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Abstract. We perform the Fourier transform in two dimensional elliptic coordinates. The case of axial symmetry allows to reduce significantly the final transformation. The obtained integral formulae generalize the Fourier-Bessel transform due to relation between elliptic and polar coordinates. We show also the alternating definition using Mathieu functions. The application of the formulae obtained is the transitions of operator representations in Quantum Mechanics.

Keywords: curvilinear coordinates, integral transform, Bessel functions, Mathieu functions, Parseval's theorem, Hankel transform.

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Introduction

Contemporary scientific and engineering research widely uses Fourier transform [1]. Great variety of examples includes digital signal processing [2], oceanology, acoustics and other applications. The methods based on Fourier transform allow to solve various differential equations that describe corresponding phenomenon.

Most widely Fourier transform is applied in rectangular coordinate system. This is caused by the fact that a lot of mathematical functions can be transformed analytically. And the second reason is the availability and the manifold of computational options. The fast algorithms [3] are especially important.

Fourier transform becomes Hankel(Fourier-Bessel) transform [1] for the circularly symmetric physical system. Fast algorithms [4], [5] are also applied in this case.

Unfortunately, for other curvilinear coordinate systems there are no so well developed computational possibilities.

The purpose of this work to find an analytically compact and useful form of Fourier transform in elliptical coordinates that extends the application of Fourier methods to this coordinate system.

As well known, elliptical coordinates generalize polar coordinates. Several authors (see [6], [7] and range of works by Badge and Khobragade, for example [8]) use the integral transform based on the solution of Mathieu equation [9]. That transform becomes Fourier-Bessel transform under the conditions of the elliptic coordinates become polar coordinates. This result allow to make a hypothesis that this transform is a Fourier transform in elliptic coordinates. This statement will also be considered in the frame of this work.

It would thus be of interest to obtain the Fourier transform directly from the coordinate conversion and compare the results with integral transformations mentioned above.

We start with general form of Fourier transform and expand the scalar product in the the exponent taking into account the elliptical coordinates. The definition of the generating function for Bessel functions leads to a mathematically compact form for Fourier transform in the axially symmetrical system. The integral kernel contains the product of the two Bessel functions of zero order.

The obtained expression possesses a number of interesting properties including the relation to Fourier-Bessel transformation, Parseval's theorem and connection with Mathieu equation.

We also present two examples of application: transitions between operator representations in Quantum Mechanics (see for example [10]) and Operational Dynamic Modelling (ODM) [11].

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Fourier transform in elliptic coordinates

We begin with definition of coordinates. The relation of rectangular coordinates (x, y) and elliptic coordinates (μ, ν) is:

$$x = a \cosh \mu \cos \nu; \quad y = a \sinh \mu \sin \nu. \tag{1}$$

The coordinates (μ, ν) vary within the following ranges:

$$\mu \geq 0; \quad 0 \leq \nu < 2\pi. \tag{2}$$

Generally, the Fourier transform in two dimensions is:

$$F(\omega) = \int_D f(\mathbf{r}) e^{-i\mathbf{r}\omega} d\mathbf{r}, \tag{3}$$

where $\mathbf{r} = \overrightarrow{\{x, y\}}$ and $\boldsymbol{\omega} = \overrightarrow{\{\omega_x, \omega_y\}}$. Here the integral is taken through the whole area $D = \{(x, y) : x \in (-\infty, +\infty), y \in (-\infty, +\infty)\}$ for rectangular coordinates and $D = \{(\mu, \nu) : \mu \in [0, +\infty), \nu \in [0, 2\pi)\}$ for elliptical coordinates. And the $\mathbf{r}\boldsymbol{\omega}$ is the scalar product of vectors $\mathbf{r}$ and $\boldsymbol{\omega}$ [12]. Expanding that scalar product in the exponent (3) we receive the formula in elliptic coordinates. The scalar product in rectangular coordinates is:

$$\mathbf{r}\boldsymbol{\omega} = x\omega_x + y\omega_y. \tag{4}$$

Then we change coordinates taking into account (1). Continuing with vector $\boldsymbol{\omega}$ we have:

$$\omega_x = b \cosh \omega_\mu \cos \omega_\nu; \quad \omega_y = b \sinh \omega_\mu \sin \omega_\nu. \tag{5}$$

