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FOURIER TRANSFORM OF RANDOM VARIABLES ASSOCIATED WITH THE MULTI-DIMENSIONAL HEISENBERG LIE ALGEBRA

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Abstract. We compute the Fourier transform (or vacuum characteristic function) of quantum random variables (observables), defined as self-adjoint finite sums of Fock space operators, satisfying the multi-dimensional Heisenberg Lie algebra commutation relations. The main tool is a splitting formula for the multi-dimensional Heisenberg group obtained by Feinsilver and Pap.

1. Bochner's Theorem and quantum random variables

A continuous function $f : \mathbb{R} \mapsto \mathbb{C}$ is *positive definite* if

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t-s)\phi(t)\bar{\phi}(s)\, dt\, ds \geq 0$$

for every continuous function $\phi : \mathbb{R} \mapsto \mathbb{C}$ with compact support. Bochner's theorem (see [3, p. 346]) states that such a function can be represented as

$$f(t) = \int_{\mathbb{R}} e^{it\lambda}\, dv(\lambda),$$

where v is a non-decreasing right-continuous bounded function. If $f(0) = 1$, then such a function v defines a probability measure on $\mathbb{R}$ and Bochner's theorem says that f is the Fourier transform of a probability measure, i.e., the characteristic function of a random variable that follows the probability distribution defined by v. Moreover, the condition of positive definiteness of f is necessary and sufficient for such a representation.

An example of such a positive definite function is provided by $f(t) = \langle \Phi, e^{itX}\Phi\rangle$ where Φ is the normalized vacuum vector of a Fock-Hilbert space $\mathcal{F}$ and X is an *observable* (self-adjoint operator on $\mathcal{F}$) also called a *quantum random variable* in which case

$$\int_{\mathbb{R}} \int_{\mathbb{R}} f(t-s)\phi(t)\bar{\phi}(s)\, dt\, ds = \| \int_{\mathbb{R}} e^{-itX}\phi(t)\, dt\ \Phi \|^2 \geq 0.$$

In this paper we consider quantum random variables X acting on the Fock space associated with the multi-dimensional Heisenberg algebra and our goal is to provide a formula for the computation of their *vacuum characteristic function* $\langle \Phi, e^{itX}\Phi\rangle$.

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2. Multi-dimensional Heisenberg algebra random variables

The multi-dimensional Heisenberg algebra is the infinite-dimensional Lie algebra with generators $\{\mathbf{1}, x_j, D_k \mid j,k \geq 1\}$ and commutation relations

$$[D_k, x_j] = \delta_{j,k}\mathbf{1}\ ,\ [D_k, \mathbf{1}] = [x_j, \mathbf{1}] = [D_k, D_j] = [x_k, x_j] = 0.$$

The following theorem was proved in [1] and is central to our approach. For convenience we keep its notation.

Theorem 1. *Let $\alpha_1 \in \mathbb{C}$, $\alpha_2, \alpha_3 \in \mathbb{C}^{n\times 1}$, $\alpha_4, \alpha_5, \alpha_6 \in \mathbb{C}^{n\times n}$, with $\alpha_4^T = \alpha_4$ and $\alpha_6^T = \alpha_6$. Let also*

$$v = \begin{pmatrix} \alpha_5 & \alpha_4 \\ -\alpha_6 & -\alpha_5^T \end{pmatrix}$$

and define the functions $P, Q, R, S : \mathbb{R} \mapsto \mathbb{C}^{n\times n}$ by

$$e^{tv} = \begin{pmatrix} P(t) & Q(t) \\ -R(t) & S(t) \end{pmatrix}.$$

Then, letting $\mathbf{1}$ denote the identity operator and using the notation

$$x_{\alpha_2} = \sum_{j=1}^n \alpha_2^j x_j\ ,\ D_{\alpha_3} = \sum_{j=1}^n \alpha_3^j D_j\ ,\ R_{\alpha_4} = \frac{1}{2}\sum_{j,k=1}^n \alpha_4^{j,k} x_j x_k,$$

