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FOURIER-STIELTJES TRANSFORMS WITH BOUNDED POWERS

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1. Denote by A the set of functions

$$f(t) = \int_{-\infty}^{\infty} e^{itx} d\sigma(x)$$

where $\sigma(x)$ is a function of bounded variation on the real line. The sum and product of functions in A belong to A, and the ring is complete under the norm

$$\|f\| = \int_{-\infty}^{\infty} |d\sigma(x)| \, .$$

The problem of this paper is to determine all the functions of A such that

$$\|f^n\|$$

is bounded as n ranges over all (positive and negative) integers. Any exponential

$$f(t) = e^{i(at+b)} \, ,$$

where a and b are real constants, has this property; the theorem to be proved states that no other such functions exist.

The interest of the authors in this question has arisen in two connections. The behavior of

$$\|f^n\|$$

for large positive integers n was investigated in [1], and it was shown that

$$\lim_{n \to \infty} \|f^n\|^{1/n} = \sup_t |f(t)|$$

provided $\sigma(x)$ contains no singular component. On the other hand, the

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problem treated here presents itself as the essential difficulty in classifying the automorphisms of the algebra of summable functions on the line, or more generally, on any locally compact abelian group [2].

2. Let Γ be the subset of A containing those functions f for which

$$M(f) = \sup \|f^n\| < \infty ,$$

where the supremum is taken over all integers. The powers of f and of $1/f$ are thus uniformly bounded over the whole axis, so that

$$|f(t)| \equiv 1 .$$

The argument of f,

$$\varphi(t) = \frac{1}{i} \log f(t) ,$$

can be chosen to be continuous. Denote by γ the totality of real functions φ so obtained from f in Γ.

For any elements f and g of A,

$$\|f \cdot g\| \leqq \|f\| \cdot \|g\| .$$

Hence the product of functions in Γ belongs to Γ, or equivalently, γ is closed under the formation of sums. Even more is true: given any φ in γ, integers A_ν, and real constants a_ν, b_ν, the sum

$$a_0 + b_0 t + \sum_{\nu=1}^{k} A_\nu \varphi(a + b_\nu t)$$

belongs to γ.

The result to be proved is the following:

THEOREM. *Every f in Γ has the form*

$$f(t) = e^{i(at+b)} \qquad (a,\, b \text{ real constants}) .$$

3. For the proof of this theorem we need the following

LEMMA. *If f and g belong to Γ and if*

$$f(t) = g(t)$$

on a set of positive measure, then the functions coincide for all t.

Set $h(t) = f(t)/g(t)$. Then h is in Γ and is equal to 1 on a closed set of positive measure, which we call E. We are to show that h is constant everywhere. Define

$$k_n(t) = \left(\frac{1+h(t)}{2}\right)^n .$$

Then
$$k(t) = \lim_{n\to\infty} k_n(t) = \begin{cases} 1 & (t \in E) \\ 0 & (t \in E') \end{cases}$$

where E' denotes the complement of E. Expand the expression for k_n by the binomial theorem and estimate the norm of the sum by the triangle inequality:
$$\| k_n \| \leqq \frac{1}{2^n} \sum_{\nu=0}^{n} \binom{n}{\nu} \| h^\nu \| \leqq \frac{M(h)}{2^n} (1+1)^n = M(h) .$$

Thus there exist functions μ_n of bounded variation such that
$$k_n(t) = \int_{-\infty}^{\infty} e^{itx} d\mu_n(x) ,$$
and
$$\int_{-\infty}^{\infty} | d\mu_n(x) | \leqq M(h) .$$

Integrating k_n over an interval $(y-\delta, y+\delta)$ and dividing by 2δ gives
$$\frac{1}{2\delta} \int_{y-\delta}^{y+\delta} k_n(t) dt = \int_{-\infty}^{\infty} e^{ixy} \frac{\sin \delta x}{\delta x} d\mu_n(x) .$$

As n increases the integrand on the left converges to $k(t)$, so the limit of the integral is the function
$$k_\delta(y) = \frac{1}{2\delta} \int_{y-\delta}^{y+\delta} k(t)\, dt .$$

