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## FOURIER SERIES OF SMOOTH FUNCTIONS ON COMPACT LIE GROUPS

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### Introduction

The purpose of the present note is to give an elementary proof of the following theorems. Any  $C^{2k}$ -function on a compact connected Lie group  $G$  can be expanded by the absolutely and uniformly convergent Fourier series of the matricial components of irreducible representations if  $2k > \frac{1}{2} \dim G$  (Theorem 1).

The Fourier transform is a topological isomorphism of  $C^\infty(G)$  onto the space  $S(D)$  of rapidly decreasing functions on the set  $D$  of the classes of irreducible representations of  $G$  (Theorem 3 and 4).

The related results which the author found in the literature are as follows. In Séminaire Sophus Lie [1] exposé 21, it was proved that any  $C^\infty$ -functions on  $G$  can be expanded by the uniformly convergent Fourier series. Zhelobenko [3] proved Theorem 3 for the group  $SU(2)$ . R.A. Mayer [4] proved that the Fourier series of any  $C^1$ -function on  $SU(2)$  is uniformly convergent but there exists a  $C^1$ -function on  $SU(2)$  whose Fourier series does not converge absolutely.

### 1. The Fourier expansion of a smooth function

Throughout this paper we use the following notations.  $G$ : a compact connected Lie group,  $G_0$ : the commutator subgroup of  $G$ ,  $T$ : a maximal toral subgroup of  $G$ ,  $l$ : the rank of  $G = \dim T$ ,  $p$ : the rank of  $G_0$ ,  $n$ : the dimension of  $G = l + 2m$ ,  $\mathfrak{g}$ : the Lie algebra of  $G$ ,  $\mathfrak{g}^c$ : the complexification of  $\mathfrak{g}$ ,  $\mathfrak{t}$ : the Lie algebra of  $T$ ,  $R$ : the root system of  $\mathfrak{g}^c$  with respect to  $\mathfrak{t}^c$ ,  $dg$ : the Haar measure on  $G$  normalized as  $\int_G dg = 1$ ,  $L^2(G)$ : the Hilbert space of the complex valued square integrable functions on  $G$  with respect to  $dg$ ,  $C^k(G)$ : the set of all  $k$ -times continuously differentiable complex valued functions on  $G$ ,  $\|A\| = \text{Tr}(AA^*)^{1/2}$ : the Hilbert-Schmidt norm of a matrix  $A$ .

In this paper, a finite dimensional continuous matricial representation of  $G$

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is simply called a representation of  $G$ . So a representation of  $G$  is a continuous and hence analytic homomorphism of  $G$  into  $GL(k, \mathbf{C})$  for some  $k \geq 1$ . For any representation  $U$  of  $G$ , the differential  $dU$  of  $U$  is defined as

$$dU(X) = \left[ \frac{d}{dt} U(\exp tX) \right]_{t=0}$$

for any  $X$  in  $\mathfrak{g}$ . The differential  $dU$  of  $U$  is a representation of the Lie algebra  $\mathfrak{g}$ . The representation  $dU$  of  $\mathfrak{g}$  is uniquely extended to a representation of the universal enveloping algebra  $U(\mathfrak{g})$  of  $\mathfrak{g}$ . This representation of  $U(\mathfrak{g})$  is also denoted by  $dU$ .

For any representation  $U$  of  $G$ , all the elements in  $dU(\mathfrak{t})$  can be transformed simultaneously into the diagonal matrices. That is, there exists a non singular matrix  $Q$  and pure imaginary valued linear forms  $\lambda_1, \dots, \lambda_k$  on  $\mathfrak{t}$  such that

$$QdU(H)Q^{-1} = \begin{pmatrix} \lambda_1(H) & & 0 \\ & \ddots & \\ 0 & & \lambda_k(H) \end{pmatrix}$$

for all  $H$  in  $\mathfrak{t}$ . The linear forms  $\lambda_1, \dots, \lambda_k$  are called the weights of  $U$ .

We fix once for all a positive definite inner product  $(X, Y)$  on  $\mathfrak{g}$  which is invariant under  $\text{Ad } G$ . The norm defined by the inner product is denoted by  $|X| = (X, X)^{1/2}$ . The inner product  $(X, Y)$  is extended to a bilinear form on the complexification  $\mathfrak{g}^c$  of  $\mathfrak{g}$ . A pure imaginary valued linear form (in particular a weight

of a representation)  $\lambda$  is identified with an element  $h_\lambda$  in  $\mathfrak{t}$  which satisfies

$$\lambda(H) = i(h_\lambda, H)$$

for all  $H$  in  $\mathfrak{t}$ . So we denote as  $\lambda(H) = i(\lambda, H)$ . Let  $\Gamma = \Gamma(G) = \{H \in \mathfrak{t} : \exp_G H = 1\}$  be the kernel of the homomorphism  $\exp_G : \mathfrak{t} \rightarrow T$ . Then  $\Gamma$  is a discrete subgroup of  $\mathfrak{t}$  of rank  $l$ . Let  $I$  be the set of all  $G$ -integral forms on  $\mathfrak{t}$ :

$$I = \{\lambda \in \mathfrak{t} : (\lambda, H) \in 2\pi\mathbf{Z} \text{ for all } H \in \Gamma\}.$$

Then the set  $I$  coincides with the set of all the weights of the representations of  $G$ . We choose once for all a lexicographic order  $\mathcal{O}$  in  $\mathfrak{t}$ . Let  $P$  be the set of positive roots with respect to the order  $\mathcal{O}$ . Then the number  $m$  of elements in  $P$  is equal to  $\frac{1}{2}(n-l)$ . Let  $B$  be the set of simple roots in  $P$ , that is,  $B$  is the set of roots in  $P$  which can not be the sum of two elements in  $P$ .  $B$  consists of exactly  $p$  elements ( $p = \text{rank } G$ ). We denote the elements of  $B$  as  $\alpha_1, \dots, \alpha_p$ .