Simple manipulations with hyperbolic functions result in the following expression instead of (4):

$$x\omega_x + y\omega_y = 0.5 \cosh(\mu + \omega_\mu) \cos(\nu - \omega_\nu) + 0.5 \cosh(\mu - \omega_\mu) \cos(\nu + \omega_\nu). \tag{6}$$

Thus, we have the final formula for Fourier transform in elliptic coordinates using the (3):

$$F(\omega_\mu, \omega_\nu) = a^2 \int_0^\infty \int_0^{2\pi} f(\mu, \nu)(\sinh^2 \mu + \sin^2 \nu) \times \\ \exp[i0.5ab(\cosh(\mu + \omega_\mu) \cos(\nu - \omega_\nu) + \cosh(\mu - \omega_\mu) \cos(\nu + \omega_\nu))] d\mu d\nu, \tag{7}$$

where $f(\mu, \nu)$ - function of the space $L^2(D)$. Inverse transform has a similar form:

$$f(\mu, \nu) = b^2 \int_0^\infty \int_0^{2\pi} F(\omega_\mu, \omega_\nu)(\sinh^2 \omega_\mu + \sin^2 \omega_\nu) \times \\ \exp[-i0.5ab(\cosh(\mu + \omega_\mu) \cos(\nu - \omega_\nu) + \cosh(\mu - \omega_\mu) \cos(\nu + \omega_\nu))] d\omega_\mu d\omega_\nu. \tag{8}$$

The case of axial symmetry

The systems that possess axial symmetry are often the most interesting, especially in optics. If $f(\mu, \nu) = f(\mu)$, then it is possible to obtain more applicable formulae instead of (7)-(8).

We use the well-known expression for Bessel function [13]:

$$e^{\frac{\rho}{2}(t - \frac{1}{t})} = \sum_{n=-\infty}^{+\infty} t^n J_n(\rho). \tag{9}$$

Also we define new variables as:

$$\rho_1 = 0.5ab \cosh(\mu + \omega_\mu); \quad \rho_2 = 0.5ab \cosh(\mu - \omega_\mu); \quad t_1 = ie^{i(\nu - \omega_\nu)}; \quad t_2 = ie^{i(\nu + \omega_\nu)}. \tag{10}$$

After insertion of (9) and (10) into (7) we have:

$$F(\omega_\mu,\omega_\nu) = a^2\int_0^\infty\int_0^{2\pi} f(\mu)\left(\sum_{n_1=-\infty}^{+\infty} i^{n_1}J_{n_1}(\rho_1)e^{in_1(\nu-\omega_\nu)}\cdot\sum_{n_2=-\infty}^{+\infty} i^{n_2}J_{n_2}(\rho_2)e^{in_2(\nu+\omega_\nu)}\right)(\sinh^2\mu+\sin^2\nu)d\mu d\nu. \tag{11}$$

Let us to consider the integration by ν. Expanding the brackets of factor $(\sinh^2\mu+\sin^2\nu)$ we have a sum of two integrals which we refer as $Int_1 + Int_2$.