The integrand of the right side is continuous and tends to zero for $x \to \pm\infty$. Since the μ_n are uniformly of bounded variation, there is a function μ of bounded variation such that the limit of the right side for some subsequence of n is
$$k_\delta(y) = \int_{-\infty}^{\infty} e^{ixy} \frac{\sin \delta x}{\delta x} d\mu(x) .$$

As δ tends to zero, the first expression of $k_\delta(y)$ tends to 0 on E' and almost everywhere to 1 on E by a classical theorem of Lebesgue. Setting
$$a(y) = \int_{-\infty}^{\infty} e^{ixy} d\mu(x)$$
we thus find that almost everywhere $k(x) = a(x)$. Being a continuous

function a will everywhere satisfy the equation $a(1-a) = 0$, which implies $a \equiv 1$, since $a = 1$ on some non-empty subset of E. Therefore $k = 1$ almost everywhere. Since E is a closed set it will therefore cover the entire real axis.

4. Let f be an arbitrary function in Γ, and for each integer n let σ_n be the function of bounded variation for which

$$f^n(t) = \int_{-\infty}^{\infty} e^{itx} d\sigma_n(x) .$$

We write M for $M(f)$. Let $\varphi(t)$ be the argument of $f(t)$, defined to be continuous, and multiply f by a constant of absolute value 1 if necessary so as to have

$$\varphi(0) = 0 .$$

From the theory of Fourier series it is well known that there exist an integer k (depending on M) and coefficients a_ν ($\nu = 0, 1, \ldots, k$) so that

$$\sum_{\nu=0}^{k} |a_\nu| = 1 ,$$

$$\sup_\theta \left| \sum_{\nu=0}^{k} a_\nu e^{i\nu\theta} \right| \leq \frac{1}{2M} .$$

For any real numbers s and t, the sum

$$\sum_{\nu=0}^{k} a_\nu e^{in\varphi(t+\nu s)} \qquad (n = \text{integer})$$

has the representation

$$\int_{-\infty}^{\infty} \sum_{\nu=0}^{k} a_\nu e^{ix(t+\nu s)} d\sigma_n(x) ,$$

from which follows

$$\left| \sum_{\nu=0}^{k} a_\nu e^{in\varphi(t+\nu s)} \right| \leq \int_{-\infty}^{\infty} | d\sigma_n(x) | \cdot \sup_x \left| \sum_{\nu=0}^{k} a_\nu e^{ix(t+\nu s)} \right|$$

$$\leq M \cdot \sup_\theta \left| \sum_{\nu=0}^{k} a_\nu e^{i\nu\theta} \right| \leq \frac{1}{2} .$$

Suppose the numbers

$$\alpha_\nu = \varphi(t+\nu s) , \quad (\nu = 0, 1, \ldots, k) ,$$

$$\alpha_{k+1} = 2\pi$$

were linearly independent over the integers. By the Kronecker approximation theorem, for any given system of real numbers $\beta_0, \ldots, \beta_k$, and $\varepsilon > 0$ there is an integer n satisfying the inequalities

$$| n\alpha_\nu - \beta_\nu | < \varepsilon \pmod{2\pi} \quad (\nu = 0, \ldots, k) .$$

So by proper choice of n,

$$\sum_{\nu=0}^{k} a_\nu e^{in\varphi(t+\nu s)}$$

can be made arbitrarily close to

$$\sum_{\nu=0}^{k} | a_\nu | = 1 .$$

But we have just found the bound $\frac{1}{2}$ for the modulus of the sum. Hence the α_ν are linearly dependent.

Thus for given t and s there are integers $A_0, \ldots, A_{k+1}$ not all zero such that

$$\sum_{\nu=0}^{k} A_\nu \varphi(t+\nu s) + 2\pi A_{k+1} = 0 .$$

We shall show that the coefficients can be chosen independent of t and s.