Let  $\lambda_1, \dots, \lambda_k$  be the weights of a representation  $U$ . Then the maximal element  $\lambda$  among  $\lambda_i$ 's in the order  $\mathcal{O}$  is called the highest weight of  $U$ . The set of all highest weights of the representations of  $G$  coincides with the set  $D$  of

all dominant  $G$ -integral forms on  $\mathfrak{t}$ :

$$D = \{\lambda \in I ; (\lambda, \alpha_i) \geq 0 \ (1 \leq i \leq p)\}.$$

Since an irreducible representation of  $G$  is uniquely determined, up to equivalence, by its highest weight (cf. Serre [2] Ch. VII Théorème 1), there exists a bijection from  $D$  onto the set  $\mathfrak{D}$  of equivalence classes of irreducible representations of  $G$ .  $\mathfrak{D}$  is identified with  $D$  by this bijection. We choose, once for all, an irreducible unitary representation  $U^\lambda$  with the highest weight  $\lambda$  for each  $\lambda$  in  $D$ . The degree  $d(\lambda)$  of the representation  $U^\lambda$  is given by Weyl's dimension formula:

$$(1.0) \quad d(\lambda) = \prod_{\alpha \in P} \frac{(\lambda + \delta, \alpha)}{(\delta, \alpha)}$$

where  $\delta = 2^{-1} \sum_{\alpha \in P} \alpha$ .

If  $G$  is abelian, the right hand side of (1.0) should be understood to express 1.

Let  $X_1, \dots, X_n$  be a basis of  $\mathfrak{g}$  and  $g_{ij} = (X_i, X_j)$  and  $(g^{ij}) = (g_{ij})^{-1}$ . Then the element  $\Delta$  defined by

$$-\Delta = \sum_{i,j=1}^n g^{ij} X_i X_j$$

in the universal enveloping algebra  $U(\mathfrak{g})$  of  $\mathfrak{g}$  is called the Casimir operator of  $\mathfrak{g}$ .  $\Delta$  is independent of the choice of the basis  $X_1, \dots, X_n$ . As an element in  $U(\mathfrak{g})$ ,  $\Delta$  is regarded as a left invariant linear differential operator on  $G$ .

Let  $u_{ij}^\lambda(g)$  be the  $(i, j)$ -element of the unitary matrix  $U^\lambda(g)$ . Then the following Lemma is well known.

**Lemma 1.1.** 1) Let  $dU^\lambda$  be the differential of the representation  $U^\lambda$ . Then we have

$$dU^\lambda(\Delta) = (\lambda, \lambda + 2\delta) \mathbf{1}.$$

2) The matricial element  $u_{ij}^\lambda$  is an eigenfunction of the Casimir operator  $\Delta$  regarded as a differential operator on  $G$ :

$$\Delta u_{ij}^\lambda = (\lambda, \lambda + 2\delta) u_{ij}^\lambda.$$

**Proof.** 1) Since the Casimir operator  $\Delta$  belongs to the center of  $U(\mathfrak{g})$ ,  $dU^\lambda(\Delta)$  is a scalar operator  $c\mathbf{1}$  by Schur's lemma. The scalar  $c$  is determined as follows. We can choose a Weyl base  $E_\alpha$  ( $\alpha \in R$ ),  $H_i$  ( $1 \leq i \leq l$ ) of  $\mathfrak{g}^c$  satisfying  $(E_\alpha, E_{-\alpha}) = 1$ ,  $(H_i, H_j) = \delta_{ij}$  and  $E_\alpha + E_{-\alpha}, i(E_\alpha - E_{-\alpha}), H_i \in \mathfrak{g}$ . Then we have

$$-\Delta = \sum_{\alpha \in R} E_{-\alpha} E_\alpha + \sum_{i=1}^l H_i^2 = \sum_{\alpha \in P} (2E_{-\alpha} E_\alpha + H_\alpha) + \sum_{i=1}^l H_i^2,$$

because  $[E_\alpha, E_{-\alpha}] = H_\alpha$  where  $H_\alpha$  is the element in the space it satisfying  $(H, H_\alpha) = \alpha(H)$  for every  $H$  in  $\mathfrak{t}^c$ . Let  $x \neq 0$  be the weight vector corresponding to the highest weight  $\lambda$ :  $dU^\lambda(E_\alpha)x = 0$  ( $\alpha \in P$ ),  $dU^\lambda(H_\alpha)x = \lambda(H_\alpha)x = -(\lambda, \alpha)x$ . Then we have

$$\begin{aligned} cx &= dU^\lambda(\Delta)x = \left\{ \sum_{\alpha \in P}^l (\lambda, \alpha) - \sum_{i=1}^l \lambda(H_i)^2 \right\} x = \{(\lambda, 2\delta) + (\lambda, \lambda)\}x \\ &= (\lambda, \lambda + 2\delta)x. \end{aligned}$$

2) For any element  $X$  in  $\mathfrak{g}$ , we have

$$(Xu_{ij}^\lambda)(g) = \left[ \frac{d}{dt} u_{ij}^\lambda(g \exp tX) \right]_{t=0} = \sum_{k=1}^{d(\lambda)} u_{ik}^\lambda(g) \left[ \frac{du_{kj}^\lambda}{dt}(\exp tX) \right]_{t=0}.$$

This equality can be expressed as

$$(XU^\lambda)(g) = U^\lambda(g)dU^\lambda(X).$$

So we get

$$(\Delta U^\lambda)(g) = U^\lambda(g)dU^\lambda(\Delta) = (\lambda, \lambda + 2\delta)U^\lambda(g)$$

by 1). q.e.d.