First integral (12) includes a factor $\sinh^2\mu$. It does not affect on integration by ν as well as $J_{n_1}(\rho_1)J_{n_2}(\rho_2)$.

$$Int_1 = a^2\int_0^\infty\int_0^{2\pi} f(\mu)\left(\sum_{n_1=-\infty}^{+\infty} i^{n_1}J_{n_1}(\rho_1)e^{in_1(\nu-\omega_\nu)}\cdot\sum_{n_2=-\infty}^{+\infty} i^{n_2}J_{n_2}(\rho_2)e^{in_2(\nu+\omega_\nu)}\right)\sinh^2\mu d\mu d\nu. \tag{12}$$

Thus, in factor $\sum_{n_1=-\infty}^{+\infty} i^{n_1}J_{n_1}(\rho_1)e^{in_1(\nu-\omega_\nu)}\cdot\sum_{n_2=-\infty}^{+\infty} i^{n_2}J_{n_2}(\rho_2)e^{in_2(\nu+\omega_\nu)}$ the multiplication of exponents is determining. Removing the parentheses in multiplying sums we receive the series of integrals of the following form:

$$a^2\int_0^\infty\int_0^{2\pi} i^{n_1}J_{n_1}(\rho_1)e^{in_1(\nu-\omega_\nu)}i^{n_2}J_{n_2}(\rho_2)e^{in_2(\nu+\omega_\nu)}\sinh^2\mu d\mu d\nu =$$
$$a^2\int_0^\infty i^{n_1+n_2}J_{n_1}(\rho_1)J_{n_2}(\rho_2)e^{-i(n_1+n_2)\omega_\nu}\sinh^2\mu d\mu\int_0^{2\pi} e^{i(n_1+n_2)\nu}d\nu. \tag{13}$$

The nonzero value is only possible when integers $n_1 = n_2 = 0$. Due to periodic exponential property the zero is obtained in another cases. Thus

$$Int_1 = 2\pi a^2\int_0^\infty f(\mu)J_0(\rho_1)J_0(\rho_2)\sinh^2\mu d\mu. \tag{14}$$

Second part of (11) includes $\sin^2\nu$:

$$Int_2 = a^2\int_0^\infty\int_0^{2\pi} f(\mu)\left(\sum_{n_1=-\infty}^{+\infty} i^{n_1}J_{n_1}(\rho_1)e^{in_1(\nu-\omega_\nu)}\cdot\sum_{n_2=-\infty}^{+\infty} i^{n_2}J_{n_2}(\rho_2)e^{in_2(\nu+\omega_\nu)}\right)\sin^2\nu d\mu d\nu. \tag{15}$$

Acting in similar manner as before we receive the following result instead of (13):

$$a^2\int_0^\infty\int_0^{2\pi} i^{n_1}J_{n_1}(\rho_1)e^{in_1(\nu-\omega_\nu)}i^{n_2}J_{n_2}(\rho_2)e^{in_2(\nu+\omega_\nu)}\sin^2\nu d\mu d\nu =$$
$$i^{n_1+n_2}J_{n_1}(\rho_1)J_{n_2}(\rho_2)e^{-i(n_1+n_2)\omega_\nu}d\mu\int_0^{2\pi} e^{i(n_1+n_2)\nu}\sin^2\nu d\nu. \tag{16}$$

Once again the only values of integers are $n_1 = 0$ and $n_2 = 0$. And we have:

$$Int_2 = a^2\pi\int_0^\infty f(\mu)J_0(\rho_1)J_0(\rho_2)d\mu. \tag{17}$$

The final result is the sum of $Int_1 + Int_2$:

$$F(\omega_\mu) = a^2\pi\int_0^\infty f(\mu)J_0(c\cosh(\mu+\omega_\mu))J_0(c\cosh(\mu-\omega_\mu))\cosh(2\mu)d\mu, \tag{18}$$

where $c = 0.5ab$. We also used the identity $2\sinh^2\mu + 1 = \cosh(2\mu)$.

The formula (18) represents Fourier transform in elliptic coordinates for the case of axial symmetry.

If we start from (8) and perform similar conversions then we obtain the inverse formula:

$$f(\mu) = b^2\pi\int_0^\infty F(\omega_\mu)J_0(c\cosh(\mu+\omega_\mu))J_0(c\cosh(\mu-\omega_\mu))\cosh(2\omega_\mu)d\omega_\mu, \tag{19}$$

which is the inverse Fourier transform in elliptic coordinates for the case of axial symmetry.

