99. *Fourier Transforms on the Cartan Motion Group*

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The purpose of the present paper is to characterize the images of some function spaces on the Cartan motion group by the Fourier transform.

1. Preliminaries. Let G_0 be a connected non-compact semisimple Lie group with finite centre and $\mathfrak{g}$ be its Lie algebra. We fix a maximal compact subgroup K of G_0. Let $\mathfrak{g}=\mathfrak{k}+\mathfrak{p}$ be the Cartan decomposition of $\mathfrak{g}$, where $\mathfrak{k}$ is the subalgebra corresponding to K. Then K operates on $\mathfrak{p}$ via the adjoint representation. Let G be the semidirect product of $\mathfrak{p}$ and K. The group G is called the Cartan motion group.

Let $\hat{\mathfrak{p}}$ be the dual space of $\mathfrak{p}$. Then K operates also on $\hat{\mathfrak{p}}$ via the contragredient representation of Ad, $\langle k\cdot\xi, X\rangle=\langle\xi, Ad(k)^{-1}X\rangle$ ($k\in K$, $\xi\in\hat{\mathfrak{p}}$ and $X\in\mathfrak{p}$). For any $\xi\in\hat{\mathfrak{p}}$ we can associate an irreducible unitary representation of $\mathfrak{p}$ by $X\to e^{i\langle\xi,X\rangle}$. We also denote it by ξ. We denote by U^{ξ} the unitary representation of G induced by $\xi\in\hat{\mathfrak{p}}$. Since the Killing form B on $\mathfrak{g}$ is positive definite on $\mathfrak{p}$, we can identify $\hat{\mathfrak{p}}$ with $\mathfrak{p}$. We denote by ξ_X the corresponding element in $\hat{\mathfrak{p}}$ to $X\in\mathfrak{p}$.

Let dk be the normalized Haar measure on K. Let $\mathfrak{H}=L^2(K)$. We denote by $\mathbf{B}(\mathfrak{H})$ the Banach space of all bounded linear operators on $\mathfrak{H}$. In $\mathfrak{p}$ and $\hat{\mathfrak{p}}$ we can define K-invariant measures which are induced by B. We normalize these measures by multiplying $(2\pi)^{-n/2}$ ($n=\dim\mathfrak{p}$) and denote them by dX and $d\xi$, respectively. We normalize the Haar measure dg on G such as $dg=dXdk$. For any $f\in L^1(G)$ we put

$$T_f(\xi)=\int_G f(g)U_g^{\xi}dg.$$

Then T_f is a $\mathbf{B}(\mathfrak{H})$-valued function on $\hat{\mathfrak{p}}$. It is called the Fourier transform of f.

2. Plancherel formula. Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{g}$ contained in $\mathfrak{p}$. Fixing a lexicographic order in the dual space of $\mathfrak{a}$, we denote by P_+ the set of all positive restricted roots of the pair $(\mathfrak{g},\mathfrak{a})$. Let $\mathfrak{a}^+$ be the positive Weyl chamber in $\mathfrak{a}$. Since the Killing form B is positive definite on $\mathfrak{a}$, B gives rise to an euclidean measure dH on $\mathfrak{a}$. Let M be the centralizer of $\mathfrak{a}$ in K. We denote by dk_M the K-invariant measure on K/M induced by $-B$. We put $vol(K/M)=\int_{K/M}dk_M$. Let

$C_c^\infty(G)$ be the space of all infinitely differentiable functions on G with compact support. Then we have the following Plancherel formula.

Theorem 1. *For any* $f \in C_c^\infty(G)$

$$\int_G |f(g)|^2 dg = (2\pi)^{-n/2} vol(K/M) \int_{\mathfrak{a}+} \| T_f(\xi_H) \|_{HS}^2 \prod_{\alpha \in P_+} |\alpha(H)| \, dH,$$

where $\| \ \|_{HS}$ *denotes the Hilbert-Schmidt norm.*

3. Paley-Wiener theorem. Let $U(\mathfrak{k}^c)$ be the universal enveloping algebra of the complexification $\mathfrak{k}^c$ of $\mathfrak{k}$. We regard any element $y \in U(\mathfrak{k}^c)$ as the right invariant differential operator on K. Then y operates on $\mathfrak{H}$ in the sense of the distributions. Let Δ be the Casimir operator of $\mathfrak{k}$. We denote by R the right regular representation of K. Let us define a compact set $\Omega(a)$ of G for any positive number a by $\Omega(a) = \{(X, k) \in G ; B(X, X)^{1/2} \leqq a\}$. We denote by $\mathbf{N}$ the set of all non-negative integers. Then we have the following Paley-Wiener theorem.

Theorem 2. *A* $\mathbf{B}(\mathfrak{H})$*-valued function* T *on* $\hat{\mathfrak{p}}$ *is the Fourier transform of* $f \in C_c^\infty(G)$ *such that* $\operatorname{supp}(f) \subset \Omega(a)$ $(a > 0)$ *if and only if it satisfies the following conditions:*

(I) *T can be extended to an entire analytic function on the complexification $\hat{\mathfrak{p}}^c$ of $\hat{\mathfrak{p}}$.*

(II) *For any K-invariant polynomial function p on $\hat{\mathfrak{p}}^c$ and for any $l, m \in \mathbf{N}$ there exists a constant $C_p^{l,m}$ such that*

$$\| p(\zeta) \Delta^l T(\zeta) \Delta^m \| \leqq C_p^{l,m} \exp a |\operatorname{Im} \zeta| \qquad (\zeta \in \hat{\mathfrak{p}}^c).$$