By Peter-Weyl's theorem, the set

$$\mathfrak{B} = \{d(\lambda)^{1/2} u_{ij}^\lambda ; \lambda \in D, 1 \leq i, j \leq d(\lambda)\}$$

is an orthonormal base of  $L^2(G)$ . Therefore any function  $f$  in  $L^2(G)$  can be expanded by a mean convergent Fourier series of  $\mathfrak{B}$ :

$$(1.1) \quad f = \sum_{\lambda \in D} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (f, u_{ij}^\lambda) u_{ij}^\lambda.$$

The precise meaning of (1.1) is given by

$$(1.2) \quad \lim_{n \rightarrow \infty} \|f - \sum_{|\lambda| \leq n} d(\lambda) \sum_{i,j=1}^{d(\lambda)} (f, u_{ij}^\lambda) u_{ij}^\lambda\|_2 = 0.$$

(1.2) is equivalent to the Parseval's equality:

$$\|f\|_2^2 = \sum_{\lambda \in D} d(\lambda) \sum_{i,j=1}^{d(\lambda)} |(f, u_{ij}^\lambda)|^2.$$

For an arbitrary function  $f$  in  $L^2(G)$ , the right hand side of (1.1) does not, in general, converge at every point of  $G$ . We shall show that if  $f$  is sufficiently smooth, then the series (1.1) converges uniformly on  $G$ . First we give a convenient expression of the series (1.1). Let  $f$  be a function in  $L^1(G)$  and  $\lambda \in D$ . Then the  $\lambda$ -th Fourier coefficient  $\mathcal{F}f(\lambda)$  of  $f$  is defined by

$$\overline{\mathcal{F}f(\lambda)} = \int_G f(g) U^\lambda(g^{-1}) dg.$$

$\mathcal{F}f(\lambda)$  is a matrix of degree  $d(\lambda)$  and its  $(i,j)$ -element  $\mathcal{F}f(\lambda)_{ij}$  is given by

$$(1.3) \quad \begin{aligned} \mathcal{F}f(\lambda)_{ij} &= \int_G f(g) u_{ij}^\lambda(g^{-1}) dg \\ &= \int_G f(g) \overline{u_{ji}^\lambda(g)} = (f, u_{ji}^\lambda). \end{aligned}$$

Therefore we have

$$(1.4) \quad \sum_{i,j=1}^{d(\lambda)} (f, u_{ij}^\lambda) u_{ij}^\lambda(g) = \sum_{i,j=1}^{d(\lambda)} \mathcal{F}f(\lambda)_{ij} u_{ij}^\lambda(g) = \text{Tr}(\mathcal{F}f(\lambda) U^\lambda(g)).$$

By (1.3) the Parseval equality can be expressed as

$$(1.5) \quad \|f\|_2^2 = \sum_{\lambda \in D} d(\lambda) \|\mathcal{F}f(\lambda)\|^2.$$

**Lemma 1.2.** *If  $f$  belongs to  $C^2(G)$ , then we have*

$$\mathcal{F}(\Delta f)(\lambda) = \omega(\lambda) \mathcal{F}f(\lambda)$$

where

$$(1.6) \quad \omega(\lambda) = (\lambda, \lambda + 2\delta).$$

**Proof.** Let  $\varphi$  and  $\psi$  be any  $C^1$ -functions on  $G$ . Then for any element  $X$  in  $\mathfrak{g}$ , we have

$$\begin{aligned} (X\varphi, \psi) &= \int_G \left[ \frac{d}{dt} \varphi(g \exp tX) \right]_{t=0} \overline{\psi(g)} dg = \frac{d}{dt} \left[ \int_G \varphi(g \exp tX) \overline{\psi(g)} dg \right]_{t=0} \\ &= \frac{d}{dt} \left[ \int_G \varphi(g) \overline{\psi(g \exp(-tX))} dg \right]_{t=0} = -(\varphi, X\psi). \end{aligned}$$

Let  $X_1, \dots, X_n$  be an orthonormal base of  $\mathfrak{g}$ :  $(X_i, X_j) = \delta_{ij}$ . Then we have  $\Delta = -\sum_{i=1}^n X_i^2$  and by the above equality we get

$$(1.7) \quad (\Delta\varphi, \psi) = (\varphi, \Delta\psi)$$

for any  $C^2$ -functions  $\varphi$  and  $\psi$ . By (1.3), (1.7) and Lemma 1.1, 2), we have

$$\mathcal{F}(\Delta f)(\lambda)_{ij} = (\Delta f, u_{ji}^\lambda) = (f, \Delta u_{ji}^\lambda) = \omega(\lambda) \mathcal{F}f(\lambda)_{ij}.$$

q. e. d.

**Lemma 1.3.** *Let  $D_0 = D - \{0\}$ . Then the series*

$$\zeta(s) = \sum_{\lambda \in D_0} (\lambda, \lambda)^{-s}$$

converges if  $2s > l$ .

Proof. Let  $I$  be the set of all  $G$ -integral form on  $\mathfrak{t}$  and  $I_0 = I - \{0\}$ . It is sufficient to prove the series

$$\sum_{\lambda \in I_0} (\lambda, \lambda)^{-s}$$

converges if  $2s > l$ . Let  $\lambda_1, \dots, \lambda_l$  be a basis of the lattice  $I$  and  $\langle x, y \rangle$  be the inner product on  $\mathfrak{t}$  defined by  $\langle \Sigma x_i \lambda_i, \Sigma y_i \lambda_i \rangle = \Sigma x_i y_i$ . Then it is well known that the series

$$(1.8) \quad \sum_{\lambda \in I_0} \langle \lambda, \lambda \rangle^{-s} = \sum_{n \in \mathbb{Z}^{l-1}} (n_1^2 + \dots + n_l^2)^{-s}$$

converges if and only if  $2s > l$ . On the other hand, there exists a positive definite symmetric operator  $A$  such that  $(x, y) = \langle Ax, y \rangle$  for all  $x, y$  in  $\mathfrak{t}$ . Let  $a$  and  $b$  be the maximal and minimal eigenvalues of  $A$ . Then we have

$$b \langle \lambda, \lambda \rangle \leq (\lambda, \lambda) \leq a \langle \lambda, \lambda \rangle$$

for all  $\lambda$  in  $\mathfrak{t}$ . Therefore the series  $\sum_{\lambda \in I_0} (\lambda, \lambda)^{-s}$  converges if and only if the series  $\sum_{\lambda \in I_0} \langle \lambda, \lambda \rangle^{-s}$  converges. So we have proved that  $\sum_{\lambda \in D_0} (\lambda, \lambda)^{-s}$  converges if  $2s > l$ .