Useful properties

Relation to Fourier-Bessel transform

Now we show how the integral transform (18)-(19) generalizes the Fourier-Bessel transform. Major axis A and minor axis B of ellipse is equal correspondingly to $A = a\cosh\mu$ and $B = a\sinh\mu$ where a is focus distance. Eccentricity is $\varepsilon = 1/\cosh\mu$. If the major axis A is constant and $\varepsilon \to 0$ then $\mu \to \infty$ and $a \to 0$. So $a\cosh\mu = a\sinh\mu \to r$. As a result we have transformation from elliptic coordinate μ to polar coordinate r. In ω-space we have the similar transformations. Therefore, ω_μ becomes ω_r. And

$$\begin{aligned}
c\cosh(\mu + \omega_\mu) = 0.5ab(\cosh(\mu)\cosh(\omega_\mu) + \sinh(\mu)\sinh(\omega_\mu)) &\to r\omega_r;\\
c\cosh(\mu - \omega_\mu) = 0.5ab(\cosh(\mu)\cosh(\omega_\mu) - \sinh(\mu)\sinh(\omega_\mu)) &\to 0;\\
J_0(c\cosh(\mu + \omega_\mu)) &\to J_0(r\omega_r);\\
J_0(c\cosh(\mu - \omega_\mu)) &\to 1;\\
a^2\cosh(2\mu)d\mu &\to 2rdr;\\
b^2\cosh(2\omega_\mu)d\omega_\mu &\to 2\omega_r d\omega_r.
\end{aligned}$$

Thus, we have well-known Fourier-Bessel transform instead (18)-(19):

$$F(\omega_r) = 2\pi \int_0^\infty f(r)J_0(r\omega_r)rdr, \quad f(r) = 2\pi \int_0^\infty F(\omega_r)J_0(r\omega_r)\omega_r d\omega_r.$$

Parseval's theorem

Now we obtain Parseval's formula [14] which is also known as Parseval's theorem. We multiply (18) by $b^2F(\omega_\mu)\cosh(2\omega_\mu)$ and integrate by ω_μ over whole definition area $[0,\infty)$:

$$\begin{aligned}
&b^2\int_0^\infty F(\omega_\mu)^2\cosh(2\omega_\mu)d\omega_\mu = \\
&\qquad c^2\pi\int_0^\infty\int_0^\infty F(\omega_\mu)f(\mu)J_0(c\cosh(\mu+\omega_\mu))J_0(c\cosh(\mu-\omega_\mu))\cosh(2\mu)\cosh(2\omega_\mu)d\mu d\omega_\mu,
\end{aligned}\tag{20}$$

The similar conversion could be performed with (19). Multiplication by $a^2 f(\mu)\cosh(2\mu)$ and integration by μ over area $[0,\infty)$) gives:

$$\begin{aligned}
&a^2\int_0^\infty f(\mu)^2\cosh(2\mu)d\mu = \\
&\qquad c^2\pi\int_0^\infty\int_0^\infty F(\omega_\mu)f(\mu)J_0(c\cosh(\mu+\omega_\mu))J_0(c\cosh(\mu-\omega_\mu))\cosh(2\mu)\cosh(2\omega_\mu)d\mu d\omega_\mu,
\end{aligned}\tag{21}$$

Comparison of (20) and (21) results in Parseval's formula:

$$a^2\int_0^\infty f(\mu)^2\cosh(2\mu)d\mu = b^2\int_0^\infty F(\omega_\mu)^2\cosh(2\omega_\mu)d\omega_\mu. \tag{22}$$

Relation to modified Mathieu equation

It is important to understand how the noted in introduction integral transformations [6]-[8] in elliptic coordinates relate to expressions (18)-(19). We try to construct differential equation based on integral kernel of (18)-(19) for that purpose.

