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FOURIER SERIES OF FUNCTIONS OF Λ-BOUNDED VARIATION

DANIEL WATERMAN[1]

ABSTRACT. It is shown that the Fourier coefficients of functions of Λ-bounded variation, $\Lambda = \{\lambda_n\}$, are $O(\lambda_n/n)$. This was known for $\lambda_n = n^{\beta+1}$, $-1 < \beta < 0$. The classes L and HBV are shown to be complementary, but L and ΛBV are not complementary if ΛBV is not contained in HBV. The partial sums of the Fourier series of a function of harmonic bounded variation are shown to be uniformly bounded and a theorem analogous to that of Dirichlet is shown for this class of functions without recourse to the Lebesgue test.

We have shown elsewhere that functions of harmonic bounded variation (HBV) satisfy the Lebesgue test for convergence of their Fourier series, but if a class of functions of Λ-bounded variation (ΛBV) is not properly contained in HBV, it contains functions whose Fourier series diverge **[1]**. We have also shown that Fourier series of functions of class $\{n^{\beta+1}\} - \mathrm{BV}$, $-1 < \beta < 0$, are (C, β) bounded, implying that the Fourier coefficients are $O(n^{\beta})$ **[2]**.

Here we shall estimate the Fourier coefficients of functions in ΛBV. Without recourse to the Lebesgue test, we shall prove a theorem for functions of HBV analogous to that of Dirichlet and also show that the partial sums of the Fourier series of an HBV function are uniformly bounded. From this one can conclude that L and HBV are complementary classes, i.e., Parseval's formula holds (with ordinary convergence) for $f \in L$ and $g \in$ HBV. We shall see that L and ΛBV are *not* complementary if ΛBV is not a subclass of HBV.

1. Definitions and results. Let f be a real function on an interval $[a, b]$, $\Lambda = \{\lambda_n\}$ a nondecreasing sequence of positive numbers such that $\sum 1/\lambda_n$ diverges, and $\{I_n\}$ a sequence of nonoverlapping intervals $I_n = [a_n, b_n] \subset [a, b]$. The function f is said to be of Λ-*bounded variation* (ΛBV) if $\sum |f(a_n) - f(b_n)|/\lambda_n$ converges for every choice of $\{I_n\}$. The supremum of these sums is called the Λ-variation of f, denoted by $V_\Lambda(f; [a, b])$. When $\Lambda = \{n\}$, the class is referred to as the functions of *harmonic bounded variation* (HBV).

We shall suppose that $[a, b] = [0, 2\pi]$ and our functions have period 2π. Two classes of functions, K and K_1, are said to be *complementary* **[4**, p. 157**]** if $f \in K$ and $g \in K_1$ implies

$$\frac{1}{\pi} \int_0^{2\pi} fg \, dx = \frac{1}{2} a_0 a_0' + \sum_1^{\infty} (a_k a_k' + b_k b_k'),$$

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where a_k, b_k are the Fourier coefficients of f and a'_k, b'_k are the Fourier coefficients of g. Here we suppose that the series on the right converges.

We shall prove the following results.

THEOREM 1. *If* $f \in \Lambda\mathrm{BV}$, *then the Fourier coefficients of* f *are* $O(\lambda_n/n)$.

THEOREM 2. *The classes* L *and* HBV *are complementary. If* $\Lambda\mathrm{BV}$ *is not contained in* HBV, *then* L *and* $\Lambda\mathrm{BV}$ *are not complementary.*

THEOREM 3. *If* $f \in \mathrm{HBV}$, *then the partial sums of its Fourier series are uniformly bounded. The series converges everywhere and converges uniformly on closed intervals of points of continuity.*

2. Proof of Theorem 1. Suppose we consider b_n. We have

$$\pi b_n = \int_0^{2\pi} f(t)\sin nt\,dt = \frac{1}{n}\sum_0^{2n-1}(-1)^k\int_0^{\pi} f\left(\frac{t+k\pi}{n}\right)\sin t\,dt$$

$$= \frac{1}{n}\int_0^{\pi}\sum_1^{2n-1}{}^{*}\left[f\left(\frac{t+(k-1)\pi}{n}\right) - f\left(\frac{t+k\pi}{n}\right)\right]\sin t\,dt,$$

where * denotes summation over *odd* indices. Hence

$$\pi|b_n| < \frac{1}{n}\int_0^{\pi}\sum_1^{2n-1}{}^{*}\left|f\left(\frac{t+(k-1)\pi}{n}\right) - f\left(\frac{t+k\pi}{n}\right)\right|dt$$

$$= \frac{1}{n}\int_0^{\pi}\sum{}^{*}\frac{1}{\lambda_k}\,|\cdots|\lambda_k\,dt.$$

Applying Abel's transformation, we see that this expression is $O(\lambda_n/n)$.

