PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 101, Number 2, October 1987

SHORTER NOTES

The purpose of this department is to publish very short papers of unusually polished character, for which there is no other outlet.

FOURIER SERIES WITH POSITIVE COEFFICIENTS

J. MARSHALL ASH,[1] MICHAEL RAINS AND STEPHEN VÁGI

ABSTRACT. Extending a result of N. Wiener, it is shown that functions on the circle with positive Fourier coefficients that are pth power integrable near 0, $1 < p \leq 2$, have Fourier coefficients in $l^{p'}$.

The following result was proved (but never published) by Norbert Wiener in the early 1950's. (See [**1**, pp. 242, 250] and [**3**].)

WIENER'S THEOREM. *If $\sum c_n e^{int}$ is the Fourier series of a function $f \in L^1(-\pi,\pi)$ with $c_n \geq 0$ for all n, and f restricted to a neighborhood $(-\delta,\delta)$ of the origin belong to $L^2(-\delta,\delta)$, then f belongs to $L^2(-\pi,\pi)$.*

A question which immediately arises in connection with this result is the following: does the theorem remain true if one replaces $L^2(-\delta,\delta)$ and $L^2(-\pi,\pi)$ in its statement respectively by $L^p(-\delta,\delta)$ and $L^p(-\pi,\pi)$, with $1 < p \leq \infty$? In 1969 Stephen Wainger showed, by ingenious counterexamples, that the answer is negative for $1 < p < 2$ [**4**]. If p is an even integer or ∞ it is very easy to see that the answer is "yes." For every other exponent between 2 and ∞ it is "no," as was shown in 1975 by Harold S. Shapiro [**3**]. These negative results have been extended to compact abelian groups [**2**]. However, the conclusion of Wiener's theorem can be stated equivalently as "then $\sum c_n^2 < \infty$." This suggested the following theorem.

THEOREM. *If $\sum c_n e^{int}$ is the Fourier series of a function $f \in L^1(-\pi,\pi)$ with $c_n \geq 0$ for all n, and f restricted to a neighborhood $(-\delta,\delta)$ of the origin belongs to $L^p(-\delta,\delta)$ with $1 < p < 2$, then $\sum c_n^{p'} < \infty$, where $p' = p/(p-1)$.*

PROOF. (See [**3**, p. 12].) Let $h(t)$ be the 2π-periodic function which for $|t| \leq \pi$ is defined by

$$h(t) = \begin{cases} 1 - |t|/\delta, & |t| \leq \delta, \\ 0, & \delta \leq |t| \leq \pi. \end{cases}$$

Received by the editors March 2, 1987.
1980 *Mathematics Subject Classification.* Primary 42A32, 42A16; Secondary 43A15.
[1] The research presented here was supported in part by a grant from the University Research Council of DePaul University.

Then $|hf| \leq \chi_{[-\delta,\delta]}|f| \in L^p(-\pi,\pi)$ and $\sum |(hf)^\wedge(n)|^{p'}$ is finite by the Hausdorff-Young inequality. (See (*) below.) We have

$$(h \cdot f)^\wedge(n) = \sum_{k+l=n} \hat{h}(k) c_l$$

where $c_l \geq 0$ for all l by hypothesis and $\hat{h}(k) \geq 0$ for all k by direct calculation. Drop all terms except the $k = 0$ one from the right side of the last equation to get

$$c_n \leq \frac{(h \cdot f)^\wedge(n)}{\hat{h}(0)} = \frac{2\pi}{\delta}(h \cdot f)^\wedge(n).$$

Take p'th powers and sum over n.

REMARKS. 1. This theorem was motivated by studying the above-mentioned counterexamples of Wainger [**4**].

2. It is very well known that the Hausdorff-Young theorem consists of two irreversible implication, one of which is

(*) if $\sum c_n e^{inx}$ is the Fourier series of a function $f \in L^p(-\pi,\pi)$, where $1 < p < 2$, then $\{c_n\} \in l^{p'}$. (See [**5**, pp. 101–103].)

Wainger's counterexamples are functions designed to satisfy the hypotheses of our theorem while violating the hypothesis of (*). Our theorem shows that they must also satisfy the conclusion of (*), and thereby gives another demonstration that the converse of (*) is false.

3. Our theorem easily extends to compact abelian groups. In the above proof simply replace $[-\pi,\pi]$ by a general compact abelian group and $[-\delta,\delta]$ by a symmetric neighborhood of the identity, note that $h = \varphi \star \tilde{\varphi}$ where $\varphi(t) = \tilde{\varphi}(-t) = 1/\sqrt{\delta}\chi_{[-\delta/2,\delta/2]}(t)$, etc.

REFERENCES

1. R. P. Boas, *Entire functions*, Academic Press, New York, 1964.
2. M. Rains, *On functions with nonnegative Fourier transforms*, Indian J. Math. **27** (1985), 41–48.
3. H. S. Shapiro, *Majorant problems for Fourier coefficients*, Quart. J. Math. Oxford (2) **26** (1975), 9–18.
4. S. Wainger, *A problem of Wiener and the failure of a principle for Fourier series with positive coefficients*, Proc. Amer. Math. Soc. 20 (1969), 16–18.
5. A. Zygmund, *Trigonometric series*, Vol. II, Cambridge Univ. Press, Cambridge, 1959.

DEPARTMENT OF MATHEMATICS, DEPAUL UNIVERSITY, CHICAGO, ILLINOIS 60614

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WESTERN ONTARIO, LONDON, ONTARIO, CANADA