**Theorem 1.** *Let  $f$  be a continuous function on a compact connected Lie group  $G$  and let  $l = \text{rank } G$ ,  $n = \dim G = l+2m$ . If  $f$  satisfies one of the following conditions (1) and (2), then the Fourier series of  $f$ ,*

$$\sum_{\lambda \in D} d(\lambda) \text{Tr}(\mathcal{F}f(\lambda) U^\lambda(g))$$

*converges to  $f(g)$  absolutely and uniformly on  $G$ :*

- (1)  $f$  is  $2k$ -times continuously differentiable and  $2k > \frac{l}{2} + m = \frac{n}{2}$ ,
- (2)  $\|\mathcal{F}f(\lambda)\| = O(|\lambda|^{-h})$  ( $|\lambda| \rightarrow \infty$ ) for some integer  $h > l + \frac{3}{2}m$ .

Proof. (1) Suppose  $f$  belongs to  $C^{2k}(G)$ . Then we have, by Lemma 1.2,

$$(1.9) \quad \mathcal{F}f(\lambda) = \omega(\lambda)^{-k} \mathcal{F}(\Delta^k f)(\lambda) \quad (\lambda \in D_0).$$

On the other hand we have an inequality

$$(1.10) \quad \omega(\lambda) = (\lambda, \lambda + 2\delta) \geq |\lambda|^2.$$

By (1.9) and (1.10), we have

$$(1.11) \quad \|\mathcal{F}f(\lambda)\| \leq \|\mathcal{F}(\Delta^k f)(\lambda)\| |\lambda|^{-2k} \text{ for all } \lambda \in D_0 = D - \{0\}.$$

Since  $(A, B) = \text{Tr}(AB^*)$  is an inner product on the space  $M_n(C)$  of the matrices

of order  $n$ , we have the Schwarz inequality

$$(1.12) \quad |\mathrm{Tr}(AB)| \leq \|A\| \|B\|.$$

Since  $U^\lambda(g)$  is a unitary matrix of order  $d(\lambda)$ , the Hilbert-Schmidt norm of  $U^\lambda(g)$  is equal to

$$(1.13) \quad \|U^\lambda(g)\| = d(\lambda)^{1/2}.$$

By (1.12), (1.13) and (1.11), we have

$$(1.14) \quad \begin{aligned} \sum_{\lambda} d(\lambda) |\mathrm{Tr}(\mathcal{F}f(\lambda) U^\lambda(g))| &\leq \sum_{\lambda} d(\lambda)^{3/2} \|\mathcal{F}f(\lambda)\| \\ &\leq \sum_{\lambda} d(\lambda)^{3/2} |\lambda|^{-2k} \|\mathcal{F}(\Delta^k f)(\lambda)\|. \end{aligned}$$

By the Schwarz inequality, the right hand side of (1.14) is

$$(1.15) \quad \leq \left( \sum_{\lambda} d(\lambda) \|\mathcal{F}(\Delta^k f)(\lambda)\|^2 \right)^{1/2} \left( \sum_{\lambda} d(\lambda)^2 |\lambda|^{-4k} \right)^{1/2}.$$

Since  $\Delta^k f \in C^0(G) \subset L^2(G)$ , we have the Parseval equality

$$(1.16) \quad \|\Delta^k f\|_2^2 = \sum_{\lambda \in D} d(\lambda) \|\mathcal{F}(\Delta^k f)(\lambda)\|^2.$$

Moreover by Weyl's dimension formula, we have for any  $\lambda \in D_0$

$$(1.17) \quad d(\lambda) \leq C(|\lambda| + |\delta|)^m \leq N |\lambda|^m$$

where  $C = \prod_{\alpha \in P} |\alpha|(\delta, \alpha)^{-1}$  and  $N$  are positive constants. By (1.16) and (1.17), the right hand side of (1.15) is

$$(1.18) \quad \leq \|\Delta^k f\|_2 (N^2 \sum_{\lambda} |\lambda|^{2m-4k})^{1/2}.$$

Since  $4k-2m > l+2m-2m = l$  by condition (1), the series in (1.18) converges (Lemma 1.3). So we have proved that the Fourier series of  $f$  converges absolutely and uniformly on  $G$ , if  $f$  satisfies the condition (1). The sum  $s(g)$  of the Fourier series of  $f$  is a continuous function and equal to  $f(g)$  almost everywhere on  $G$  by the Parseval equality. Since  $f$  and  $s$  are continuous, the sum  $s(g)$  is equal to  $f(g)$  everywhere on  $G$ .

If a function  $f$  satisfies the condition (2), then there exists a positive constant  $M$  such that

$$(1.19) \quad \|\mathcal{F}f(\lambda)\| \leq M |\lambda|^{-h} \quad \text{for all } \lambda \in D_0.$$

So we have

$$(1.20) \quad \begin{aligned} \sum_{\lambda} d(\lambda) |\mathrm{Tr}(\mathcal{F}f(\lambda) U^\lambda(g))| &\leq \sum_{\lambda} d(\lambda)^{3/2} \|\mathcal{F}f(\lambda)\| \\ &\leq L \sum_{\lambda} (|\lambda| + |\delta|)^{3m/2} |\lambda|^{-h} \end{aligned}$$

where  $L = M(\prod_{\alpha \in P} |\alpha|(\delta, \alpha)^{-1})^{3/2}$  is a positive constant. Therefore the series on the right hand side of (1.20) converges if  $h - 3m/2 > l$ , i.e.,  $h > l + 3m/2$  (Lemma 1.3). q.e.d.