$$\rho_{\alpha_5} = \frac{1}{2}\sum_{j,k=1}^n \alpha_5^{j,k}\left(x_j D_k + D_k x_j\right)\ ,\ \Delta_{\alpha_6} = \frac{1}{2}\sum_{j,k=1}^n \alpha_6^{j,k} D_j D_k$$

for $t \in \mathbb{R}$ sufficiently close to 0 we have

$$e^{t\left(\alpha_1\mathbf{1} + x_{\alpha_2} + D_{\alpha_3} + R_{\alpha_4} + \rho_{\alpha_5} + \Delta_{\alpha_6}\right)} = e^{A_1(t)\mathbf{1}} e^{x_{A_2(t)}} e^{D_{A_3(t)}} e^{R_{A_4(t)}} e^{\rho_{A_5(t)}} e^{\Delta_{A_6(t)}},$$

where the functions $A_1 : \mathbb{R} \mapsto \mathbb{C}$, $A_2, A_3 : \mathbb{R} \mapsto \mathbb{C}^{n\times 1}$ and $A_4, A_5, A_6 : \mathbb{R} \mapsto \mathbb{C}^{n\times n}$ are given by

$$A_1(t) = \alpha_1 t + \frac{1}{2}\begin{pmatrix} \alpha_3^T & \alpha_2^T \end{pmatrix} \frac{e^{tv} - \mathbf{1} - tv}{v^2} \begin{pmatrix} \alpha_2 \\ -\alpha_3 \end{pmatrix} + \frac{1}{2} A_2(t)^T A_3(t),$$

$$\begin{pmatrix} A_2(t) \\ -A_3(t) \end{pmatrix} = \frac{e^{tv} - \mathbf{1}}{v} \begin{pmatrix} \alpha_2 \\ -\alpha_3 \end{pmatrix},$$

$$A_4(t) = Q(t)S(t)^{-1}\ ,\ A_5(t) = -\log S(t)^T\ ,\ A_6(t) = S(t)^{-1}R(t),$$

and the functions $\frac{e^{tv}-\mathbf{1}}{v}$ and $\frac{e^{tv}-\mathbf{1}-tv}{v^2}$ are understood in the sense of power series, i.e.,

$$\frac{e^{tv} - \mathbf{1}}{v} = t\sum_{k=0}^{\infty} \frac{(tv)^k}{(k+1)!}\ ,\ \frac{e^{tv} - I - tv}{v^2} = t^2 \sum_{k=0}^{\infty} \frac{(tv)^k}{(k+2)!}.$$

Lemma 1. *Let D, x, and h satisfy the Heisenberg algebra commutation relations $[D, x] = h$ and $[D, h] = [x, h] = 0$. Then, for all $t, a \in \mathbb{R}$*

$$e^{tD} e^{ax} = e^{ax} e^{tD} e^{ath}, \tag{2.1}$$

$$[e^{tD}, x^n] = \sum_{k=0}^{n-1} \binom{n}{k} t^{n-k} h^{n-k} x^k e^{tD}, \tag{2.2}$$

$$e^{tD} e^{ax^2} = e^{a(th+x)^2} e^{tD}. \tag{2.3}$$

Proof. The proof of (2.1) can be found in [2]. For (2.2), using (2.1) and Leibniz's rule for derivatives we have

$$[e^{tD}, x^n] = e^{tD}x^n - x^n e^{tD} = \frac{\partial^n}{\partial a^n}|_{a=0} e^{tD}e^{ax} - x^n e^{tD}$$

$$= \frac{\partial^n}{\partial a^n}|_{a=0} e^{ax}e^{tD}e^{ath} - x^n e^{tD} = \sum_{k=0}^{n} \binom{n}{k} t^{n-k}h^{n-k}x^k e^{tD} - x^n e^{tD}$$

$$= \sum_{k=0}^{n-1} \binom{n}{k} t^{n-k}h^{n-k}x^k e^{tD}.$$

To prove (2.3) we notice that

$$e^{tD}e^{ax^2} = e^{tD}\sum_{n=0}^{\infty}\frac{a^n x^{2n}}{n!} = \sum_{n=0}^{\infty}\frac{a^n}{n!}e^{tD}x^{2n}$$

