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FOURIER TRANSFORMS AND MEASURE-PRESERVING TRANSFORMATIONS[1]

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ABSTRACT. There exists a continuous function f on the real line, vanishing at infinity, such that, for every measure-preserving transformation h, the composition $f\circ h$ fails to be a Fourier transform. This fact is a consequence of a theorem about measurable functions which is obtained from the theory of idempotents.

When G is a locally compact abelian group, and Γ is its dual group, let $A(G)$ denote the algebra of Fourier transforms of elements of $L^1(\Gamma)$, as described in Rudin's book [**9**, Chapter 1]. Let Z, R and T denote respectively the integer group, the real number system, and the circle group.

Jean-Pierre Kahane [**4**] adapted the work of P. J. Cohen and H. Davenport [**3**] to show that there is a function f in $C_0(Z)$ such that for every permutation p of the integers, $f\circ p$ fails to be in $A(Z)$. In this paper, the following result is obtained in a similar way.

THEOREM 1. *There is a function f in $C_0(R)$ such that for every measure-preserving transformation $h:R\to R$, $f\circ h$ fails to be in $A(R)$.*

Theorem 1 is a consequence of the stronger Theorem 2 below, which concerns measurable functions, not just continuous ones. If S is a Lebesgue-measurable set, let $|S|$ denote the measure of S. Let $L_0(R)$ denote the class of Lebesgue-measurable functions f such that $|\{x:|f(x)|>\varepsilon\}|$ is finite for every $\varepsilon>0$.

THEOREM 2. *For every positive number s, there exist small positive numbers $\alpha=\alpha(s)$ and $\varepsilon=\varepsilon(s)$ such that if $f\in L_0(R)$ and if*

$$|\{x:\varepsilon<|f(x)|<1\}|<\alpha\,|\{x:|f(x)|\geqq 1\}|,$$

then there is a discrete measure $\mu\in M(R)$ such that $\|\hat{\mu}\|_\infty\leqq 1$ and $|\int f\,d\mu|>s$.

Theorem 2 implies that if $f\in L_0(R)$ and

$$(1)\qquad |\{x:\varepsilon(s)<s^{1/2}|f(x)|<1\}|<\alpha(s)\,|\{x:s^{1/2}|f(x)|\geqq 1\}|,$$

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then there is a discrete measure μ such that $\|\hat{\mu}\|_\infty \leqq 1$ and $|\int f\,d\mu| > \sqrt{s}$. It is easy to construct a function $f \in C_0(R)$ such that (1) is satisfied for a sequence of values of s tending to ∞. If h is a measure-preserving transformation, then $f \circ h$ also satisfies (1) for the same values of s. If $M_1(R)$ denotes the space of discrete finite measures on R, then

$$\sup\left\{\left|\int f \circ h \, d\mu\right| : \mu \in M_1(R),\ \|\hat{\mu}\|_\infty \leqq 1\right\} = \infty.$$

Therefore $f \circ h$ cannot belong to $A(R)$, since

$$\left|\int g \, d\mu\right| \leqq \|g\|_{A(R)} \|\hat{\mu}\|_\infty \text{ for } g \in A(R), \qquad \mu \in M(R).$$

Thus Theorem 1 follows from Theorem 2.

This work was done while trying to answer a question attributed to N. N. Lusin ([**1**, Volume 1, p. 330] or [**2**, p. 168]), which concerns homeomorphisms instead of measure-preserving transformations:

> Is it true that for every continuous function f on the circle group T there is a homeomorphism φ from T onto T such that $f \circ \varphi \in A(T)$?

Kahane's result, cited above, is that with Z in the role of T, the answer is no. The answer for T or R is not known. For related work see [**5**] or [**6**, VII. 9], and [**7**] and [**8**].

We do not know how to prove a satisfactory analogue of Theorem 2 for the case of the circle group. We offer the following conjecture: For every $s>0$, there exist small positive numbers $\alpha=\alpha(s)$ and $\varepsilon=\varepsilon(s)$ such that if f is a measurable function on T and if

$$|\{x : \varepsilon < |f(x)| < 1\}| < \alpha \cdot \min\{|\{x : |f(x)| \geqq 1\}|, |f^{-1}(0)|\},$$

then there is a discrete measure $\mu \in M(T)$ such that $\|\hat{\mu}\|_\infty \leqq 1$ and $|\int f\,d\mu| > s$. It would follow from this result, of course, that there is a continuous function on T of which no measure-preserving rearrangement is in $A(T)$.