The direct substitution of $W = J_0(c\cosh(z))$ satisfies the following differential equation:

$$\frac{\partial^2 W}{dz^2} - \frac{2}{\sinh(2z)}\frac{\partial W}{\partial z} + c^2\sinh^2(z)W = 0. \tag{23}$$

We define W_1 as solution (23) when $z = \mu + \omega_\mu$ and W_2 as solution (23) when $z = \mu - \omega_\mu$. Thus, we have:

$$\frac{\partial^2 W_1}{\partial \mu^2} - \frac{2}{\sinh(2\mu + 2\omega_\mu)} \frac{\partial W_1}{\partial \mu} + c^2 \sinh^2(\mu + \omega_\mu) W_1 = 0, \tag{24a}$$

$$\frac{\partial^2 W_2}{\partial \mu^2} - \frac{2}{\sinh(2\mu - 2\omega_\mu)} \frac{\partial W_2}{\partial \mu} + c^2 \sinh^2(\mu - \omega_\mu) W_2 = 0. \tag{24b}$$

Here we use the identity $dz = d(\mu + \omega_\mu) = d(\mu - \omega_\mu) = d\mu$, taking into account that ω_μ is a parameter.

We multiply (24a) by W_2 and (24b) by W_1. After this we make a summation of the resulted equations. Using the relations for derivatives $(W_1 W_2)' = W_1' W_2 + W_1 W_2'$ and $(W_1 W_2)'' = W_1'' W_2 + 2W_1' W_2' + W_1 W_2''$ we can obtain possible differential equations for the function $V = W_1 W_2$.

We select two forms. First one includes first derivative of V:

$$\frac{d^2 V}{d\mu^2} + \left(\frac{2}{\sinh(2\mu + 2\omega_\mu)} - \frac{2}{\sinh(2\mu - 2\omega_\mu)} \right) \frac{dV}{d\mu} + c^2(\sinh^2(\mu + \omega_\mu) + \sinh^2(\mu - \omega_\mu))V = v_1(\mu), \tag{25a}$$

$$v_1(\mu) = \frac{2}{\sinh(2\mu + 2\omega_\mu)} \frac{dW_2}{d\mu} W_1 - \frac{2}{\sinh(2\mu - 2\omega_\mu)} \frac{dW_1}{d\mu} W_2 + 2 \frac{dW_1}{d\mu} \frac{dW_2}{d\mu}. \tag{25b}$$

The second one is:

$$\frac{d^2 V}{d\mu^2} + c^2(\sinh^2(\mu + \omega_\mu) + \sinh^2(\mu - \omega_\mu))V = v_2(\mu), \tag{26a}$$

$$v_2(\mu) = \frac{2}{\sinh(2\mu + 2\omega_\mu)} \frac{dW_1}{d\mu} W_2 - \frac{2}{\sinh(2\mu - 2\omega_\mu)} \frac{dW_2}{d\mu} W_1 + 2 \frac{dW_1}{d\mu} \frac{dW_2}{d\mu}. \tag{26b}$$

Left part of (26a) is referred to the form of modified Mathieu equation:

$$\frac{d^2 V}{d\mu^2} - (z - 2q \cosh(2\mu))V = 0. \tag{27}$$

Here $z = c^2$, $q = 0.5c^2 \cosh(2\omega_\mu)$.

The obtained results (25) or (26) show that the integral kernel of (18)-(19) cannot be a solution of modified Mathieu equation since the nonzero right part.

Alternating definition using Mathieu functions

However, the solution of Mathieu equation (27) could also be useful because it is considered in elliptical coordinates. This differential equation plays a major role in elliptic coordinate system. For example, Helmholtz equation could be solved with splitting into Mathieu equations in these coordinates [9].

The another form of (18) and (19) can be obtained using Mathieu functions. The main idea is based on the relation [15]:

$$e^{k \cos(z) \cos(\theta)} = \sum_{n=0}^{\infty} A_n \, ce_n(z, q) ce_n(\theta, q). \tag{28}$$

Here $q = k^2/32$ and ce_n - Mathieu function.