$$= \sum_{n=0}^{\infty}\frac{a^n}{n!}\left([e^{tD}, x^{2n}] + x^{2n}e^{tD}\right) = \sum_{n=0}^{\infty}\frac{a^n}{n!}[e^{tD}, x^{2n}] + \sum_{n=0}^{\infty}\frac{a^n}{n!}x^{2n}e^{tD}$$

which using (2.2) is

$$= \sum_{n=0}^{\infty}\frac{a^n}{n!}\sum_{k=0}^{2n-1}\binom{2n}{k} t^{2n-k}h^{2n-k}x^k e^{tD} + e^{ax^2}e^{tD}$$

$$= \sum_{n=0}^{\infty}\frac{a^n}{n!}\left(\sum_{k=0}^{2n}\binom{2n}{k} t^{2n-k}h^{2n-k}x^k - x^{2n}\right)e^{tD} + e^{ax^2}e^{tD}$$

$$= \left(\sum_{n=0}^{\infty}\frac{a^n}{n!}(th+x)^{2n} - \sum_{n=0}^{\infty}\frac{a^n}{n!}x^{2n}\right)e^{tD} + e^{ax^2}e^{tD}$$

$$= \left(e^{a(th+x)^2} - e^{ax^2}\right)e^{tD} + e^{ax^2}e^{tD} = e^{a(th+x)^2}e^{tD}.$$

□

Our goal is to compute the moment generating and characteristic functions, $\langle \Phi, e^{tX}\Phi\rangle$ and $\langle \Phi, e^{itX}\Phi\rangle$, respectively, of the random variable

$$X = \alpha_1\mathbf{1} + x_{\alpha_2} + D_{\alpha_3} + R_{\alpha_4} + \rho_{\alpha_5} + \Delta_{\alpha_6},$$

where Φ is the Fock vacuum vector with $\langle\Phi, \Phi\rangle = 1$. We assume that for all $n > 0$, $D_n\Phi = 0$ which implies that for all $a_n \in \mathbb{C}$

$$e^{a_n D_n}\Phi = \sum_{k=0}^{\infty}\frac{a_n^k}{k!}D_n^k\Phi = \Phi.$$

Moreover, we assume that for all $n > 0$, $(D_n)^* = x_n$, $(x_n)^* = D_n$. In view of this assumption we have that for each $i, j > 0$, $D_i = a_i$ and $x_j = a_j^\dagger$ where $[a_i, a_j^\dagger] = \delta_{i,j}$ and $a_i^* = a_i^\dagger$, i.e., a_i and $a_i^\dagger$ are a Boson pair.

The operator

$$Z = \alpha_1\mathbf{1} + x_{\alpha_2} + D_{\alpha_3} + R_{\alpha_4} + \rho_{\alpha_5} + \Delta_{\alpha_6}$$

is self-adjoint if and only if

$$\alpha_1 \in \mathbb{R}\,,\ \alpha_2 = \alpha_3^*\,,\ \alpha_6 = \alpha_4^*\,,\ \alpha_5 = \alpha_5^*,$$

where $(\cdots)^*$ denotes conjugate transpose. Since in Theorem 1 we have assumed that $\alpha_4^T = \alpha_4$ and $\alpha_6^T = \alpha_6$, the condition $\alpha_6 = \alpha_4^*$ reduces to $\alpha_6 = \overline{\alpha_4}$.

The Fock space corresponding to the multi-dimensional Heisenberg algebra would be defined as the Hilbert space completion of the exponential vectors

$$\psi_a := e^{\sum_{i=1}^{\infty} a_i x_i}\Phi = \prod_{i=1}^{\infty} e^{a_i x_i}\Phi \ , \ a = (a_1, a_2, ...) \in l_2(\mathbb{C}),$$

with respect to the inner product

$$\langle \psi_a, \psi_b \rangle := \langle \prod_{i=1}^{\infty} e^{a_i x_i}\Phi, \prod_{i=1}^{\infty} e^{b_i x_i}\Phi \rangle = \prod_{i=1}^{\infty} e^{\overline{a_i} b_i} = e^{\sum_{i=1}^{\infty} \overline{a_i} b_i}.$$