3. Proof of Theorem 2. Let us suppose that $f \in L$, $g \in \mathrm{HBV}$, and $S_n(g, x)$ is the nth partial sum of the Fourier series of g. Then

$$\Delta_n = \left|\frac{1}{\pi}\int_0^{2\pi} fg\,dx - \left[\frac{1}{2}a_0a'_0 + \sum_1^n (a_ka'_k + b_kb'_k)\right]\right|$$

$$= \frac{1}{\pi}\left|\int_0^{2\pi}(g - S_n(g))f\,dx\right|.$$

If we assume Theorem 3, then $S_n(g) \to g$ everywhere and $S_n(g)$ is uniformly bounded. Applying the dominated convergence theorem, we have $\Delta_n \to 0$. Thus L and HBV are complementary.

We now assume that $\Lambda\mathrm{BV}$ is not contained in HBV and show that, under this assumption, there is an $f_0 \in L$ and a $g_0 \in \Lambda\mathrm{BV}$ such that $\{\int_0^{2\pi} f_0S_n(g_0)dx\}$ is a divergent sequence.

Our assumption is equivalent to the existence of a nonincreasing sequence of positive numbers a_n such that $\sum a_n/\lambda_n$ converges, but $\sum a_n/n$ diverges. Let $g_n(x)$ be a function of period 2π defined in $[0, 2\pi]$ to be a_i for $(2i-2)\pi < (n + \frac{1}{2})x < (2i-1)\pi$, $i = 1, \ldots, n+1$, and 0 elsewhere. Clearly $g_n \in \Lambda\mathrm{BV}$. Now $\Lambda\mathrm{BV}$ is a Banach space with norm [3]

$$\|g\|_\Lambda = |g(0)| + V_\Lambda(g; [0, 2\pi]).$$

We have

$$V_\Lambda(g_n; [0, 2\pi]) = \sum_1^{n+1} a_i(1/\lambda_{2i-1} + 1/\lambda_{2i}) \leqslant 2 \sum_1^{n+1} a_i/\lambda_i.$$

Hence

$$\|g_n\|_\Lambda \leqslant 2 \sum_1^{\infty} a_n/\lambda_n = C < \infty$$

for every n. Now

$$\sup_x |S_n(g_n, x)| \geqslant |S_n(g_n, 0)| = \frac{1}{\pi} \left| \int_0^{2\pi} g_n(t) \frac{\sin(n + \frac{1}{2})t}{\sin \frac{1}{2}t} dt \right|$$

$$= \frac{1}{\pi} \left| \sum_0^{2n} \frac{(-1)^k}{n + \frac{1}{2}} \int_0^{\pi} g_n\left(\frac{t + k\pi}{n + \frac{1}{2}}\right) \frac{\sin t}{\sin((t + k\pi)/(2n + 1))} dt \right|$$

$$\geqslant \frac{2}{\pi} \sum_1^{n+1} a_i \int_0^{\pi} \frac{\sin t}{t + (2i - 2)\pi} dt$$

$$\geqslant \frac{4}{\pi^2} \sum_1^{n+1} a_i/(2i - 1) > \frac{2}{\pi^2} \sum_1^{n+1} a_i/i,$$

implying that $\sup_x |S_n(g_n, x)| \neq O(1)$.

Let $P_n(f)$ be the continuous linear functional on L defined by

$$P_n(f) = \int_0^{2\pi} f S_n(g_n)\, dx.$$

Then

$$\|P_n\| = \sup_x |S_n(g_n, x)| \neq O(1),$$

implying that there is an $f_0 \in L$ such that

$$P_n(f_0) \neq O(1).$$

Let $Q_n(g)$ be the continuous linear functional on ΛBV defined by

$$Q_n(g) = \int_0^{2\pi} f_0 S_n(g)\, dx.$$

Then

$$\|Q_n\| \geqslant |Q_n(g_n)|/\|g_n\|_\Lambda \geqslant |P_n(f_0)|/C \neq O(1).$$

Hence there is a g_0 in ΛBV such that $Q_n(g_0) \neq O(1)$, implying that $\{\int_0^{2\pi} f_0 S_n(g_0)\, dx\}$ diverges.

4. Proof of Theorem 3. Suppose $f \in$ HBV. If $S_n(x)$ denotes the nth partial sum of the Fourier series of f, then for any $\delta > 0$,

$$S_n(x) - f(x) = \frac{1}{\pi} \int_0^{\delta} (f(x + t) + f(x - t) - 2f(x)) \frac{\sin nt}{t} dt + o(1)$$

uniformly in x. Since $f(x+0)$ and $f(x-0)$ exist at each point, we may assume that $f(x) = \frac{1}{2}[f(x+0) + f(x-0)]$ for each x. Therefore we may write the integral above as

$$\int_0^\delta (f(x+t) - f(x+0)) \frac{\sin nt}{t}\, dt + \int_0^\delta (f(x-t) - f(x-0)) \frac{\sin nt}{t}\, dt.$$

We consider only the first of these. The other may be treated in an analogous manner.