**Corollary to Theorem 1.** *If  $f$  is a  $C^{2k}$ -function on  $G$ , then we have  $\|\mathcal{F}f(\lambda)\| = o(|\lambda|^{-2k})$  ( $|\lambda| \rightarrow \infty$ ), that is,*

$$\lim_{|\lambda| \rightarrow \infty} |\lambda|^{2k} \|\mathcal{F}f(\lambda)\| = 0.$$

**Proof.** By the inequality (1.11), we have

$$(1.21) \quad |\lambda|^{2k} \|\mathcal{F}f(\lambda)\| \leq \|\mathcal{F}(\Delta^k f)(\lambda)\|.$$

Since  $\Delta^k f$  belongs to  $C^0(G) \subset L^2(G)$ , we have

$$(1.22) \quad \lim_{|\lambda| \rightarrow \infty} \|\mathcal{F}(\Delta^k f)(\lambda)\| = 0$$

by the Parseval equality (1.16). (1.21) and (1.22) prove the Corollary.

## 2. Fourier coefficients of a smooth function

**Theorem 2.** *Let  $G$  be a compact connected Lie group and  $D$  be the set of all dominant  $G$ -integral forms on the Lie algebra  $\mathfrak{t}$  of a maximal toral subgroup  $T$  of  $G$ . Let  $U^\lambda$  be an irreducible unitary representation of  $G$  with the highest weight  $\lambda \in D$  and  $d(\lambda)$  be the degree of  $U^\lambda$ . Then we have the following inequality for every  $X$  in the Lie algebra  $\mathfrak{g}$  of  $G$ :*

$$(2.1) \quad \begin{aligned} \|dU^\lambda(X)\|^2 &\leq d(\lambda)|\lambda|^2|X|^2 \quad \text{for any } \lambda \in D \text{ and} \\ \|dU^\lambda(X)\|^2 &\leq N|\lambda|^{m+2}|X|^2 \quad \text{for any } \lambda \in D_0 \end{aligned}$$

where  $N$  is a positive constant and  $m$  is the number of the positive roots.

**Proof.** First we show that the inequality (2.1) is valid for every  $X$  in  $\mathfrak{g}$  if (2.1) is valid for every  $X$  in the Cartan subalgebra  $\mathfrak{t}$ . Since every element  $X$  in  $\mathfrak{g}$  is conjugate to an element  $H$  in  $\mathfrak{t}$ , that is, there exists an element  $g$  in  $G$  such that  $(\text{Ad } g)X = H$ , we have

$$(2.2) \quad \|dU^\lambda(H)\| = \|U^\lambda(g)dU^\lambda(X)U^\lambda(g^{-1})\| = \|dU^\lambda(X)\| \quad \text{and}$$

$$(2.3) \quad |H| = |X|.$$

The equalities (2.2) and (2.3) prove that if the inequality (2.1) is valid for any  $H$  in  $\mathfrak{t}$ , then (2.1) is valid for every  $X$  in  $\mathfrak{g}$ .

Now let  $X$  be any element in  $\mathfrak{t}$  and  $W(\lambda)$  be the set of weights in the representation  $U^\lambda$ . Then the linear transformation  $dU^\lambda(X)$  is represented by a diagonal matrix whose diagonal elements are  $\{i(\mu, X) : \mu \in W(\lambda)\}$  with respect to some orthonormal base of the representation space. Therefore we have

$$(2.4) \quad \|dU^\lambda(X)\|^2 = \sum_{\mu \in W(\lambda)} |i(\mu, X)|^2 \leq \sum_{\mu \in W(\lambda)} |\mu|^2 |X|^2.$$

On the other hand every weight  $\mu$  in  $W(\lambda)$  has the form

$$(2.5) \quad \mu = \lambda - \sum_{i=1}^p m_i \alpha_i,$$

where  $m_i$ 's are non negative integers. (cf. Serre [2] Ch. VII Théorème 1). If  $\mu \in W(\lambda)$  is dominant, that is,  $(\mu, \alpha_i) \geq 0$  ( $l \leq i \leq p$ ), then we have by (2.5)

$$(2.6) \quad |\mu|^2 \leq |\mu|^2 + \sum_{i=1}^p m_i(\mu, \alpha_i) = (\lambda, \mu) = |\lambda|^2 - \sum_{i=1}^p m_i(\lambda, \alpha_i) \leq |\lambda|^2.$$

Since every weight  $\mu$  in  $W(\lambda)$  is conjugate to a dominant weight in  $W(\lambda)$  under the Weyl group, (cf. Serre [2] Ch. VII-12 Remarque), we have the inequality

$$(2.7) \quad |\mu| \leq |\lambda| \quad \text{for all } \mu \in W(\lambda)$$

by (2.6). The inequalities (2.4) and (2.7) prove the inequality

$$(2.8) \quad \|dU^\lambda(X)\|^2 \leq d(\lambda) |\lambda|^2 |X|^2.$$

Since the degree  $d(\lambda)$  of  $U$  is given by Weyl's dimension formula

$$d(\lambda) = \prod_{\alpha \in P} (\lambda + \delta, \alpha) (\delta, \alpha)^{-1},$$

$d(\lambda)$  is estimated by (1.17) as

$$(2.9) \quad d(\lambda) \leq C(|\lambda| + |\delta|)^m \leq N |\lambda|^m \quad \text{for any } \lambda \in D_0$$

where  $C$  and  $N$  are positive constants and  $m$  is the number of positive roots. So we have proved Theorem 2 completely.

**Lemma 2.1.** *Let  $G$  be a connected Lie group and  $\mathfrak{g}$  be the Lie algebra of  $G$ . Moreover let  $f$  be a complex valued function on  $G$ , and  $k$  be a positive integer. Then the function  $f$  belongs to  $C^k(G)$  if and only if*

$$(Xf)(g) = \left[ \frac{d}{dt} f(g \exp tX) \right]_{t=0}$$

*can be defined for every  $X$  in  $\mathfrak{g}$  and  $g$  in  $G$ , and it belongs to  $C^{k-1}(G)$ .*

**Proof.** If a function  $f$  belongs to  $C^k(G)$ , then  $\varphi(g, t) = f(g \exp tX)$  belongs to  $C^k(G \times \mathbf{R})$ . So  $(Xf)(g) = \frac{\partial \varphi}{\partial t}(g, 0)$  exists and belongs to  $C^{k-1}(G)$ .