Throughout this paper we will make repeated use of the fact that for all group elements g

$$\langle \Phi, e^{a_n x_n} g\Phi \rangle = \langle (e^{a_n x_n})^* \Phi, g\Phi \rangle = \langle e^{\bar{a}_n D_n}\Phi, g\Phi \rangle = \langle \Phi, g\Phi \rangle.$$

Lemma 2. *For $i, j, k \in \{1, 2, ..., n\}$ and $a_i, b_{j,k} \in \mathbb{C}$*

$$\langle \Phi, \prod_{i=1}^{n} e^{a_i D_i} \prod_{j,k=1\, ,\, j\neq k}^{n} e^{b_{j,k} x_j x_k}\Phi \rangle = e^{\sum_{i,j=1,\, i\neq j}^{n} a_i a_j b_{i,j}}.$$

Proof. Let $A = \sum_{i=1}^{n} a_i D_i$ and $B = \sum_{j,k=1\, ,\, j\neq k}^{n} b_{j,k} x_j x_k$. Then

$$C := [A, B] = \sum_{i=1}^{n} \sum_{j,k=1\, ,\, j\neq k}^{n} a_i b_{j,k} [D_i, x_j x_k] = \sum_{i=1}^{n} \sum_{j,k=1\, ,\, j\neq k}^{n} a_i b_{j,k} c_{i,j,k},$$

where

$$c_{i,j,k} = [D_i, x_j x_k] = \begin{cases} 0 & \text{if } i \neq j, i \neq k, \\ x_k & \text{if } i = j, i \neq k, \\ x_j & \text{if } i = k, i \neq j. \end{cases}$$

Clearly $[B, C] = 0$ and

$$E := [A, C] = \sum_{m=1}^{n} \sum_{i=1}^{n} \sum_{j,k=1\, ,\, j\neq k}^{n} a_m a_i b_{j,k} [D_m, c_{i,j,k}]$$

$$= \sum_{m=1}^{n} \sum_{i=1}^{n} \sum_{j,k=1\, ,\, j\neq k}^{n} a_m a_i b_{j,k} d_{m,i,j,k},$$

where

$$d_{m,i,j,k} = \begin{cases} 1 & \text{if } m = k \neq i = j \text{ or } m = j \neq i = k, \\ 0 & \text{otherwise.} \end{cases}$$

Thus

$$E = 2 \sum_{i,j=1,\, i\neq j}^{n} a_i a_j b_{i,j}.$$

Using the expansion

$$e^A B e^{-A} = e^{\mathrm{ad}A} B = B + [A, B] + \frac{1}{2}[A, [A, B]] + 0 + \cdots$$

$$= B + C + \frac{1}{2}E$$

we find that

$$e^A B^n e^{-A} = \left(e^A B e^{-A}\right)\left(e^A B e^{-A}\right)\cdots\left(e^A B e^{-A}\right) = \left(B + C + \frac{1}{2}E\right)^n$$

and so

$$e^A e^B e^{-A} = \sum_{n=1}^{\infty} \frac{1}{n!} e^A B^n e^{-A} = \sum_{n=1}^{\infty} \frac{1}{n!}\left(B + C + \frac{1}{2}E\right)^n = e^{B+C+\frac{1}{2}E}.$$

Thus

$$\begin{aligned}\langle \Phi, \prod_{i=1}^n e^{a_i D_i} \prod_{j,k=1\,,\,j\neq k}^n e^{b_{j,k} x_j x_k}\Phi\rangle &= \langle \Phi, e^A e^B \Phi\rangle = \langle \Phi, e^A e^B e^{-A}\Phi\rangle \\ &= \langle \Phi, e^{B+C+\frac{1}{2}E}\Phi\rangle = \langle \Phi, e^{B+C} e^{\frac{1}{2}E}\Phi\rangle = e^{\frac{1}{2}E}\langle e^{B^*+C^*}\Phi, \Phi\rangle \\ &= e^{\frac{1}{2}E}\langle\Phi,\Phi\rangle = e^{\frac{1}{2}E} = e^{\sum_{i,j=1,\, i\neq j}^n a_i a_j b_{i,j}}.\end{aligned}$$