Substitution of (28) into (7) gives:

$$F(\omega_\mu, \omega_\nu) = \int_0^\infty \int_0^{2\pi} f(\mu, \nu) a^2 (\sinh^2 \mu + \sin^2 \nu) \sum_{n=0}^{\infty} A_n \, Ce_n(\mu + \omega_\mu, q) \, ce_n(\nu - \omega_\nu, q) \times$$
$$\sum_{m=0}^{\infty} B_m \, Ce_m(\mu - \omega_\mu, q) \, ce_m(\nu + \omega_\nu, q) d\mu d\nu, \tag{29}$$

where Ce_n is modified Mathieu function and $k = 0.5iab$.

If $f(\mu, \nu) = f(\mu)$, then the formula (29) do not allow exclude directly the argument ν as it was done for axial symmetry case using Bessel functions. Thus, the previous definition for Fourier transform in elliptic coordinates (18) is preferable for us.

Possibilities for application

The presented formulae (18) and (19) allow to consider Fourier transform methods in elliptical coordinate system. However, the particular examples of its application are not available in analytical form. At present moment there are no variety of functions which could be transformed analytically. We believe that certain progress in this question will be achieved in nearest future due to the continuous developing of the Bessel functions theory.

Therefore numerical methods could be critically important. The application of approximations of (18) and (19) resulted in integration of highly oscillating functions. This means using of Filon's type methods which are rather slow.

This reasoning shows that a good perspective for further study is to present a fast algorithm which can combine accuracy properties with economic loads of computational resources.

The discussion of such algorithms it is required of the separate consideration and is not included in this work.

Representations in Quantum Mechanics

Returning to the section topic the coordinate and momentum representations in quantum mechanics are the possible field for the supposed integral transformation (18) and (19). As well-known in coordinate representation we have the definitions for the operators acting on the wave function ϕ

$$\hat{x}\phi(\mathbf{x}) \quad = x\phi(\mathbf{x}),\ \hat{x} \text{ - operator of } x\text{-coordinate}, \quad \hat{p}\phi(\mathbf{x}) \quad = -i\hbar\nabla\phi(\mathbf{x}),\ \hat{p} \text{ -operator of momentum } \mathbf{p},$$

and in momentum representation we have

$$\hat{x}\phi(\mathbf{p}) \quad = i\hbar\nabla\phi(\mathbf{p}), \quad \hat{p}\phi(\mathbf{p}) \quad = p\phi(\mathbf{p}).$$

The direct and reverse transitions between these representations could be done using Fourier transform:

$$\phi(\mathbf{x}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int_{R^3} e^{\frac{i}{\hbar}\mathbf{xp}}\phi(\mathbf{p})d\mathbf{p}, \quad \phi(\mathbf{p}) = \frac{1}{(2\pi\hbar)^{\frac{3}{2}}} \int_{R^3} e^{-\frac{i}{\hbar}\mathbf{xp}}\phi(\mathbf{x})d\mathbf{x}.$$

The case of axial symmetry in elliptical coordinates automatically involves the formulas (18) and (19) for that type transitions.

Operator representation in Operational Dynamic Modelling

The similar application is ODM. This study allow to obtain the Wigner function in curvilinear coordinates.

The general description of ODM consider the using of new auxiliary operators $\hat{\theta}^\nu$ and $\hat{\lambda}^\nu$ besides $\hat{x}^\nu$ and $\hat{p}^\nu$. Here and below the $^\nu$ and $_\mu$ are the covariant and contravariant indices. Thus we have new commutation relations:

$$[\hat{x}^\mu, \hat{p}_\nu] = i\hbar\delta^\mu_\nu, \quad [\hat{x}^\mu, \hat{\lambda}_\nu] = i\delta^\mu_\nu, \quad [\hat{p}_\mu, \hat{\theta}^\nu] = i\delta^\nu_\mu, \quad [\hat{\lambda}_\mu, \hat{\theta}^\nu] = 0. \tag{30}$$