Conversely suppose that  $Xf$  is defined and belongs to  $C^{k-1}(G)$  for every  $X \in \mathfrak{g}$ . Then for any real number  $t$ ,  $(df/dt)(g \exp tX)$  exists and is equal to  $(Xf)(g \exp tX)$ . Moreover for any element  $h$  in  $G$ ,  $(df/dt)(g \exp tXh)$  exists and is equal to

$$(2.10) \quad \begin{aligned} \frac{d}{dt}f(g \exp tXh) &= \frac{d}{dt}f(gh \exp (t \operatorname{Ad} h^{-1}X)) \\ &= ((\operatorname{Ad} h^{-1}X)f)(g \exp tXh). \end{aligned}$$

Let  $X_1, X_2, \dots, X_n$  be a base of  $\mathfrak{g}$  and

$$\varphi(t) = \varphi(t_1, \dots, t_n) = \exp tX_1 \cdots \exp t_n X_n.$$

Then  $\varphi$  is an analytic diffeomorphism of an open neighbourhood  $W$  of 0 in  $\mathbf{R}^n$  onto an open neighbourhood  $V$  of the identity element  $e$  in  $G$ . Let

$$(2.11) \quad (\operatorname{Ad}(\exp t_1 X_1 \cdots \exp t_n X_n)^{-1}) X_i = \sum_{j=1}^n a_{ij}(t) X_j.$$

Then  $a_{ij}(t) = a_{ij}(t_1, \dots, t_n)$  is an analytic function on  $\mathbf{R}^n$ . Let  $g$  be a fixed element in  $G$ . Then the mapping  $g\varphi(t) \mapsto t = (t_1, \dots, t_n)$  defines a local coordinates on  $gV$ , the canonical coordinates of the second kind. Let  $\partial/\partial t_i$  be the partial derivative with respect to  $t_i$  just introduced. Then by the equalities (2.10) and (2.11),  $\frac{\partial f}{\partial t_i}(g\varphi(t))$  exists and is equal to

$$(2.12) \quad \begin{aligned} \frac{\partial}{\partial t_i}f(g\varphi(t)) &= [\operatorname{Ad}(\exp t_{i+1} X_{i+1} \cdots \exp t_n X_n)^{-1} X_i] f(g\varphi(t)) \\ &= \sum_{j=1}^n a_{ij}(0, \dots, 0, t_{i+1}, \dots, t_n) (X_j f)(g\varphi(t)). \end{aligned}$$

By the assumption, the right hand side of (2.12) regarded as a function of  $t$  is a  $C^{k-1}$ -function on  $W$ . So  $f$  is a  $C^k$ -function on  $gV$ . Since  $g$  is arbitrary, this proves that  $f$  is a  $C^k$ -function on  $G$ .

**Lemma 2.2.** *Let  $G, \mathfrak{g}, f, k$  be as in Lemma 2.1. Then  $f$  is a  $C^k$ -function on  $G$  if and only if  $X_k X_{k-1} \cdots X_1 f$  can be defined and is continuous for any  $k$  elements  $X_1, \dots, X_k$  in  $\mathfrak{g}$ .*

**Proof.** This Lemma is easily proved by the induction with respect to  $k$  using Lemma 2.1.

**Theorem 3.** *For any continuous function  $f$  on a compact connected Lie group  $G$ , the following two conditions (1) and (2) are mutually equivalent.*

- (1)  $f$  is a  $C^\infty$ -function on  $G$ .  
(2) The Fourier coefficients  $\mathcal{F}f(\lambda)$  is rapidly decreasing:  $\lim_{|\lambda| \rightarrow \infty} |\lambda|^h \|\mathcal{F}f(\lambda)\| = 0$  for every non negative integer  $h$ .

Proof. (1) $\Rightarrow$ (2). This part of Theorem 3 is proved in Corollary to Theorem 1.

(2) $\Rightarrow$ (1). Suppose that  $\mathcal{F}f(\lambda)$  is rapidly decreasing. Then  $f$  satisfies the condition (2) in Theorem 1. So the Fourier series of  $f$  converges uniformly to  $f$ . Thus for every  $g \in G$ ,  $X \in \mathfrak{g}$  and  $t \in \mathbf{R}$  we have

$$(2.13) \quad f(g \exp tX) = \sum_{\lambda \in D} d(\lambda) \text{Tr}(\mathcal{F}f(\lambda) U^\lambda(g \exp tX)).$$

The series obtained from the right hand side of (2.13) by termwise differentiation with respect to the variable  $t$  is

$$(2.14) \quad \sum_{\lambda \in D} d(\lambda) \text{Tr}(\mathcal{F}f(\lambda) U^\lambda(g \exp tX) dU^\lambda(X)).$$

By Theorem 2 and the rapidly decreasingness of  $\mathcal{F}f(\lambda)$ , the series (2.14) converges absolutely and uniformly with respect to  $t$ , when  $t$  runs through any bounded set in  $\mathbf{R}$ . Therefore the series (2.13) can be differentiated termwise and the function  $f(g \exp tX)$  is differentiable with respect to  $t$ . So

$$(Xf)(g) = \left[ \frac{d}{dt} f(g \exp tX) \right]_{t=0}$$

is defined and equal to

$$(2.15) \quad \sum_{\lambda \in D} d(\lambda) \text{Tr}(\mathcal{F}f(\lambda) U^\lambda(g) dU^\lambda(X)).$$

Since (2.15) is uniformly convergent on  $G$ , the sum  $Xf$  is a continuous function on  $G$ . Therefore  $f$  is a  $C^1$ -function by Lemma 2.1.

By the same argument,  $X_1 \cdots X_k f$  is defined and continuous for any  $k \in N$  and  $X_1, \dots, X_k \in \mathfrak{g}$  and it has the following uniformly convergent expansion;

$$(2.15) \quad (X_1 \cdots X_k f)(g) = \sum_{\lambda \in D} d(\lambda) \text{Tr}(\mathcal{F}f(\lambda) U^\lambda(g) dU^\lambda(X_1) \cdots dU^\lambda(X_k)).$$

So  $f$  is a  $C^k$ -function for any  $k \in N$  by Lemma 2.2, i.e.,  $f$  is a  $C^\infty$ -function on  $G$ .