□

Theorem 2. *In the notation of Theorem* 1*, for* $t \in \mathbb{R}$ *sufficiently close to* 0*, the moment generating and characteristic functions of the random variable*

$$Z = \alpha_1 \mathbf{1} + x_{\alpha_2} + D_{\alpha_3} + R_{\alpha_4} + \rho_{\alpha_5} + \Delta_{\alpha_6}$$

are, respectively,

$$\langle \Phi, e^{tZ}\Phi\rangle = e^{A_1(t)} e^{\frac{1}{2}\sum_{j=1}^n A_5^{j,j}(t) + A_4^{j,j}(t) A_3^j(t)^2} e^{\frac{1}{2}\sum_{m,k=1,\, m\neq k}^n A_3^m(t) A_3^k(t) A_4^{m,k}(t)}$$

and

$$\langle \Phi, e^{itZ}\Phi\rangle = e^{A_1(it)} e^{\frac{1}{2}\sum_{j=1}^n \left(A_5^{j,j}(it) + A_4^{j,j}(it) A_3^j(it)^2\right)} e^{\frac{1}{2}\sum_{m,k=1,\, m\neq k}^n \left(A_3^m(it) A_3^k(it) A_4^{m,k}(it)\right)}.$$

Proof. By Theorem 1

$$\begin{aligned}&\langle \Phi, e^{t\left(\alpha_1 \mathbf{1} + x_{\alpha_2} + D_{\alpha_3} + R_{\alpha_4} + \rho_{\alpha_5} + \Delta_{\alpha_6}\right)}\Phi\rangle \\ &= \langle \Phi, e^{A_1(t)\mathbf{1}} e^{x_{A_2(t)}} e^{D_{A_3(t)}} e^{R_{A_4(t)}} e^{\rho_{A_5(t)}} e^{\Delta_{A_6(t)}}\Phi\rangle \\ &= e^{A_1(t)}\,\langle \Phi, e^{x_{A_2(t)}} e^{D_{A_3(t)}} e^{R_{A_4(t)}} e^{\rho_{A_5(t)}} e^{\Delta_{A_6(t)}}\Phi\rangle \\ &= e^{A_1(t)}\,\langle \left(e^{x_{A_2(t)}}\right)^* \Phi, e^{D_{A_3(t)}} e^{R_{A_4(t)}} e^{\rho_{A_5(t)}} e^{\Delta_{A_6(t)}}\Phi\rangle \\ &\text{(using } \left(e^{x_{A_2(t)}}\right)^*\Phi = e^{\sum_{j=1}^n \bar{A}_2^j(t) D_j}\Phi = \textstyle\prod_{j=1}^n e^{\bar{A}_2^j(t) D_j}\Phi = \Phi \\ &\text{and } e^{\Delta_{A_6(t)}}\Phi = e^{\frac{1}{2}\sum_{j,k=1}^n A_6^{j,k}(t) D_j D_k}\Phi = \Phi\ ) \\ &= e^{A_1(t)}\langle \Phi, e^{D_{A_3(t)}} e^{R_{A_4(t)}} e^{\rho_{A_5(t)}}\Phi\rangle.\end{aligned}$$

Moreover,

$$\begin{aligned}\rho_{A_5(t)} &= \frac{1}{2}\sum_{j,k=1}^n A_5^{j,k}(t)\,(x_j D_k + D_k x_j) \\ &= \frac{1}{2}\left(\sum_{1\le j\neq k\le n} A_5^{j,k}(t)\,2x_j D_k + \sum_{j=1}^n A_5^{j,j}(t)\,(x_j D_j + D_j x_j)\right) \\ &= \frac{1}{2}\left(\sum_{1\le j\neq k\le n} A_5^{j,k}(t)\,2x_j D_k + \sum_{j=1}^n A_5^{j,j}(t)\,(x_j D_j + [D_j, x_j] + x_j D_j)\right)\end{aligned}$$