It remains to prove Theorem 2. The next two results are from [**3**], and we omit the proof of the first one.

LEMMA 1. *Let $m_1, \cdots, m_r$ be integers, and let $z_1, \cdots, z_r$ be numbers of modulus* 1, *where $r \geqq 3$. If g is a trigonometric polynomial, $|g(x)| \leqq 1$ for all real x, and*

$$G(x) = g(x)\left\{1 - 2r^{-2} - r^{-3}\sum_{i<j} \bar{z}_i z_j e(m_i x - m_j x)\right\} + r^{-5/2}\sum_j \bar{z}_j e(m_j x)$$

(*where $e(t)$ means $e^{2\pi i t}$*), *then $|G(x)| \leqq 1$ for all real x.*

LEMMA 2. *Let P, Q and q be sets of integers, $Q\cap q=\varnothing$, $Q=\{n_j\}_{j=1}^N$, $n_1>n_2>\cdots>n_N$. For $p\in P$, let $N(p)$ be the number of integers in $Q\cup q$ that are greater than or equal to p. Let r be an integer such that*

$$r+\frac{r(r-1)}{2}\sum_{p\in P}N(p)<N. \tag{2}$$

Then there is a subset $\{m_j\}_{j=1}^r$ of Q such that $m_1>m_2>\cdots>m_r$,

$$p+m_i-m_j\notin Q\cup q \quad \text{if } p\in P \text{ and } i<j, \tag{3}$$

$$m_j=n_{t(j)} \quad \text{where } t(j)\leqq j+\frac{j(j-1)}{2}\sum_{p\in P}N(p). \tag{4}$$

PROOF. The m_j's may be chosen inductively. Let $m_1=n_1$. Having chosen m_{j-1}, let m_j be the largest integer in Q that is less than m_{j-1} and satisfies (3). Condition (3) rules out at most $(j-1)\sum_{p\in P}N(p)$ integers, and therefore

$$t(j)-t(j-1)\leqq 1+(j-1)\sum_{p\in P}N(p).$$

Statement (4) follows. Condition (2) assures that the process may be repeated r times.

LEMMA 3. *For every positive number s, there exist small positive numbers $a=a(s)$ and $\varepsilon=\varepsilon(s)$ such that, for all sufficiently large integers N, the following conditions hold. Let Q and q be disjoint sets of integers, Q containing N elements, q containing no more than aN elements. Let c be a function on Z such that $|c(n)|\geqq 1$ for $n\in Q$ and $|c(n)|<\varepsilon$ for $n\notin Q\cup q$. Then there exists a trigonometric polynomial g such that $\|g\|_{L^\infty(T)}\leqq 1$ and $|\sum_{n\in Z}c(n)\hat{g}(n)|>s$.*

PROOF. Let r be an integer, $\sqrt{r}>5s$. Choose a and ε so that

$$0<a<r^{-3r^2-2}, \qquad 0<\varepsilon<\sqrt{r}/(20\cdot 3^r). \tag{5}$$

It suffices to find a polynomial g with these properties:

(i) $\|g\|_{L^\infty(T)}\leqq 1$,

(ii) $\hat{g}(n)=0$ for $n\in q$,

(iii) $\sum\{|\hat{g}(n)|:n\notin Q\cup q\}<3^r$,

(iv) $\sum_{n\in Q}c(n)\hat{g}(n)>\sqrt{r}/4$.

It follows from the last three conditions that

$$\left|\sum_{n\in Z}c(n)\hat{g}(n)\right|>(\sqrt{r}/4)-\varepsilon 3^r>\sqrt{r}/5>s.$$