Let E_A be the closed set of points (t, s) in the plane for which the form above vanishes, A denoting the set of its coefficients A_ν. Each pair (t, s) falls into at least one E_A, and there are only denumerably many sets of coefficients; hence at least one E_A has positive plane measure. That is, for certain integers $B_0, \ldots, B_{k+1}$ not all zero the function

$$\psi(t, s) = \sum_{\nu=0}^{k} B_\nu \varphi(t+\nu s) + 2\pi B_{k+1}$$

vanishes for all pairs (t, s) belonging to a set E_B of positive plane measure. Considered as a function of either variable alone, $\psi(t, s)$ belongs to γ. By the Fubini theorem, there is an s_0 such that (t, s_0) is in E_B for all t in a set of positive linear measure, so that $\psi(t, s_0)$ vanishes on a set of positive measure. By the lemma, it vanishes identically. Again by the Fubini theorem, the set of s_0 for which this argument applies has positive linear measure. For every fixed t then, $\psi(t, s)$ vanishes on a set of positive measure and hence identically. We have shown that $\psi(t, s)$ vanishes for all values of the variables.

Since $\varphi(0) = 0$, the coefficient B_{k+1} vanishes. To complete the proof of the theorem, we have to show that only a linear function can satisfy the difference equation

$$\sum_{\nu=0}^{k} B_\nu \varphi(t+\nu s) \equiv 0$$

and belong to γ. Now any Fourier-Stieltjes transform is uniformly continuous, and so the same holds for the functions of γ. This implies in particular that

$$\varphi(t) = O(|t|) \quad (|t| \to \infty) .$$

We shall prove that under this restriction all the solutions of the difference equation are linear.

For $\varepsilon > 0$ define the function

$$\varphi_\varepsilon(t) = \int_{-\infty}^{\infty} \varphi(t-\xi) \frac{e^{-\xi^2/\varepsilon^2}}{\pi^{1/2}\varepsilon} d\xi .$$

It is easy to verify that φ_ε satisfies the same difference equation as φ; φ_ε has derivatives of all orders; and $\varphi_\varepsilon(t)$ converges to $\varphi(t)$ uniformly on bounded sets as $\varepsilon \to 0$. For any non-negative integer p, the Taylor theorem gives

$$\varphi_\varepsilon(t+\nu s) = \sum_{m=0}^{p} \frac{\varphi_\varepsilon^{(m)}(t)(\nu s)^m}{m!} + O(s^{p+1}) .$$

Choose p to be the smallest integer for which

$$\sum_{\nu=0}^{k} B_\nu \nu^p \neq 0 ,$$

interpreting this to mean

$$\sum_{\nu=0}^{k} B_\nu \neq 0$$

for $p = 0$. The existence of p is assured by the fact that not all the B_ν are zero.

Inserting the expression for φ_ε in the difference equation,

$$\sum_{\nu=0}^{k} B_\nu \left\{ \sum_{m=0}^{p} \frac{\varphi_\varepsilon^{(m)}(t)(\nu s)^m}{m!} + O(s^{p+1}) \right\} = 0 .$$

Rearranging terms and using the fact that

$$\sum_{\nu=0}^{k} B_\nu \nu^m = 0 \quad (m < p)$$

we find

$$\frac{\varphi_\varepsilon^{(p)}(t) s^p}{p!} \sum_{\nu=0}^{k} B_\nu \nu^p + O(s^{p+1}) \equiv 0 .$$

Holding t fixed and letting s tend to zero, we find that

$$\varphi_\varepsilon^{(p)}(t) \equiv 0 \,.$$

Thus φ_ε is a polynomial of degree at most $p-1$. But φ is the limit of φ_ε, uniformly on bounded sets, so φ is itself a polynomial. The restriction on the rate of growth of φ and our normalization $\varphi(0) = 0$ imply $\varphi(t) = at$.

5. For any locally compact abelian group G, the notions of Fourier-Stieltjes transform and spectral norm can be defined, and the problem solved here can be posed anew.

THEOREM. *If G is a locally compact abelian and connected group, and f a Fourier-Stieltjes transform defined on G whose positive and negative powers are bounded in spectral norm, then f is a continuous character of G.*

A corollary of this result is a solution of the isomorphism problem for the corresponding class of group algebras.

THEOREM. *If G and H are locally compact abelian groups, and if at least one of G and H has connected dual, then G and H are topologically isomorphic provided $L(G)$ and $L(H)$ are algebraically isomorphic.*

The proof of the first theorem can be reduced to the case of the real line already considered by the consideration of one-parameter subgroups, and the second theorem follows immediately by virtue of the results of [2].

REFERENCES

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