$$= \frac{1}{2}\left(\sum_{1\le j\ne k\le n} A_5^{j,k}(t)\,2x_jD_k + \sum_{j=1}^{n} A_5^{j,j}(t)\,(2x_jD_j+1)\right)$$

$$= \sum_{1\le j\ne k\le n} A_5^{j,k}(t)x_jD_k + \sum_{j=1}^{n} A_5^{j,j}(t)x_jD_j + \frac{1}{2}\sum_{j=1}^{n} A_5^{j,j}(t)$$

so we have that

$$e^{\rho_{A_5(t)}}\Phi = e^{\frac{1}{2}\sum_{j=1}^{n} A_5^{j,j}(t)} e^{\sum_{1\le j\ne k\le n} A_5^{j,k}(t)x_jD_k + \sum_{j=1}^{n} A_5^{j,j}(t)x_jD_j}\Phi$$

$$= e^{\frac{1}{2}\sum_{j=1}^{n} A_5^{j,j}(t)} e^{\sum_{1\le j,k\le n} A_5^{j,k}(t)x_jD_k}\Phi = e^{\frac{1}{2}\sum_{j=1}^{n} A_5^{j,j}(t)}\Phi = \prod_{j=1}^{n} e^{\frac{1}{2}A_5^{j,j}(t)}\Phi$$

because $\sum_{1\le j,k\le n} A_5^{j,k}(t)x_jD_k\Phi = 0$ implies $e^{\sum_{1\le j,k\le n} A_5^{j,k}(t)x_jD_k}\Phi = \Phi$. Thus

$$\langle \Phi, e^{t\left(\alpha_1\mathbf{1} + x_{\alpha_2} + D_{\alpha_3} + R_{\alpha_4} + \rho_{\alpha_5} + \Delta_{\alpha_6}\right)}\Phi\rangle$$

$$= e^{A_1(t)}\prod_{j=1}^{n} e^{\frac{1}{2}A_5^{j,j}(t)}\langle \Phi, e^{D_{A_3(t)}}e^{R_{A_4(t)}}\Phi\rangle.$$

Using

$$D_{A_3(t)} = \sum_{j=1}^{n} A_3^{j}(t)D_j\ ,\ R_{A_4(t)} = \frac{1}{2}\sum_{j,k=1}^{n} A_4^{j,k}(t)x_jx_k$$

we have that

$$e^{D_{A_3(t)}}e^{R_{A_4(t)}} = e^{\sum_{j=1}^{n} A_3^{j}(t)D_j} e^{\frac{1}{2}\sum_{j,k=1}^{n} A_4^{j,k}(t)x_jx_k}$$

which, since the operators in each exponent commute, splits into

$$\prod_{j=1}^{n} e^{A_3^{j}(t)D_j} \prod_{J,K=1}^{n} e^{\frac{1}{2}A_4^{J,K}(t)x_Jx_K}$$

$$= \prod_{j=1}^{n} e^{A_3^{j}(t)D_j} \prod_{m=1}^{n} e^{\frac{1}{2}A_4^{m,m}(t)x_m^2} \prod_{J,K=1\,,\,J\ne K}^{n} e^{\frac{1}{2}A_4^{J,K}(t)x_Jx_K}$$

$$= \prod_{j=1}^{n}\left(e^{A_3^{j}(t)D_j}e^{\frac{1}{2}A_4^{j,j}(t)x_j^2}\right) \prod_{J,K=1\,,\,J\ne K}^{n} e^{\frac{1}{2}A_4^{J,K}(t)x_Jx_K}$$

(by Lemma 1 (iii))

$$= \prod_{j=1}^{n}\left(e^{\frac{1}{2}A_4^{j,j}(t)(A_3^{j}(t)+x_j)^2}e^{A_3^{j}(t)D_j}\right) \prod_{J,K=1\,,\,J\ne K}^{n} e^{\frac{1}{2}A_4^{J,K}(t)x_Jx_K}.$$