Require $N>a^{-1}$. Let Q be enumerated: $n_1>n_2>\cdots>n_N$. We shall construct a sequence of polynomials g_k, all satisfying conditions (i) and (ii), beginning with $g_0(x)=|c(n_1)|e(n_1x)/c(n_1)$. Finally, we shall let g be g_k for a suitable value of k (namely, $k=r^2$). Suppose that g_{k-1} has been defined. Let P_{k-1} be the set of its frequencies:

$$g_{k-1}(x) = \sum_{p\in P_{k-1}} \hat{g}_{k-1}(p)e(px).$$

If

$$r + \frac{r(r-1)}{2} \sum_{p\in P_{k-1}} N(p) < N, \tag{6}$$

then Lemma 2, with P_{k-1} in the role of P, may be applied to obtain a set $\{m_{ki}\}_{i=1}^r\subset Q$. Let $z_i=c(m_{ki})/|c(m_{ki})|$ and let

$$\begin{aligned} g_k(x) = g_{k-1}(x)&\left\{1 - 2r^{-2} - r^{-3}\sum_{i<j} \bar{z}_i z_j e(m_{ki}x - m_{kj}x)\right\} \\ &+ r^{-5/2}\sum_j \bar{z}_j e(m_{kj}x). \end{aligned} \tag{7}$$

By Lemma 1, g_k is bounded by one since g_{k-1} is. The frequencies of g_k are the integers in the set

$$P_k = P_{k-1} \cup (P_{k-1} + \{m_{ki} - m_{kj}: i<j\}) \cup \{m_{ki}\},$$

and hence

$$\begin{aligned} \sum_{p\in P_k} N(p) &\leqq \sum_{p\in P_{k-1}} N(p) + \sum_{i<j}\sum_{p\in P_{k-1}} N(p) + \sum_{j=1}^{r}(t(j) + aN) \\ &\leqq \left(1 + \frac{r(r-1)}{2}\right)\sum_{p\in P_{k-1}} N(p) \\ &\quad + \sum_j \left[j + \frac{j(j-1)}{2}\sum_{p\in P_{k-1}} N(p) + aN\right] \\ &\leqq \frac{r(r+1)}{2} + \left[1 + \frac{r(r-1)}{2} + \frac{r(r+1)(2r+1)}{12} - \frac{r(r+1)}{4}\right] \\ &\quad \sum_{p\in P_{k-1}} N(p) + raN \\ &< (r^3/2)\sum_{p\in P_{k-1}} N(p) + raN. \end{aligned}$$

Since $P_0=\{n_1\}$ and $N(n_1)\leqq aN$, by induction we obtain that $\sum_{p\in P_k} N(p)< r^{3k}aN$. By the restriction (5) on the choice of a, and for all $N>a^{-1}$, (6) is satisfied for $k\leqq r^2$. Let

$$I_k = \sum_{n\in Q} c(n)\hat{g}_k(n).$$

Then $I_0=1$ and $I_k \geqq (1-2r^{-2})I_{k-1}+r^{-3/2}$. By induction,

$$I_k \geqq (\sqrt{r}/2) - (1 - 2r^{-2})^k((\sqrt{r}/2) - 1).$$

Therefore when $k=r^2$, $I_k \geqq (\sqrt{r}/2)(1-e^{-2}) > \sqrt{r}/4$, so that (iv) is established for $g=g_k$. For every k,

$$\begin{aligned}\sum_{n\in Z} |\hat{g}_k(n)| &\leqq \sum |\hat{g}_{k-1}(n)|\,(1 - 2r^{-2} + r(r-1)/2r^3) + r^{-3/2} \\ &\leqq \sum |\hat{g}_{k-1}(n)|\,(1 + 1/r) \\ &< (1 + 1/r)^k.\end{aligned}$$

When $k=r^2$, this quantity is still less than 3^r, and (iii) follows for $g=g_k$. Lemma 3 is proved.

PROOF OF THEOREM 2. Given s, let a and ε be obtained as in Lemma 3, and let $\alpha=a/4$. Let $F=\{x:\varepsilon<|f(x)|<1\}$, $E=\{x:|f(x)|\geqq 1\}$, and suppose that $|F|<(a/4)|E|$. We must show the existence of a suitable μ.