### 3. The topology of $C^\infty(G)$ and $S(D)$

Let  $G$  be a compact connected Lie group as before. The space  $C^\infty(G)$  of all complex valued  $C^\infty$ -functions on  $G$  is topologized by the family of seminorms:

$$(3.0) \quad \{p_U(f) = \|Uf\|_\infty : U \in \mathbf{U}(\mathfrak{g})\}.$$

$C^\infty(G)$  is a complete locally convex topological vector space by this topology. It is clear that the topology of  $C^\infty(G)$  is coincides with the one which is determined by the subfamily of seminorms:

$$(3.1) \quad \{p_{X_1 \cdots X_k}(f) = \|X_1 \cdots X_k f\|_\infty : k = 0, 1, 2; \dots, X_1, \dots, X_k \in \mathfrak{g}\}.$$

Let  $S(D)$  be the space of matrix valued functions  $F$  on the lattice  $D$  which satisfies the following two conditions:

(1)  $F(\lambda)$  belongs to the space  $M_{d(\lambda)}(\mathbb{C})$  of complex matrices of order  $d(\lambda)$  for each  $\lambda \in D$ .

(2)  $F(\lambda)$  is a rapidly decreasing function of  $\lambda$ : i.e.,  $\lim_{|\lambda| \rightarrow \infty} |\lambda|^k \|F(\lambda)\| = 0$  for all  $k \in \mathbb{N}$ .

In the following, we use the inner product  $(X, Y)$  which satisfies the following condition:

$$(3.3) \quad (\lambda, \lambda) \geq 1 \text{ for all } \lambda \in D_0 = D - \{0\}.$$

The vector space  $S(D)$  is topologized by the family of seminorms

$$\{q_s(F) = \max_{\lambda \in D} |\lambda|^s \|F(\lambda)\| : s \geq 0\}.$$

By the condition (3.3), we get the following inequality for the seminorms on  $S(D)$ :

$$(3.4) \quad q_s(F) \leq q_t(F) \quad \text{if } 0 < s \leq t$$

for all  $F$  in  $S(D)$ .

Using these topologies, the result in Theorem 3 can be reformulated more precisely in the following Theorem 4.

**Theorem 4.** *The Fourier transform  $\mathcal{F}: f \mapsto \mathcal{F}f$  is a topological isomorphism of  $C^\infty(G)$  onto  $S(D)$ .*

**Proof.** By Theorem 3, the Fourier transform  $\mathcal{F}$  maps  $C^\infty(G)$  into  $S(D)$ . Since any continuous function  $f$  on  $G$  is uniquely determined by its Fourier coefficients  $\mathcal{F}f(\lambda)$  by (1.5), the mapping  $\mathcal{F}$  is injective. The mapping  $\mathcal{F}$  is also surjective. Let  $F$  be a function in  $S(D)$ . Then the series

$$(3.5) \quad \sum_{\lambda \in D} d(\lambda) \operatorname{Tr}(F(\lambda) U^\lambda(g))$$

converges uniformly on  $G$ , because the function  $F$  satisfies the condition (2) in Theorem 1. Let  $f(g)$  be the sum of the series (3.5). Then  $f$  is a continuous function on  $G$  and the Fourier transform  $\mathcal{F}f$  of  $f$  coincides with the original function

$F$  by the orthogonality relations. Since  $F(\lambda) = \mathcal{F}f(\lambda)$  is rapidly decreasing, the function  $f$  is a  $C^\infty$ -function on  $G$  by Theorem 3. Thus we have proved that the Fourier transform  $\mathcal{F}$  is a linear isomorphism of  $C^\infty(G)$  onto  $S(D)$ .

Now we shall prove that the Fourier transform  $\mathcal{F}$  is a homeomorphism. First we show that  $\mathcal{F}$  is continuous. Since  $(\Delta^k f)(\lambda) = \omega(\lambda)^k \mathcal{F}f(\lambda)$  (Lemma 1.2), we have

$$(3.6) \quad \begin{aligned} \omega(\lambda)^k \|\mathcal{F}f(\lambda)\| &= \|\mathcal{F}(\Delta^k f)(\lambda)\| \leq \int_G |\Delta^k f(g)| \|U^\lambda(g^{-1})\| dg \\ &\leq d(\lambda)^{1/2} \|\Delta^k f\|_\infty. \end{aligned}$$

Since  $|\lambda|^2 \leq \omega(\lambda)$  and there exists a constant  $M > 0$  such that  $d(\lambda)^{1/2} \leq M |\lambda|^{m/2}$  for all  $\lambda \in D_0$ , we have

$$(3.7) \quad |\lambda|^{2k-m/2} \|\mathcal{F}f(\lambda)\| \leq M \|\Delta^k f\|_\infty$$

by (3.6). Therefore we have

$$(3.8) \quad q_{2k-m/2}(\mathcal{F}f) \leq M \|\Delta^k f\|_\infty$$

for all  $f$  in  $C^\infty(G)$  and all  $k > \frac{1}{4}m$ . Since  $k$  can be taken arbitrarily large, we have proved by (3.4) and (3.8) that for any  $s > 0$  there exists an integer  $k > 0$  such that the inequality

$$(3.9) \quad q_s(\mathcal{F}f) \leq M \|\Delta^k f\|_\infty.$$

is valid. On the other hand, since  $\|\mathcal{F}f(0)\| \leq \|f\|_\infty$  by the definition of  $\mathcal{F}f$ , we have

$$(3.10) \quad q_0(\mathcal{F}f) \leq \|\mathcal{F}f(0)\| + \max_{\lambda \in D_0} \|\mathcal{F}f(\lambda)\| \leq \|f\|_\infty + M \|\Delta^k f\|_\infty$$

for  $k > \frac{1}{4}m$  by (3.3) and (3.7). The inequalities (3.8) and (3.10) prove that the Fourier transform  $\mathcal{F}$  is a continuous mapping of  $C^\infty(G)$  into  $S(D)$ .