Introducing also operators:

$$\hat{x}^\mu = \hat{X}^\mu - \tfrac{\hbar}{2}\hat{\Theta}^\mu; \quad \hat{p}^\mu = \hat{P}_\mu + \tfrac{\hbar}{2}\hat{\Lambda}^\mu; \quad \hat{\lambda}_\mu = \hat{\Lambda}_\mu; \quad \hat{\theta}^\mu = \hat{\Theta}^\mu;$$

in X-Θ representation the explicit form of these classical operators is:

$$\hat{X}^\mu = X^\mu, \quad \hat{\Lambda}_\mu = -i\tfrac{\partial}{\partial X^\mu}, \quad \hat{P}_\mu = i\tfrac{\partial}{\partial \Theta^\mu}, \quad \hat{\Theta}^\mu = \Theta^\mu.$$

X-P representation gives:

$$\hat{X}^\mu = X^\mu, \quad \hat{\Lambda}_\mu = -i\tfrac{\partial}{\partial X^\mu}, \quad \hat{P}_\mu = P_\mu, \quad \hat{\Theta}^\mu = -i\tfrac{\partial}{\partial P^\mu}.$$

Obviously the Λ-P and Λ-Θ representations could be shown in the similar way.

Applying the coordinate system allow to obtain exact form of these operators. For Wigner function ψ the transition between representations undergo through Fourier transform [16].

Let us see the transition between X-Θ to X-P representation. Applying this we should write the following for the general ψ-function in rectangular coordinate system:

$$\psi(X_1, X_2, P_1, P_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \psi(X_1, X_2, \Theta_1, \Theta_2)e^{-i(P_1\Theta_1 + P_2\Theta_2)}d\Theta_1 d\Theta_2.$$

Here $X_1 = x$, $X_2 = y$, $P_1 = p_x$ and $P_2 = p_y$. As it was done previously in (8) in elliptical coordinates this will be:

$$\psi(X_1, X_2, P_1, P_2) = b^2 \int_0^\infty \int_0^{2\pi} \psi(X_1, X_2, \Theta_1, \Theta_2)(\sinh^2\Theta_1 + \sin^2\Theta_2)\times$$
$$\exp[-i0.5ab(\cosh(P_1 + \Theta_1)\cos(P_2 - \Theta_2) + \cosh(P_1 - \Theta_1)\cos(P_2 + \Theta_2))]d\Theta_1 d\Theta_2,$$

where $X_1 = \mu$, $X_2 = \nu$, $P_1 = p_\mu$, $P_2 = p_\nu$, a and b are the focuses. In the case of axial symmetry we receive the result accordingly to (19):

$$\psi(X_1, P_1) = b^2\pi \int_0^\infty \psi(X_1, \Theta_1) J_0(0.5ab\cosh((P_1 + \Theta_1))) J_0(0.5ab\cosh(P_1 - \Theta_1))\cosh(2\Theta_1)d\Theta_1.$$

Another transitions can be easily obtained in the same manner.

Conclusions

We have received a form of Fourier transform in two dimensional elliptic coordinates for the case of axial symmetry. The resulted formulae (18)-(19) contain multiplication of Bessel functions of zero order that generalize the well-known Fourier-Bessel transform. Also we have shown the Parseval's theorem for this case.

Moreover, we have considered connection of the obtained integral transformation with Mathieu equation. Practically the hypothesis about the identity of Fourier transform and transforms based on the solution of Mathieu equation has not proved. We also could not receive connection formula between these transformations. However, instead of Bessel functions we can use Mathieu functions for the Fourier transform in elliptical coordinates.

The application of considered integral transform can extend the existing Fourier transform techniques in curvilinear coordinates. The good example is the coordinate and momentum representations in Quantum Mechanics. The obtained formulae (18)-(19) allow directly to do the transitions between these representations.

Another example of application is the Operational Dynamic Modelling which is also considered the different operator representations to obtain Wigner function. And once again the transitions between these representations in elliptical coordinates resulted in integrals (18)-(19) for case of axial symmetry.

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