Consider first the case when $|\{x\in E\cup F:|x|>b\}|=0$ for some finite b. Both the hypothesis, that $|F|<(a/4)|E|$, and the desired conclusion are invariant under the change from $f(x)$ to $f(2bx-b)$, and therefore we may suppose without loss of generality that $E\cup F\subseteq(0, 1]$. Let $\eta>0$. There is a set $U\subset(0, 1]$ which is the union of a finite number of open intervals, and such that the measure of the symmetric difference $E\nabla U$ is less than η. If J is sufficiently large, then

$$\left|\frac{1}{J}\sum_{j=0}^{J-1}\chi_U(x + j/J) - |U|\right| < \eta \quad \text{for all } x \in [0, 1/J].$$

For arbitrary J,

$$\int_0^1 \chi_{E\nabla U}(x)\,dx = \int_0^{1/J}\sum_{j=0}^{J-1}\chi_{E\nabla U}(x + j/J)\,dx = |E\nabla U| < \eta$$

and

$$\int_0^1 \chi_F(x)\,dx = \int_0^{1/J}\sum_{j=0}^{J-1}\chi_F(x + j/J)\,dx = |F| < (a/4)\,|E|.$$

Therefore there exists an x such that both

$$\sum_{j=0}^{J-1}\chi_{E\nabla U}(x + 1/j) < 2J\eta \quad \text{and} \quad \sum_{j=0}^{J-1}\chi_F(x + j/J) < J(a/2)\,|E|.$$

Therefore for every sufficiently large J, there is an x such that

$$\left|\frac{1}{J}\sum_{j=0}^{J-1}\chi_E(x + j/J) - |E|\right| < 3\eta \tag{8}$$

and

$$\left|\frac{1}{J}\sum_{j=0}^{J-1}\chi_F(x+j/J)\right| < (a/2)\,|E|. \tag{9}$$

Let $Q=\{j:0\leqq j<J$ and $x+j/J\in E\}$. Then Q has N elements, where $N>J(|E|-3\eta)$, so that by taking J sufficiently large, we can make N sufficiently large in the sense of Lemma 3. By choosing η sufficiently small and using (8) and (9), we may ensure that the set $q=\{j:0\leqq j<J$ and $x+j/J\in F\}$ has fewer than aN elements. By Lemma 3, there is a polynomial $g(t)=\sum_j \hat{g}(j)e(jt)$ such that $|\sum_{j\in Z}\hat{g}(j)f(x+j/J)|>s$ and $|g(t)|\leqq 1$ for all t. Let μ be the measure that places mass $\hat{g}(j)$ at $x+j/J$. Then $|\int f\,d\mu|>s$ and $|\hat{\mu}(t)|=|\sum_j \hat{g}(j)e(-t(x+j/J))|=|g(-t/J)|\leqq 1$. Theorem 2 is proved in the case when $|(E\cup F)\backslash[-b,b]|=0$ for some b, and in particular for all f with compact support.

Now to prove the theorem in the case of arbitrary f, let E and F be defined as before. Given s, let α and ε be such that, whenever $|\{x:\varepsilon<|g(x)|<1\}|<\alpha|\{x:|g(x)|\geqq 1\}|$ and g is measurable and has compact support, then there is a measure ν with finite support such that $\|\hat{\nu}\|_\infty\leqq 1$ and $|\int g\,d\nu|>3s$. Suppose now that $|F|<\alpha|E|$. For $c>0$, let $V=V_c$ be the function in $A(R)$ defined so that $V(x)=1$ for $|x|\leqq c$, $V(x)=0$ for $|x|\geqq 2c$, and $V(x)$ is linear on $[-2c,-c]$ and on $[c,2c]$. Then $\|V\|_{A(R)}<3$. For c sufficiently large,

$$|\{x:\varepsilon<|V(x)f(x)|<1\}|<\alpha\,|\{x:|V(x)f(x)|\geqq 1\}|,$$

and therefore there is a measure ν with finite support Y such that $\|\hat{\nu}\|_\infty\leqq 1$ and $|\int Vf\,d\nu|>3s$. Let $A(Y)$ denote the algebra of restrictions to Y of elements of $A(R)$, with norm

$$\|g\|_{A(Y)}=\sup\left\{\left|\int g\,d\mu\right|:\mu\in M(Y),\ \|\hat{\mu}\|_\infty\leqq 1\right\}.$$

Thus $\|Vf\|_{A(Y)}>3s$. But $\|Vf\|_{A(Y)}\leqq 3\|f\|_{A(Y)}$. Hence $\|f\|_{A(Y)}>s$, so that there is a measure $\mu\in M(Y)$ such that $\|\mu\|_\infty\leqq 1$ and $|\int f\,d\mu|>s$.

Theorem 2 is proved.

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