Letting $h(t) = f(x+t) - f(x+0)$, we see that

$$\left| \int_0^{\pi/n} h(t) \frac{\sin nt}{t}\, dt \right| < \pi \sup_{0<t<\pi/n} |h(t)| = o(1)$$

for each x and uniformly on closed intervals of points of continuity. This expression is also uniformly bounded. Then

$$\int_{\pi/n}^\delta h(t) \frac{\sin nt}{t}\, dt = \sum_1^N \int_{k\pi/n}^{(k+1)\pi/n} h(t) \frac{\sin nt}{t}\, dt + \int_{(N+1)\pi/n}^\delta \cdots$$
$$= I_1 + I_2,$$

where $N + 1 = [n\delta/\pi]$. Clearly $I_2 = o(1)$ uniformly in x and

$$I_1 = \int_0^\pi \sum_1^N h\left(\frac{t + k\pi}{n}\right)(-1)^k \frac{\sin t}{t + k\pi}\, dt.$$

For even N, the absolute value of the integrand here is dominated by

$$\left| \sum_1^{N-1}{}^* \left[h\left(\frac{t+k\pi}{n}\right) \frac{1}{t+k\pi} - h\left(\frac{t+(k+1)\pi}{n}\right) \frac{1}{t+(k+1)\pi} \right] \right|,$$

where * again indicates summation over odd indices. If N is odd, then

$$\int_{N\pi/n}^{(N+1)\pi/n} h(t) \frac{\sin nt}{t}\, dt = o(1)$$

just as I_2 did, and, by removing this term, we reduce the problem to one in which the sum has an even number of terms. We shall therefore assume N to be even. The general term of the sum under consideration equals

$$\left[h\left(\frac{t+k\pi}{n}\right) - h\left(\frac{t+(k+1)\pi}{n}\right) \right] \frac{1}{t+k\pi}$$
$$- h\left(\frac{t+(k+1)\pi}{n}\right) \left[\frac{1}{t+k\pi} - \frac{1}{t+(k+1)\pi} \right].$$

Given $\epsilon >$ and choosing N_0 such that $\Sigma_{N_0+1}^\infty 1/k^2 < \epsilon$, we have

$$\left| \sum_1^{N-1}{}^* h\left(\frac{t+(k+1)\pi}{n}\right) \left[\frac{1}{t+k\pi} - \frac{1}{t+(k+1)\pi} \right] \right|$$
$$\leqslant \sum_1^{N-1}{}^* \left| f\left(x + \frac{t+(k+1)\pi}{n}\right) - f(x+0) \right| / k^2 = \sum_1^{N_0}{}^* + \sum_{N_0+1}^{N-1}{}^*$$

and the second sum is bounded by $2\epsilon \sup|f(x)|$.The first sum is bounded by

$$\sup_{0<t\leqslant(N_0+2)\pi/n} |f(x+t)-f(x+0)| \cdot \sum_1^{N_0}{}^* 1/k^2$$

which is $o(1)$ as $n\to\infty$ for each x, is $o(1)$ uniformly in x in any closed interval of points of continuity, and is bounded uniformly in n and x.

Finally we have

$$\left|\sum_1^{N-1}{}^* \left[h\left(\frac{t+k\pi}{n}\right)-h\left(\frac{t+(k+1)\pi}{n}\right)\right]\frac{1}{t+k\pi}\right|$$

$$\leqslant \sum_1^{N-1}{}^* \left|f\left(x+\frac{t+k\pi}{n}\right)-f\left(x+\frac{t+(k+1)\pi}{n}\right)\right|/k \leqslant V_H\left(\bar{f};\left[x,x+\delta\right]\right),$$

where $\bar{f}(t)$ is $f(t)$ on $(x, x+\delta]$ and $\bar{f}(x)=f(x+0)$. Now

$$V_H\left(\bar{f};\left[x,x+\delta\right]\right)<\epsilon$$

if δ is sufficiently small, since $\bar{f}$ is continuous on the right at x [3]. If f is continuous of each point of a closed interval I, then we may choose $\delta>0$ such that $V_H(f;[x,x+\delta])<\epsilon$ for every $x\in I$. Clearly the sum is bounded by $V_H(f;[0,2\pi])$ for every n and x.

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Department of Mathematics, Syracuse University, Syracuse, New York 13210