprocally, assume that (48) takes place for all $\mu^2 \in \mathcal{M}_1$, then using (47) with $\epsilon = 1$ it follows that μ^1 minimizes the energy functional since $J(\mu^2-\mu^1) \geq 0$ for all $\mu^1,\mu^2 \in \mathcal{E}$.

Now, let $\overline{\mu} \in \mathcal{M}_1$ be a solution of the equilibrium potential problem determined by $\mathcal{C}$ on the system of intervals $F_j\,, j = 1, \ldots, N$. That is

$$W_j^{\overline{\mu}}(x) = w_j^{\overline{\mu}}\,, \qquad x \in \operatorname{supp}(\overline{\mu}_j)\,,$$

where $w_j^{\overline{\mu}} = \inf\{W_j^{\overline{\mu}}(x) : x \in F_j\}$. Hence, for all $\mu \in \mathcal{M}_1$,

$$\int W^{\overline{\mu}}(x) d(\mu - \overline{\mu})(x) = \sum_{j=1}^{N} \int W^{\overline{\mu}_j}(x) d(\mu_j - \overline{\mu}_j)(x) \geq \sum_{j=1}^{N} w_j^{\overline{\mu}} - w_j^{\overline{\mu}} = 0$$

and it follows that $\overline{\mu}$ minimizes the energy functional. With this we conclude the comments on Lemma 1.

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