Thus

$$\langle \Phi, e^{t\left(\alpha_1\mathbf{1} + x_{\alpha_2} + D_{\alpha_3} + R_{\alpha_4} + \rho_{\alpha_5} + \Delta_{\alpha_6}\right)}\Phi\rangle$$

$$= e^{A_1(t)}\prod_{j=1}^{n} e^{\frac{1}{2}A_5^{j,j}(t)}$$

$$\times\langle \Phi, \prod_{j=1}^{n}\left(e^{\frac{1}{2}A_4^{j,j}(t)(A_3^{j}(t)+x_j)^2}e^{A_3^{j}(t)D_j}\right) \prod_{J,K=1\,,\,J\ne K}^{n} e^{\frac{1}{2}A_4^{J,K}(t)x_Jx_K}\Phi\rangle$$

$$= e^{A_1(t)}\prod_{j=1}^{n} e^{\frac{1}{2}A_5^{j,j}(t)}$$

$$\times\langle\prod_{j=1}^{n} e^{\frac{1}{2}\overline{A_4^{j,j}}(t)(\overline{A_3^j}(t)+D_j)^2}\Phi, \prod_{j=1}^{n} e^{A_3^j(t)D_j} \prod_{J,K=1\,,\,J\neq K}^{n} e^{\frac{1}{2}A_4^{J,K}(t)x_Jx_K}\Phi\rangle.$$

Since, for each $j=1,2,...,n$

$$e^{\frac{1}{2}\overline{A_4^{j,j}}(t)(\overline{A_3^j}(t)+D_j)^2}\Phi = e^{\frac{1}{2}\overline{A_4^{j,j}}(t)\overline{A_3^j}(t)^2} e^{\frac{1}{2}\overline{A_4^{j,j}}(t)D_j^2} e^{\overline{A_4^{j,j}}(t)\overline{A_3^j}(t)D_j}\Phi$$
$$= e^{\frac{1}{2}\overline{A_4^{j,j}}(t)\overline{A_3^j}(t)^2}\Phi$$

we find that

$$\langle\Phi, e^{t\left(\alpha_1\mathbf{1}+x_{\alpha_2}+D_{\alpha_3}+R_{\alpha_4}+\rho_{\alpha_5}+\Delta_{\alpha_6}\right)}\Phi\rangle$$
$$= e^{A_1(t)}\prod_{j=1}^{n} e^{\frac{1}{2}A_5^{j,j}(t)}\prod_{j=1}^{n} e^{\frac{1}{2}A_4^{j,j}(t)A_3^j(t)^2}$$
$$\times\langle\Phi, \prod_{j=1}^{n} e^{A_3^j(t)D_j} \prod_{J,K=1\,,\,J\neq K}^{n} e^{\frac{1}{2}A_4^{J,K}(t)x_Jx_K}\Phi\rangle$$
$$= e^{A_1(t)}\prod_{j=1}^{n} e^{\frac{1}{2}A_5^{j,j}(t)+\frac{1}{2}A_4^{j,j}(t)A_3^j(t)^2}$$
$$\times\langle\Phi, \prod_{j=1}^{n} e^{A_3^j(t)D_j} \prod_{J,K=1\,,\,J\neq K}^{n} e^{\frac{1}{2}A_4^{J,K}(t)x_Jx_K}\Phi\rangle$$

(which by Lemma 2 is)

$$= e^{A_1(t)}e^{\frac{1}{2}\sum_{j=1}^{n}\left(A_5^{j,j}(t)+A_4^{j,j}(t)A_3^j(t)^2\right)}e^{\sum_{I,J=1,\,I\neq J}^{n}A_3^I(t)A_3^J(t)\frac{1}{2}A_4^{I,J}(t)}$$
$$= e^{A_1(t)}e^{\frac{1}{2}\sum_{j=1}^{n}\left(A_5^{j,j}(t)+A_4^{j,j}(t)A_3^j(t)^2\right)}e^{\frac{1}{2}\sum_{I,J=1,\,I\neq J}^{n}A_3^I(t)A_3^J(t)A_4^{I,J}(t)}.$$

Replacing t by it we obtain the formula for the characteristic function. □

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