Next we shall prove that the inverse Fourier transform  $\mathcal{F}^{-1}: \mathcal{F}f \rightarrow f$  is continuous. Since  $|\lambda|^2 \leq \omega(\lambda)$  and there exists a constant  $M > 0$  such that  $d(\lambda) \leq M^2 |\lambda|^m$ , the series

$$(3.11) \quad \sum_{\lambda \in D_0} d(\lambda)^{(3+k)/2} \omega(\lambda)^{-s}$$

converges to a positive real number  $K$  if  $s > 2^{-1}l + 4^{-1}(k+3)m$  by Lemma 1.3. Let  $k$  be a positive integer and  $X_1, \dots, X_k$  be  $k$  elements in  $\mathfrak{g}$ . Then by (2.15) and Theorem 2, we have the inequality

$$(3.12) \quad \|X_1 \cdots X_k f\|_\infty \leq \sum_{\lambda \in D_0} d(\lambda)^{(3+k)/2} |\lambda|^k \|\mathcal{F}f(\lambda)\| |X_1| \cdots |X_k|$$

$$\begin{aligned}
&= |X_1| \cdots |X_k| \sum_{\lambda \in D_0} d(\lambda)^{(3+k)/2} \omega(\lambda)^{-s} |\lambda|^k \|\mathcal{F}(\Delta^s f)(\lambda)\| \\
&\leq K |X_1| \cdots |X_k| q_k(\mathcal{F}(\Delta^s f)) \\
&= K |X_1| \cdots |X_k| \operatorname{Max}_{\lambda \in D} \omega(\lambda)^s |\lambda|^s \|\mathcal{F}f(\lambda)\|.
\end{aligned}$$

if  $s > \frac{1}{2}l + \frac{1}{4}(k+3)m$ . Since  $\omega(\lambda) = (\lambda, \lambda+2\delta) \leq |\lambda|^2 + 2|\lambda||\delta|$

and  $\omega(\lambda)^s \leq \sum_{r=0}^s {}_s C_r |\lambda|^{s+r} (2|\delta|)^{s-r}$ ,

we have

$$(3.13) \quad \|X_1 \cdots X_k f\|_\infty \leq K |X_1| \cdots |X_k| \sum_{r=0}^s {}_s C_r (2|\delta|)^{s-r} q_{s+n+r}(\mathcal{F}f).$$

Similarly we have the inequality

$$\begin{aligned}
(3.14) \quad \|f\|_\infty &\leq \|\mathcal{F}f(0)\| + \sum_{\lambda \in D_0} d(\lambda) |\operatorname{Tr}(\mathcal{F}f(\lambda) U^\lambda(g))| \\
&\leq q_0(\mathcal{F}f) + \sum_{\lambda \in D_0} d(\lambda)^{3/2} \|\mathcal{F}f(\lambda)\| \\
&\leq q_0(\mathcal{F}f) + \sum_{\lambda \in D_0} d(\lambda)^{3/2} \omega(\lambda)^{-s} \|\mathcal{F}(\Delta^s f)(\lambda)\| \\
&\leq q_0(\mathcal{F}f) + K \operatorname{Max}_{\lambda \in D_0} \omega(\lambda)^s \|\mathcal{F}f(\lambda)\| \\
&\leq q_0(\mathcal{F}f) + K \sum_{r=0}^s {}_s C_r (2|\delta|)^{s-r} q_{s+r}(\mathcal{F}f).
\end{aligned}$$

for  $s > \frac{1}{2}l + \frac{3m}{4}$ .

The inequalities (3.13) and (3.14) prove that the inverse Fourier transform  $\mathcal{F}^{-1}: \mathcal{F}f \rightarrow f$  is a continuous mapping from  $S(D)$  into  $C^\infty(G)$ . q.e.d.

**Corollary to Theorem 4.** *The topology of  $C^\infty(G)$  defined by the family of seminorms (3.0) (or (3.1)) coincides with the topology defined by the family of seminorms*

$$\{r_m(f) = \|\Delta^m f\|_\infty; m = 0, 1, 2, \dots\}.$$

Proof. This Corollary is clear from the inequalities (3.10) and (3.9) and Theorem 4.

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*Added in proof*

The Fourier series in Theorem 1 is obtained from the series (1.1) by first taking the partial sum  $\sum_{i,j=1}^{d(\lambda)}$ . However we can prove that the original series (1.1) converges absolutely and uniformly if  $f$  belongs to  $C^{2k}(G)$  and  $2k > \frac{n}{2}$ .

This fact can be seen from the following inequalities:

$$\begin{aligned}
& \sum_{\lambda \in D_0} \sum_{i,j=1}^{d(\lambda)} d(\lambda) |(f, u_{ij}^\lambda)| |u_{ij}^\lambda(g)| \leq \sum_{\lambda \in D_0} \sum_{i,j=1}^{d(\lambda)} d(\lambda) |\lambda|^{-2k} |(\Delta^k f, u_{ij}^\lambda)| |u_{ij}^\lambda(g)| \\
& \leq \left( \sum_{\lambda \in D_0} \sum_{i,j=1}^{d(\lambda)} d(\lambda) |(\Delta^k f, u_{ij}^\lambda)|^2 \right)^{1/2} \left( \sum_{\lambda \in D_0} \sum_{i,j} d(\lambda) |\lambda|^{-4k} |u_{ij}^\lambda(g)|^2 \right)^{1/2} \\
& \leq \|\Delta^k f\|_2 \left( \sum_{\lambda \in D_0} d(\lambda)^2 |\lambda|^{-4k} \right)^{1/2} \leq \|\Delta^k f\|_2 N \left( \sum_{\lambda \in D_0} |\lambda|^{2m-4k} \right)^{1/2}.
\end{aligned}$$