(III) *For any $k \in K$*

$$T(k \cdot \zeta) = R_k T(\zeta) R_k^{-1} \qquad (\zeta \in \hat{\mathfrak{p}}^c).$$

4. Fourier transforms of rapidly decreasing functions. Let $Y_1, \cdots, Y_\delta$ ($\delta = \dim K$) be a fixed basis of $\mathfrak{k}$. Then the set $\{y(m) = Y_1^{m_1} \cdots Y_\delta^{m_\delta} ; m = (m_1, \cdots, m_\delta) \in \mathbf{N}^\delta\}$ forms a basis of $U(\mathfrak{k}^c)$ by the Birkhoff-Witt theorem. Let $X_1, \cdots, X_n$ be an orthonormal basis of $\mathfrak{p}$ with respect to B. And let $\Xi_1, \cdots, \Xi_n$ be its dual basis of $\hat{\mathfrak{p}}$. Making use of the coordinate systems with respect to these bases, we define differential operators D_X^α on $\mathfrak{p}$ and D_ξ^α on $\mathfrak{p}$ for any $\alpha = (\alpha_1, \cdots, \alpha_n) \in \mathbf{N}^n$ by $D_X^\alpha = \left(\frac{\partial}{\partial x_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial x_n}\right)^{\alpha_n}$ and $D_\xi^\alpha = \left(\frac{\partial}{\partial \xi_1}\right)^{\alpha_1} \cdots \left(\frac{\partial}{\partial \xi_n}\right)^{\alpha_n}$, respectively. We put $|X|^2 = B(X, X)^{1/2}$ for $X \in \mathfrak{p}$ and put $|\xi|^2 = B(X, X)^{1/2}$ for $\xi = \xi_X \in \hat{\mathfrak{p}}$. Let λ and μ be the left and right regular representations of G, respectively, and also we denote by the same notations the corresponding representations of the universal enveloping algebra on the space of C^∞-vectors.

Let $\mathcal{S} = \mathcal{S}(G)$ be the set of all those functions f on G satisfying the following conditions:

(i) f is of class C^∞,

(ii) for any $\alpha \in \mathbf{N}^n$, $\beta \in \mathbf{N}$ and $m, m' \in \mathbf{N}^\delta$ there exists a constant $C_{\alpha,\beta}^{m,m'}$ such that

$$|(1+|X|^2)^{\beta}(D_X^{\alpha}\lambda(y(m))\mu(y(m'))f)(X,k)| \leqq C_{\alpha,\beta}^{m,m'}$$

for all $(X,k) \in G$.

Such functions are called rapidly decreasing. We topologize $\mathcal{S}$ by the system of semi-norms of the form

$$\gamma_{\alpha,\beta}^{m,m'}(f) = \sup_{(X,k)\in G} |(1+|X|^2)^{\beta}(D_X^{\alpha}\lambda(y(m))\mu(y(m'))f)(X,k)|,$$

where $\alpha \in \mathbf{N}^n$, $\beta \in \mathbf{N}$ and $m, m' \in \mathbf{N}^s$. Let $\hat{\mathcal{S}}$ be the set of all $\mathbf{B}(\mathfrak{H})$-valued function T on $\hat{\mathfrak{p}}$ satisfying the following conditions:

(i)$^\wedge$ T is a $\mathbf{B}(\mathfrak{H})$-valued C^∞ function on $\hat{\mathfrak{p}}$,

(ii)$^\wedge$ for any $\alpha \in \mathbf{N}^n$, $\beta \in \mathbf{N}$ and $m, m' \in \mathbf{N}^s$ there exists a constant $\hat{C}_{\alpha,\beta}^{m,m'}$ such that

$$\|(1+|\xi|^2)^{\beta} y(m) D_{\xi}^{\alpha} T(\xi) y(m')\| \leqq \hat{C}_{\alpha,\beta}^{m,m'}$$

for all $\xi \in \hat{\mathfrak{p}}$,

(iii)$^\wedge$ for any $k \in K$

$$T(k\cdot\xi) = R_k T(\xi) R_k^{-1} \qquad (\xi \in \hat{\mathfrak{p}}).$$

We topologize $\hat{\mathcal{S}}$ by the system of semi-norms of the form

$$\hat{\gamma}_{\alpha,\beta}^{m,m'}(T) = \sup_{\xi\in\hat{\mathfrak{p}}} \|(1+|\xi|^2)^{\beta} y(m) D_{\xi}^{\alpha} T(\xi) y(m')\|,$$

where $\alpha \in \mathbf{N}^n$, $\beta \in \mathbf{N}$ and $m, m' \in \mathbf{N}^s$. Then we can prove that $\mathcal{S}$ and $\hat{\mathcal{S}}$ are Fréchet spaces.

Theorem 3. *The Fourier transform $f \rightarrow T_f$ is a topological isomorphism from $\mathcal{S}$ onto $\hat{\mathcal{S}}$.*

5. Appendix. If the group G_0 in § 1 is $SO_0(n,1)$, the Cartan motion group G is the euclidean motion group. K. Okamoto and the author proved Theorem 2 for the euclidean motion group in [1]. M. Sugiura proved Theorem 3 without topology for the euclidean motion group. The detailed proofs of the present paper will appear elsewhere.

Reference

[1] K. Kumahara and K. Okamoto: An analogue of the Paley-Wiener theorem for the euclidean motion group. Osaka J. Math., **10**, 77–92 (1973).